Set Theory
Master program Logic and algorithms
Exam problems, February 2018

Problem 0. [(ZF)] Let A be a non-empty set of ordinal numbers. Prove that $\bigcap A$ is the least element of A.

Problem 1. [(ZFC)] Let $\mathcal{A} = \langle A, \leq \rangle$ be a linearly ordered set. Prove that $\mathcal{A}$ is woset if and only if there is no strictly decreasing ω-sequence in $\mathcal{A}$.

Problem 2. [(ZF)] Let α be an ordinal. By (primitive) recursion define the ω-sequence $\{\alpha_n\}_{n<\omega}$ as follows: $\alpha_0 = \aleph_\alpha$, $\alpha_{S(n)} = \aleph_{\alpha_n}$ for $n < \omega$. Let $\beta = \bigcup\{\alpha_n \mid n < \omega\}$. Prove that:
(1) $\beta = \aleph_\beta$;
(2) the sequence $\{\alpha_n\}_{n<\omega}$ is strictly monotone;
(3) $\beta = \mu\gamma[\alpha \leq \gamma \,\&\, \gamma = \aleph_\gamma]$.

Problem 3. Let F be an ordinal operation, i.e. F is a formula operation and $\forall\alpha \mathrm{Ord}(F(\alpha))$. F is called:

- *monotone* (or *strictly increasing*) if $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;
- *continuous* if $\mathrm{Lim}(\alpha) \Rightarrow F(\alpha) = \bigcup\{F(\beta) \mid \beta < \alpha\}$;
- *normal* if it is monotone and continuous.

Prove that the $\aleph$ is a normal operation (without loss of generality assume that the value of this operation for non-ordinals is $\varnothing$).
Prove that if F is a normal operation then:

1. $\forall\alpha\exists\beta(\alpha < \beta \,\&\, F(\beta) = \beta$;
2. $\neg\exists A\forall\gamma(\gamma \in A \iff \gamma = F(\gamma))$.

Problem 4. [(ZFC)] Let $\mathcal{A} = \langle A, \preceq \rangle$ be a poset. A set B, $B \subseteq A$, is called *cofinal* in $\mathcal{A}$ if $(\forall a \in A)(\exists b \in B)(a \preceq b)$.
Prove that if $\mathcal{A}$ is linearly ordered then there exists a woset $\langle B, \preceq \rangle$ such that B is cofinal in $\mathcal{A}$

Problem 5. For each limit ordinal α the ordinal $\mathrm{cf}(\alpha)$ is defined as the least ordinal γ such that there exists a nondecreasing γ-sequence of ordinals $\beta_0, \ldots, \beta_\xi, \ldots, \xi < \gamma$ such that $\alpha = \bigcup\{\beta_\xi \mid \xi < \gamma\}$.
Prove that:

1. $\forall\alpha(\mathrm{Lim}(\alpha) \Rightarrow \mathrm{Card}(\mathrm{cf}(\alpha)))$;

2. $\forall\alpha(\mathrm{Lim}(\alpha) \Rightarrow \mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha))$;
3. $\forall\alpha(\mathrm{Lim}(\alpha) \Rightarrow \mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha))$;
4. [(ZFC)] $\forall\alpha(\mathrm{Suc}(\alpha) \Rightarrow \mathrm{cf}(\aleph_\alpha) = \aleph_\alpha)$

Problem 6. [(ZFC)] A subset C of $\aleph_1$ is called *closed* if the supremum of any ω-sequence $\{\alpha_n\}_{n<\omega}$ of ordinals from C belongs to C also, i.e.

$$(\forall n < \omega)(\alpha_n \in C \Rightarrow \bigcup\{\alpha_n \mid n < \omega\} \in C.$$

Let A be at most countable nonempty set of closed unbounded in $\aleph_1$ sets of ordinals from $\aleph_1$. Prove that $\bigcap A$ is uncountable, closed and unbounded in $\aleph_1$.

Problem 7. A binary relation R on a set A is called *well-founded* if every non-empty subset B of A has an R-*minimal* element, that is $x_0 \in B$ such that there is no $x \in B$ with $\langle x, x_0\rangle \in R$.

(A) [(ZF)] Prove the following induction principle on the well-founded relations. Let $\varphi(x, \overline{u})$ be a set-theoretical property. Let R be a well-founded relation on a set A. Let the parameters $\overline{u}$ be fixed. If

$$(\forall x \in A)(\forall y(\langle y, x\rangle \in R \Rightarrow \varphi(y, \overline{u})) \Rightarrow \varphi(x, \overline{u})),$$

then $(\forall x \in A)\varphi(x, \overline{u})$.

(B) [(ZF)] Let R be a binary relation on A. Prove that if there exists a function f such that $\mathrm{Dom}(f) = A$, $\forall x(x \in \mathrm{Range}(f) \Rightarrow \mathrm{Ord}(x))$ and $(\forall x \in A)(\forall y \in A)(\langle x, y\rangle \in R \Rightarrow f(x) < f(y))$ then R is a well-founded relation on A.

(C) [(ZF)] Let R be a well-founded relation on A. Define by (primitive) recursion a transfinite sequence $A_0, \ldots, A_\alpha, \ldots$ as follows:

$A_0 = \varnothing$,
$A_{S(\alpha)} = \{y \mid \forall x(\langle x, y\rangle \in R \Rightarrow x \in A_\alpha)\}$,
$\mathrm{Lim}(\alpha) \Rightarrow A_\alpha = \bigcup\{A_\beta \mid \beta < \alpha\}$.

Prove that:

1. $\forall\alpha\forall\beta(\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta)$;
2. $\exists\alpha(A_\alpha = A_{S(\alpha)})$;
3. $\exists\alpha(A_\alpha = A)$;
4. if the function f is defined for each $a \in A$ as follows
   $f(a) = \mu\alpha[a \in A_{S(\alpha)}]$,
   then for any $a \in A$ it holds $f(a) = \bigcup\{S(f(x)) \mid \langle x, a\rangle \in R\}$ and $\mathrm{Range}(f) = \mu\alpha[A_\alpha = A]$.