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Dynamical Stabilization of an Unstable Equilibrium in Chemical and Biological Systems

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Abstract—The dynamics of two-component diffusion-reaction systems is considered. Using wellknown models from population dynamics and chemical physics, it is shown that for certain parameter values the systems exhibit a rather unusual behaviour: a locally unstable equilibrium may become stable during a certain transition process. Both the analytical and numerical investigations of this phenomenon are presented in one and two spatial dimensions. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Diffusion-reaction systems have been attracting significant interest during the last decades because of their numerous applications in chemistry and chemical physics [1], biology [2], ecology [3,4], and many other scientific fields [5]. Particularly, instabilities and related transition processes are thought to be responsible for the formation of spatial "dissipative structures" in a chemical reactor [6], in a cell community (morphogenesis) [7], and in population dynamics [8].

Now days, a number of different theoretical tools for the investigation of pattern formation processes are known [9–11]. However, a widely used analytical technique is still based on partial differential equations. In many cases, the dynamics of a diffusion-reaction system is described by the following two equations:

$$\frac{\partial u(\mathbf{r},t)}{\partial t} = D_u \nabla^2 u(\mathbf{r},t) + f(u,v), \tag{1}$$

$$\frac{\partial v(\mathbf{r},t)}{\partial t} = D_v \nabla^2 v(\mathbf{r},t) + g(u,v), \qquad (2)$$

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where t is the time, **r** is the position, ∇^2 is the Laplace operator, and the functions f and g describe the local kinetics. Here and further on, we will refer to the dynamical variables u and v as the concentrations of the interacting components and to the coefficients D_u and D_v as the corresponding diffusivities. It should be noted, however, that the particular meaning of the quantities in equations (1),(2) can be somewhat different in different problems. While in chemical applications D_u and D_v are the usual molecular diffusivities, in problems of population dynamics these coefficients describe the intensity of mixing either due to the self-motion of animals [3,12] or due to turbulence, e.g., in case of plankton populations [3,13,14].

The dynamics of system (1),(2) is to a large extent controlled by the properties of the "reduced" system, i.e., equations (1),(2) without diffusion

$$\dot{u} = f(u, v), \qquad \dot{v} = g(u, v), \tag{3}$$

because of the evident relation between the stationary solutions of equations (3) and the homogeneous stationary states of the full system (1),(2). However, the dynamics of the full system is remarkably more rich. Since the famous paper of Turing [7], it is well known that a linearly stable stationary point of the reduced system (3) may become unstable in the full system (1),(2). Then, after the homogeneity is broken due to the linear Turing instability, the nonlinear interactions between the components drive the system into the formation of standing spatial patterns [6]. This is an irreversible process; i.e., the broken homogeneity is never restored unless the parameters of the system are changed so that, at least, the instability conditions are not met anymore.

In a somewhat more general sense, a kind of inverse process may occur. We show here that for certain parameter values, an "anti-Turing" phenomenon takes place: a locally unstable equilibrium of the system (3) can be made dynamically stable in the full diffusion-reaction system (1),(2). In this case, for certain times and lengths, the formation of spatial patterns is suppressed and the homogeneity is restored.

The structure of the paper is as follows. In the second section, an example of a biological system is given exhibiting the dynamical stabilization of an unstable equilibrium. Both analytical and numerical results are presented. In the third section, the results are extended to the 2-D case. In the fourth section, a chemical system described by the well-known Gray-Scott model is considered. It is shown that, in spite of significantly different local kinetics of the system, its spatiotemporal dynamics can also follow the dynamical stabilization scenario. In the last section, some open problems arising in connection with this new phenomenon, are discussed and an ecological example is given where the dynamical stabilization may be underlying the system dynamics.

2. A BIOLOGICAL SYSTEM: A PREY-PREDATOR COMMUNITY

Population dynamics is one of the fields of traditional and successful applications of diffusionreaction systems [2,4,8,14,15]. Although the spatial mixing of the system components, i.e., biological species in this case, is typically caused by the self-motion of the organisms or by the specific properties of the environment (e.g., marine turbulence in case of plankton systems) and not by diffusion in the usual physicochemical meaning, the mathematical description of the mixing stays much the same [3]. Choosing a proper parameterization for the biological "reactions", i.e., for the processes of replication, predation, and mortality, one can arrive at the following equations for the key species, cf. [2,4]:

$$\frac{\partial u}{\partial t} = \nabla^2 u + u(1-u) - \frac{u}{u+h}v,\tag{4}$$

$$\frac{\partial v}{\partial t} = \nabla^2 v + k \frac{u}{u+h} v - mv.$$
⁽⁵⁾

Dimensionless quantities have been introduced, and the details are omitted here in order to be brief; cf. [16]. $D_u = D_v = D$ is suggested for simplicity; however, the main results do not depend on this assumption.

A crucial point, affecting the type of the spatiotemporal system dynamics, is the choice of the initial conditions. Actually, the formation of the dissipative patterns resulting from Turing instability occurs when the initial distribution of the concentrations consists of the homogeneous stationary states *plus* a perturbation with certain wavelengths; otherwise no spatial structure will emerge. Another example is given by the propagation of diffusive fronts which is only possible in case of finite initial conditions. In this paper, the considerations are restricted to the case when at the beginning of the process the prey is spatially uniformly distributed at the level of the carrying capacity, $u(\mathbf{r}, t) \equiv 1$, whereas the predator is inhabiting a finite region. These initial conditions correspond to the problem of biological invasions [4].

Thus, we begin with 1-D problem described by the following equations:

$$u_t = u_{xx} + u(1-u) - \frac{u}{u+h}v,$$
(6)

$$v_t = v_{xx} + k \frac{u}{u+h} v - mv, \tag{7}$$

and

$$u(x,0) = 1, \quad \forall x; \qquad v(x,0) = V_0, \quad \text{if } |x| \le \frac{\Delta}{2}; \qquad v(x,0) = 0, \quad \text{if } |x| > \frac{\Delta}{2}.$$
 (8)

Since the dynamics of the system does not show any significant dependence on the form of the finite initial distribution of the predator, the simplest form of v(x, 0) is chosen here.

Another important point is the number and the type of stationary states of the reduced system. It is readily seen that the phase plane (u, v) of equations (4),(5) without diffusion terms has the following structure: under the condition h < (1 - p)/p, p = m/k, there are three stationary points in the physically meaningful region $u \ge 0$, $v \ge 0$, namely, (0,0) (trivial extinction), (1,0) (predator extinction), and (u_*, v_*) (coexistence), where

$$u_* = \frac{ph}{1-p}, \qquad v_* = (1-u_*)(h+u_*).$$
 (9)

The trivial solutions (0,0) and (1,0) are always saddle-points, whereas the nontrivial point (u_*, v_*) can be either focus or node, stable, or unstable, depending on the problem parameters; cf. Figure 1 and see [16,17] for more details.

The distinctions in the dynamics of system (6),(7) for different values of k, p, h can be associated with the change of the stability of the nontrivial stationary state (u_*, v_*) which takes place when

$$h = h_{\rm cr}(p) = \frac{1-p}{1+p},\tag{10}$$

which corresponds to Curve 2 in Figure 1. Note that the position of Curves 1 and 2 does not depend on k. For all parameter values when the state (u_*, v_*) is unstable, i.e., below Curve 2, it is surrounded by a stable limit cycle.

The full account of possible dynamical regimes of system (6),(7) with finite initial conditions can be found in [17] or in [18] for the case of a somewhat different parameterization of the biological processes. Typically, the solution of problem (6)–(8) evolves, after a certain transition time, to the propagation of stationary diffusive fronts. In case the stationary state (u_*, v_*) is locally stable, the dynamics of these fronts is something one can expect: they "switch" the system from the homogeneous stationary unstable state $u \equiv 1$, $v \equiv 0$ (after some damped oscillations if (u_*, v_*) is a stable focus [2,19]) to the homogeneous stable state $u \equiv u_*, v \equiv v_*$.

However, the situation becomes less expected when the point (u_*, v_*) becomes unstable, i.e., when the parameters cross the Hopf bifurcation curve $h = h_{cr}(p)$ in the plane (p, h). The results



Figure 1. A map in the (p, h) parametric plane of the prey-predator system (4), (5): the nontrivial "coexistence" state exists for the parameters below Curve 1 and loses its stability when crossing Curve 2; the critical relation (17) for the dynamical stabilization is shown by Curve 3 for k = 0.1, Curve 4 for k = 0.4, and Curve 5 for k = 1.2.

of numerical experiments show that in this case, for parameter values "not very far" from the critical relation (10), the diffusive fronts still work as switching waves but switching the system to the state $u \equiv u_*, v \equiv v_*$ which is now locally unstable. Typical wave profiles are presented in Figure 2 (since the problem (6)-(8) is symmetrical with respect to x = 0, only half of the numerical domain is shown).

One can see that, after rather strong oscillations at the front of the wave, there comes the region (from approximately x = 200 to x = 420 in Figure 2, top, and from x = 400 to x = 900 in Figure 2, bottom) where the concentrations u(x,t) and v(x,t) nearly reach their stationary (but unstable!) values u_* , v_* —the dynamical stabilization takes place. We want to stress, based on our numerical results, that this unstable "plateau" exists during a remarkably long time before it is finally displaced by the irregular spatiotemporal oscillations [16]. Moreover, the length of the plateau grows with time; cf. top and bottom of Figure 2.

In order to better understand this phenomenon, the following points must be addressed.

- (i) For which restrictions on the parameter values can the dynamical stabilization of the unstable equilibrium occur? There must be some restrictions because the stabilization does not appear for arbitrary sets of parameters.
- (ii) How does the length of the plateau change (increase) with time?

The first of these problems has been considered in [17], where the conditions for dynamical stabilization were related to the change of the type of the nontrivial stationary state in fourdimensional phase space generated by system (6),(7) in case of stationary wave propagation. However, the results obtained in this way do not allow us to make any estimates concerning the length of the plateau. Besides, the approach developed in [17] somewhat lacks physical lucidity, and that might make the interpretation of the results difficult. In this paper, another physically clear approach to deal with problems (i),(ii) is proposed. Both the restrictions on the problem parameters and the equation describing the growth of the plateau length with time will appear quite naturally as a result of the comparison between the speed of different diffusive fronts.



Figure 2. The profiles of the concentration of prey u (Curve 1) and predator v (Curve 2) calculated at t = 600 (top) and t = 1200 (bottom) for parameters k = 0.4, p = 0.15, and h = 0.6 and the initial conditions (8) with $V_0 = 1.2$ and $\Delta = 100$. The arrows show the direction of the diffusive front propagation. The region of homogeneity in the middle corresponds to the locally unstable "coexistence" state.

The idea of the method is as follows. Results of computer simulations [17,18,20] show that, generally, the stabilization of a locally unstable state occurs behind the stationary diffusive fronts (which may have an oscillating "structure", cf. Figure 2), travelling with a certain constant speed c. The minimal possible value c_{\min} can be obtained considering the properties of the problem solution in the vicinity of the steady (1,0) state in R^4 phase space [2,17],

$$c_{\min} = \left[2\left(T_0 + \sqrt{\Delta_0}\right)\right]^{1/2},\tag{11}$$

with $\Delta_0 = T_0^2 - 4\delta_0$. T_0 and δ_0 are trace and determinant of the Jacobi matrix, respectively. We want to mention that, although c_{\min} is only the exact lower bound of the possible values of the speed of the front and not its actual value, it provides a good estimate for the actual value c. Moreover, it holds quite often $c = c_{\min}$ [2].

Furthermore, if (u_*, v_*) is unstable, the propagation of the diffusive fronts, no matter if with or without the dynamical stabilization in the wake, is followed by a region occupied by spatiotemporal oscillations [16,17,20–22], typically irregular. The remarkable thing is that in case of the formation of the unstable plateau, there exists a distinct boundary or interface at any time, separating the plateau from the region with irregular oscillations [16,22]. Our numerical results show that the size of the region always grows with time, and the interface propagates with a constant speed. Considering the travelling wave solutions of equations (6),(7) far ahead of the interface, where they can be regarded as small perturbations of the stationary state $u \equiv u_*, v \equiv v_*$, we arrive at the following estimate for the speed w of the interface [22]:

$$w_{\min} = \sqrt{2T_1} \tag{12}$$

in case of an unstable focus, and

$$w_{\min} = \left[2\left(T_1 + \sqrt{\Delta_1}\right)\right]^{1/2} \tag{13}$$

in case of an unstable node. Here $\Delta_1 = T_1^2 - 4\delta_1$ where T_1 and δ_1 are the trace and the determinant of the matrix of the system in the vicinity of the coexistence state (u_*, v_*) . Again, although equations (12),(13) give only the minimal possible value of the speed, it is in an excellent agreement with numerical results [22].

Numerical results indicate that the dynamical stabilization is unlikely to be observed if (u_*, v_*) is an unstable node; this is also in agreement with the results of the bifurcation analysis [17]. Hence, the further considerations treat the case (u_*, v_*) as an unstable focus.

Now, the domain where the dynamical stabilization may take place is bounded by two moving boundaries: the leading edge propagating with a constant speed c and the interface between the plateau and the region of irregular oscillations propagating with a speed w. The development of the plateau is thus controlled by the relation between the values of c and w. Obviously, in case w < c, the length of the domain grows with time as (c - w)t. Let us note, however, that since the leading front behaves as a stationary travelling wave, its form and "width" (i.e., the size of the region occupied by the regular damping oscillations, cf. Figure 2 between x = 700 and x = 850) do not change with time. Then, the increase of the length of the domain locked between the two moving fronts can only mean the increase of the length of the plateau. Thus, we obtain

$$L_{\text{plateau}} = (c - w)t + L_0, \tag{14}$$

where L_0 is a constant.

On the other hand, in case w > c the dynamical stabilization can hardly be observed. The length of the plateau, if it happens to appear as a result of certain specific initial conditions, would decrease with time until, finally, the region of spatiotemporal oscillations would start immediately after the stationary travelling front; cf. Figure 3. Unlike the case shown in Figure 2, the "nucleus" of the unstable plateau which can be seen just behind the damping oscillations at the front does not grow with time.

Thus, we arrive at a simple necessary condition for the dynamical stabilization

$$w < c. \tag{15}$$

However, relations (14) and (15) are still not very useful because the actual values of speed are not known. Assuming that the fronts propagate with the minimal possible speed, as it usually



Figure 3. The profiles of the concentration of prey u (Curve 1) and predator v (Curve 2) calculated at t = 1000 for parameters k = 0.4, p = 0.5, and h = 0.25. No plateau is formed.

takes place, and taking into account equations (11) and (12), from (15) we obtain the following critical relation between the problem parameters:

$$T_1 = T_0 + \sqrt{\Delta_0}.\tag{16}$$

If the nonlinearities in the equations are chosen as in equations (6),(7), one can easily obtain that $T_1 = -2\delta_0$ (note that $\delta_0 < 0$ because the state "prey only" is a saddle-point) and, finally,

$$\frac{p}{1-p}\left[(1-h) - p(1+h)\right] = -2k\left(p - \frac{1}{1+h}\right).$$
(17)

The critical relation (17) is shown in Figure 1 by the dashed line for the values of parameter k = 0.1 (Curve 3), k = 0.4 (Curve 4), and k = 1.2 (Curve 5). The domain in the (p, h) parameter plane where one can expect the dynamical stabilization of an unstable equilibrium in the wake of the travelling diffusive front is on the left-hand side of the dashed line and below the Hopf bifurcation Curve 2.

3. DYNAMICAL STABILIZATION IN TWO SPATIAL DIMENSIONS

The results of the previous section were obtained for a spatially one-dimensional diffusionreaction system. However, the dynamics of natural systems is usually higher dimensional. In this connection, and also in order to show that the dynamical stabilization is not an exotic but a rather typical phenomenon in diffusion-reaction systems, it seems important to know whether the previous results can be extended to a more-dimensional case.

Note that, strictly speaking, this extention is not a formal routine and the results can hardly be foreseen. The matter is that the increase of the number of the spatial dimensions not only makes the dynamics of the system more complex, but may lead to suppression of the regimes which would be dominant in the system with fewer dimensions. This is just the situation described above: the increase of the number of spatial dimensions from 0 (cf. equations (3)) to 1 (equations (6),(7)) makes the unstable equilibrium dynamically stable.

Let us mention that, in some cases, the dimensionality of the system dynamics depends on the scale of the processes under consideration. For instance, the spatiotemporal functioning of a plankton community is three dimensional if considered on scale $L \leq L_0$ where L_0 is the thickness of the upper productive ocean layer, but becomes effectively two dimensional on scale $L \gg L_0$.

Taking into account that the 3-D case is much more complicated for computer simulations and for the visualization of the results, we restrict ourselves to the 2-D case here. Figure 4 shows the snapshots of the prey spatial distribution (the distribution of predator is qualitatively similar) obtained numerically for a prey-predator community described by equations (4),(5) where now u = u(x, y, t), v = v(x, y, t), and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The growing inner grey ring is easily recognized as the dynamically stabilized unstable coexistence region.

Thus, the phenomenon of the dynamical stabilization of an unstable equilibrium exists also in a 2-D diffusion-reaction system. As it was in 1-D case, the size of the unstable plateau grows with time. However, to the case of cylindrical diffusive fronts, neither condition (14) nor (15) is not directly applicable because they were obtained for the plane waves. Particularly, condition (14) now gives only rough estimates of the parameter values where the dynamical stabilization may be observed.



Figure 4. The 2-D spatial distribution of prey calculated at equidistant moments of time, parameter values are the same as in Figure 2. Black color corresponds to the no-species state, while in white areas the prey is at its carrying capacity. The grey color corresponds to the unstable coexistence state.

4. A CHEMICAL SYSTEM: THE GRAY-SCOTT MODEL

In the previous sections, the possibility of the dynamical stabilization of an unstable equilibrium has been shown by computer experiments. Furthermore, the restrictions on the parameter values necessary for the stabilization have been obtained analytically. It has been demonstrated that the phenomenon is not sensitive to the dimensionality of the system and can be observed in 1-D and 2-D cases.

A prey-predator system has been chosen as local kinetics to demonstrate the stabilization by computer experiments. Now, another question naturally arises: how strongly does the existence of this phenomenon depend on the type of the local kinetics? Although it is shown in [17,18] that the dynamical stabilization is robust with respect to some variations in the form of the nonlinearities in equations (1),(2) (cf. also [20]), a certain doubt may still exist. The matter is that any realistic parameterization of the prey-predator interactions, in spite of the details, should allow for, at least, two principal features:

- (i) f(u = 0, v) = 0, g(u, v = 0) = 0, and
- (ii) for large values of u the predation must show a tendency to saturation; cf. equations (4),(5).

These features impose certain constraints both on the structure of the phase plane of the reduced system (3), and on the spatiotemporal dynamics of the full system (1),(2).

To check the robustness of the results with respect to the type of local interactions, another field of application of equations (1),(2) is considered. A system of two chemical reactants is free from the above limitations and can possess quite different local kinetics. As a particular example, the well-known Gray-Scott model [1] is chosen, describing an autocatalytic reaction in an open 1-D flow reactor,

$$u_t = u_{xx} + F(1 - u) - uv^2, (18)$$

$$v_t = v_{xx} + uv^2 - (F+k)v,$$
(19)

with accordingly chosen dimensionless variables; see [23] for details. Here u(x,t) and v(x,t) are the concentration of the substrate and the autocatalyst, respectively, F is the flow rate, and k is the effective rate constant for the decay of the autocatalyst. As in previous cases, we assume $D_u = D_v$ for simplicity.

Equations (18),(19) have been investigated in many papers; e.g., see [24,25]. A very brief summary of the results is only given here as far as they will be needed for the further considerations. One can easily see that under the limitation $d = 1 - 4(F + k)^2/F > 0$, there are three stationary points: "substrate only" (1,0) and two nontrivial coexistence states (u_s, v_s) ("substrate dominated") and (u_a, v_a) ("autocatalyst dominated") where

$$u_s = \frac{1+\sqrt{d}}{2}, \qquad v_s = \left(\frac{F}{F+k}\right)\frac{1-\sqrt{d}}{2},\tag{20}$$

$$u_a = \frac{1 - \sqrt{d}}{2}, \qquad v_a = \left(\frac{F}{F + k}\right) \frac{1 + \sqrt{d}}{2}.$$
 (21)

When crossing the critical curve d = 0 in the (k, F)-plane (Curve 1 in Figure 5) towards smaller values of k (i.e., from right to left), the two nontrivial states appear through a saddle-node bifurcation, the "autocatalyst dominated" state being an unstable node.

The "substrate only" state is always stable and the "substrate dominated" state is always unstable. A change in the local dynamics can be associated with the change of the type of the "autocatalyst dominated" state, first of all, with the change of its stability which takes place when

$$\frac{k-F}{k+F} = \sqrt{d},\tag{22}$$



Figure 5. A map in the parametric (k, F) plane. Curve 1 bounds the domain where the two nontrivial states exist, Curve 2 is the Hopf bifurcation line, and Curve 3 corresponds to the critical relation for the dynamical stabilization.

i.e., when crossing Curve 2 in Figure 5. The Hopf bifurcation, which takes place when crossing the Curve (22), is supercritical for $k < k_{cr}$ (where k_{cr} is estimated as about 0.035, cf. [24]) and only in this case a stable limit cycle appears. Otherwise, i.e., for $k_{cr} < k < 0.625$, no limit cycle arises and any trajectory starting in the vicinity of the "autocatalyst dominated" state after a number of expanding convolutions is finally attracted to the "substrate only" state.

This brief consideration of the local dynamics of the Gray-Scott model provides "input" information for the investigation of the spatiotemporal dynamics of the distributed system (18),(19). Namely, accounting for the results of the previous sections, it is now obvious that, as far as one is concerned with the possibility to observe the dynamical stabilization, the values of F and kin equations (18),(19) should be chosen from the domain between the Hopf bifurcation curve and the saddle-node bifurcation curve where the "autocatalyst dominated" state is unstable, i.e., from the narrow strip between Curves 1 and 2 in Figure 5.

Thus, the structure of the local phase plane of the Gray-Scott model is essentially different from the one of the prey-predator system (4),(5). Quite naturally, it results in a significantly different behaviour of the diffusive fronts; cf. [17,18,23–25]. And this difference makes it probably even more remarkable that the phenomenon of dynamical stabilization also occurs in the system described by equations (18),(19). Figure 6 shows the profiles of the concentrations u and v at t = 3200 calculated for the initial conditions (8) with $\Delta = 400$ and $V_0 = 0.1$ for parameter values F = 0.015 and k = 0.04. Only half of the domain is shown.

Again, after promptly damping oscillations behind the leading edge, there comes a region where substrate and autocatalyst are distributed homogeneously at the level corresponding to the locally unstable state $u_a = 0.28$, $v_a = 0.196$. To stress the distinctions from the cases considered in Sections 2 and 3, it should be noted that the unstable focus (u_a, v_a) is not surrounded by a stable limit cycle now. Moreover, since the "substrate only" state is stable for all parameter values, the propagation of the diffusive front followed by the dynamical stabilization, with the length of the unstable plateau growing with time, means "switching" the system from the stable state to an unstable one, a situation that seems quite exotic if considered in a general physical context.



Figure 6. The profiles of the concentration of the substrate u (Curve 1) and the autocatalyst v (Curve 2) calculated at t = 3200 for parameters F = 0.015 and k = 0.04. The plateau behind the oscillating front appears as a result of the dynamical stabilization of the unstable "autocatalyst dominated" state.

The next step to be done is to examine the restrictions on the problem parameters F and k resulting from the conditions for dynamical stabilization; cf. equations (15) and (16). It is readily seen that for the Gray-Scott model (18),(19) $T_0 + \sqrt{\Delta_0} = -2F$, whereas $T_1 = (F + k) - Fu_a^{-1}$. Accounting for (21), equation (16) takes the following form:

$$\frac{F+k}{3F+k} = \sqrt{d}.$$
(23)

The critical relation (23) is shown in Figure 5 by Curve 3. Note that Curve 3 is situated in the domain where the "autocatalyst dominated" state is stable. Thus, in this case the necessary condition (15) of the dynamical stabilization is fulfilled for all parameter values from the domain where this state is unstable (cf. the strip between Curves 1 and 2) and does not bring any additional restraints on the values of the problem parameters.

5. DISCUSSION AND CONCLUSIONS

In this paper, a new facet of the dynamics of a two-component diffusion-reaction system has been demonstrated: the dynamical stabilization of an unstable equilibrium resulting in the formation of a homogeneous spatial distribution of the interacting components, a "plateau" at the level corresponding to a locally unstable steady state. It has been shown that this is not an exotic but a rather typical phenomenon occurring both in one and two spatial dimensions and in systems with essentially different local kinetics, i.e., with different structure of the local phase plane. The size of the dynamically stabilized homogeneous region is growing with time according to equation (14).

A simple necessary condition for the dynamical stabilization has been suggested, cf. equation (15), which imposes certain constraints on the parameter values. It must be mentioned, however, that equation (15) gives only a necessary condition. Particularly, concerning the Gray-Scott model, the absence of additional restrictions does not mean that the dynamical stabilization can be observed for all the parameter values where the "autocatalyst dominated" state is unstable. To obtain more detailed information about possible constraints, the semiempirical "physical" approach considered in this paper should be complemented by the results of a strict bifurcation analysis; cf. [17]. Ideally, a rigorous mathematical investigation of the problem should also include an explicit analytical proof of the existence of a separatrix in the R^4 phase space of the system corresponding to the stationary travelling front. For a somewhat different problem, such consideration is done in [19,26], whereas it is still an open problem for equations (4),(5) or (18),(19).

Concerning a probable application of the results to natural systems, another open problem is how the phenomenon can be modified in the presence of noise. Our tentative numerical results show that for a periodically applied perturbation, the plateau can survive if the period is sufficiently large and the amplitude of the perturbation is sufficiently small. Under the influence of a perturbation, the length of the plateau may decrease, and in some cases it may be broken into a few parts separated by regions with irregular spatiotemporal oscillations. However, this problem needs more careful consideration and will be subject of a separate paper.

In conclusion, it should be noted that the existence of this phenomenon may shed a new light on some ecological problems. Particularly, it is shown in [27] that the temporary behaviour of the concentrations of key species in a biological community sometimes exhibits "intermittence": an oscillatory behaviour gives way to a quasi-stationary state of the community which is followed again by the oscillations with a period of alternation much less than one year so that it can unlikely be related to the seasonal changes. Now, accounting for the results of this paper, one can consider the situation when an "observer" is taking measurements of the species concentrations in a fixed point in front of the population wave; cf. Figure 2. Then his account of the temporal dynamics of the community in a given point would be very similar to the one reported in [27]. The intermittent temporal behaviour of the community would arise as a result of a biological invasion combined with the dynamical stabilization behind the front. An indirect proof for this explanation can be also found in [28] where it is shown how a spatial structure of a community may result in complex temporal dynamics.

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