

01.10.2013г.

IR
реален свят

M
модел



A B $\overline{AB}$

Лекции по Мат. Моделиране Марков

$$m, \frac{d^2}{dt^2} = \ddot{x}^{(t)} \quad x(t)$$

$$F(x(t), \dot{x}(t))$$

$$\dot{x}(t) = \frac{d}{dt} x(t)$$

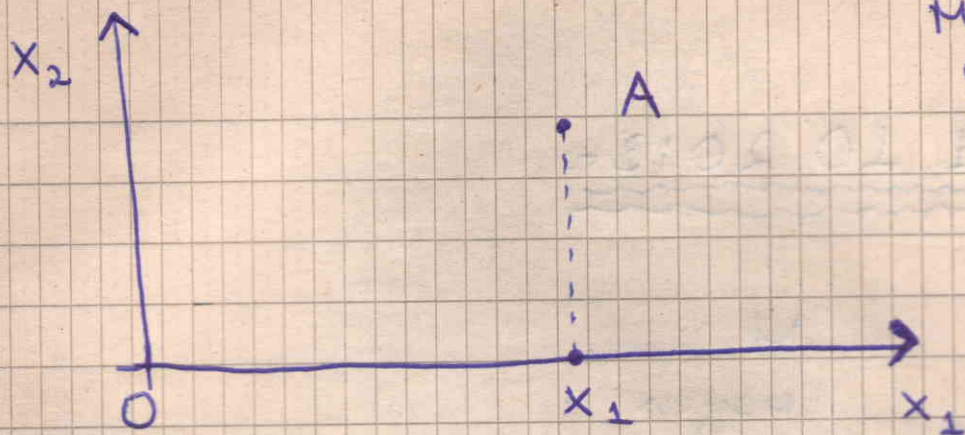
$$m \frac{d^2}{dt^2} x_1(t) = F_1(x(t), \dot{x}(t))$$

$$m \frac{d^2}{dt^2} x_2(t) = F_2(x(t), \dot{x}(t))$$

$$m \frac{d^2}{dt^2} x_3(t) = F_3(x(t), \dot{x}(t))$$

M - маса

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$



M, L, T
основные
величины

$$[S] = L^2$$

$$[V] = L^3$$

! калькулятор

$$l = 2m = 100 \times 2cm$$

$$1m = 100 \times 1cm$$

$$\bar{L} = \kappa L$$

$$\bar{e} = \kappa e$$

$$S = 4m^2 = 10^4 \times 4cm^2$$

$$1m^2 = 100 \times 100 = 10^4 cm^2$$

$$\bar{S} = \kappa^2 S$$

$$\bar{V} = \kappa^3 V$$

$$[v] = \frac{L}{T}$$

$$[a] = \frac{L}{T^2}$$

$$v = 60 km/h$$

$$\bar{V} = \frac{60.1000}{60.3600} =$$

$$1 \text{ km}^3 = 10^3 \text{ m}^3$$

$$1 \text{ h} = 3.6 \cdot 10^3$$

$$\bar{V} = \frac{60.1000}{3600} = 16,6...6 \text{ m/s}$$

$$\bar{L} = \lambda_L L$$

$$\bar{T} = \lambda_T T$$

$$\bar{V} = \frac{\lambda_L}{\lambda_T} V$$

$$[q] = T^x L^p M^x \quad (*)$$

$$\bar{M} = \lambda_M M$$

$$\bar{q}_i = \lambda_T^x \lambda_L^p \lambda_M^x q_i$$

$$2 \text{ m}$$

$$200 \text{ cm}$$

$$3 \text{ m}$$

$$300 \text{ cm}$$

$$q = f(r)$$

$$\frac{\bar{q}_1}{\bar{q}_2} = \frac{q_1}{q_2} \leftarrow \forall \text{ car or bug } (*)$$

$$\bar{q}_2 = f(\lambda_L r_2)$$

$$q_1 = f(r_1)$$

$$\bar{q}_1 = f(\lambda_L r_1)$$

$$q_2 = f(r_2)$$

$$\frac{q_2}{q_1} = \frac{q_2}{q_1} \Rightarrow \frac{f(x_{p_2})}{f(x_{p_1})}$$

$$\frac{= f(p_2)}{f(p_1)} = \text{const}$$

$$\frac{f(\kappa p_2)}{f(p_2)} = \frac{f(\kappa p_1)}{f(p_1)} \quad \text{не зависит от } p$$

" $\psi(\kappa)$

$$P_2 = \mu P_1$$

$$\frac{f(x \mid \mu, \sigma^2)}{f(\mu, \sigma^2)} = \psi(x)$$

$$\frac{\frac{f(x \mid \mu, \sigma)}{f(x)}$$

$$\psi(\lambda\mu) = \psi(\lambda)\psi(\mu)$$

е диференц.

$$\lambda \psi'(\lambda \mu) = \psi(\lambda) \psi'(\mu)$$

$$P = 1$$

$$\frac{x v'(x) = a v(x)}{a = v'(1)}$$

$$v(1) = 1$$

$$\frac{v'(x)}{v(x)} = \frac{a}{x} \quad \int \frac{1}{x} dx$$

$$\ln v(x) = a \ln x + c$$

$$\frac{d}{dx} \ln v(x) = a \frac{d}{dx} \ln x$$

$$\ln v(x) = a \ln x + c$$

$$v(x) = c x^a$$

$$v(x) = c x^a$$

$$v'(x) = c a x^{a-1}$$

$$x \cdot c a x^{a-1} = a c x^a$$

$$x = a$$

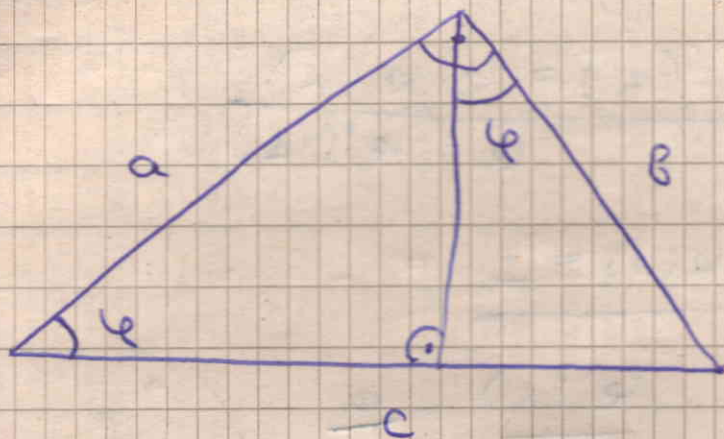
$$v(x) = x^a$$

$$q = f(e) = f(e \cdot 1) =$$

$$= v(e) \cdot f(1) =$$

$$= e^a f(1)$$

$$q = c e^a$$



$$s = s(c, \psi)$$

$$s = c^2 g(\psi) = a^2 g(\psi) + b^2 g(\psi)$$

$$c^2 = a^2 + b^2$$

11. $h = \frac{1}{2} g t^2$

$$L = \frac{L}{T^2} T^2$$

$$h - \frac{1}{2} g t^2 = 0$$

$$\bar{L} = \kappa_L L$$

$$\bar{T} = \kappa_T T$$

$$\bar{h} = \kappa_h h$$

$$g = \frac{\kappa_g}{\kappa_t^2} g$$

$$\bar{t} = \kappa_t t$$

$$\bar{h} - \frac{1}{2} \bar{g} \bar{t}^2 = \exp\left(h - \frac{1}{2} g t^2\right) = 0$$

запозна се закон
при смена на
масаба
инвариантност

$$1 - \frac{1}{2} \left(\frac{g t^2}{h} \right) = 0$$

$$1 - \frac{q}{2} = 0$$

(визуал)

$$f(p_1, \dots, p_k, q_1, \dots, q_m) = 0$$

$$[p_i] = A_i$$

$$[p_k] = A_k$$

независими

$$[q_i] = A_1^{d_1^i} A_2^{d_2^i} \dots A_k^{d_k^i}$$

$$[q_m] = A_1^{d_1^m} \dots A_k^{d_k^m} \quad d_i^j \rightarrow 0$$

$$\bar{A}_i = \lambda_i A_i, \quad i = 1, \dots, k$$

$$\lambda_i > 0$$

$$f(p_1, \dots, p_k, q_1, \dots, q_m)$$

~~scribbled out text~~

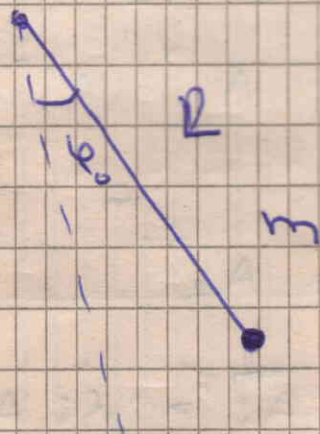
$$[\bar{p}_1] =$$

$$f(p_1 x_1, \overset{p_2 x_2}{x_1^{d_1}}, \dots, \overset{p_k x_k}{x_k^{d_k}}, x_1^{d_1'}, x_2^{d_2'}, \dots, x_k^{d_k'}, q_1, \dots, q_m) = 0$$

$$x_i = 1/p_i$$

$$f(1, 1, 1, \dots, 1, q_1 / (p_1^{d_1'} p_2^{d_2'} \dots p_k^{d_k'}), \dots, q_m / (p_1^{d_1''} \dots p_k^{d_k''})) = 0$$

$$f(1, 1, 1, \dots, \tau_1, \dots, \tau_m) = 0$$



$$t, L, g, m, l_0$$

$$f(t, L, g, m, l_0) = 0$$

масата не участва,
не се компенсира

$$t^{\alpha} \epsilon^{\beta} g^{\gamma} = h(\psi_0)$$

$$t \epsilon^{\beta} g^{\gamma} = h(\psi_0)$$

$$500 \alpha = 1$$

$$t \epsilon^{\beta} g^{\gamma} = h(\psi_0)$$

$$\pi L^{\beta} [g]^{\gamma} = 0$$

$$\pi L^{\beta} \frac{L^{\gamma}}{T^{2\gamma}} = 0$$

$$\Delta = 2\gamma$$

$$\gamma = \frac{1}{2}$$

$$\beta = -\gamma$$

$$\beta = -\frac{1}{2}$$

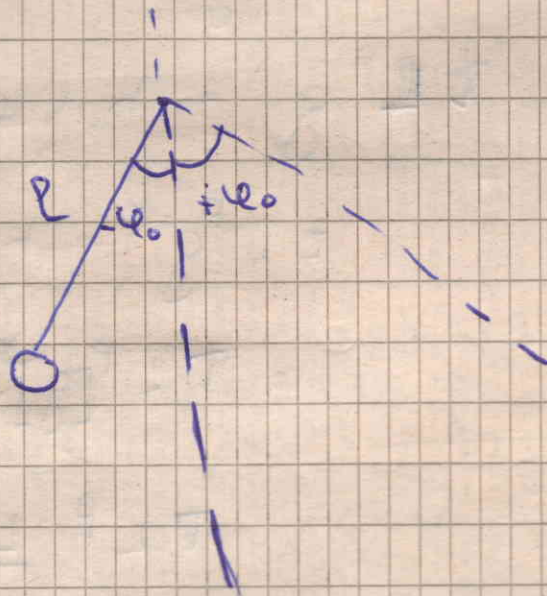
$$t \left(\frac{g}{\epsilon} \right)^{\frac{1}{2}} = h(\psi_0)$$

$$h(\psi_0)$$

08.10.2013г.

Лекция

m, l, T, φ_0



$$f(m, l, T, \varphi_0) =$$

$$[m] = M$$

$$[l] = L$$

$$[T] = T$$

$$[\varphi_0]$$

$$f(m, l, T, \varphi_0, g) = 0$$

$$[g] = \frac{L}{T^2}$$

$$h(\pi, \varphi_0) = 0$$

$$\pi = m^2 e^{\beta} T^{\gamma} g^{\delta}$$

$$m^2 e^{\beta} T^{\gamma} g^{\delta} = \phi(\psi_0) \rightarrow$$

$$\Rightarrow L = 0$$

нмаша как во да
уравновесии т

$$L^{\beta} T^{\gamma}$$

$$T L^a g^{\gamma} = \psi(\psi_0)$$

~~$$T L^a g^{\gamma} = \psi(\psi_0)$$~~

$$T \cdot L^a \cdot \frac{L^b}{T^{2b}}$$

$$\begin{cases} a+b=0 \\ 1-2b=0 \end{cases}$$

$$b = \frac{1}{2}$$

$$a = -\frac{1}{2}$$

$$\frac{L^{a+b}}{T^{2b-1}}$$

$$t e^a g^b = \psi(\psi_0) \quad A \quad \pm \psi(\psi_0)$$

$$t \sqrt{\frac{g}{t}} = \psi(\psi_0) = \psi(0) + \psi'(0) \psi_0$$

$$+ \frac{1}{2!} \psi''(0) \psi_0^2$$

$$+ o(\psi_0^2)$$

$$\Rightarrow t = A \sqrt{\frac{p}{g}} + o(\epsilon_0)$$

$$t = A \sqrt{\frac{p}{g}} + o(\epsilon_0)$$

$$m \ddot{x}(t) = F(x(t), \dot{x}(t))$$



$$F(x) = -kx$$

$$m \frac{d^2}{dt^2} = -kx(t)$$

$$x(0) = x_0$$

$$\frac{d^2}{dt^2} = -c(x(t))$$

$$c = -\frac{k}{m} > 0$$

$$\begin{aligned} g(x) &= \cos x \\ g'(x) &= -\sin x \\ g''(x) &= -\cos x \end{aligned}$$

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \end{aligned}$$

$$f(x) = \sin 2x$$

$$f''(x) = -2^2 \sin 2x$$

$$g(x) = \cos 2x$$

$$g''(x) = -2^2 \cos(2x)$$

$$g(x) = X_0 \cos(\sqrt{c}x)$$

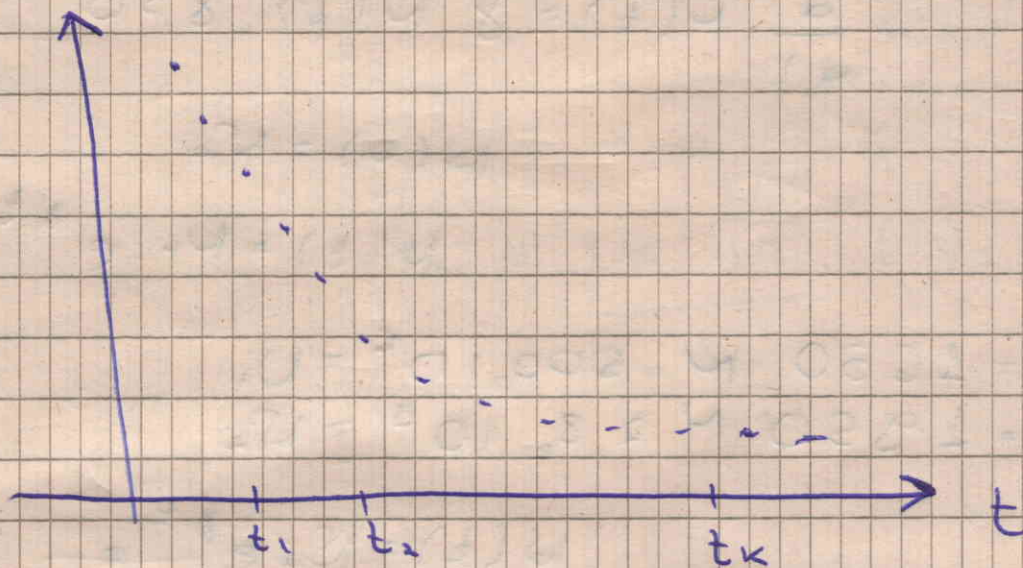
~~$$g(t) = X_0 \cos$$~~

$$x(t) = X_0 \cos \sqrt{\frac{k}{m}} t$$

$$x(t+T) = x(t)$$

∇

$$x \cos\left(\sqrt{\frac{k}{m}} t + 2\pi\sqrt{\frac{k}{m}}\right) \rightarrow \dots$$



$N(t)$ - брой нелеруду

$$N(t) = N_0 e^{-\lambda t}$$

- намя-
ване на
нелерудите

n_i - брой нелеруду в
момента t_i

~~W(t) = N(t) e^{-\lambda t}~~

$$\sum_{i=1}^n (n_i - N_0 e^{-\lambda t_i})^2 \rightarrow \min$$

$$\left| \begin{array}{l} \frac{d}{dt} N(t) = -\lambda N_0 e^{-\lambda t} = -\lambda N(t) \\ N(0) = N_0 \end{array} \right.$$

$$\begin{aligned} \frac{d}{dt} N(t) &= \lambda N(t) - \beta N(t) = \\ &= \underbrace{(\lambda - \beta)}_{\gamma} N(t) \end{aligned}$$

$$\frac{d}{dt} N(t) = \gamma N(t), \gamma > 0$$

$$N(0) = N_0$$

$$N(t) = N_0 e^{\gamma t}$$

$$Y_1 = 1650 \text{ N } 500 \cdot 10^6 = N_1$$

$$Y_2 = 1950 \text{ N } 2,8 \cdot 10^9 = N_2$$

$$N(t) = N_0 e^{\gamma t}$$

$$\frac{d}{dt} N(t) = \frac{N_0 e^{\gamma t} \gamma}{\gamma}$$

$$Y_3 = ? \sim 25 \cdot 10^9 = N_3$$

$$N(t) = N_0 e^{\lambda t}$$

$$N_2 = N_1 e^{\lambda (Y_2 - Y_1)}$$

$$\frac{N_2}{N_1} = e^{\lambda (Y_2 - Y_1)}$$

$$\lambda = \frac{\ln N_2 - \ln N_1}{Y_2 - Y_1}$$

$$N_3 = N_1 \cdot e^{\lambda \cdot (Y_3 - Y_1)}$$

$$\ln \frac{N_3}{N_1} = \lambda \cdot (Y_3 - Y_1)$$

$$Y_3 = Y_1 + \frac{\ln \left(\frac{N_3}{N_1} \right)}{\lambda}$$

$$= Y_1 + \frac{\ln \frac{N_3}{N_1}}{\ln \frac{N_2}{N_1}} (Y_2 - Y_1)$$

$$\begin{array}{l} 1,6094 \\ 1,7228 \end{array}$$

$$Y_3 = Y_1 + \frac{\ln \frac{N_3}{N_1}}{\ln \frac{N_2}{N_1}} (Y_2 - Y_1)$$

$$\frac{N_3}{N_1} = \frac{2,5 \cdot 10^9}{500 \cdot 10^6} = \frac{2,5 \cdot 10}{500} \quad \left| \frac{\ln 5}{\ln 5,6} \right| \quad \frac{2,8 \cdot 10^3}{500}$$

~~2031~~
2331

закон за радиоактивност

$$\frac{d}{dt} N(t) = -\lambda N(t)$$

$$N(t) = N_0 e^{-\lambda t}$$

$$\frac{N}{2} = N e^{-\lambda T}$$

$$e^{-\lambda T} = \frac{1}{2}$$

$$-\lambda T = \ln \frac{1}{2}$$

$$T = - \frac{\ln \frac{1}{2}}{\lambda} =$$

$$= \frac{\ln 2}{\lambda}$$

$$\frac{3}{4} N = N e^{-\lambda T_2}$$

$$e^{-\lambda T_2} = \frac{3}{4}$$

$$-\lambda T_2 = \ln 3 - \ln 4$$

$$T_2 = \frac{\ln \frac{4}{3}}{\lambda}$$

$$T = \frac{\ln \frac{4}{3}}{\ln 2} T_2$$

C^{14}

$$T = 5600 \text{ r}$$

$$\frac{N}{5} = N e^{-\lambda T}$$

$$-\lambda T = \ln \frac{1}{5}$$

$$T = \frac{\ln 5}{\lambda}$$

$$\lambda = \frac{\ln 2}{T}$$

$$\Rightarrow T = \frac{\ln 5}{\ln 2} T$$

$$\frac{d}{dt} N(t) = -\lambda N(t)$$

$$N(t_0) = N_0$$

~~with λ~~
no

$$N(t) = N_0 \cdot e^{-\lambda(t-t_0)} \quad \underline{\underline{t=t_0}}$$

$$N(t_0) =$$

$$\underline{\underline{t=t_0}}$$

$$\frac{d}{dt} N(t) = -\lambda N(t) + p = -\lambda (N(t) - \frac{p}{\lambda})$$

$$N(t_0) = N_0$$

$$N(t) = N_0 \cdot e^{-\lambda(t-t_0)} + \frac{p}{\lambda}$$

~~with λ~~

$$M(t)$$

$$\frac{d}{dt} \left(N(t) - \frac{P}{\gamma} \right)$$

$$\frac{d}{dt} M(t) = -\gamma M(t)$$

$$M(t_0) = N(t_0) - \frac{P}{\gamma}$$

$$M(t) = M(t_0) \cdot e^{-\gamma(t-t_0)}$$

$$N(t) = \frac{P}{\gamma} + \left(N(t_0) - \frac{P}{\gamma} \right) e^{-\gamma(t-t_0)}$$

автономна
система

$$\frac{d}{dt} N(t) = f(N(t))$$

$$N(0) = N^*$$

$$f(N^*) = 0$$

$$N(t) \equiv N^* \text{ е решение}$$

Единствено реш.

$f(x)$ с Липшицова

$$|f(x_1) - f(x_2)| \leq L |x_1 - x_2|$$

Ако е диф.

$\rightarrow$ sup на производна-та

$$N(t) = N^* + \varepsilon(t)$$

$$N(0) = N^* + \varepsilon(0) = \varepsilon_0$$

$$\frac{d}{dt} N(t) = \frac{d}{dx} (N^* + \varepsilon(t)) = \frac{d}{dt} \varepsilon(t)$$

||

$$\begin{aligned} f(N(t)) &= f(N^* + \varepsilon(t)) = \\ &= f(N^*) + f'(N^*) \varepsilon(t) + o(\varepsilon(t)) \end{aligned}$$

$$\frac{d\varepsilon(t)}{dt} = f'(N^*) \varepsilon(t)$$

~~_____~~

$$\varepsilon(t) = \cancel{\varepsilon_0 e^{f'(N^*)t}} \varepsilon_0 e^{f'(N^*)t}$$

$$f'(N^*) > 0$$

$$\begin{aligned} \varepsilon(t) &\rightarrow \infty \\ t &\rightarrow \infty \end{aligned}$$

$$f'(N^*) < 0$$

$$\begin{aligned} \varepsilon(t) &\rightarrow 0 \\ t &\rightarrow \infty \end{aligned}$$

$$f'(N^*) = 0$$

$$\frac{d}{dt} N(t) = f(N(t))$$

$$f(N^*) = 0$$

$N(t)$ — решение

$$\frac{d^2}{dt^2} N(t) = \frac{d}{dt} \left(\frac{d}{dt} N(t) \right) =$$

$$= \frac{d}{dt} f(N(t)) =$$

$$= f'(N(t)) \cdot \frac{dN(t)}{dt} = f'(N(t)) \cdot f(N(t))$$

15.10.2013

$$\dot{x}(t) = \lambda x(t) \quad x(0) = x_0$$

$$x(t) = x_0 \cdot e^{\lambda t}$$

$$\frac{d}{dx} x(t) = \lambda(t) x(t) + p(t)$$

$$x(t_0) = x_0$$

$$\frac{d}{dx} x(t) - \lambda(t) x(t) = p(t)$$

$$e^{-\int_{t_0}^t \lambda(s) ds} \cdot \left[\frac{d}{dx} x(t) - \lambda(t) x(t) \right] =$$

$$= b(t) \cdot e^{-\int_{t_0}^t \lambda(s) ds}$$

$$\frac{d}{dx} \left(x(t) \cdot e^{-\int_{t_0}^t \lambda(s) ds} \right) = e^{-\int_{t_0}^t \lambda(s) ds} b(t)$$

|| $\int_{t_0}^t$ gberne compatu

$$x(t) e^{-\int_{t_0}^t \lambda(s) ds} - x_0 = \int_{t_0}^t e^{-\int_{t_0}^s \lambda(n) dn} b(s) ds$$

$$| \cdot e^{\int_{t_0}^t \lambda(s) ds}$$

$$x(t) = e^{\int_{t_0}^t \lambda(s) ds} x_0 + \int_{t_0}^t e^{\int_{t_0}^s \lambda(n) dn} b(s) ds$$

! да се
знае

$$x(t) = e^{\int_{t_0}^t \lambda(s) ds} x_0 + \int_{t_0}^t e^{\int_s^t \lambda(n) dn} b(s) ds$$

Закон за изстигване на телата (Нютон)

$$\frac{d}{dt} \theta(t) = -\lambda (\theta(t) - T) =$$

$$= -\lambda \theta(t) + \lambda T$$

T - температура ок. ср

$$\theta(0) = \theta_0$$

$$\theta(t) = e^{-\int_{t_0}^t \lambda(s) ds} \theta_0 + \int_{t_0}^t e^{-\int_s^t \lambda(n) dn} \lambda(s) ds =$$

$$= e^{-\int_{t_0}^t \lambda(s) ds} \theta_0 + \int_{t_0}^t e^{-\int_s^t \lambda(n) dn}$$

$$= -\lambda (\theta(t) - T) =$$

~~$\theta(t) = e^{-\lambda t} \theta_0 + \int_0^t e^{-\lambda t} \lambda T dt$~~

$$= \underbrace{-\lambda \theta(t)} + \underbrace{\lambda T}$$

$$\rightarrow \theta(t) = e^{-\lambda t} \theta_0 + \int_0^t e^{-\lambda t} \lambda T dt$$

$$\theta(t) = e^{-\lambda t} \theta_0 + e^{-\lambda t} \lambda T \int_0^t dt$$

$$\theta(t) =$$

$$\frac{d}{dt} (\theta(t) - T) = -\lambda (\theta(t) - T)$$

$$\frac{d}{dt} z(t) = -\lambda z(t)$$

$$\theta(t) - T = z(t) = e^{-\lambda t} (\theta_0 - T)$$

$$\underset{40^\circ}{\theta(t)} = \underset{20^\circ C}{T} + e^{-\lambda t} (\underset{90^\circ}{\theta_0} - \underset{20^\circ}{T})$$

$$\lambda, (t)$$

$$\lambda = ?$$

Don.: $\lambda_1 = ?$
 $\lambda_2 = ?$

$$t_1 = 6$$

$$T_1 = 23^\circ$$

$$T_2 = 16^\circ$$

$$t_2 = 4$$

$$\lambda_1 = 0,47 \text{ or } 0,23$$

$$\lambda_2 = 0,08$$

$$e^{-\lambda_1 t_1} (90 - 23) = 40.23$$

$$e^{-\lambda_1 \cdot 6} = \frac{17}{67}$$

$$e^{-\lambda_2 t_2} (90 - 16) = 40.16$$

$$e^{-\lambda_2 \cdot 4} = \frac{24}{74}$$

Свободно падане

$$m \frac{dV(t)}{dt} = mg - m\kappa V^2(t)$$

$$\frac{dV(t)}{dt} = g - \kappa V^2(t)$$

$$- \frac{d(g - \kappa V^2(t))}{dt} = g - \kappa V^2(t)$$

$$V(0) = 0$$

~~$$V(t) = \dots$$~~

~~$$V(t) = \dots$$~~

$$V(t) = e^{-\kappa t} (V_0 - g/\kappa) + g/\kappa$$

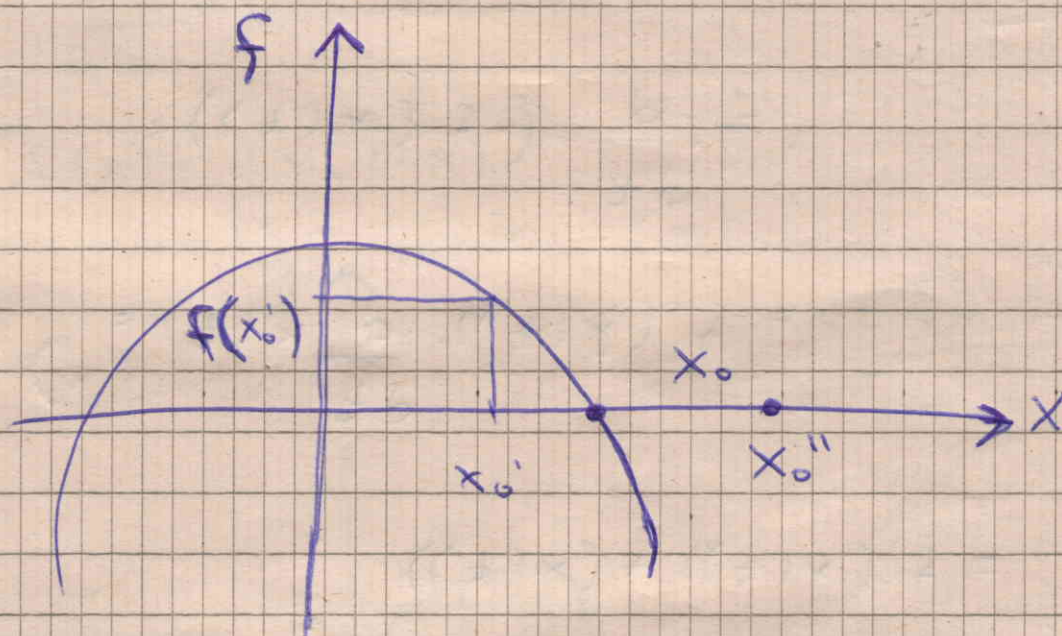
$$m \frac{dV(t)}{dt} = mg - m\kappa^2 V^2$$

$$\frac{d}{dt} V(t) = g - \kappa^2 V^2$$

$$f(x) = g - k^2 x^2 = 0 \quad x_{1,2} = \pm \frac{\sqrt{g}}{k}$$

$$V_0 = \frac{\sqrt{g}}{k}$$

$$V(t) \equiv V_0$$



$$\frac{d}{dx} x(t) = f(x(t))$$

$$x(t_0) = x_0$$

$$\frac{d}{dt} x(t) \equiv 0 = f(x_0)$$

$$f(x_0')$$

$$\neq f(x_0') > 0$$

"

$$\frac{d}{dt} x(t_0)$$

$$\frac{d}{dt} x(t) = f(x(t))$$

$$\frac{d^2}{dt^2} x(t) = \frac{d}{dt} \left(\frac{d}{dt} x(t) \right) =$$

$$= \frac{d}{dt} (f(x(t))) =$$

$$= f'(x(t)) \cdot \frac{d}{dt} x(t) =$$

$$= f'(x(t)) f(x(t))$$

изтъкнала
вдълбната

Популация N вида

Епидемия

$x(t)$ - болни

$y(t)$ - здрави

$$N = x(t) + y(t)$$

$$x(t) = X(t) / N$$

$$y(t) = Y(t) / N$$

$$\frac{d}{dt} x(t) = 2x(t)y(t) = 2x(t)(1-x(t))$$

$$y(t) = 1 - x(t)$$

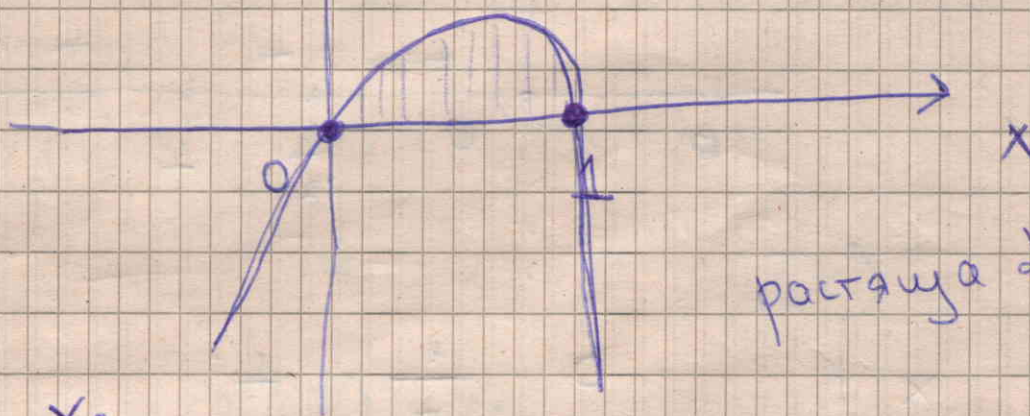
$$x(t) + y(t) = 1$$

$$\frac{d}{dt} x(t) = 2x(t)(1-x(t))$$

$$f(x) = 2x(1-x)$$

y

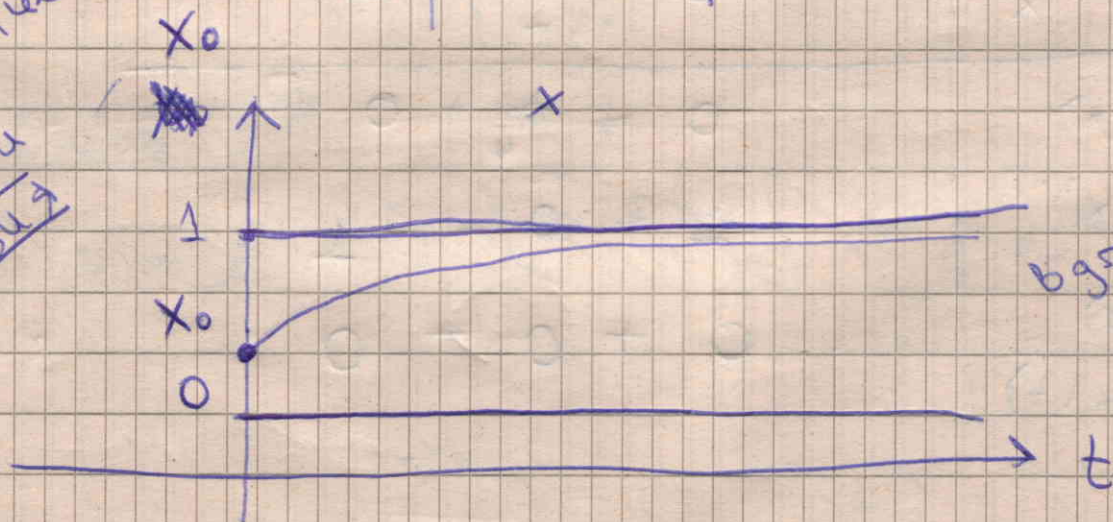
$$\underline{x \in [0, 1]}$$



растуща f-я

решение

начальн
услови



вдълбочната?

$$x = x^2 \quad x = 2x$$

$$x = x^2$$

$$\frac{d(y(t))}{y(t)} = 2x_0 e^{-ut} dt$$

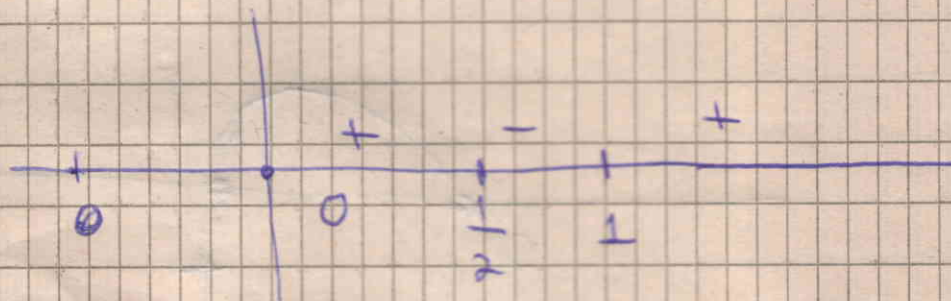
$$\ln(y(t)) = \frac{e^{-ut}}{-u} \cdot 2x_0$$

$$\frac{d^2}{dx^2} x(t) = f(x(t)) \quad f'(x(t)) =$$

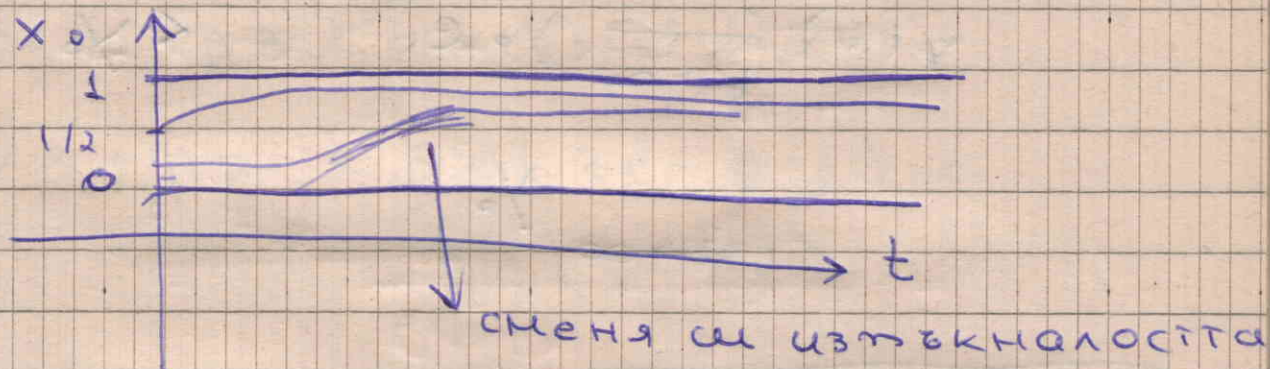
$$f(x) = 2x(1-x)$$

$$f'(x) = 2(1-2x)$$

$$= 2^2 x(1-x)(1-2x)$$



x	0	$\frac{1}{2}$	1
$f(x)$	0	$+\frac{1}{4}+$	0
$f'(x)$	1	0	-1
$\frac{d^2}{dt^2} x(t)$	0	0	0



$$\frac{d}{dt} x(t) = -\mu x(t)$$

$$x(0) = x_0$$

$$x(t) = x_0 e^{-\mu t}$$

$$\frac{d}{dt} y(t) = -2 y(t) x(t) =$$

$$= -2 x_0 e^{-\mu t} y(t)$$

$$y(0) = y_0$$

~~$y(t) = y_0 e^{-2 x_0 \int_0^t e^{-\mu \tau} d\tau}$~~

~~$y(t) = y_0 e^{-2 x_0 \frac{1 - e^{-\mu t}}{\mu}}$~~

~~$y(t) = y_0 e^{-\frac{2 x_0}{\mu} (1 - e^{-\mu t})}$~~

$$x(t) = x_0 e^{-\mu t}$$

$$y(t) = y_0 e^{\frac{2}{\mu} x_0 (e^{-\mu t} - 1)}$$

$$y(t) \xrightarrow{t \rightarrow \infty} y_0 e^{-\lambda x_0 / M} < y_0$$

$$\xrightarrow{\mu \rightarrow \infty} y_0$$

$$\frac{d}{dx} y(t) = -\mu y(t) - \beta y \rightarrow \text{болни}$$

y - е на здравите

$$\frac{d}{dt} N(t) = -\mu N(t) - \nu \beta y$$

$$\nu \in (0, 1)$$

част от заразените
умират

$$\frac{d}{dt} \frac{y(t)}{N(t)} = \frac{\dot{y}(t)}{N(t)} - \frac{y(t) \dot{N}(t)}{N^2(t)} =$$

//

$$\frac{d}{dt} z(t) = - \frac{(\mu + \beta) y(t)}{N(t)} - \frac{y(t)}{N(t)} \left[\frac{-\mu N(t)}{N(t)} - \nu \beta y \right]$$

$$= -\mu z - \beta z - z(t) [-\mu + \nu \beta z(t)] =$$

$$z(t) = \frac{y(t)}{N(t)}$$

$$\begin{aligned}
 &= -\mu z(t) - \beta z + \mu z(t) - z^2 \sqrt{\beta} - \\
 &= +z^2 \sqrt{\beta} - \beta z = \beta z (z\sqrt{\beta} - 1) = \\
 &= -\beta z (1 - z\sqrt{\beta})
 \end{aligned}$$

$$z(0) = 1$$

$$\frac{d}{dt} z(t) = -\beta z(t) (1 - z(t)\sqrt{\beta})$$

$$z(0) = 1$$

$$\frac{d}{dt} z(t) = -\beta z(t) + \sqrt{\beta} z^2(t)$$

$$- \frac{1}{z(t)} \frac{d}{dt} z(t) = - \frac{\beta}{z(t)} + \sqrt{\beta}$$

$$x(t) = \frac{1}{z(t)}$$

$$\left| \frac{d}{dt} x(t) = +\beta x(t) \sqrt{\beta} \right.$$

$$x(0) = 1$$

$$x(t) = e^{\int_{t_0}^t -\beta ds} x_0 + \int_{t_0}^t e^{\int_s^t -\beta ds} \sqrt{\beta} ds$$

$$x(t) = e^{-\beta t} x_0 + \int_{t_0}^t e^{-\beta t} \nu \beta dt =$$

$$= e^{-\beta t} x_0 + \nu \beta e^{-\beta t} =$$

$$= e^{-\beta t} (x_0 + \nu \beta)$$

$$\frac{d}{dt} (x(t) \beta - \nu \beta) = \beta (\beta x(t) - \nu \beta)$$

$$\nu(t) = x(t) \beta - \nu \beta$$

$$\frac{d \nu(t)}{\nu(t)} = \beta dt$$

$$\frac{e^{\beta t} + \nu \beta}{\nu \beta}$$

$$e^{\beta t}$$

$$\ln(\nu(t)) = \beta t + \ln \nu \beta$$

$$\frac{e^{\beta t} + \nu \beta}{\nu \beta}$$

$$\frac{\beta e^{\beta t} + \nu \beta}{\beta} = e^{\beta t} + \nu$$

$$\beta e^{\beta t} + \nu \beta = \beta e^{\beta t} + \nu \beta$$

$$e^{\beta t} + \nu$$

$$\frac{x(t)}{\nu(t)} = \left(\frac{e^{\beta t} + \nu \beta}{\beta} \right) - \nu \beta = e^{\beta t}$$

$$\beta e^{\beta t} + \nu$$

$$\beta e^{\beta t} + \nu \beta - \nu \beta$$

$$e^{\beta t} + \nu$$

$$\beta e^{\beta t}$$

22.10.2013r.

$$\frac{\beta e^{\beta t} (1-\nu)}{\beta e^{\beta t} (1-\nu)}$$

$$\frac{\beta e^{\beta t}}{\beta (\nu + e^{\beta t} (1-\nu))}$$

$$-\nu\beta = \frac{\beta e^{\beta t} (1-\nu)}{1 - \nu e^{\beta t}}$$

$$\begin{cases} \dot{x}(t) = \beta x(t) - \nu\beta \\ x(0) = 1 \end{cases}$$

$$x(t) = \frac{1}{z(t)}$$

$$\beta e^{\beta t} = \beta \cdot e^{\beta t}$$

$$\beta (\nu + e^{\beta t} (1-\nu)) =$$

$$x(t) = e^{\int_0^t \beta ds} \cdot \underset{x_0}{1} + \int_0^t e^{\int_s^t \beta ds} (-\nu\beta) ds$$

$$\frac{\beta e^{\beta t} (1-\nu)}{\beta (\nu + e^{\beta t} (1-\nu))} = e^{\beta t}$$

$$\frac{e^{\beta(t-s)}}{1 - e^{\beta t}}$$

$$= e^{\beta t} + \nu \int_0^t e^{\beta(t-s)} \cdot d(\beta(t-s)) =$$

$$= e^{\beta t} + \nu (1 - e^{\beta t}) =$$

$$= \nu + e^{\beta t} (1-\nu)$$

$$\nu + \beta e^{\beta t} - \nu\beta$$

$$\dot{x}(t) = (1-\nu) e^{\beta t} \cdot \beta =$$

$$= e^{\beta t} \beta - \nu e^{\beta t} \beta$$

$$\beta x(t) - \nu\beta$$

$$z(t) = \frac{1}{\nu + e^{\beta t} (1-\nu)} \xrightarrow[t \rightarrow \infty]{} 0$$

$$z(t) = \frac{1}{\nu + e^{\beta t} (1-\nu)}$$

$$t = 20$$

$$\nu = \beta = \frac{1}{8}$$

процент здрави
20-годишници

$$\frac{1}{\frac{1}{8} + e^{\frac{1}{8} \cdot 20} \cdot \frac{7}{8}} = \frac{1}{\frac{1}{8} + e^{\frac{5}{2}} \cdot \frac{7}{8}} = \dots$$

$$\frac{64,3}{100}$$

концентрации
p q

$$\dot{x}(t) = 2$$

концентрация
пропорц. на реагиращите в-ва

$$\begin{cases} \dot{x}(t) = 2(p - x(t))(q - x(t)) \\ x(0) = 0 \end{cases}$$

$$f(x) = 2(p-x)(q-x)$$

	$-\infty$	p	$\frac{p+q}{2}$	q	$+\infty$
f	+	0	-	0	+
f'	-	-	0	+	+
$\ddot{x}$		$f \cdot f' = -2x^2 + 2(p+q)x - 2pq$			

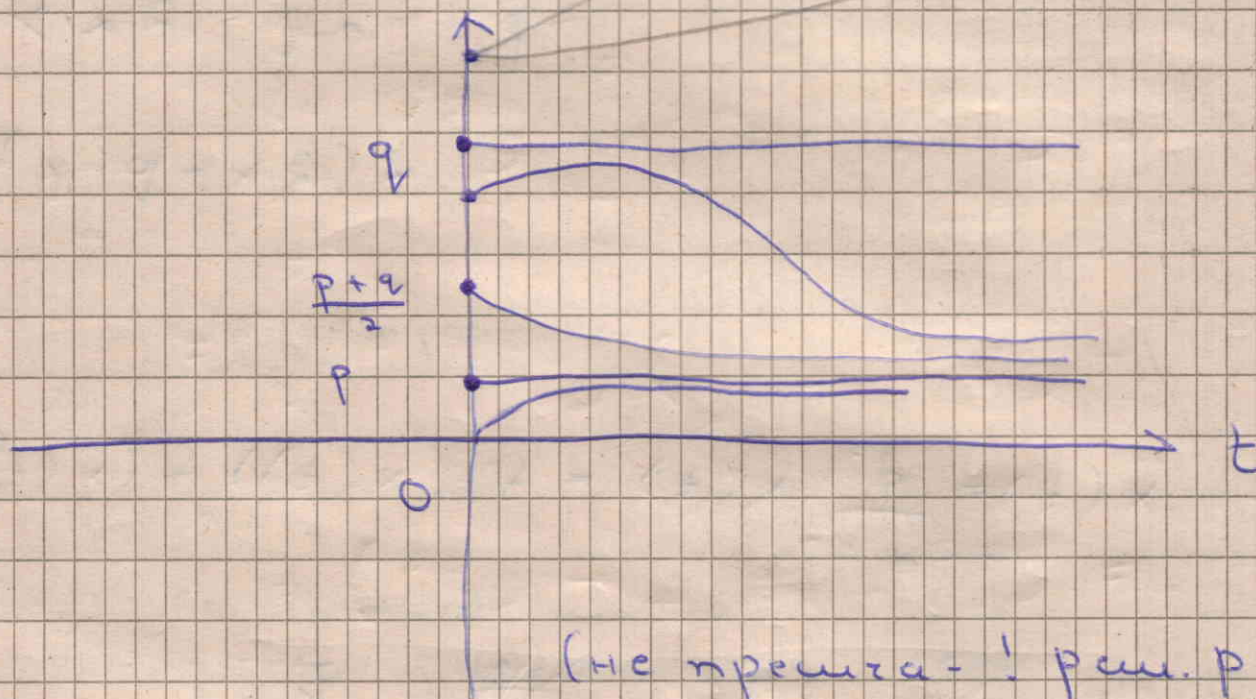
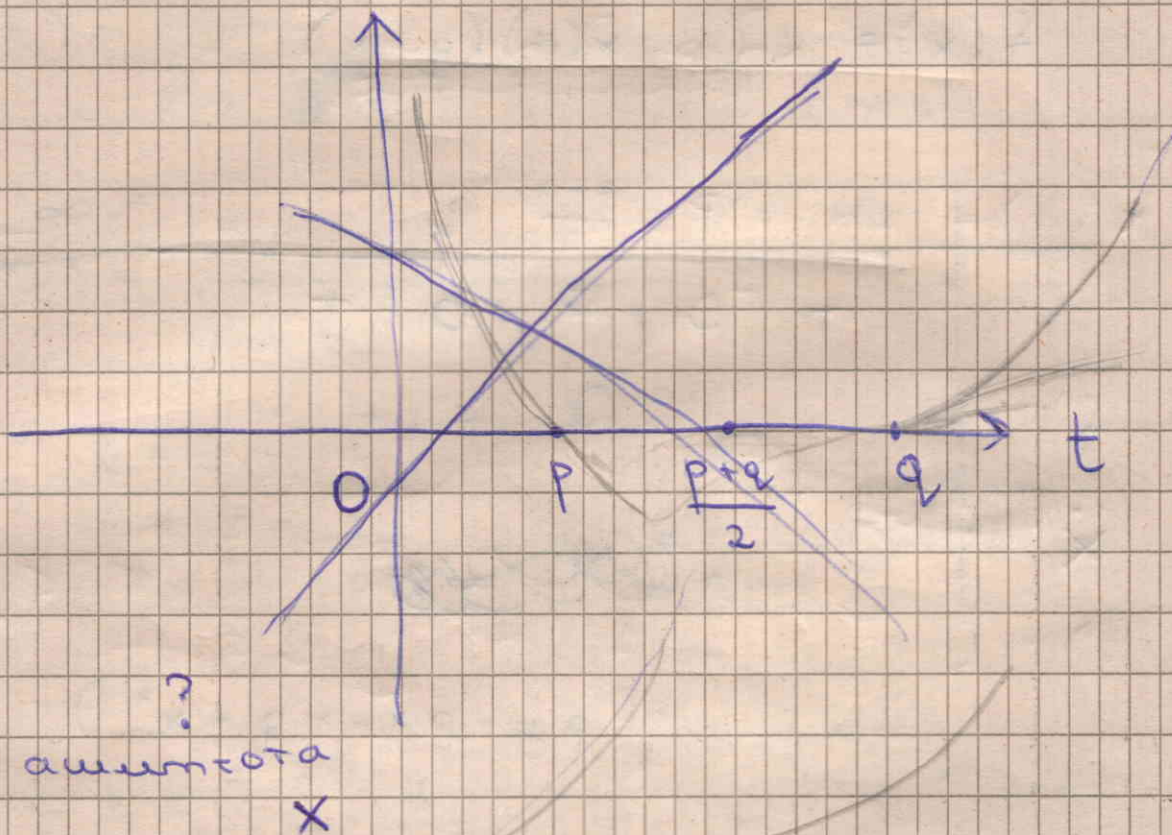
$$d(pq - px - xq + x^2)$$

$$-p - q + 2x$$

$$2(2x - p - q)$$

$$\ddot{x}(t) = \frac{d^2}{dt^2} x(t) = f'(x(t)) f(x(t))$$

	p	$\frac{p+q}{2}$	q
f	0	-	0
f'	-	0	+
$\ddot{x}$	-	+	-
$f \cdot f'$	+	-	-



(не прешга - ! раш. p const)

$p=q \rightarrow p=q$ единотвено решение

★

$$\begin{cases} \dot{N}(t) = 2 N(t) (1 - L(N)) \\ N(0) = N_0 > 0 \end{cases}$$

$$L(0) = 0$$

$$L'(N) > 0$$

$$\lim_{N \rightarrow \infty} L(N) = \infty$$

$$f(N) = 2N(1 - L(N))$$

$$N^* \quad L(N^*) = 1$$

единственное

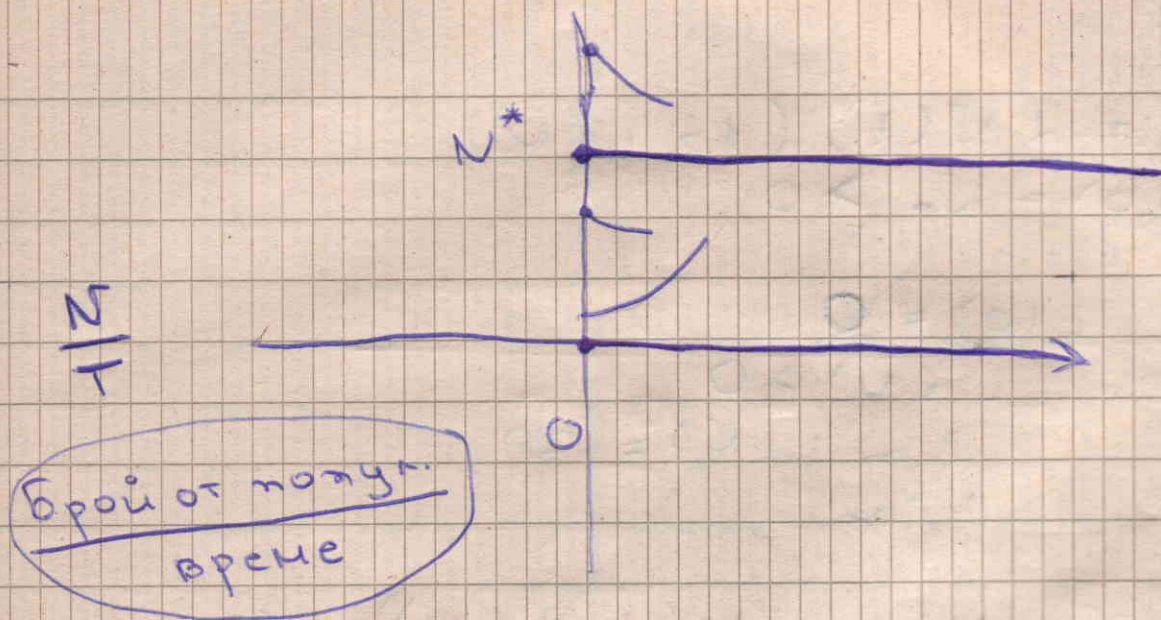
$$\begin{matrix} N=0 \\ N^* \end{matrix}$$

$$\begin{aligned} f(N) &= f(N^*) + f'(N^*)(N - N^*) \\ &\quad + o(N - N^*) \end{aligned}$$

$$\begin{aligned} f'(N) &= 2[(1 - L(N)) - N L'(N)] = \\ &= 2(1 - L(N) - N L'(N)) \end{aligned}$$

$$f'(0) = 2 > 0$$

$$f'(N^*) = -2N^* L'(N^*)$$



$$\dot{N}(t) = \lambda N(t) \left(1 - \frac{N(t)}{K}\right)$$

модел на Ферхюс

↓
максимално
количество
популации

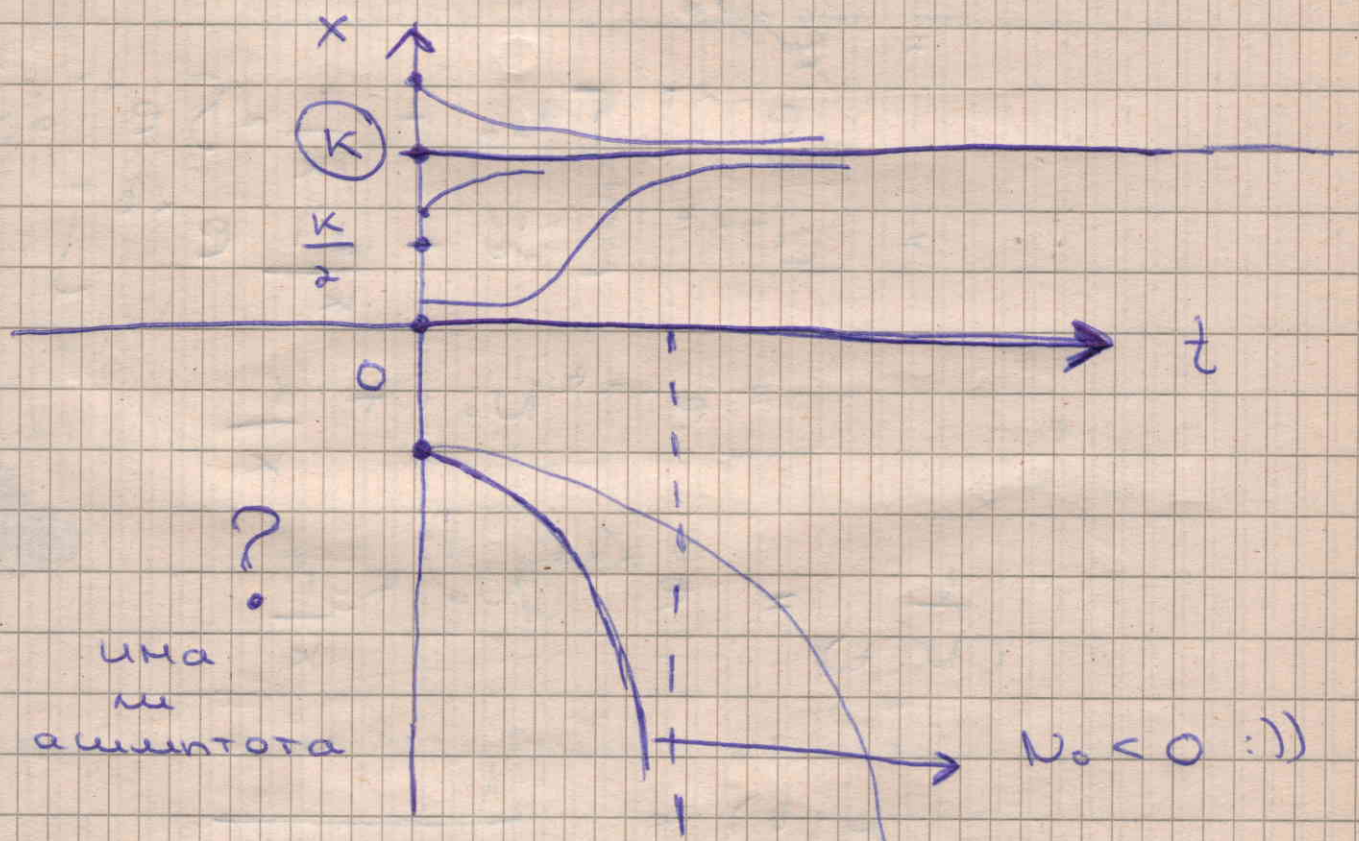
K - брой индивиди

$\lambda > 0$

$$f(x) = x \left(1 - \frac{x}{K}\right)$$

	0	$\frac{K}{2}$	K	
f	-	0	+	+
f'	+		+	-
<u>$f \cdot f'$</u>	-		+	-
	$x - \frac{1}{K}x^2$	$1 - \frac{2}{K}x$		

$$f'(x) = 1 - \frac{2x}{K}$$



$$\dot{N}(t) = 2N(t) \left(1 - \frac{N(t)}{K} \right) =$$

$$\dot{N}(t) = 2N(t) - \frac{2N^2(t)}{K}$$

~~scribble~~ $z = \frac{1}{N(t)}$ $z(0) = \frac{1}{N_0}$

$$-\frac{1}{z^2} z' = \frac{2}{z} - \frac{2}{z^2 K}$$

$$-z' = 2z - \frac{2}{K}$$

$$\dot{N}(t) = -\frac{1}{z^2} z'$$

$$z' = -2z + \frac{2}{K}$$

$$e^{-\lambda t} \left[\frac{1}{N_0} + \int_0^t \frac{\lambda}{k} e^{\lambda s} ds \right]$$

$$Z = e^{-\lambda t} \left[\frac{1}{N_0} + \int_0^t \frac{\lambda}{k} e^{\lambda s} ds \right]$$

$$= e^{-\lambda t} \left[N_0 + \frac{1}{k} \lambda \int_0^t e^{\lambda x} dx \right] =$$

$$= e^{-\lambda t} \left[N_0 + \frac{\lambda}{k} e^{\lambda t} \right] =$$

$$= e^{-\lambda t} N_0 + \frac{1}{k}$$

$$\frac{1}{N(t)} = e^{-\lambda t} N_0 + \frac{1}{k}$$

$$N(t) = \frac{1}{e^{-\lambda t} N_0 + \frac{1}{k}}$$

~~$$Z(t) = e^{-\lambda t} \left[\frac{1}{N_0} + \int_0^t \frac{\lambda}{k} e^{\lambda s} ds \right]$$~~

$$Z(t) = \frac{1}{N_0} e^{-\lambda \int_0^t \lambda s} + \int_0^t \frac{\lambda}{k} e^{-\lambda \int_s^t \lambda s}$$

$$\Rightarrow z(t) = e^{-\lambda t} \left(\frac{1}{N_0} - \frac{1}{K} \right) + \frac{1}{K}$$

$$z(t) = \frac{1}{K} \left(1 - \left(1 - \frac{K}{N_0} \right) e^{-\lambda t} \right)$$

$\frac{1}{z(t)}$
 $N(t) = \frac{K}{1 - \left(1 - \frac{K}{N_0} \right) e^{-\lambda t}}$ $\xrightarrow[t \rightarrow \infty]{K}$

$$N_0 > 0$$

за крайню или безкрайню време

$$\eta = \frac{N}{K}$$

$$\dot{\eta} = \lambda \eta (1 - \eta)$$

$$t = \frac{\ln 2 \tilde{\tau}}{\lambda}$$

$$e^{-\lambda t} = e^{-\ln 2 \tilde{\tau}} = 2^{-\tilde{\tau}}$$

$$\lambda = \frac{1}{2} \quad N_0 = \frac{K}{4}$$

$$N(t) = \frac{3}{4} K$$

$$t = ?$$

$$\frac{1}{1 - \left(1 - \frac{K}{4}\right)} 2^{-t} = \frac{3}{4}$$

$$(1 - (-3)) \cdot 2^{-t} = \frac{4}{3}$$

$$2^{2-t} = \frac{4}{3}$$

$$2-t = \log_2 \frac{4}{3}$$

$$t = 2 - \log_2 \frac{4}{3}$$

$$t = \frac{2 \ln 2}{\ln 2} (2 - \log_2 \frac{4}{3})$$

$$t = 2 \ln 2 (2 - \log_2 \frac{4}{3})$$

$$2 \ln 2 (2 - (\log_2 4 - \log_2 3))$$

$$= 2 \ln 2 (2 - 2 + \log_2 3) =$$

$$= 2 \ln 2 \log_2 3$$

$$t = 2 \ln 3$$

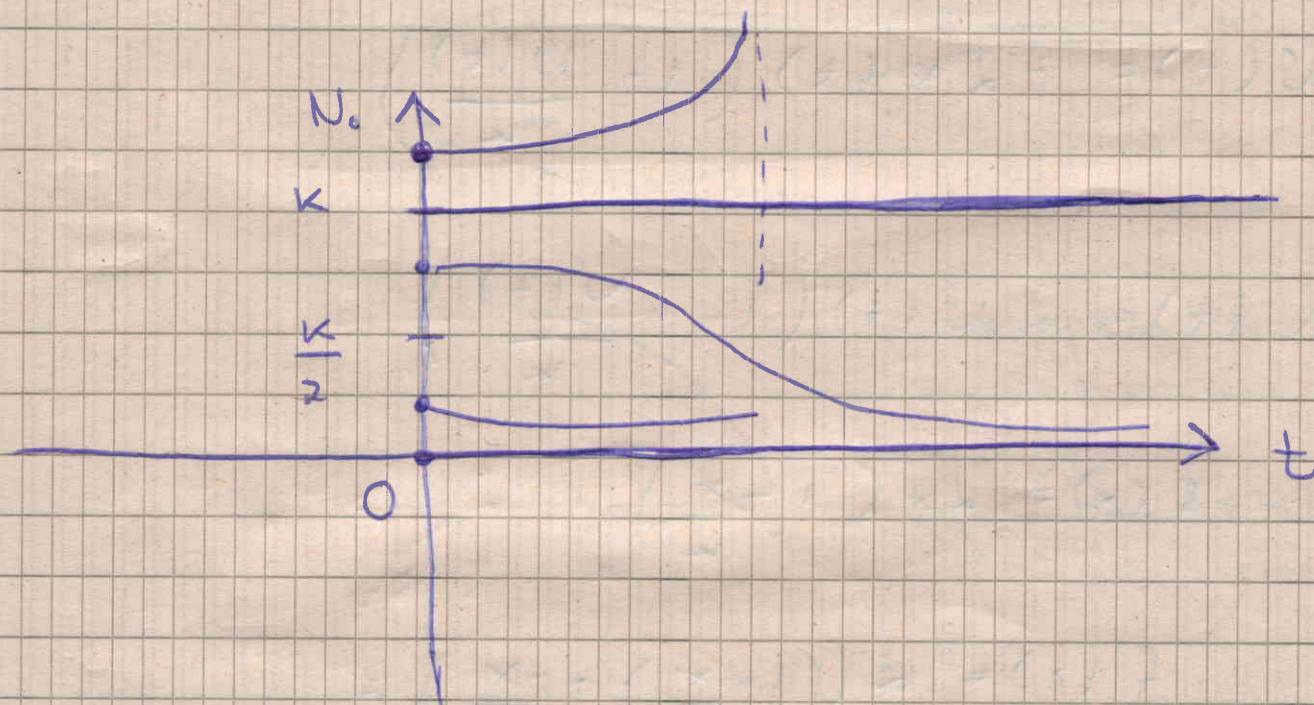
$$t = 4 \ln 3 ?$$

неотрицателна
изменчивост

$$\dot{N}(t) = -2N(t) \left(1 - \frac{N(t)}{K}\right)$$

популация, която намалява във времето

$$\Rightarrow N(t) = \frac{K}{1 - \left(1 - \frac{K}{N_0}\right) e^{2t}}$$



	0	$\frac{K}{2}$	K
f	+	0	-
f'	-	-	+
$f \cdot f'$	-	+	-

$N_0 > K \rightarrow$

$$\frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{2t}}$$

$$\dot{N}(t) = \underbrace{-2}_{\text{добавиме}} N(t) \left(1 - \frac{N(t)}{k}\right) \left(1 - \frac{N(t)}{L}\right)$$

не може да расте
неогр.

29.10.2013г.

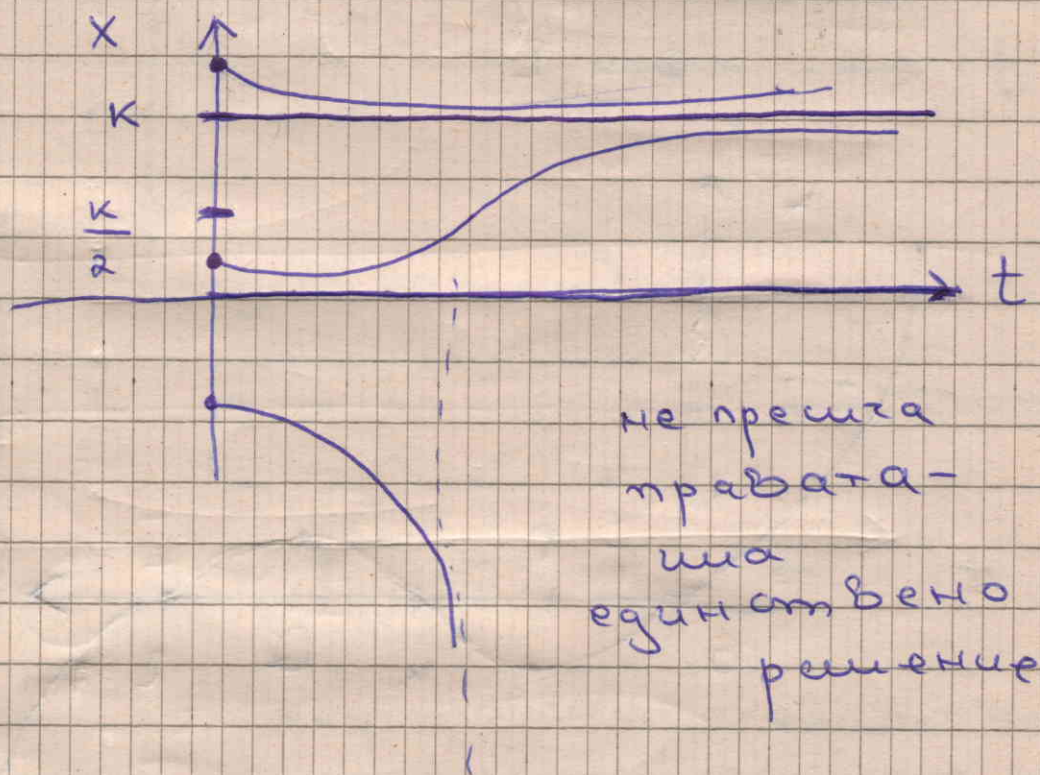
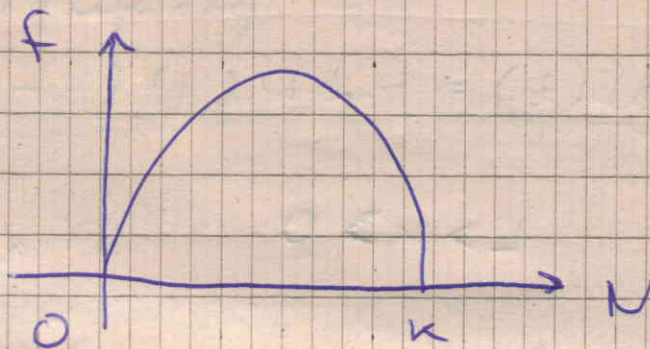
$$\dot{N}(t) = -2 N(t) \left(1 - \frac{N(t)}{k}\right)$$

$$\dot{N}(t) = 2 N(t) \left(1 - \frac{N(t)}{k}\right)$$

$$f(x) = 2x \left(1 - \frac{x}{k}\right)$$

$$\begin{aligned} f'(x) &= 2 \left(\left(1 - \frac{x}{k}\right) + -\frac{x}{k} \right) = \\ &= 2 \left(1 - \frac{2x}{k} \right) \end{aligned}$$

	N	0	$\frac{k}{2}$	k	
$\dot{x} = f$	-	+	+	-	
f'	+	+	-	-	
$f \cdot f' = \ddot{x}$	-	+	-	+	



$$\dot{M}(t) = -2 M(t) \left(1 - \frac{M(t)}{K} \right) \quad M(0) =$$

$$\dot{N}(t) = 2 N(t) \left(1 - \frac{N(t)}{K} \right) \quad N(0) = N_0$$

? връзка M/N M и N

c - - верт. асимптота
разширява се неогр.

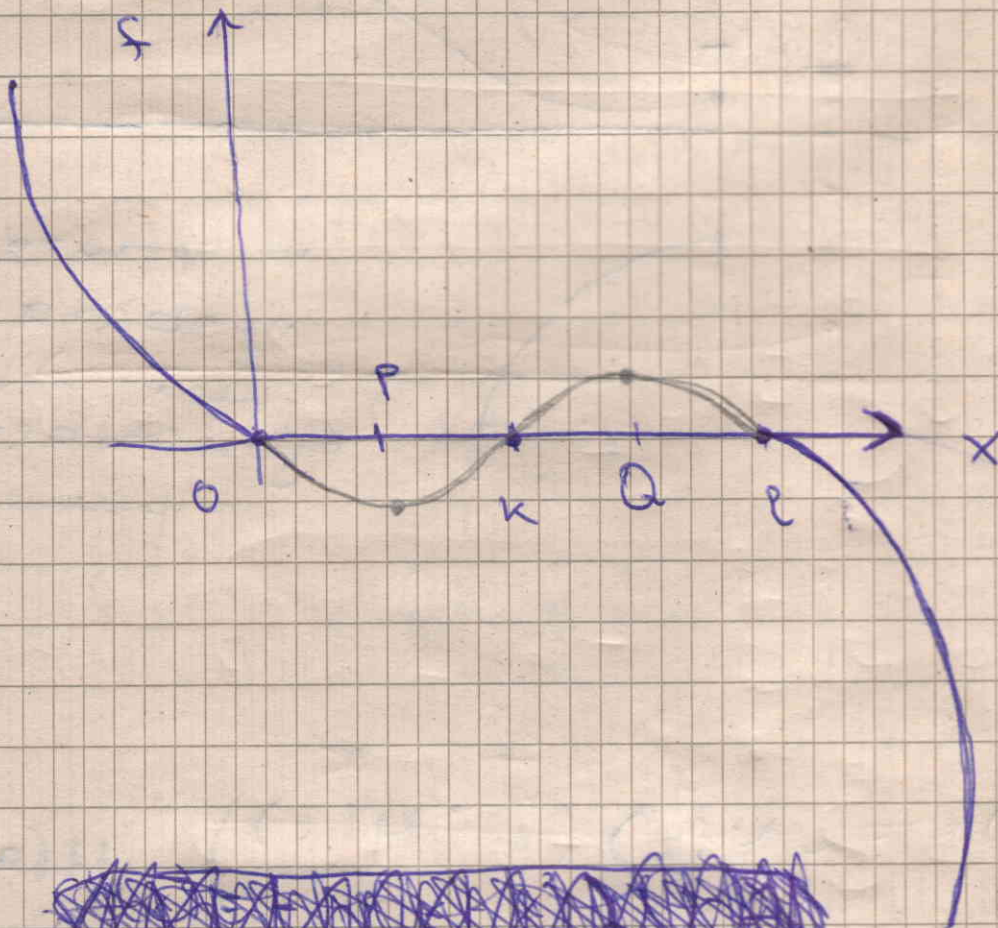
$$N(t) = -2N(t) \left(1 - \frac{N(t)}{k}\right) \cdot \left(1 - \frac{N(t)}{L}\right)$$

$$L > k > 0$$

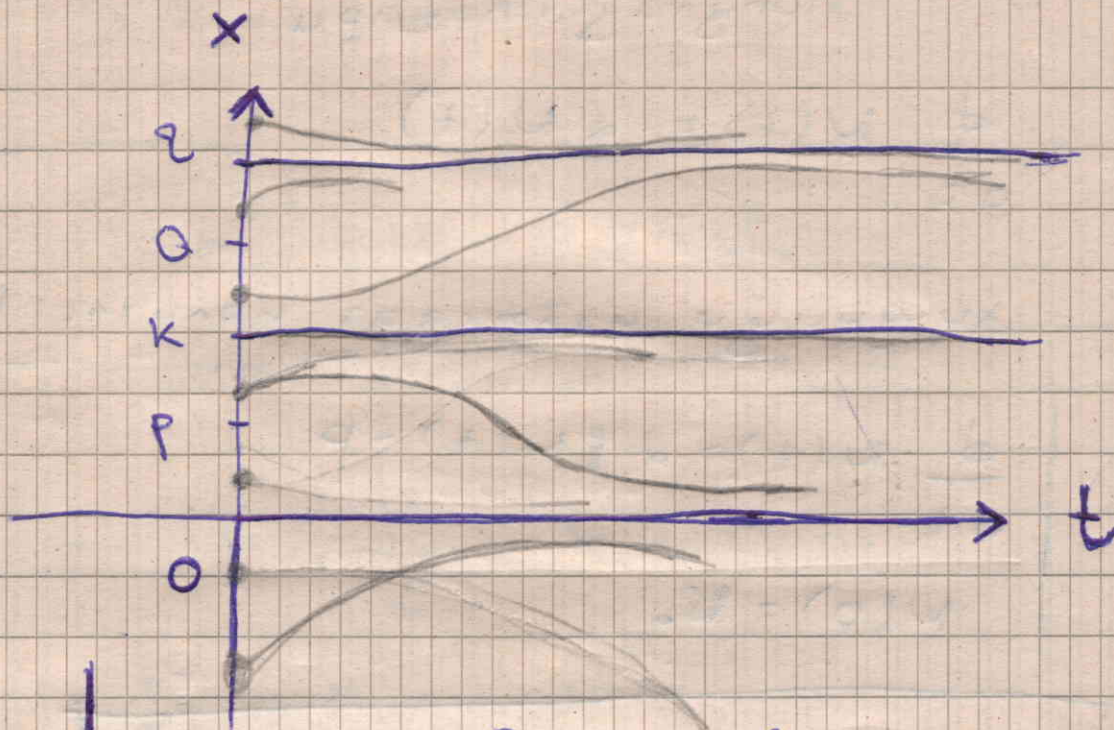
$$\underline{\underline{\lambda > 0}}$$

$$f(x) = -x \left(1 - \frac{x}{k}\right) \left(1 - \frac{x}{L}\right)$$

$$\underline{\underline{0, k, L}}$$



P, Q - нули
производна



		0	P	K	Q	2						
$\dot{x}$	f	+	0	-	-	-	0	+	+	+	0	-
	f'	-	-	-	0	+	+	0	-	-	-	-
$\ddot{x}$		-	+	-	-	+	+	-	-	+	-	+

(K-устойчива стойность на популацията (с + и без 2))

K-не е устойчива
K-граница на оцеляване
P-устойчива равновесна точка

Популяция Комари

$$\frac{d}{dt} N(t) = \lambda N(t)$$

хищници измислят количество p

$$\frac{d}{dt} N(t) = \lambda N(t) - p$$

$$N(0) = N_0$$

- линейни у-я-да се сметне
- груп-качествено

$$N(t) = e^{\int_0^t \lambda ds} \left[N_0 - \int_0^t e^{-\int_0^s \lambda dn} p ds \right] =$$

$$= e^{\lambda t} \left[N_0 - \int_0^t e^{\lambda(t-s)} p ds \right] =$$

$$= e^{\lambda t} \left[N_0 - p \int_0^t e^{\lambda(t-s)} ds \right] =$$

$$= e^{\lambda t} \left[N_0 - p \int_0^t (1-\lambda) d e^{\lambda(t-s)} \right] =$$

$$= e^{\lambda t} \left[N_0 + p \lambda e^{\lambda t} \right]$$

$$- \int_0^t e^{\lambda s} \frac{P}{2} ds = + \frac{P}{2} \int_0^t e^{\lambda(t-s)} \lambda(t-s) ds$$

$$N(t) = e^{\lambda t} \left\{ N_0 - \frac{P}{2} \right\} + \frac{P}{2}$$

$$1. N_0 = \frac{P}{2} \quad N(t) = \text{const}$$

$$2. N_0 > \frac{P}{2} \quad N(t) \xrightarrow{t \rightarrow \infty} \infty$$

$$3. N_0 < \frac{P}{2}$$

$$-Ae^{\lambda t} + B = 0$$

$$e^{\lambda t} = \frac{B}{A}$$

$$\lambda t = \ln \frac{B}{A}$$

$$t = \frac{\ln \frac{B}{A}}{\lambda} > 0$$

$\Rightarrow$ за время t_0 масса 0

$$t_0 = \frac{1}{\lambda} \ln \frac{\frac{P}{2}}{\frac{P}{2} - N_0} > 0$$

1 юли - 15 септ.
46 дни

заг.

200 000 комара
за 1 седмица се удвояват
20 000 за 1 денонощие

$$N(t) = e^{\lambda t} \left(N_0 - \frac{P}{2} \right) + \frac{P}{2}$$

$$N_0 = 200\,000$$

$$P = 20\,000$$

$$t_0 = 7.24 \text{ h}$$

$$N(t) = e^{\lambda t} \left(N_0 - \frac{P}{2} \right) + \frac{P}{2}$$

$$N(t) = N_0 e^{\lambda t} \quad \text{нма хищници}$$

$$2N_0 = N_0 e^{\lambda \cdot 7}$$

$$\lambda = [\text{s}]^{-1}$$

$$e^{\lambda \cdot 7} = 2$$

$$7\lambda = \ln 2$$

$$\lambda = \frac{\ln 2}{7} = 0,099$$

$$? \quad N_0 = \frac{P}{2}$$

$$N_0 > \frac{P}{2}$$

$$N_0 < \frac{P}{2}$$

202020

2020

4,6
2

$$N_0 < \frac{p}{2}$$

$$t_0 = \frac{1}{2} \ln \left(\frac{\frac{p}{2}}{\frac{p}{2} - N_0} \right) = \text{scribbled out}$$

$$t_0 = 46,5 \text{ g.HU}$$

$$\frac{d}{dt} N(t) = 2 N(t) \left(1 - \frac{N(t)}{K} \right) - p$$

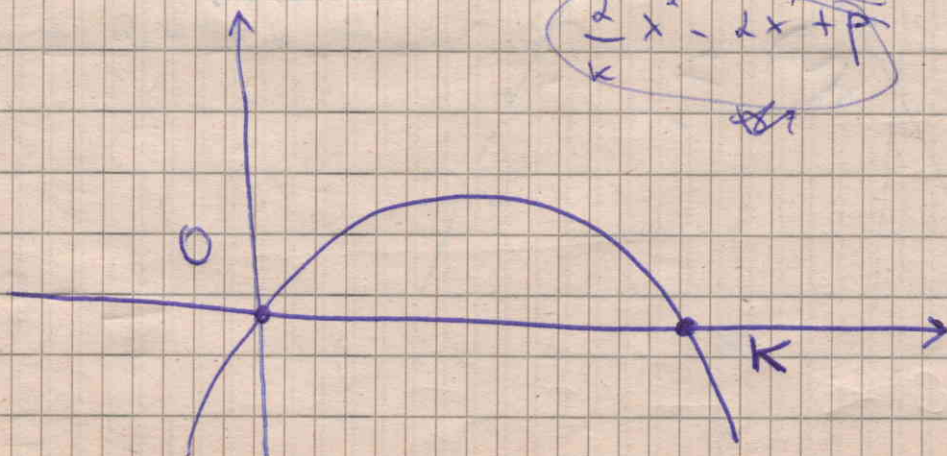
$$f(x) = 2x \left(1 - \frac{x}{K} \right) - p$$

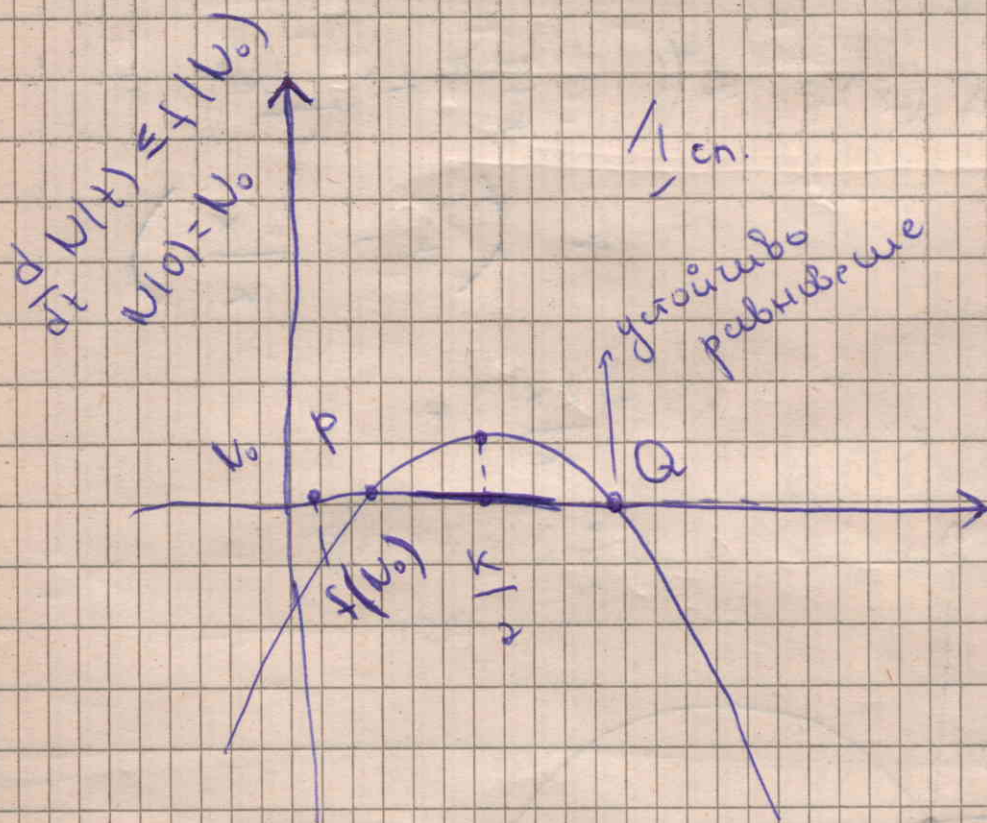
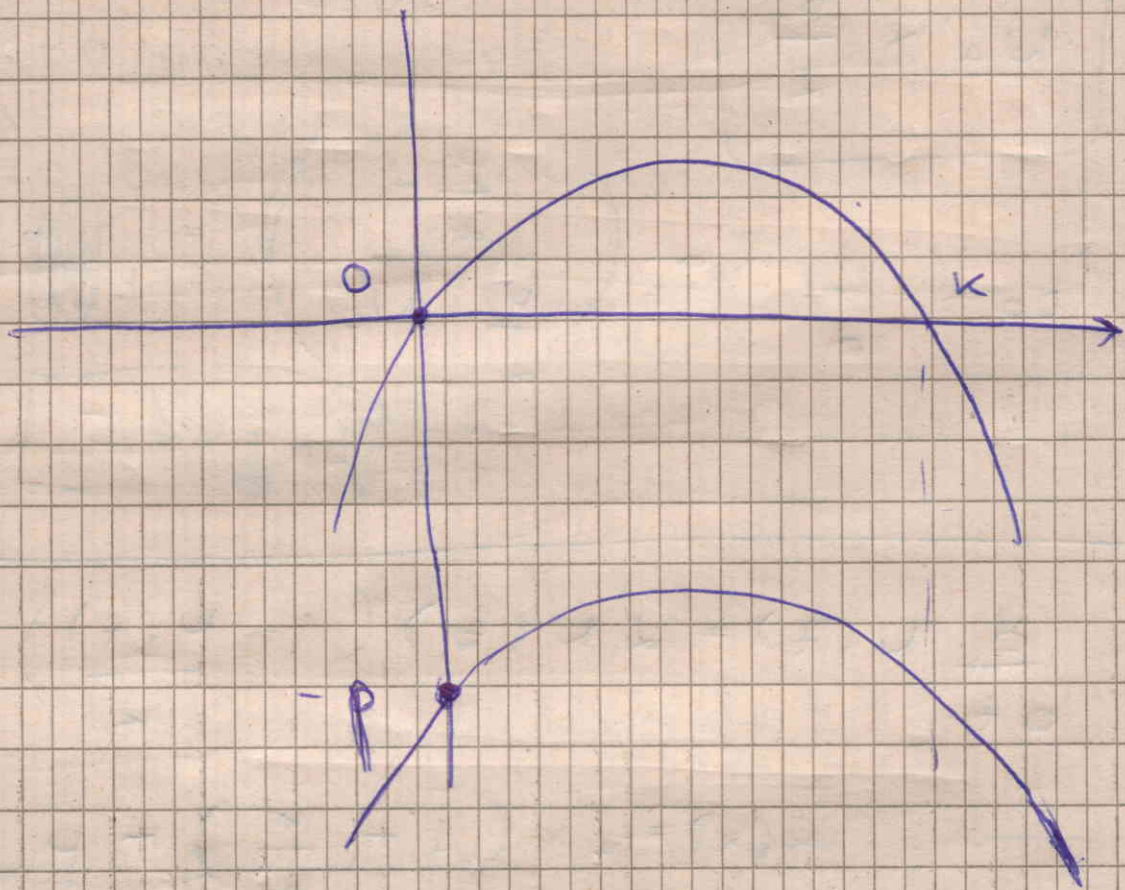
$$f'(x) = 2 \left(\left(1 - \frac{x}{K} \right) + x \left(-\frac{1}{K} \right) \right) =$$

$$= 2 \left(1 - \frac{2x}{K} \right)$$

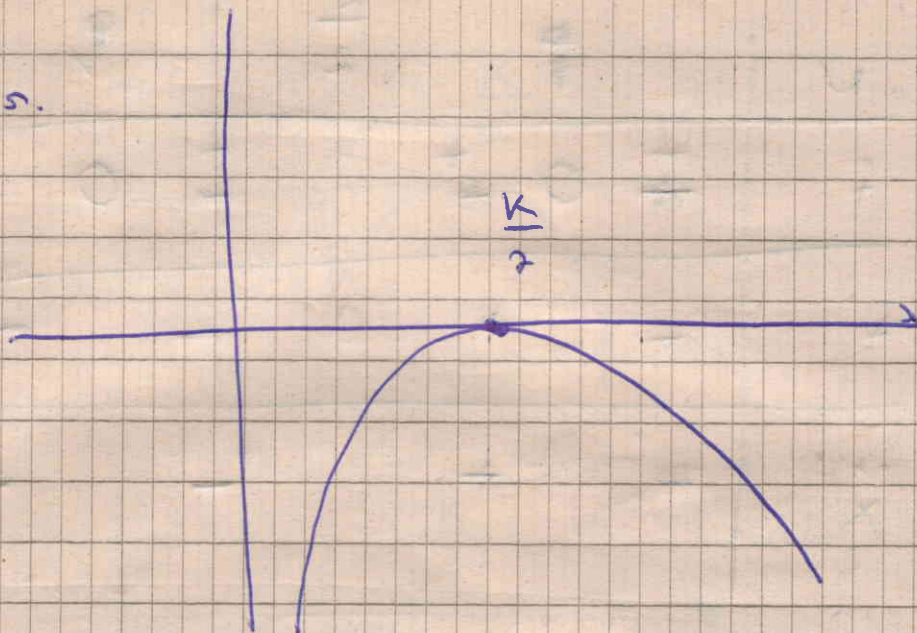
$$2x - \frac{2x^2}{K} - p$$

$$\frac{2}{K} x^2 - 2x + p$$

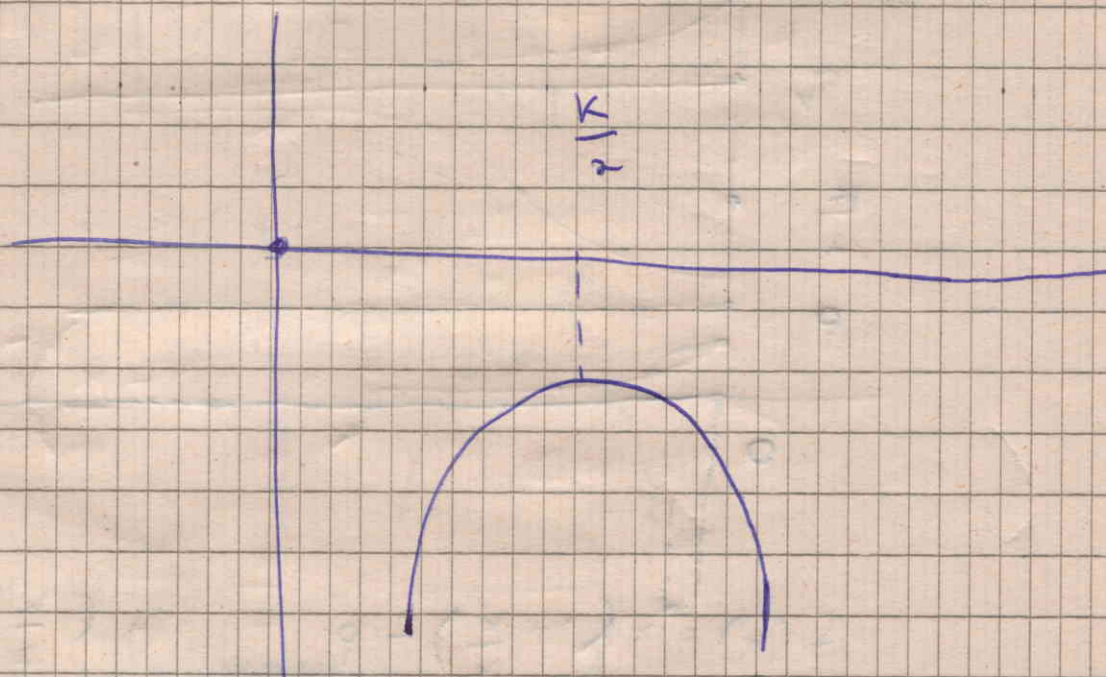




III сн.



III сн.



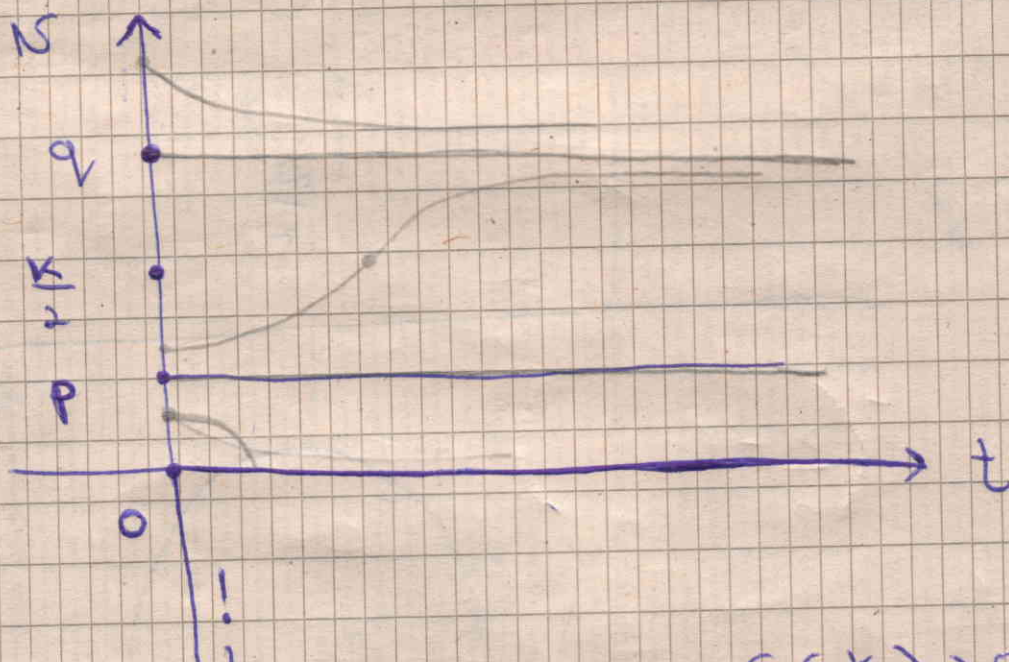
Винаги върхът е $\frac{k}{2}$

$$f(x) = x \left(1 - \frac{x}{k}\right) - p$$

$$f'(x) = 1 - \frac{2x}{k} = 0 \rightarrow x = \frac{k}{2}$$

1. con.

	N	P	$\frac{k}{2}$	Q		
f	-	0	+	+	0	-
f'	+	+	0	-	-	-
$\ddot{x}$	-	+	-	+	+	-



$$f(x) = 2x \left(1 - \frac{x}{k}\right) - p \quad f\left(\frac{k}{2}\right) > 0$$

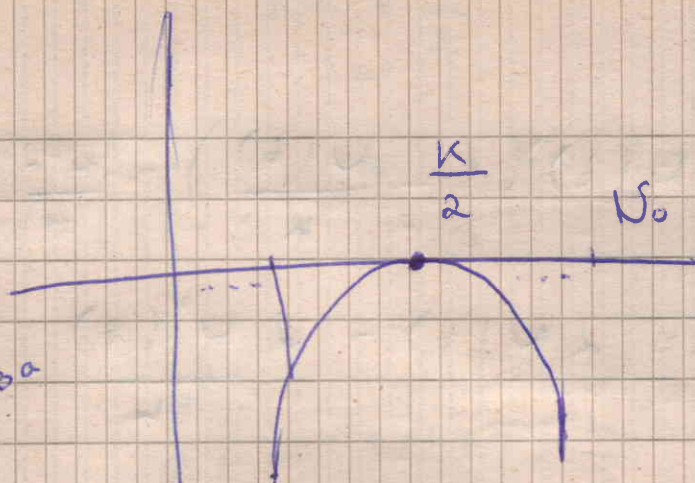
$$f'(x) = 2 \left(1 - \frac{2x}{k}\right) \quad f\left(\frac{k}{2}\right) = \frac{k}{2} \left(1 - \frac{k/2}{k}\right) - p =$$

$$= \frac{k}{2} \left(1 - \frac{1}{2}\right) - p = \frac{k}{4} - p$$

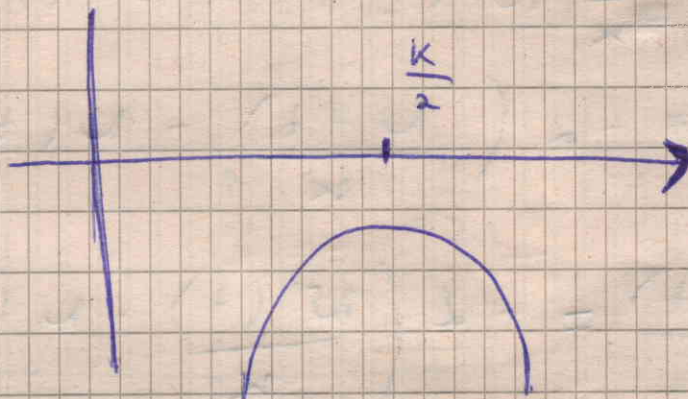
$$\frac{k}{4} > p$$

I сн.

реш.
узелва

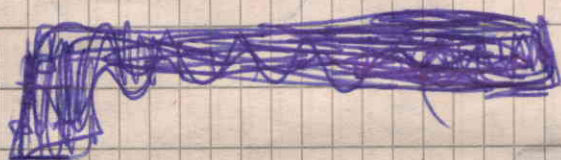
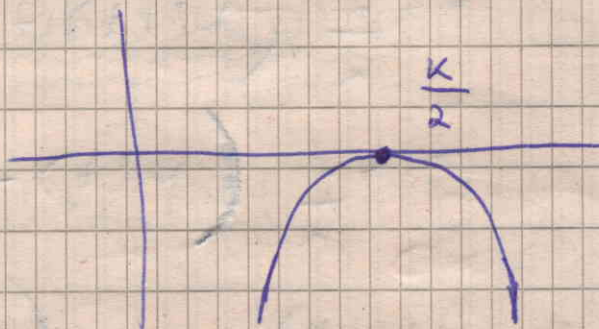


III сн.



за кр. време
популацията
узелва

II сн.



$$\frac{d}{dt} N(t) = 2N(t) \left(1 - \frac{N(t)}{k} \right) - p$$

$$f\left(\frac{k}{2}\right) = 0$$

$$\rightarrow \frac{2k}{4} - p = 0$$

$$p = \frac{2k}{4}$$

$$\frac{d}{dt} N(t) = 2N(t) \left(1 - \frac{N(t)}{k} \right) - \frac{k}{4}$$

$$2N(t) \left(1 - \frac{N(t)}{K}\right) - \frac{K^2}{4} =$$

$$= 2N(t) - \frac{2N^2(t)}{K} - \frac{K^2}{4} =$$

$$= -\frac{2}{K} N^2(t) + 2N(t) - \frac{K^2}{4} =$$

$$= -2 \left(\frac{N^2(t)}{K} - N(t) + \frac{K}{4} \right)$$

$$\frac{dN(t)}{dt} = -2 \left(\frac{N^2(t)}{K} - N(t) + \frac{K}{4} \right) =$$

$$= -\frac{2}{K} \left(N^2(t) - N(t)K + \frac{K^2}{4} \right) =$$

$$= -\frac{2}{K} \left(N(t) - \frac{K}{2} \right)^2$$

$$\Rightarrow \frac{dN(t)}{dt} = -\frac{2}{K} \left(N(t) - \frac{K}{2} \right)^2$$

$$V = \frac{K}{2}$$

~~$$\frac{d}{dt} \left(N(t) - \frac{K}{2} \right)^2 = 0$$~~

$$\left(N(t) - \frac{K}{2} \right)^2 = 0$$

$$2 \left(N(t) - \frac{K}{2} \right) \cdot \frac{dN(t)}{dt} = 0$$

$$N_0 \neq \frac{k}{2}$$

$$-\frac{1}{\left(N(t) - \frac{k}{2}\right)^2} \cdot \frac{d}{dt} \left(N(t) - \frac{k}{2}\right) = -\frac{2}{k}$$

$$\int_0^t \frac{d}{dt} \frac{1}{N(t) - \frac{k}{2}} = -\frac{2}{k}$$

$$\frac{1}{N(t) - \frac{k}{2}} = -\frac{2}{k}$$

$$N(t) - \frac{k}{2} = -\frac{k}{2}$$

$$N(t) = \frac{k}{2} + \frac{k}{2} = k$$

$$N(t) = \frac{k}{2} + \frac{k}{2} = k$$

05.11.2013r.

$$\dot{N}(t) = 2N(t) \left(1 - \frac{N(t)}{k}\right) - \frac{2k}{4}$$

$$N(0) = N_0$$

$$\dot{N}(t) = -\frac{2}{k} \left(N(t) - \frac{k}{2}\right)^2 =$$

$$= -\frac{2}{k} \left(\frac{N(t)}{k} - \frac{1}{2}\right)^2$$

$$\frac{d}{dt} \frac{N(t)}{k} = -2 \left(\frac{N(t)}{k} - \frac{1}{2}\right)^2$$

$$\frac{d}{dt} \left(\frac{v(t)}{k} - \frac{1}{2} \right) = -2 \left(\frac{v(t)}{k} - \frac{1}{2} \right)^2$$

$$x(t) = \frac{v(t)}{k} - \frac{1}{2}$$

$$\frac{d}{dt} x(t) = -2x^2(t)$$

$$\frac{d}{dt} \left(\frac{1}{x(t)} \right) = 2$$

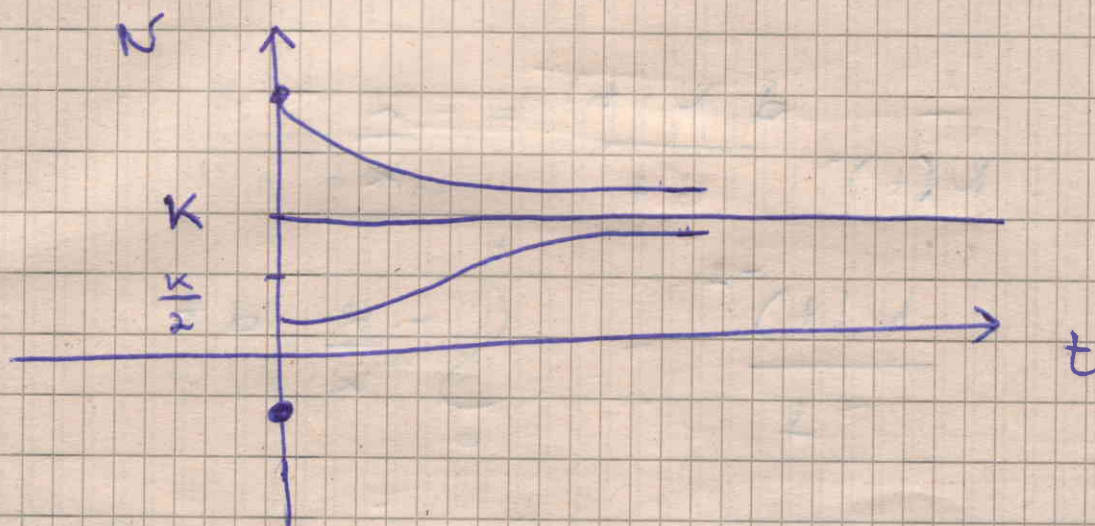
$$y(t) = \frac{1}{x(t)}$$

$$\left\{ \begin{array}{l} \frac{d}{dt} y(t) = 2 \\ y(0) = \frac{1}{x(0)} = \frac{1}{\frac{v_0}{k} - \frac{1}{2}} \end{array} \right.$$

~~1/2~~

$$y(t) = y_0 + 2t$$

$$\frac{d}{dt} N(t) = \alpha N(t) \left(1 - \frac{N(t)}{K}\right)$$



$K = 14 \cdot 10^6$ кг риба (количество риба, към което популацията се стреми)

$$\alpha = 0,7$$

Рибек, който лови риба

$$\frac{d}{dt} N(t) = \alpha N(t) \left(1 - \frac{N(t)}{K}\right) - \beta N(t)$$

$$\frac{d}{dt} N(t) = (\alpha - \beta) N(t) - \frac{\alpha}{K} N^2(t)$$

1 сл. Ако $\alpha = \beta$

$$\frac{d}{dt} N(t) = -\frac{\alpha}{K} N(t)^2$$

$$\frac{d}{dt} N(t) = -\frac{\lambda}{k} N(t)^2$$

$$\frac{1}{N(t)^2} \frac{d N(t)}{dt} = -\frac{\lambda}{k}$$

$$\frac{N(t)^{-1}}{-1} = \int_0^t -\frac{\lambda}{k} dt$$

$$-\frac{1}{N(t)} = -\frac{\lambda t}{k}$$

$$\frac{1}{N(t)} = \frac{\lambda}{k} t$$

$$N(0) = N_0 > 0$$

$$N(0) = 0 \text{ e per.}$$

$$Y(t) = \frac{1}{N(t)}$$

$$\frac{d}{dt} Y(t) = \frac{\lambda}{k}$$

$$Y(t) = Y(0) + \frac{\lambda t}{k}$$

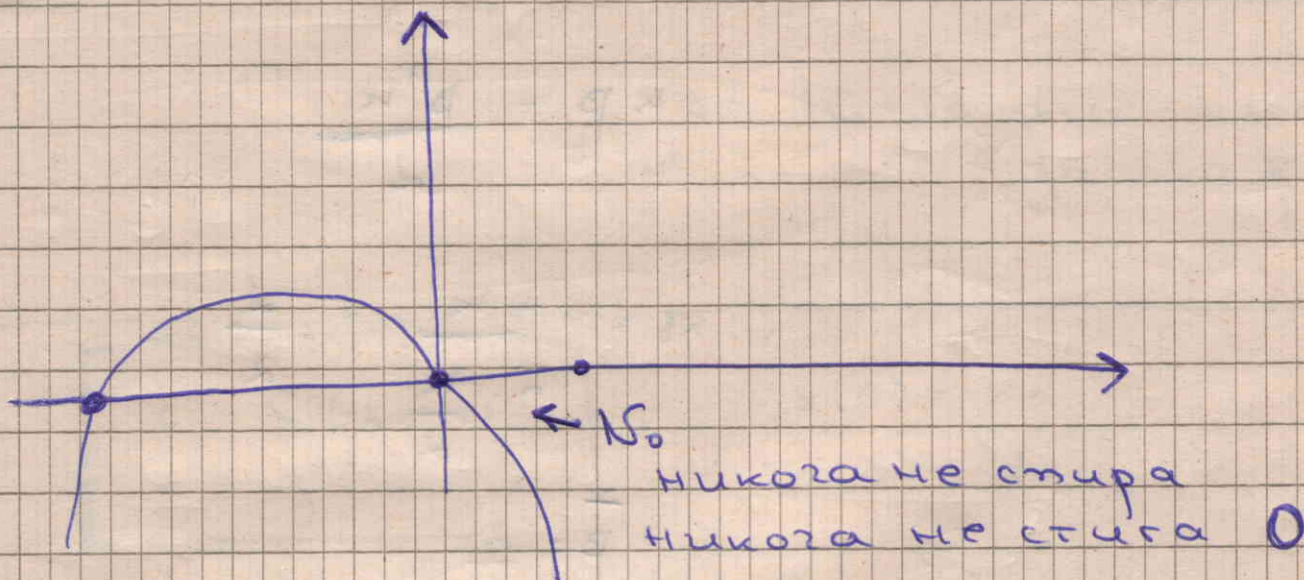
$$N(t) = \frac{1}{\gamma(0) + \frac{t\alpha}{\kappa}} = \frac{1}{\frac{1}{N_0} + \frac{t\alpha}{\kappa}}$$

$$= \frac{\kappa N_0}{\kappa + \alpha t N_0}$$

$$t \rightarrow \infty$$

$$N(t) \rightarrow 0$$

II сл. $\alpha < \beta$

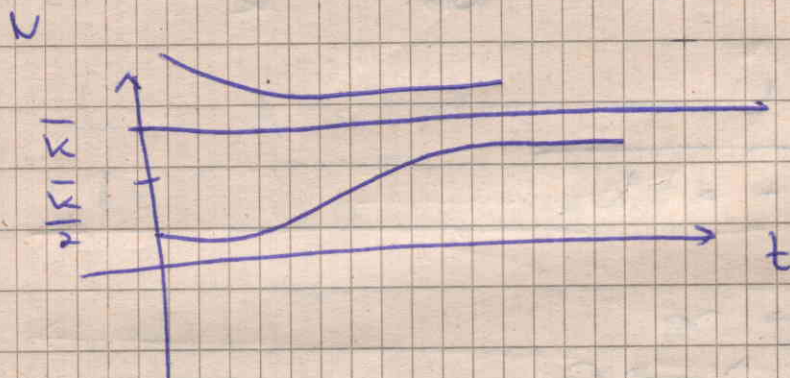


III сл. $\alpha > \beta$

$$\frac{d}{dt} N(t) = (\alpha - \beta) N(t) \left[\frac{1 - \alpha N(t)}{(\alpha - \beta) \kappa} \right] =$$

$$= (\alpha - \beta) N(t) \left[1 - \frac{\alpha N(t)}{(\alpha - \beta) \kappa} \right]$$

същото поведение за $\bar{k} = \left(\frac{2-\beta}{2}\right) k$



ще ловим риба $\beta \bar{k} = \left(1 - \frac{\beta}{2}\right) \beta k$

? максимизиране

$$k\beta - \frac{\beta^2 k}{2}$$

$$x_B = \frac{-k}{-2 \cdot \frac{k}{2}} = \frac{1}{2}$$

$$\bar{\beta} = \frac{2}{2}$$



$$\frac{1}{2} \cdot \frac{2}{2} k$$

$$\bar{\beta} k = \frac{2}{2} k$$

макс. кол. улова

? колко риба е останала

$$\max = \frac{2k}{4}$$

~~$$\frac{2k}{4} - \left(1 - \frac{B}{2}\right) Bk = \frac{2k}{4} - Bk + \frac{B^2 k}{2}$$~~

$$\bar{B} = 0,35$$

~~$$\bar{k} = 7 \cdot 10^6$$~~ ^{останали}

$$\bar{B}\bar{k} = 2,45 \cdot 10^6 \text{ } \forall \text{ година}$$

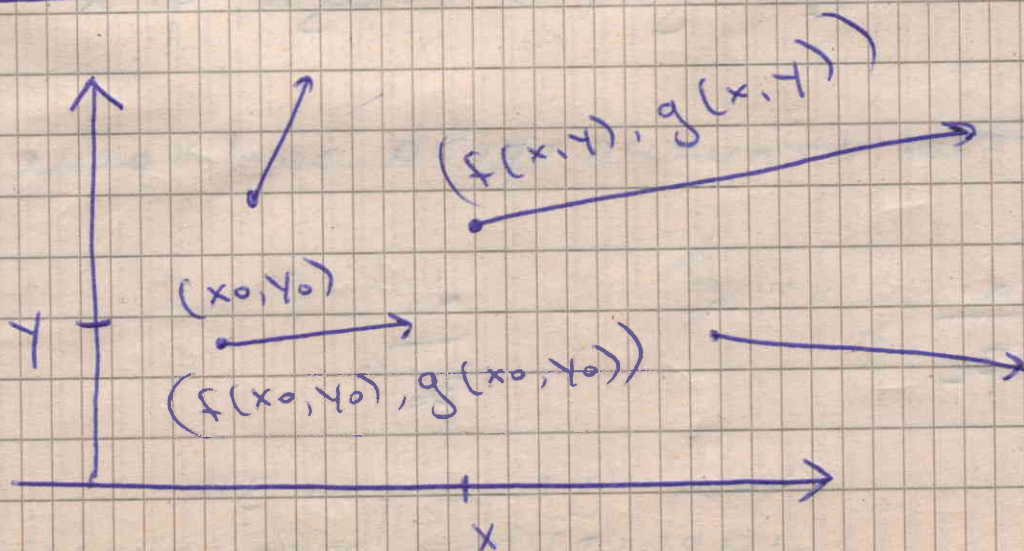
улов

за 10 години - няма да има риба

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

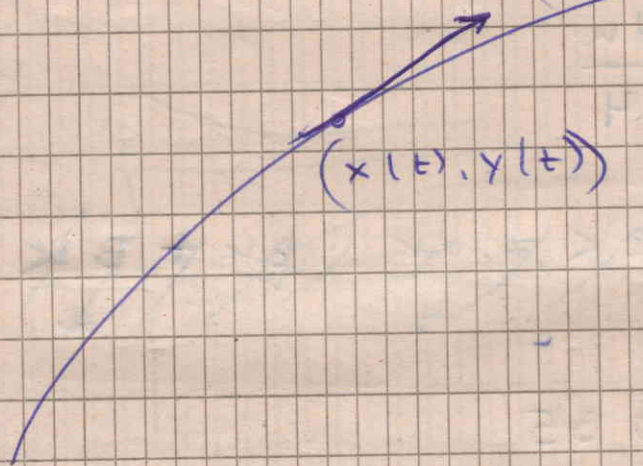
$$x(0) = x_0$$

$$y(0) = y_0$$



допирателна

$(x(t), y(t))$ - крива $\left(\frac{d}{dt} x(t), \frac{d}{dt} y(t)\right)$



||
 $(f(x(t), y(t)), g(x(t), y(t)))$

$$\frac{d}{dt} N_1(t) = \left(\lambda_1 - \beta_1 F_1(N_1(t), N_2(t)) \right) N_1(t)$$

$$\frac{d}{dt} N_2(t) = \left(\lambda_2 - \beta_2 F_2(N_1(t), N_2(t)) \right) N_2(t)$$

коэффициент

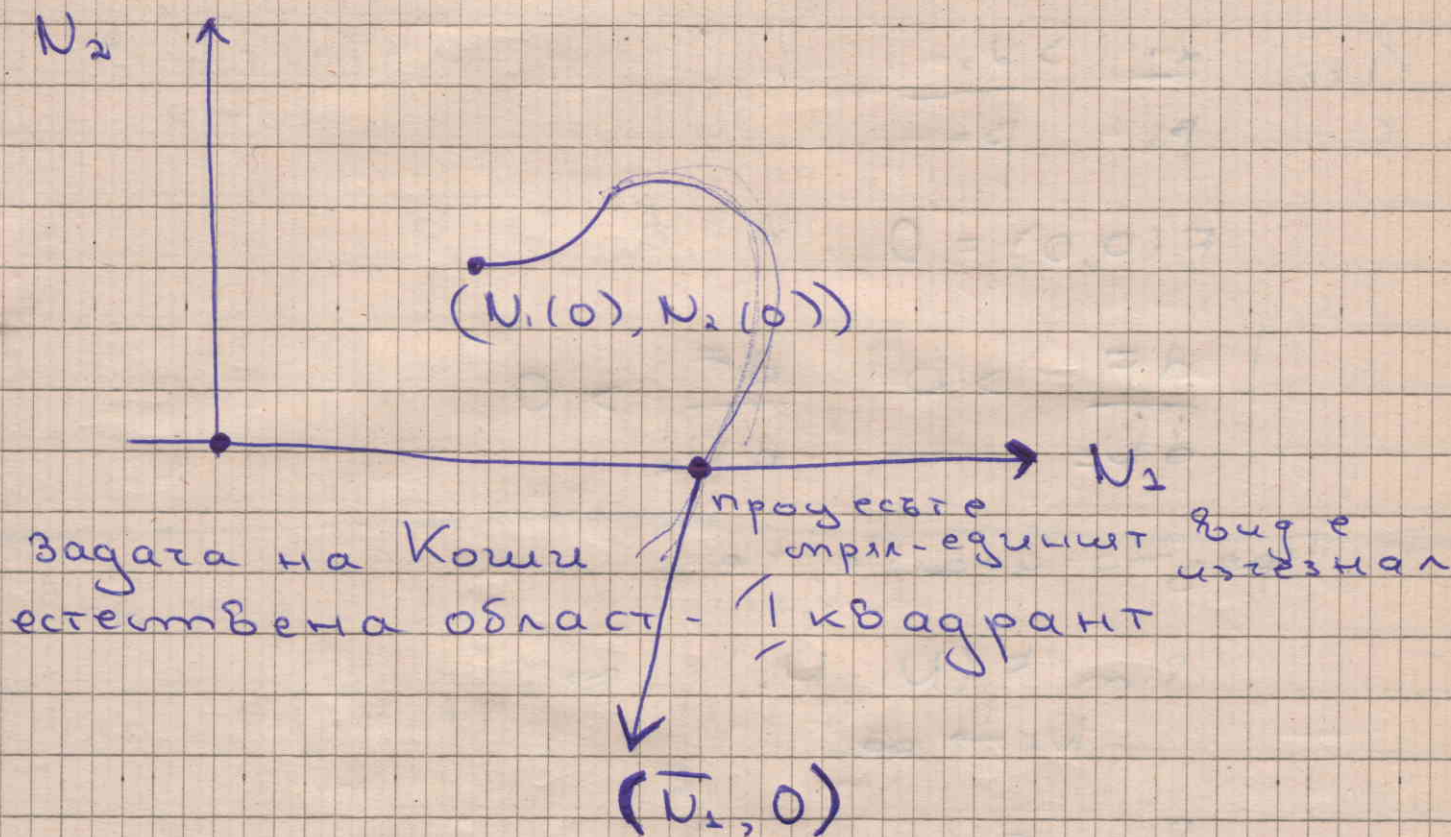
$F_1 = F_2$ една и съща ф-я

F-кол. налична трябва за 1 време

$$\nearrow \frac{\lambda_1}{\beta_1}$$

$$\frac{\lambda_2}{\beta_2}$$

кога ще спрат да нарастват популациите



$$\Rightarrow N_2(t) \equiv 0 \quad N_1(0) = \bar{N}_1$$

$$N_2(0) = 0$$

Th за $\exists$ и !

Инвариантност - решението с начална точка в Θ е в същото Θ

I сл. $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$

II сл. $\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2}$

III сл. $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$

$$\frac{1}{\beta_1} > \frac{1}{\beta_2}$$

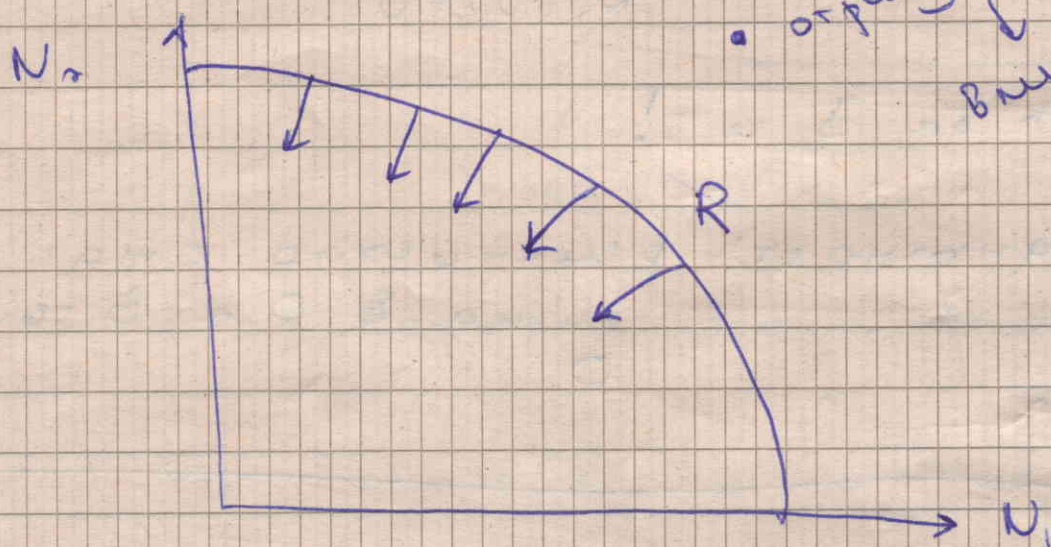
$$F(0,0) = 0$$

$$\left| \frac{dF}{dN_1} > 0 \quad \frac{dF}{dN_2} > 0 \right.$$

Предположим, что

$$\lim_{N_1 \rightarrow \infty} F(N_1, N_2) = \infty$$

$$\lim_{N_2 \rightarrow \infty} F(N_1, N_2) = \infty$$



∃ R:

$$\frac{d}{dt} N_1(t) > 0$$

$$\frac{d}{dt} N_2(t) < 0$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

$$\forall N \exists R$$

$$x > R \\ f(x) > N$$

Ако коеф. $> 0 \rightarrow$ растат

$$\text{III} \text{ сн. } F(\bar{U}_1, \bar{U}_2) = \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$

$$\exists R > 0 \quad \alpha_i - \beta_i F(U_1, U_2) < 0 \quad i = 1, 2 \\ \|U_1, U_2\| > R$$

$$\text{III} \text{ сн. } F(\bar{U}_1, \bar{U}_2) = \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$

$$\Rightarrow \|U_1, U_2\| < R$$

$$U_1(t) \equiv \bar{U}_1 \quad \text{за } \forall t \\ U_2(t) \equiv \bar{U}_2$$

Предполагаме, че г.р. е знакка
 $\Rightarrow$ реш. е !

$$\text{I. cr. } \frac{\lambda_1}{\beta_1} > \frac{\lambda_2}{\beta_2}$$

$$N_1(0) = N_1^0 > 0$$

$$N_2(0) = N_2^0 > 0$$

$$1 - \beta_2$$

$$\frac{1}{N_1(t)} \frac{d}{dt} N_1(t) = \lambda_1 - \beta_1 F(N_1(t), N_2(t))$$

$$1 - \beta_1$$

$$\frac{1}{N_2(t)} \frac{d}{dt} N_2(t) = \lambda_2 - \beta_2 F(N_1(t), N_2(t))$$

$$\frac{d}{dt} \ln N_1(t)$$

$$\frac{d}{dt} \ln N_2(t)$$

$$\beta_1 \frac{d}{dt} \ln N_2(t) - \beta_2 \frac{d}{dt} \ln N_1(t) =$$

$$= \beta_1 \lambda_2 - \lambda_1 \beta_2 = \gamma < 0$$

$$\frac{d}{dt} \left(\ln N_2(t)^{\beta_1} - \ln N_1(t)^{\beta_2} \right) = \gamma$$

$$\frac{d}{dt} \ln \frac{N_2^{\beta_1}(t)}{N_1^{\beta_2}(t)} = \gamma$$

$$\ln \frac{N_2^{\beta_1}(t)}{N_1^{\beta_2}(t)} = \gamma t - \ln \frac{N_2^{\beta_1}(0)}{N_1^{\beta_2}(0)} = \gamma t$$

$$\ln \frac{N_2^{\beta_1}(t)}{N_1^{\beta_2}(t)} \cdot \frac{N_1^{\beta_2}(0)}{N_2^{\beta_1}(0)} = \gamma t$$

$$\frac{N_2^{\beta_1}(t) \cdot N_1^{\beta_2}(0)}{N_1^{\beta_2}(t) \cdot N_2^{\beta_1}(0)} = e^{\gamma t}$$

$$\frac{N_2^{\beta_1}(t)}{N_1^{\beta_2}(t)} = \frac{N_2^{\beta_1}(0)}{N_1^{\beta_2}(0)} e^{\gamma t} \rightarrow 0 \quad t \rightarrow \infty$$

$$N_2^{\beta_1}(t) = \frac{N_2^{\beta_1}(0)}{N_1^{\beta_2}(0)} \cdot N_1^{\beta_2}(t) e^{\gamma t} \leq$$

~~$N_2^{\beta_1}(t)$~~

$$\leq \frac{N_2^{\beta_1}(t)}{N_1^{\beta_2}(t)} \quad C$$

$$R^{\beta_2} = C > 0$$

единица вид ще излезне

III сл. $\rightarrow \gamma = 0$

$$\dots = 0$$

$$N_2^{\beta_1}(t) = \frac{N_2^{\beta_1}(0) N_1^{\beta_2}(t)}{N_1^{\beta_1}(0)}$$

така връзката и у 2-та
вида
нема смисъл от система



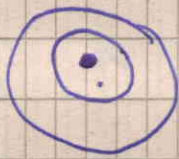


Модель Хищник-Жертва Лотка-Вольтера

$$\frac{d}{dt} x(t) = f(x(t)) \quad x(t) \in \mathbb{R}^n$$

$f(x_0) = 0$ равновесная точка

$x(t) \equiv x_0$ — решение



∃ по-малка ок-т

асимптотична устойчивост
устойчивост

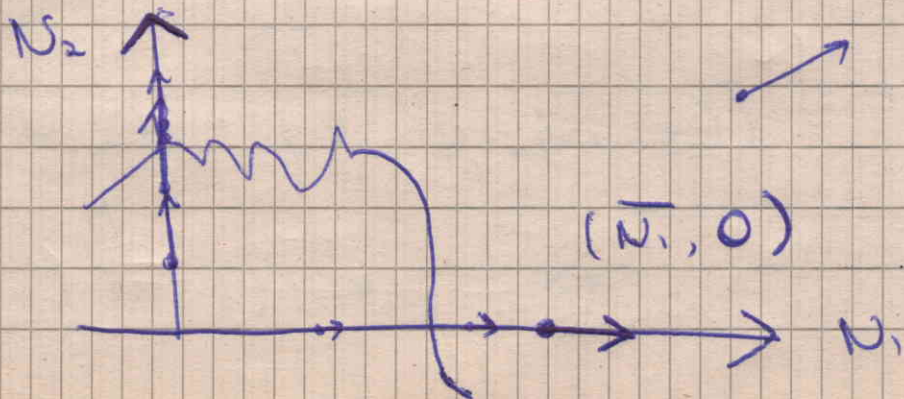
жертва $\frac{d}{dt} N_1(t) = (\alpha_1 - \beta_1 N_2(t)) N_1(t)$

$$\frac{d}{dt} N_2(t) = -(\alpha_2 - \beta_2 N_1(t)) N_2(t)$$

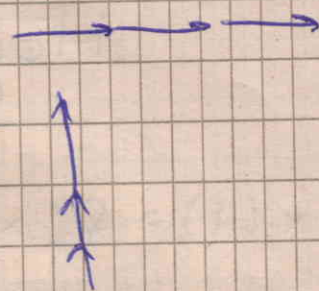
+ жертва

повече жертви —
хищници!))

система
хищник-жертва



Ако наг β_1 абсциса



$\Rightarrow$ не може да пресече

? равновесия



$$\left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1} \right)$$

$$(0,0)$$

N_2



изтъкнало $\odot$

N_1

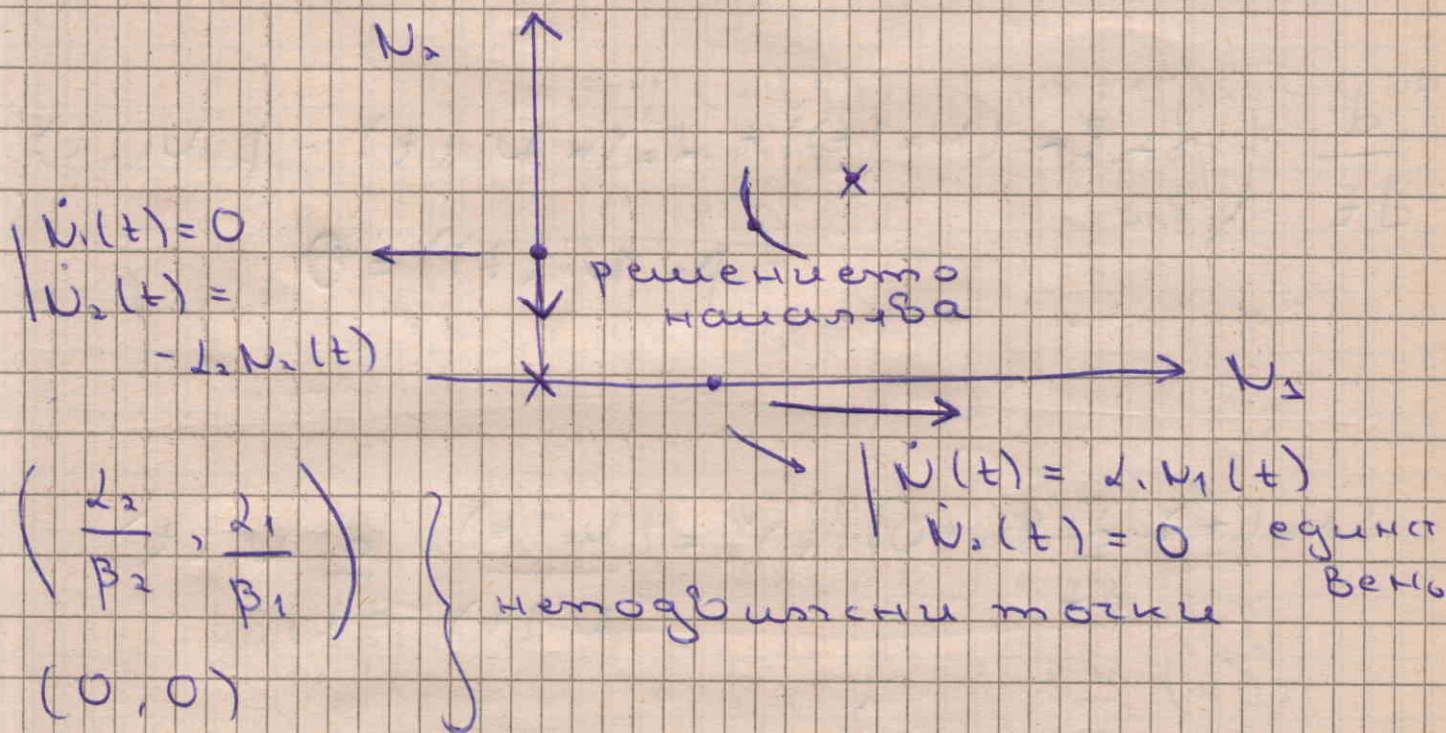
12.11.2013г.

гравитаци

$$\frac{d}{dt} N_1(t) = (\lambda_1 - \beta_1 N_2(t)) N_1(t)$$

хищници

$$\frac{d}{dt} N_2(t) = -(\lambda_2 - \beta_2 N_1(t)) N_2(t)$$



$$\frac{1}{N_1(t)} \frac{d}{dt} N_1(t) = (\lambda_1 - \beta_1 N_2(t))$$

$$\frac{d}{dt} \ln(N_1(t)) = \lambda_1 - \beta_1 N_2(t)$$

$$1: \lambda_1 - \beta_1 N_2(t)$$

$$\frac{d}{dt} \ln(N_2(t)) = -(\lambda_2 - \beta_2 N_1(t))$$

$$\lambda_2 \frac{d}{dt} \ln(N_1(t)) - \beta_2 N_1(t) \frac{d}{dt} \ln(N_1(t)) = 0$$

$$+ \lambda_1 \frac{d}{dt} \ln(N_2(t)) - \beta_1 \overset{N_2(t)}{N_2(t)} \frac{d}{dt} \ln(N_2(t))$$

$$\frac{d}{dt} \left(\lambda_1 \ln(N_2(t)) + \lambda_2 \ln(N_1(t)) - \beta_1 N_2(t) - \beta_2 N_1(t) \right) = 0$$

$$N_1(t) \frac{d}{dt} \ln(N_1(t)) = \frac{N_1(t)}{N_1(t)} \frac{d}{dt} N_1(t)$$

$$V(N_1, N_2) = \lambda_1 \ln N_2 + \lambda_2 \ln N_1 - \beta_1 N_2 - \beta_2 N_1 \quad (*)$$

~~✗~~

$$x \in \mathbb{R}^n$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\frac{d}{dt} x(t) = f(x(t))$$

$$x(0) = x_0 \in \mathbb{R}^n$$

$$x: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$$

$$V: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\frac{d}{dt} V(x(t)) = \frac{d}{dt} V(x(t)) \cdot \frac{d}{dt} x(t) =$$

$$= \underbrace{V'(x(t))}_{\text{вектор}} \cdot \underbrace{f(x(t))}_{\text{вектор}} - \text{число}$$

$$\boxed{\dot{V}(x) = V'(x) f(x)}$$

$$V'(x) = \left(\frac{\partial V}{\partial x_1}(x), \frac{\partial V}{\partial x_2}(x), \dots \right)$$

градиент

$$\dots \frac{\partial V}{\partial x_n}(x))$$

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

$V(x(t))$ - число

⊙

$$\psi(N_1, N_2) = 0$$

ψ -дифференциал във вътрешността

$$\lim_{N_2 \rightarrow 0^+} \psi(N_1, N_2) = -\infty$$

$$\lim_{N_1 \rightarrow 0^+} \psi(N_1, N_2) = -\infty$$

$$\lim_{N_2 \rightarrow +\infty} \psi(N_1, N_2) = -\infty$$

$$2, \ln N_2 - \beta_1 N_2 = -N_2 \left(\beta_1 - 2, \frac{\ln N}{N} \right) \xrightarrow[N \rightarrow \infty]{} -\infty$$

$$\lim_{N_1 \rightarrow \infty} \psi(N_1, N_2) = -\infty$$

ψ
? минимите на нуло-решения

$$\frac{\partial \psi}{\partial N_1}(N_1, N_2) = \frac{\alpha_2}{N_1} - \beta_2$$

$$\frac{\partial^2 \psi}{\partial N_1^2}(N_1, N_2) = -\frac{\alpha_2}{N_1^2}$$

$$\frac{\partial \mathcal{L}}{\partial N_2}(N_1, N_2) = \frac{\alpha_1}{N_2} - \beta_1$$

$$\frac{\partial^2 \mathcal{L}}{\partial N_2^2}(N_1, N_2) = -\frac{\alpha_1}{N_2^2}$$

$$\frac{\partial^2 \mathcal{L}}{\partial N_1 \partial N_2} = 0$$

$$\begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial N_1^2} & \frac{\partial^2 \mathcal{L}}{\partial N_1 \partial N_2} \\ \frac{\partial^2 \mathcal{L}}{\partial N_1 \partial N_2} & \frac{\partial^2 \mathcal{L}}{\partial N_2^2} \end{pmatrix} \stackrel{?}{\geq} 0$$

изтъкнала $- > 0$ и $\text{Det} > 0$

Деф f изтъкнала

$$f: S \rightarrow \mathbb{R}$$

$\downarrow$
изтъкнала

$x_1, x_2 \in S$ произволни

$$\lambda \in (0, 1)$$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq$$

$$\lambda f(x_1) + (1-\lambda)f(x_2)$$

Ако $-f$ е изтъкнала, f - вдлъбната

$$\frac{dz}{d\beta_i} = \beta_i$$

$$\begin{pmatrix} -\frac{\alpha_2}{N_1^2} & 0 \\ 0 & -\frac{\alpha_1}{N_1^2} \end{pmatrix}$$

- ψ -изпъкнала
 ψ -вдлъбната

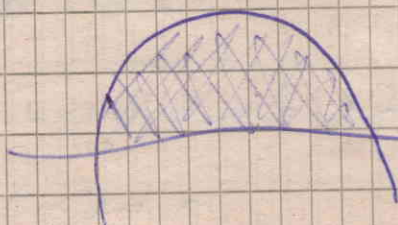
$\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}$ - тах на ψ
 (нумерат се
 производните)

Други няма
 По гр. няма тах, $\psi \rightarrow -\infty$

? Линии на ниво на ψ -решения

$$x_1 \in S \text{ } \exists x_2$$

$$S_2 = \left\{ x \in \mathbb{R}^n \mid \psi(x) \geq 2 \right\}$$



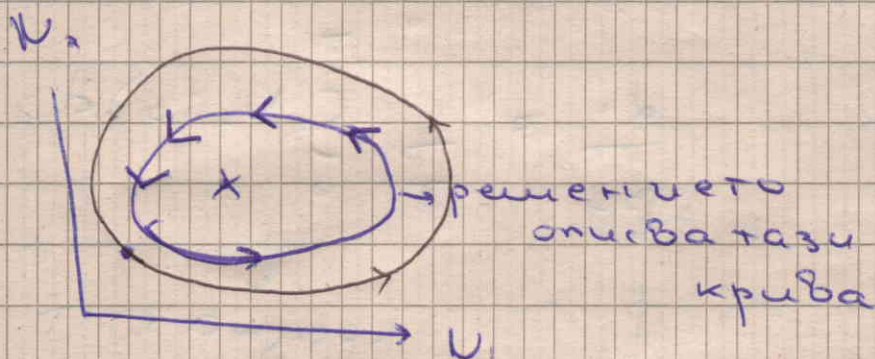
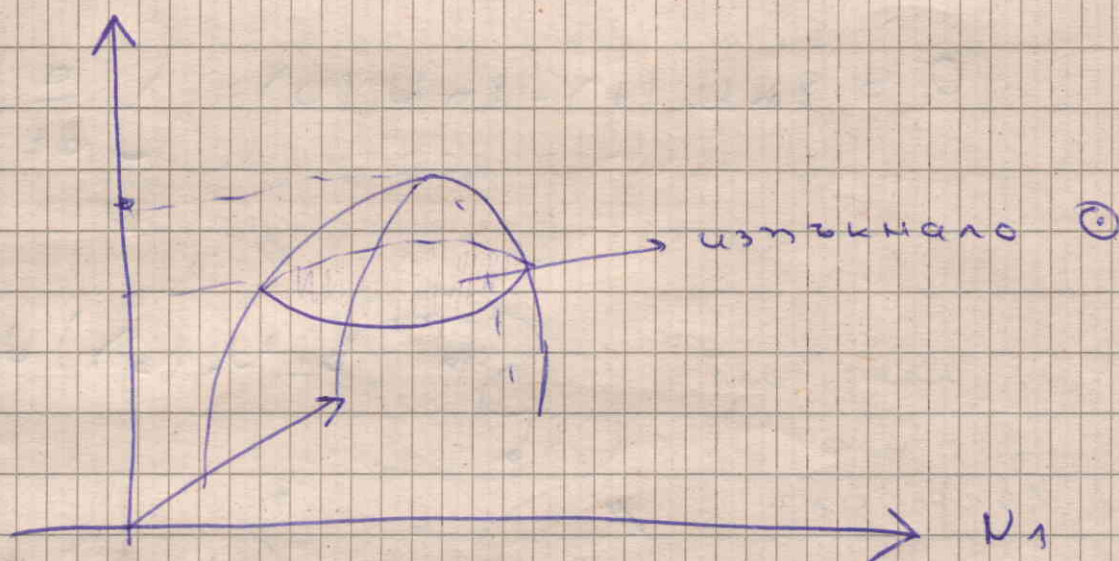
изпъкнало $\odot$
 проскупи

$$S_\beta = \{x \in \mathbb{R}^n \mid f(x) \geq \beta\}$$

по деф. вгълбнатата ф-я

$$\Rightarrow \geq \lambda\beta + (1-\lambda)\beta = \beta \in S_\beta$$

$\Rightarrow$ изтъкнало $\odot$



не, защото
не
спира

? може ли да се смени посоката
друга крива - не се пресичат

T-перуог

$$N_1(T) = N_1(0)$$

$$\frac{1}{N_1(t)} \frac{d}{dt} N_1(t) = \lambda_1 - \beta_1 N_2(t)$$

||

$$\frac{d}{dt} \ln(N_1(t))$$

$$0 = \ln N_1(t) - \ln N_1(0) = \int_0^T \frac{d}{dt} \ln(N_1(s)) ds$$

$$\int_0^T (\lambda_1 - \beta_1 N_2(s)) ds$$

$$\lambda_1 T - \beta_1 \int_0^T N_2(s) ds = 0$$

$$\frac{\lambda_1}{\beta_1} = \frac{1}{T} \int_0^T N_2(s) ds$$

↓
средна стит

↓
стационарно
решение

$$m \vec{a} = F$$

$$F = F(\vec{r}, \vec{v})$$

$$\frac{d}{dt} x(t) = F(x(t))$$

$$F(x_0) = 0$$

$$x(0) = x_0$$

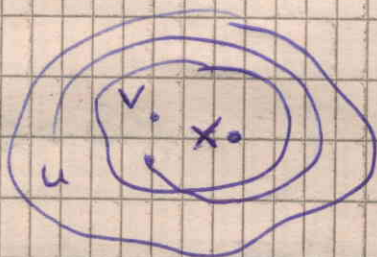
устойчиво x_0
равновесие

$x(t) \equiv x_0$ е решение

Def.

Устойчивость

Вокрестности на x_0
и

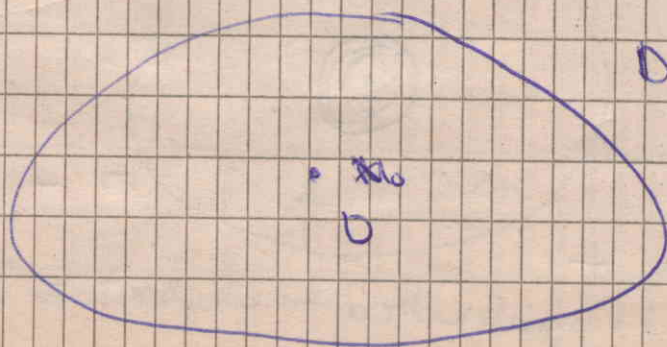


можем да
намерим
околност V

$x_0 \in V \subset U$, т.е.

произволно раш.
да не напускам

$$V: D \rightarrow R$$



устойчива
по
Ляпунову

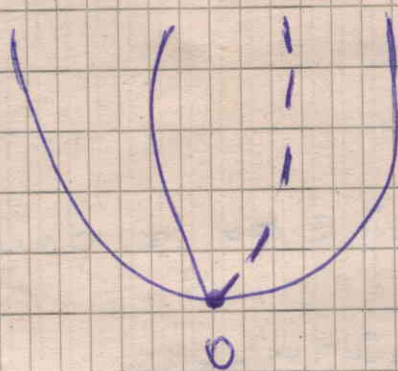
$$1) V(0) = 0$$

$$2) V(x) > 0$$

$$x \neq 0$$

$$3) \dot{V}(x) = V'(x) f(x) \leq 0$$

устойчива



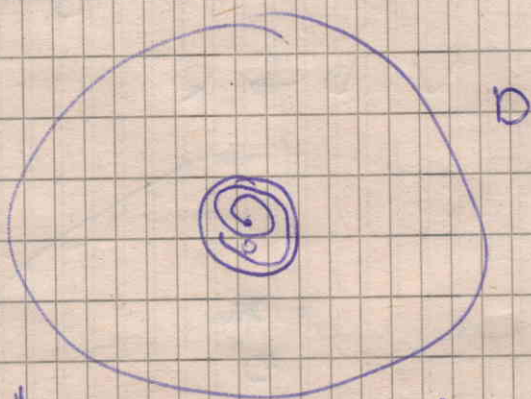
наша

$$x_0 = 0 \text{ (Коорд. с-м)}$$

$$\delta > 0$$

$$\|x\| < \delta$$

$$\lim_{t \rightarrow \infty} x(t) \Rightarrow 0$$

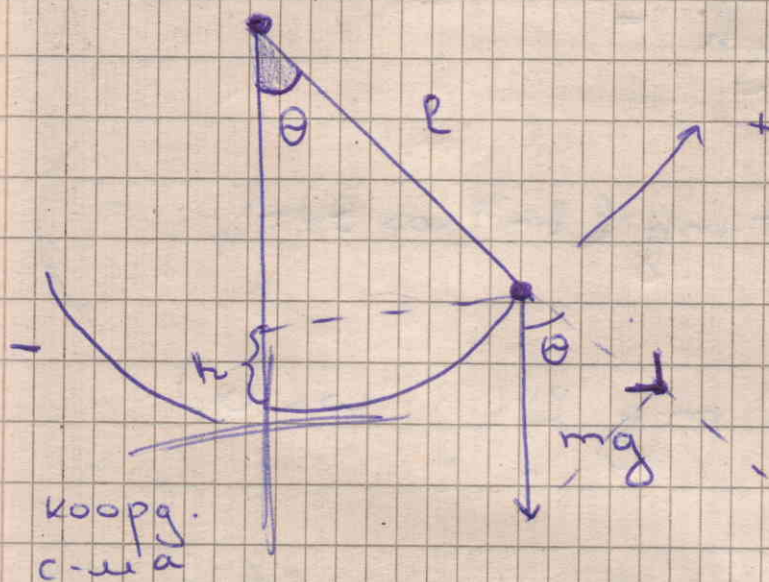


устойчива +

x_0 - привлекта

$$\dot{V}(x) = V'(x) f(x) < 0$$

Махало



коорд.
с-ма

$mg \sin \theta$ големина

$m l \ddot{\theta}$

↓
ускорение

$$m l \ddot{\theta} = - mg \sin \theta$$

$$\ddot{\theta} = - \frac{g}{l} \sin \theta \quad \text{ако няма триене}$$

$$x = \theta$$

$$\dot{\theta} = y$$

$$\dot{x} = y$$

$$\dot{y} = - \frac{g}{l} \sin \theta = - \frac{g}{l} \sin x$$

ф.з.

експерим. - ф.з. на Лепонт V

$$E = \frac{m\dot{\theta}^2}{2} + mgh =$$

$$= \frac{m}{2} l^2 \dot{\theta}^2 + mg(l - l \cos \theta) =$$

$$= \frac{m l^2 \dot{\theta}^2}{2} + mgl(1 - \cos \theta) =$$

$$= \frac{m l^2 \dot{\gamma}^2}{2} + mgl(1 - \cos x)$$

~~Energy~~

$$V(x, \gamma) = \left(\frac{m l^2 \dot{\gamma}^2}{2} + mgl(1 - \cos x) \right)$$

$$\dot{x} = \gamma$$

$$\dot{\gamma} = -\frac{g}{l} \sin x$$

$$f\left(-\frac{g}{l} \sin x\right)$$

$$V'_x(x, \gamma) = \cancel{\frac{m l^2 \dot{\gamma}^2}{2}}$$

$$\cancel{\frac{m l^2 \cdot 2\gamma \dot{\gamma}}{2}} + mgl \sin x$$

$$V_y'(x, y) = \frac{m l^2 y}{2} = m l y \cdot y' = m l.$$

$$\left(m g l \sin x, m l^2 y \right) \begin{pmatrix} y \\ -\frac{g}{l} \sin x \end{pmatrix} =$$

$$m g l \sin x y - m l^2 y y' - \frac{g}{l} \sin x$$

$$= \cancel{m g l \sin x y} + \cancel{m l^2 y y'} - \frac{g}{l} \sin x$$

$$\dot{V}(x, y) = V'(x, y) f(x, y) =$$

$$= \left(m g l \sin x, m l^2 y \right) \begin{pmatrix} y \\ -\frac{g}{l} \sin x \end{pmatrix} =$$

$$= m g l \sin x y - m g l \sin x y = 0$$

33 E

+ m p u e n e

$$m l \ddot{\theta} = - m g \sin \theta - k l \dot{\theta}$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l} \sin x - \frac{k}{3l} y \end{cases}$$

$$(mg l \sin x, m l^2 \dot{y}) \begin{pmatrix} y \\ -\frac{g}{l} \sin x - \frac{k}{3l} y \end{pmatrix} =$$

$$= m g l y \sin x - m l^2 y \frac{g}{l} \sin x -$$

$$- \cancel{m l^2 y} \frac{k}{\cancel{3l}} y =$$

$$= \cancel{m g l y \sin x} - \cancel{m g l y \sin x} -$$

$$- k l^2 y^2 = -k l^2 y^2$$

$\neq 0$

$y \neq 0$

тогда
измен
равновесие

??? Т.не В.рши
работа

19.11.2013г.

$$\frac{d}{dt} x(t) = f(x(t))$$

$x(0) = x_0$ равновесная точка

$$f(x_0) = 0$$

$$\frac{d}{dt} x(t) = f(x(t), u(t))$$

$$x(0) = x_0$$

$$f(x_0, u_0) = 0$$

$$x_0 \in \mathbb{R}^n$$

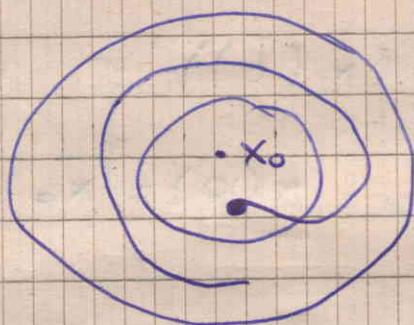
$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

Функция на Ляпунов

$$V: D \rightarrow \mathbb{R}$$

$$x_0 \in D \subset \mathbb{R}^n$$

Устойчивость по Ляпунову



не напуска
големата

асимптотично

ако x_0 е
приближава



$1 \times n$

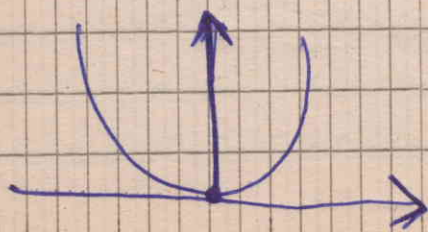
$n \times 1$

$$\dot{V}(x) = V'(x) \cdot f(x)$$

градиент
вектор-ред

$$\frac{d}{dt} V(x(t)) = V'(x(t)) \cdot f(x(t))$$

Устойчивость по Ляпунову



$$\dot{V}(x) \leq 0$$

$$V(0) = 0$$

$$V(x) > 0$$

траекторията
намалява

$$x \neq 0$$

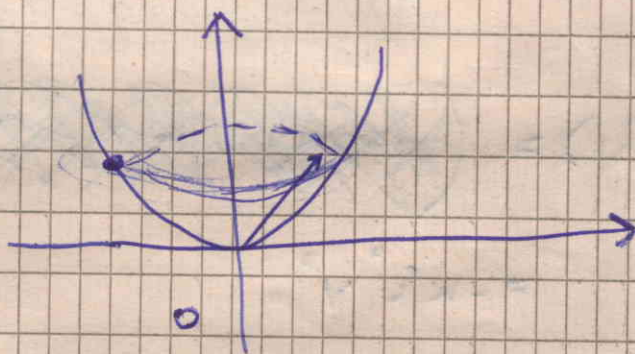
$$\begin{cases} \frac{d}{dt} x(t) = y(t) \\ \frac{d}{dt} y(t) = -x(t) \end{cases}$$

Функция на Ляпунов

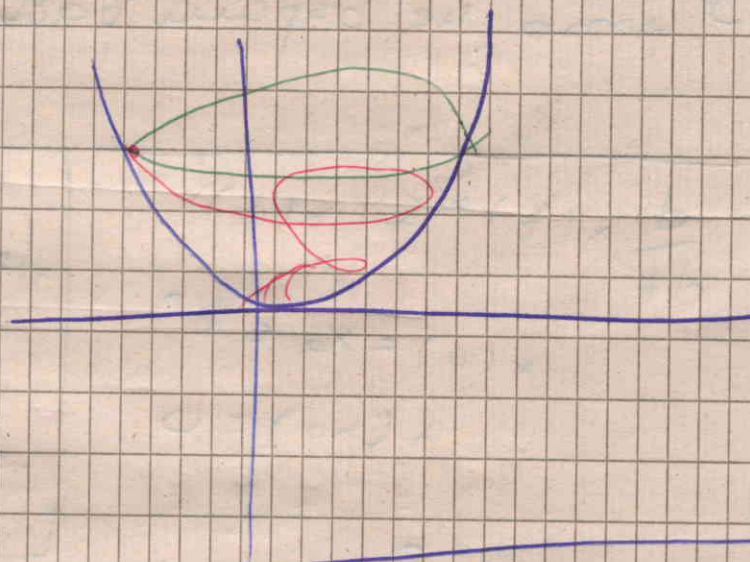
$$V(x, y) = x^2 + y^2$$

$$\dot{V}(x, y) = (2x, 2y) \cdot \begin{pmatrix} y \\ -x \end{pmatrix} = 0$$

устойчивость



Асимптотическая устойчивость:



$$\frac{d}{dt} x(t) = y(t)$$

$$\frac{d}{dt} y(t) = -\frac{g}{\ell} \sin x(t) - \frac{k}{3\ell} y(t)$$

$$V(x, y) = \frac{m\ell^2 y^2}{2} + mg\ell(1 - \cos x)$$

$$E = \frac{mv^2}{2} + mgh$$

$$\dot{V}(x, y) = \left(\text{scribbled out} \right) \\ = -k\ell^2 y^2$$

$$\dot{V}(x, y) = -k\ell^2 y^2$$

Мф. эта не верши работа тут :(

$$\frac{d}{dt} x(t) = f(x(t))$$

$$x(0) = x_0 \in \mathbb{R}^n$$

$$f(x_0) = 0$$

$x_0 \in D$
орг. и затв.

$\psi(t, x_0)$ е решение

$$\frac{d}{dt} \psi(t, x_0) = f(\psi(t, x_0))$$

f -непрерывната

$x_0 \in D \quad I_0 = (-\epsilon_0, \epsilon_0)$

реш. $\exists$ в единичен интервал

$$\varphi(t, x_0), t \in I_0$$

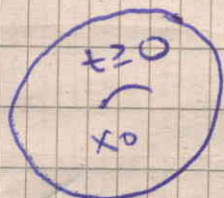
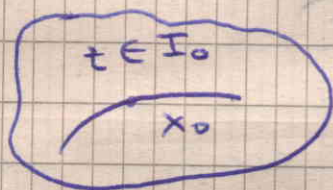
$$\varphi(0, x_0) = x_0$$

$$\text{Ако от } x_0 \in D \Rightarrow \varphi(t, x_0) \in D \quad t \in I_0$$

D -инвариантно

$$x_0 \in D \Rightarrow \varphi(t, x_0) \in D$$

$$t \geq 0, \\ t \in I_0$$



само
напрям
положителна
инвариантност

$$x_0 \in D \Rightarrow \varphi(t, x_0) \in D$$

$$t \leq 0 \\ t \in I_0$$

отриц.
инвариантност

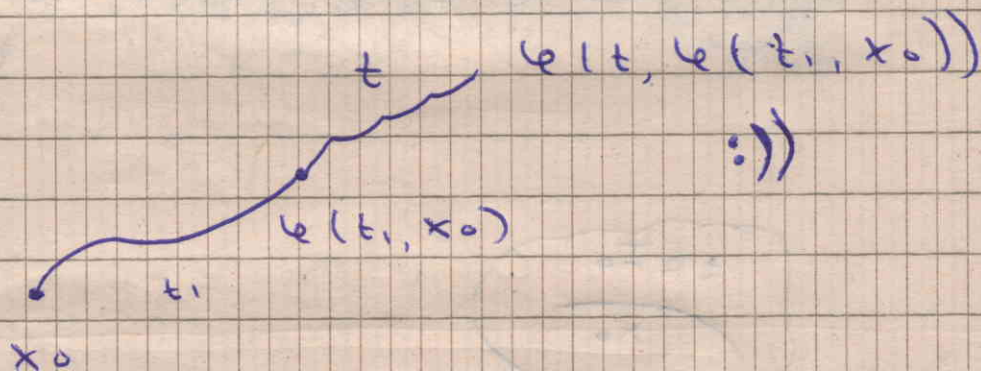


$\psi(t_1, x_0)$ - стоимость в момента t_1
 производя от $t - x_0$ в
 момента 0

$\psi(t, \psi(t_1, x_0))$ стоимость в
 момента t $\psi(t_1, x_0)$
 произв. от ~~x_0~~
 момента 0

Итак! решение

$$A(t) = \psi(t, \psi(t_1, x_0)) = \psi(t + t_1, x_0) = B(t)$$



?
 Д-во: $\begin{cases} = \text{произв.} \\ > 0 \text{ на вся точка} \end{cases}$

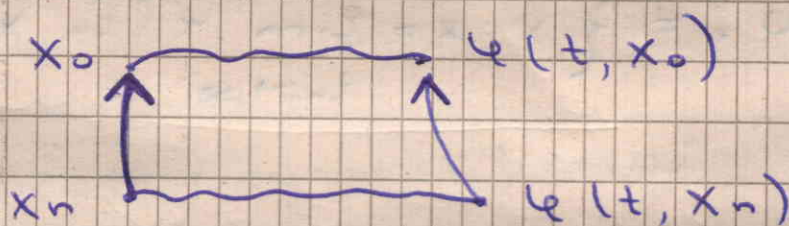
$$t=0 \text{ н.с.} = \psi(t_1, x_0)$$

$$g.c. = \psi(t_1, x_0)$$

$$\frac{d}{dt} A(t) = \text{[scribbled out]} f(A(t))$$

$$\frac{d}{dt} B(t) = f(B(t)) \Rightarrow \text{съвпада}$$

$$\Rightarrow \boxed{\varphi(t, \varphi(t_1, x_0)) = \varphi(t+t_1, x_0)}$$

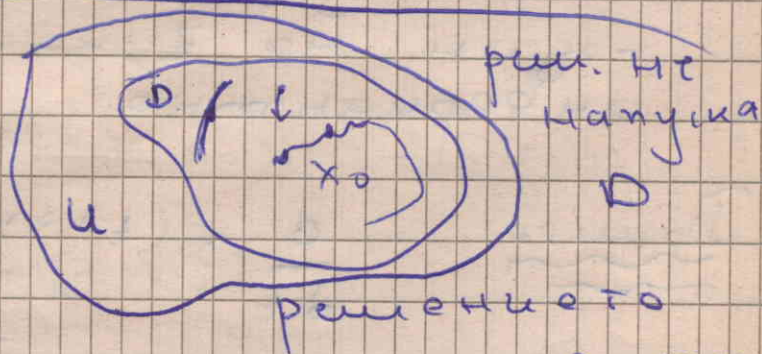


непрекъсната зависимост
по начални
данни

$$\lim_{x_n \rightarrow x_0} \varphi(t, x_n) \rightarrow \varphi(t, x_0)$$

$x_0 \in D$ - затв.
- отк.
- инв.

$$x_0 \in D \subset U$$



решението
се движи
вътре
(0, \tau_0)
не се пресича

$$v: u \rightarrow \mathbb{R}$$

$$\dot{V}(x) \leq 0$$

не расте по
траекторията

$$\psi(t, x_0) \quad t \geq 0$$

$$D \ni \omega := \left\{ x : x = \lim_{t_k \rightarrow \infty} \psi(t_k, x_0) \right\}$$

Оотт.
на
съставане
на
траекторията
 $\in D$

$$Z := \{ z : \dot{V}(z) = 0 \}$$

Принцип на La Salle

Ще съдържа в най-големо
подмнож. на Z , което е положително
инвариантно

Пример:

$$\frac{d}{dt} x(t) = y(t)$$

$$\frac{d}{dt} y(t) = -\frac{g}{l} \sin x - \frac{\kappa}{3} y(t)$$

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$y \in (-\infty; +\infty) \quad \left(\text{но е огр.} \right. \\ \left. \text{не знаем} \right. \\ \left. \text{къде} \right)$$

$$\dot{V}(x, y) = -k e^2 y^2$$

$$Z = \{ (x, y) : y = 0 \}$$

$$\frac{d}{dt} y(t) = 0 = -\frac{g}{e} \sin x(t) - \frac{k}{m} \cdot 0$$

$$x = \text{const}$$

$$\boxed{x=0}$$

единствено

за да е инвариантно

$\Rightarrow$ устойчиво :))

! Д-во: $\{V(\psi(t, x_0)), t \geq 0\}$ $\mathbb{R}$ числа
ограничени
(в $\mathbb{R}$ - огр. за тв.)

$$\Rightarrow \exists c_0 \in \{V(\psi(t, x_0)), t \geq 0\}$$

$$h(t) = V(\psi(t, x_0))$$

$$\frac{d}{dt} h(t) = \dot{V}(\psi(t, x_0)) \leq 0$$

$$V'(\psi(t, x_0)) \cdot \dot{\psi}(t, x_0)$$

$\Rightarrow 0$ -то ма $\inf \bar{C}$

$\bar{C} + \frac{1}{n}$ не е долна граница

$\Rightarrow \exists t_n$

$$\bar{C} \leq v(\psi(t_n, x_0)) < \bar{C} + \frac{1}{n}$$

$\psi(t_n, x_0) \in D$ - открито и затворено.

$\Rightarrow \exists \bar{x} \in D$

сх. поредица

$$\psi(t_{n_k}, x_0) \xrightarrow[k \rightarrow \infty]{} \bar{x} \in D$$

зр. преход

$$v(\bar{x}) = \bar{C}$$

Взимаме произволна точка

$$z \in W \Rightarrow \text{от } \underline{\text{def}}$$

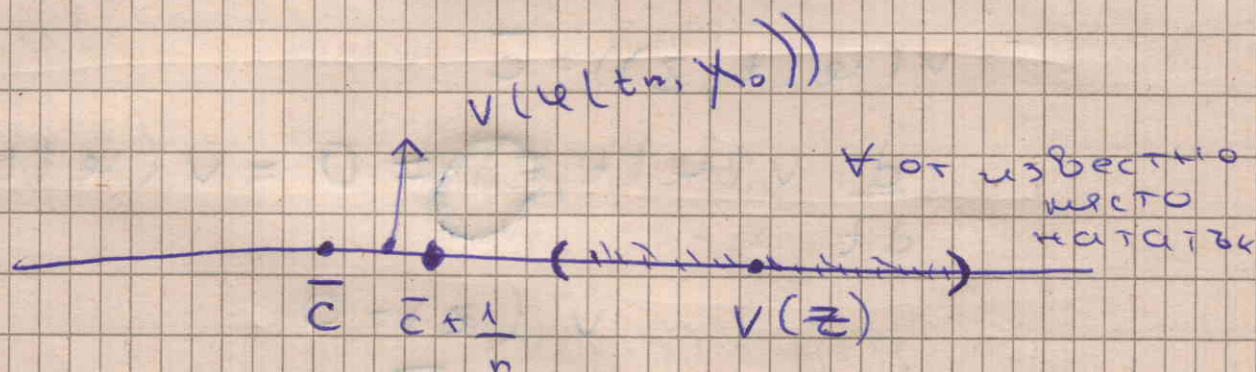
$$z = \lim_{t_s \rightarrow \infty} \psi(t_s, x_0)$$

$$v(z) = \lim_{t_s \rightarrow \infty} v(\psi(t_s, x_0))$$

$$\bar{C} \leq v(z)$$

? monce $\bar{c} < v(z)$

Don., $\bar{c} < v(z)$



$\exists s_0 \quad t_s > t_{s_0}$
 $v(\varphi(t_s, x_0))$

$t_s > \max(t_n, t_{s_0})$

~~$v(t_s)$~~

$v(\varphi(t_s, x_0)) \leq v(\varphi(t_n, x_0))$

Начинаем
 proof

X

∴))

Зна

$$z \in \omega \rightarrow v(z) = \bar{c}$$

$$\varphi(t, z), \quad t \geq 0$$

$$v(\varphi(t, z)) = \bar{c}$$

$$\frac{d}{dt} v(\varphi(t, z)) = 0 = \dot{v}(\varphi(t, z))$$

$$\dot{v}(z) = 0$$

$$\Rightarrow z \in Z$$

$z \in \omega$ произвольна



от def ω

$$z = \lim_{t_k \rightarrow \infty} \varphi(t_k, x_0)$$

$$\varphi(t, z) =$$

$$\varphi(t + t_k, x_0) = \varphi(t, \underbrace{\varphi(t_k, x_0)}_{z}) \rightarrow \varphi(t, z)$$



$$z$$

$$t_k \rightarrow \infty$$

$$t + t_k \rightarrow +\infty$$

$$\Rightarrow \varphi(t, z) \in \omega \quad ;)$$

=> ω инвариантно

$$\left| \begin{array}{l} \frac{d}{dt} \\ \frac{d}{dt} \end{array} \right. \begin{array}{l} N_1(t) = (\lambda_1 - \beta_1 N_2(t)) N_1(t) \\ N_2(t) = -(\lambda_2 - \beta_2 N_1(t)) N_2(t) \end{array}$$

$$\begin{array}{l} \dot{x} = y - b \\ \dot{y} = -x + a \end{array}$$

$$V(x, y) = x^2 + y^2$$

$$V(x, y) = (x-a)^2 + (y-b)^2$$

$$\dot{V}(x, y) = (2(x-a), 2(y-b))$$

$$(y-b, -x+a) = 0$$

)))

$$\dot{x} = (y-b)x\lambda$$

$$\dot{y} = (-x+a)y\beta$$

$$\dot{x} = (y-b)x\lambda$$

$$\dot{y} = (-x+a)y\beta$$

$$V(x, y) = \beta(x-a)^2 + \lambda(y-b)^2$$

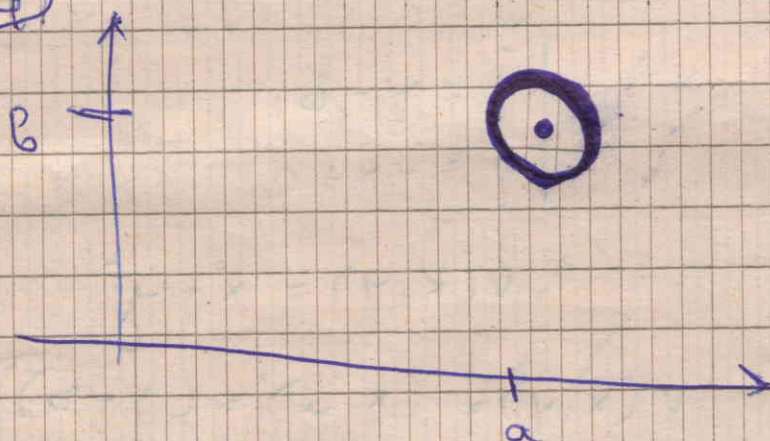
$$\dot{V}(x, y) = 0$$

$$\begin{aligned}\dot{x} &= (y - \beta) \times \alpha \\ \dot{y} &= (-x + a) \times \beta\end{aligned}$$

$$\alpha > 0$$

$$\beta > 0$$

в окрестности на a и β



$$\frac{d}{dt} x(t) = \alpha x(t) (y(t) - y^*)$$

$$\frac{d}{dt} y(t) = \beta y(t) (-x(t) + x^*)$$

(x^*, y^*) - равновесна точка



27.11.2013г.

Упражнение

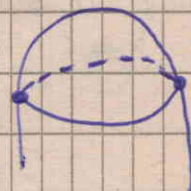
$$\dot{x} = x(a - by)$$

x-исертва

$$\dot{y} = -y(c - dx)$$

$$\left(\frac{c}{d}, \frac{a}{b}\right) \text{ равновесия}$$

$$(0,0)$$



не може да има др. ф-я с това св-во

$$\dot{x} = a - by$$

$$V(x, y) = 2\left(x - \frac{c}{d}\right)^2$$

$$\dot{y} = -y(c - dx)$$

$$V(x, y) = 2\left(x - \frac{c}{d}\right)^2 + p\left(y - \frac{a}{b}\right)^2$$

$$\dot{V}(x, y) = \left(22\left(x - \frac{c}{d}\right), 2p\left(y - \frac{a}{b}\right)\right) \cdot$$

$$\begin{pmatrix} a - by \\ -(c - dx) \end{pmatrix} =$$

$$= 22\left(x - \frac{c}{d}\right)(a - by) - 2p\left(y - \frac{a}{b}\right)(c - dx) = 0$$

= 0

$$= 2\alpha \frac{(dx - c)}{d} (a - by) - 2\beta \frac{(yb - a)}{b} (c - bx)$$

$$\frac{(2\alpha xd - 2\alpha c)(a - by)}{d}$$

$$\frac{(2\alpha xda - 2\alpha xdb y - 2\alpha ca + 2\alpha cby)}{d}$$

$$- \frac{2\beta}{b} (ybc - yb^2x - ac + abx) = 0$$

$$\begin{aligned} \alpha &= d \\ \beta &= b \end{aligned}$$

$$\dot{x} = x b (\bar{y} - y)$$

$$(\bar{x}, \bar{y}) = \left(\frac{c}{d}, \frac{a}{b} \right)$$

$$\dot{y} = -d y (\bar{x} - x)$$

$$v(x, y) = \alpha \int_{\bar{x}}^x \frac{z - \bar{x}}{z} dz + \beta \int_{\bar{y}}^y \frac{y_0 z - y}{y z} dz =$$

$$= \alpha (x - \bar{x}) - \alpha \bar{x} \ln |\bar{x} - \bar{x}| +$$

$$+ \beta |y - \bar{y}| - \beta$$

$$\dot{V}(x, y) = \left(\alpha \frac{(x - \bar{x})}{x} + \beta \frac{(y - \bar{y})}{y} \right) \begin{pmatrix} x(a - by) \\ -y(c - dx) \end{pmatrix}$$

$$= \alpha (x - \bar{x})(a - by) - \beta (y - \bar{y})(c - dx) =$$

$$= \alpha \left(x - \frac{c}{d} \right) (a - by) - \beta \left(y - \frac{a}{b} \right) (c - dx)$$

$$= \cancel{2\alpha\alpha\alpha\alpha\alpha}$$

$$\alpha = d$$

$$\beta = b$$

$$= 0$$

$$V(x, y) = d \int_{\bar{x}}^x \frac{z - \bar{x}}{z} dz + b \int_{\bar{y}}^y \frac{z - \bar{y}}{z} dz =$$

$$= d |x - \bar{x}| - d \bar{x} \ln |x - \bar{x}|$$

$$+ b |y - \bar{y}| - b \bar{y} \ln |y - \bar{y}| =$$

$$= d |x - \bar{x}| - d \bar{x} (\ln |x| - \ln |\bar{x}|)$$

$$+ b |y - \bar{y}| - b \bar{y} (\ln |y| - \ln |\bar{y}|)$$

$$= d |x - \bar{x}| - c (\ln |x| - \ln |\bar{x}|)$$

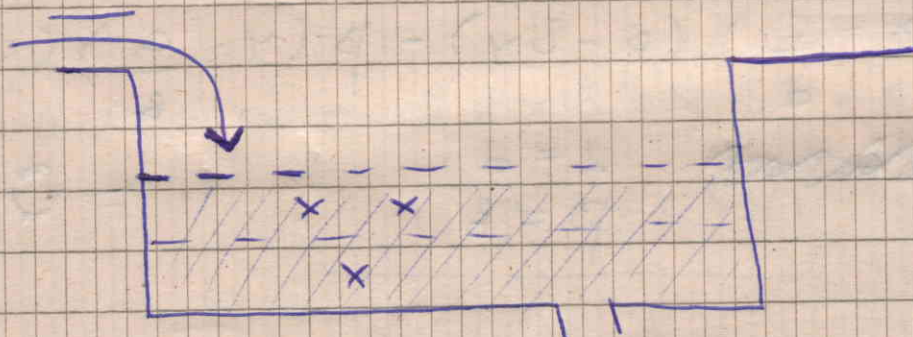
$$+ b |y - \bar{y}| - a (\ln |y| - \ln |\bar{y}|) =$$

$$= (dx - c) - c \ln \frac{x}{\bar{x}} + (by - a) - a \ln \frac{y}{\bar{y}}$$

$$V(x, y)$$

$$= dx - c \ln x + by - a \ln y + \text{const} :))$$

Биопроцеси



колкото влиза, толкова излиза за 1 време
субстрат - такаъв зашкрител
x-животинки, пречистват водата

V-обем за 1 време обем D

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = \text{промяна на организмите за 1 време}$$

$$= \underbrace{\mu(s(t))}_{\text{зависим от концентрацията}} N(t) - D \cdot x(t)$$

излиза от тръбата

μ - брой λ организми

λ - концентрация на организмите

$$\mu = \lambda V$$

s - субстрат

S

$$x(0) = x_0 > 0$$

$$S(0) = S_0 > 0$$

V

$$\frac{x(t+\Delta t)V - x(t)V}{\Delta t} = \mu(s(t)) \cdot N(t) - D x(t)$$

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = \mu(s(t)) \cdot x(t) - \frac{D}{V} x(t)$$

$$\dot{x}(t) = x(t) (\mu(s(t)) - \alpha), \quad \alpha \in (0, 1)$$

↓
когато е
хомогенен
разтвор

$$\frac{S(t+\Delta t) - S(t)}{\Delta t} = D S^0 - D S(t) -$$

↓ ↓
входна изходна

$$- K \cdot \mu(s(t)) \cdot x(t) V$$

$$\frac{S(t+\Delta t)V - S(t)V}{\Delta t} = D(S^0 - S(t)) - K \mu(s(t)) x(t) V$$

$$\frac{d}{dt} S(t) = \dot{S}(t) = u(S^0 - S(t)) - K \mu(s(t)) x(t)$$

$$\frac{d}{dt} x(t) = x(t) (\mu(s(t)) - \alpha)$$

пр. преход

$\frac{d}{dt} = \frac{d}{dt}$

$$\mu(0) = 0$$

$\mu(s) > 0, s > 0$ ша храна
ша пшвотнн

$$\mu(s) = \frac{as}{b+s} \quad s \geq 0$$

Monod



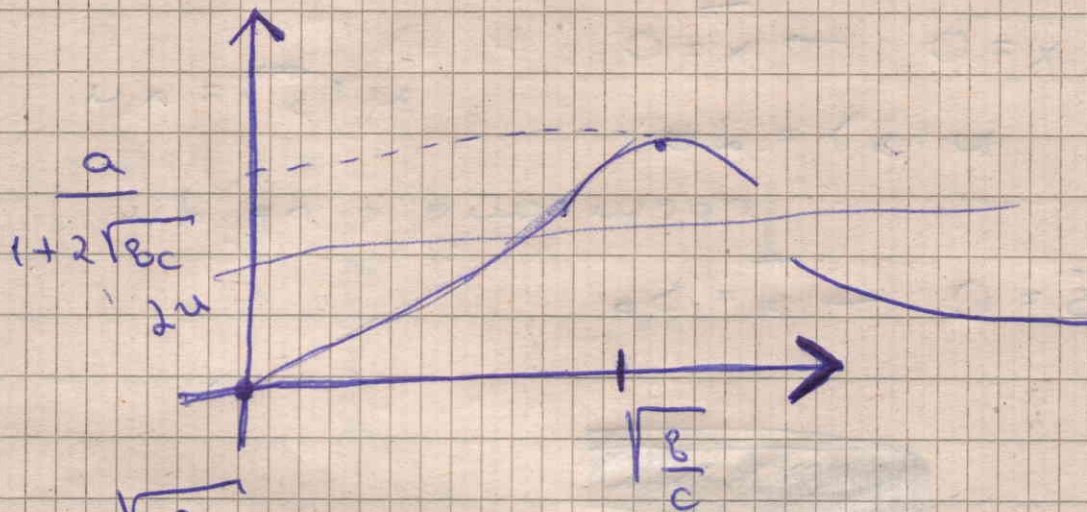
$$\mu'(s) = \frac{a(b+s) - as}{(b+s)^2} = \frac{ab}{(b+s)^2} > 0$$

функция на разтег

$$\mu(s) = \frac{as}{b+s+cs^2}$$

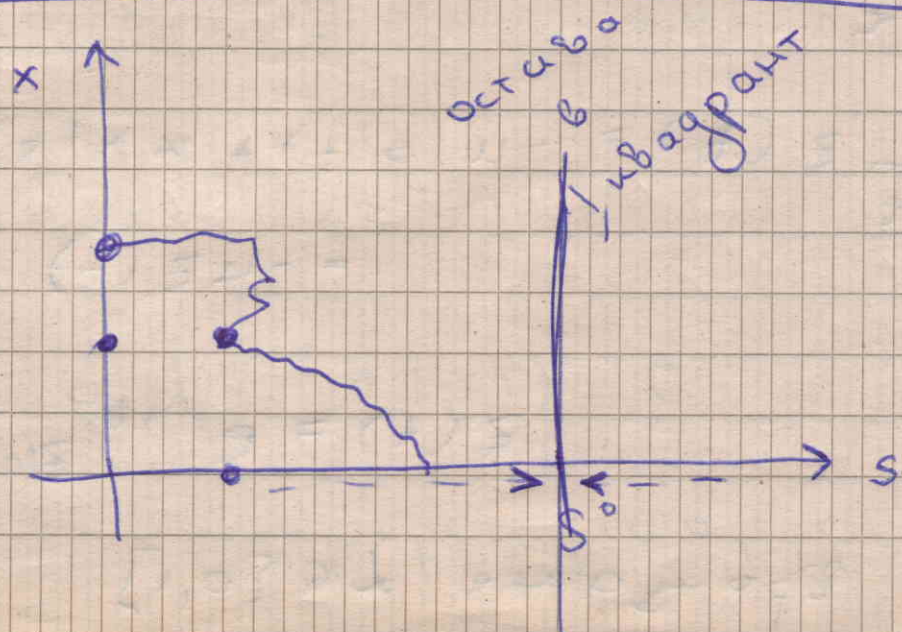
$$\begin{aligned} \mu'(s) &= \frac{a(b+s+cs^2) - as(1+2cs)}{(b+s+cs^2)^2} = \\ &= \frac{ab - acs^2}{(b+s+cs^2)^2} \end{aligned}$$

$$ab = acs^2$$



$$S_{max} = \sqrt{\frac{b}{c}}$$

$$\begin{aligned} M(s) &= \frac{a\sqrt{b}}{\sqrt{c}\left(\frac{\sqrt{b}}{\sqrt{c}} + b + \frac{c\sqrt{b}}{c}\right)} = \frac{a\sqrt{b}}{\sqrt{b} + b\sqrt{c} + b\sqrt{c}} \\ &= \frac{a\sqrt{b}}{\sqrt{b}(1 + \sqrt{bc} + \sqrt{bc})} \\ &= \frac{a}{1 + 2\sqrt{bc}} \end{aligned}$$



$$\dot{x} = 0 \rightarrow \bar{x} = 0$$

$$\mu(\bar{s}) = 2u$$

$$\mu(\bar{s}) = 2u$$

превращение с нр. $2u$

$$\dot{s} = 0 \rightarrow \bar{s} = s_0$$



$$\mu(\bar{s}) = 2u$$

$$\bar{x} =$$

$$u(\bar{s} - s_0) - \kappa 2u \bar{x} = 0$$

заместитель

взв
второго

$$\bar{x} = \frac{\bar{s} - s_0}{\kappa 2}$$



$$z = s + \kappa x - s_0$$

$$\frac{d}{dt} x(t) = \dot{x}(t) = x(t) (\mu(s(t)) - u)$$

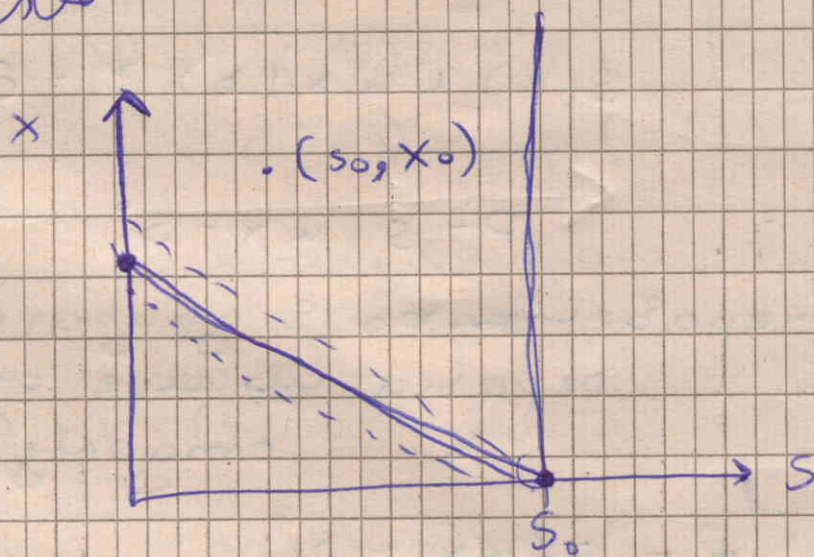
$$\begin{aligned} \frac{d}{dt} z(t) &= -u(s(t) + \kappa x(t) - s_0) = \\ &= -u z(t) \end{aligned}$$

$$z(t) = e^{-ut} \quad z(0) \rightarrow 0 \quad t \rightarrow \infty$$

время $\lambda \in [0, 1]$

$$P = S + KX$$

$$\frac{d}{dt} P(t) \neq$$



$$\begin{aligned} \frac{d}{dt} P(t) &= -u (s(t) + Kx(t) - s^0) = \\ &= -us(t) - uKx(t) + us^0 \geq \\ &\geq -us(t) - uKx(t) + s^0u = \\ &= -uP(t) \end{aligned}$$

$$\frac{d}{dt} P(t) + uP(t) \geq 0 \quad e^{ut}$$

$$e^{ut} \frac{d}{dt} P(t) + u e^{ut} P(t) \geq 0$$

$$\frac{d}{dt} (e^{ut} P(t)) \geq 0$$

$$e^{ut} P(t) \geq e^{u \cdot 0} P(0)$$

$$P(t) \geq e^{-\mu t} P(0) \\ \geq S(t) + \kappa x(t)$$

$$S(t) + \kappa x(t) \geq -\varepsilon$$

$$P = S + \kappa x - S^0$$

? неравенство в другую сторону
ограничиваеме отгоре и
отдолу

$$q = S + \kappa x - \frac{S^0}{2}$$

$$\begin{aligned} \frac{d}{dt} q &= -\mu(S(t)) - 2\mu\kappa x(t) + \mu S^0 \leq \\ &\leq -2\mu S(t) - 2\mu\kappa x(t) + \mu S^0 = \\ &= -2\mu \left(S(t) + \kappa x(t) - \frac{S^0}{2} \right) \end{aligned}$$

$$S(t) + \kappa x(t) - \frac{S^0}{2} = \varepsilon(\kappa) \leq$$

$$e^{-2\mu t} \varepsilon(0) \rightarrow 0 \quad t \rightarrow \infty$$

S^0 - начална кощ.
 S_0 - точка

04.12.2013г.

$$s(0) > 0$$

$$x(0) > 0$$

$$\dot{x}(t) = x(t)(\mu(s(t)) - d_u)$$

$$\dot{s}(t) = (\mu(s^0 - s(t)) - \kappa \mu(s(t))) x(t)$$

x-конц биомаса

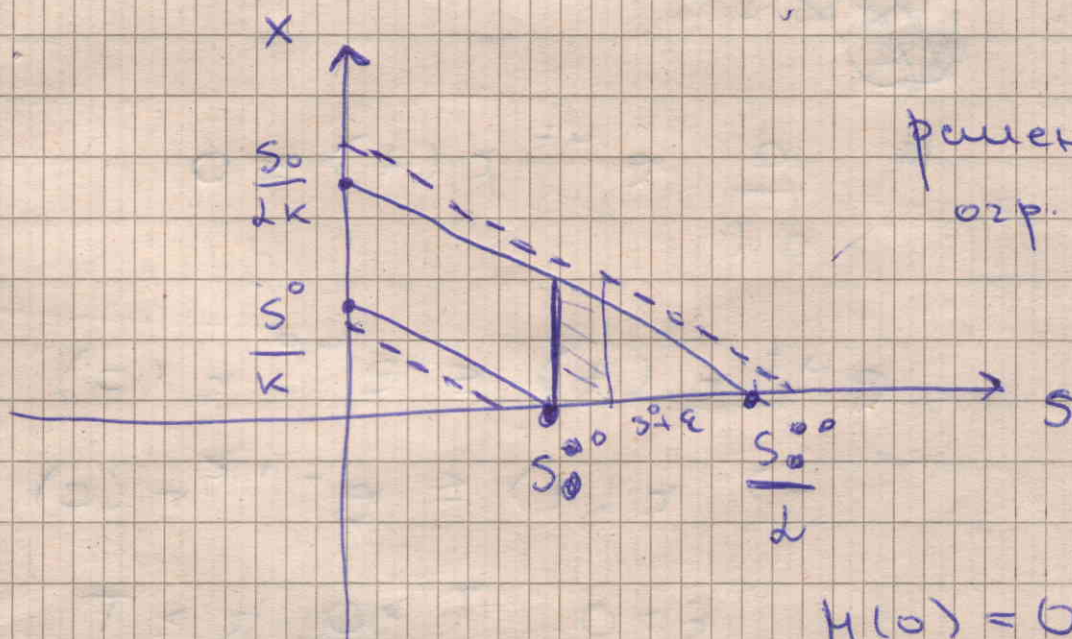
s-устройство

$$s(t) + \kappa x(t)$$

$$\text{за } \forall \varepsilon > 0 \exists T > 0$$

$$t \geq T$$

$$s^0 - \varepsilon \leq s(t) + \kappa x(t) \leq \frac{s^0}{2} + \varepsilon$$



решен. е
опр. μ/γ

$$\mu(0) = 0$$

загдка

Donc, on a $\forall t \quad s(t) \geq s^0$

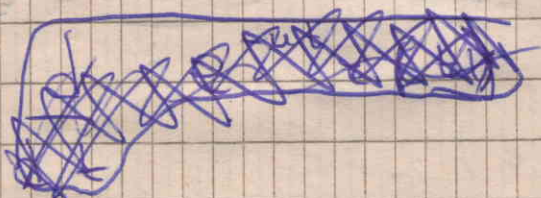
$$\frac{d}{dt} s(t) \leq -u(s(t) - s^0)$$

$$\frac{d}{dt} (s(t) - s^0) \leq -u(s(t) - s^0)$$

$$\frac{d}{dt} z(t) \leq -u z(t) \quad | \cdot e^{+ut}$$

$$e^{-ut} \frac{d}{dt} z(t) \leq -u z(t) e^{+ut}$$

$$-u z(t) e^{-ut} - e^{-ut} \frac{d}{dt} z(t) \geq 0$$



$$\frac{d}{dt} e^{ut} z(t) \leq 0$$

$$e^{ut} z(t) \leq e^{u0} z(0)$$

$$\Rightarrow z(t) \leq e^{-ut} z(0)$$

$$\varepsilon > 0 \quad T > 0 \quad t \geq T$$

$$s(t) - s^0 \leq \varepsilon$$

u-kon. oδeu
μ-paxe

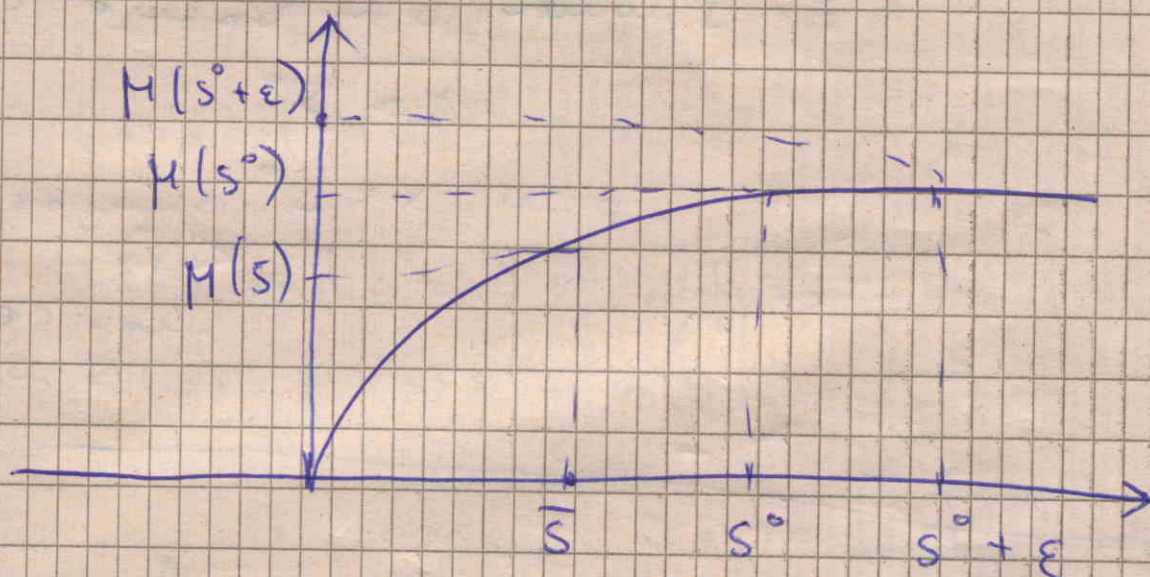
$$\dot{x}(t) = x(t) \cdot (\mu(s) - 2u)$$

$$\mu(\bar{s}) = 2u$$

$$\bar{x} = \frac{s^0 - \bar{s}}{k_2}$$

$$\mu(\bar{s})$$

изображение $\bar{s} < s^0$



$$\Rightarrow \dot{x}(t) \geq x(t) \eta \quad \eta > 0$$

$$\left\{ \begin{array}{l} x \text{ non neg} \\ s \in [s^0, s^0 + \varepsilon] \end{array} \right\}$$

$$\dot{x}(t)$$

$$\dot{x}(t) - \eta x(t) \geq 0 \quad | e^{-\eta t}$$

$$\frac{d}{dt} e^{-nt} x(t) \geq 0$$

$$e^{-nt} x(t) \geq e^{-nt} x(0)$$

$$x(t) \geq e^{nt} x(0) \rightarrow +\infty$$

~~X~~

$\Rightarrow$ Существует время t

$$x < S^0$$

реш. $x < S^0$, поэтому

не
может
 $S = S^0$

Что определим ф.н.д. на Ланднов

$$V(x, s)$$

$$\int_{\bar{x}}^x \frac{\eta - \bar{x}}{\eta} d\eta$$

$$V(x, s) = \int_{\bar{s}}^s \alpha(z) dz + \int_{\bar{x}}^x \frac{\eta - \bar{x}}{\eta} d\eta$$

s^0 - бузаны убоура

$$\dot{V}(x, s) = \cancel{Q(s) - Q(\bar{s})} +$$

$$\dot{V}(x, s) = \left(\cancel{Q(s)} \cdot \frac{x - \bar{x}}{x} \right) \cdot \begin{pmatrix} u(s^0 - s(t)) - k\mu(s)x \\ x(\mu(s) - 2u) \end{pmatrix}$$

$$= Q(s) [u(s^0 - s) - k\mu(s)x] +$$

$$+ (x - \bar{x})(\mu(s) - 2u) = 0$$

$$= [Q(s)u(s^0 - s) - \bar{x}(\mu(s) - 2u)] - Q(s)k\mu(s)x + x(\mu(s) - 2u)$$

$$\text{Потому } Q(s) = \frac{\bar{x}(\mu(s) - 2u)}{u(s^0 - s)}$$

$$x(\mu(s) - 2u) \left[1 - \frac{Q(s)k\mu(s)x}{\mu(s) - 2u} \right]$$

$\begin{matrix} \downarrow \neq \\ 0 & 0 \end{matrix}$
 при $s \neq s^0$

$$\mu(s) - 2u = \begin{cases} 0 & s = \bar{s} \\ < 0 & s < \bar{s} \\ > 0 & s > \bar{s} \end{cases}$$

$$\underline{\underline{s < s^0}}$$

$$= x(\mu(s) - 2u) \left[1 - \frac{\bar{x}(\mu(s) - 2u)K}{u(s^0 - s)(\mu(s) - 2u)} \right] =$$

$$= x(\mu(s) - 2u) \left[1 - \frac{\bar{x}K}{u(s^0 - s)} \right] =$$

$$= x(\mu(s) - 2u) \left[1 - \frac{s^0 - s}{K \cdot 2u(s^0 - s)} \right] =$$

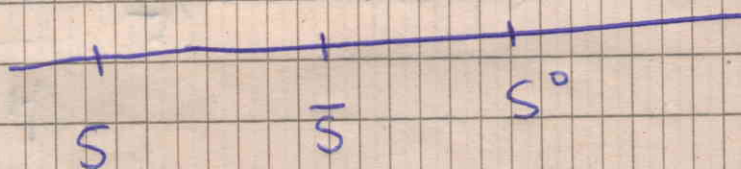
$$= x(\mu(s) - 2u) \left[1 - \frac{1}{2u} \right]$$

$$\alpha \underline{\underline{s < \bar{s}}}$$

$$\mu(s) - 2u < 0$$

$$\frac{\mu(s)}{2u} < 1$$

$$\frac{s^0 - \bar{s}}{s^0 - s}$$



~~if $s < \bar{s}$~~

$$\dot{V}(s, x) < 0$$

|| on.

$$\underline{\bar{s} < s < s^0}$$

$$\Rightarrow \dot{v}(s, x) > 0$$

$$\left(\mu(s) - 2u > 0 \right.$$

$$\frac{\mu(s)}{2u} > 1$$

$$\frac{s^0 - \bar{s}}{s^0 - s} > 0$$

$$\begin{array}{c} \text{---} \\ | \quad | \quad | \\ s^0 \quad \bar{s} \quad s^0 \\ \Rightarrow \dot{v}(s, x) < 0 \end{array}$$

$$\boxed{\begin{array}{l} \dot{v}(s, x) = 0 \\ 1) s = \bar{s} \\ 2) x = 0 \end{array}}$$

$$s = \bar{s}$$

$$\Rightarrow \bar{x} = \frac{s^0 - \bar{s}}{k\alpha}$$

$$? x \rightarrow 0$$

Принцип
La Salle
зазатв. @

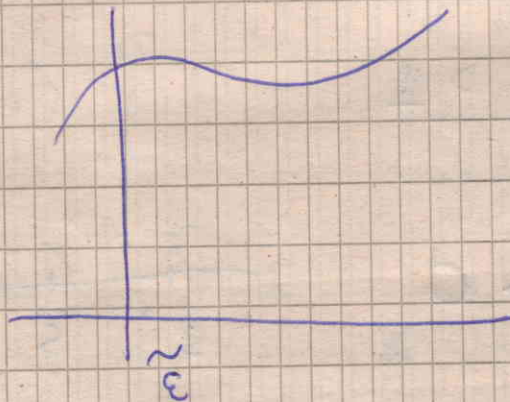
$$s^0 - \varepsilon \leq s(t) + kx(t) \leq \frac{s^0 + \varepsilon}{2}$$

Dom., ze

$$x(t) < \tilde{\varepsilon} > 0$$

$$\text{Ako } x(\hat{t}) = \tilde{\varepsilon} \\ \dot{x}(t) > 0$$

$$\Rightarrow \dot{x}(t) > 0$$



$$s(\hat{t}) \geq s^0 - \tilde{\varepsilon} - \kappa x(\hat{t}) = s^0 - \tilde{\varepsilon} (1 + \kappa) \\ > \bar{s} + \tilde{\varepsilon}$$

$$\tilde{\varepsilon} < \frac{s^0 - \bar{s}}{\kappa + 1}$$

$$x(\hat{t} + \Delta t) = x(\hat{t}) + \Delta t \cdot \dot{x}(\hat{t}) + o(\Delta t)$$

$$= x(\hat{t}) + \Delta t \left(\dot{x}(\hat{t}) + o(\Delta t) \right)$$

$$\frac{\Delta x}{\Delta t} \Rightarrow \dot{x}(\hat{t})$$

~~0~~

Лема на Барбала

$\gamma: [0; +\infty) \rightarrow \mathbb{R}$ равномерно непрекъснат

$$\exists \int_0^{\infty} \gamma(s) ds = \lim_{T \rightarrow \infty} \int_0^T \gamma(s) ds$$

за $\forall \varepsilon > 0 \exists T > 0$

$$\lim_{t \rightarrow \infty} \gamma(t) = 0$$

$$t > T \implies |\gamma(t)| < \varepsilon$$

$t \in [0, \infty)$

$$\forall \varepsilon > 0 \exists \delta > 0 : \text{if } |t - t_0| < \delta \implies |\gamma(t) - \gamma(t_0)| < \varepsilon$$

$$\forall \varepsilon > 0 \exists \delta > 0 : \text{if } |t_1 - t_2| < \delta \implies |\gamma(t_1) - \gamma(t_2)| < \varepsilon$$

Д-во: Доп. противното

$$\implies \exists \varepsilon_0 \text{ за } \forall n \in \mathbb{N} \exists t_n \xrightarrow{n \rightarrow \infty} \infty$$

$$|\gamma(t_n)| > \varepsilon_0$$

отравн.
непр.

$$\exists \delta > 0 \text{ if } |t - t_n| < \delta \implies |\gamma(t) - \gamma(t_n)| > \frac{\varepsilon_0}{2}$$

н.с.р. Δ

$$|y(t)| \geq |y(t_n)| - |y(t) - y(t_n)| > \epsilon_0$$

$$|y(t_n)| = |y(t_n) - y(t) + y(t)|$$

$$|y(t_n)| = |y(t_n) - y(t) + y(t)| \leq \\ \leq |y(t_n) - y(t)| + |y(t)|$$

$$> \epsilon_0 - \frac{\epsilon_0}{2} = \frac{\epsilon_0}{2}$$

$$|y(t)| \geq \frac{\epsilon_0}{2}$$

$$t \in (t_n - \delta, t_n + \delta)$$

$$f(t) \rightarrow c \\ t \rightarrow \infty$$

$$\epsilon > 0$$

$$T > 0$$

$$t > T \\ |f(t) - c| < \epsilon$$

$$t_1 > T$$

$$t_2 > T$$

$$|f(t_1) - f(t_2)| = |f(t_1) - c + c - f(t_2)| \\ \leq |f(t_1) - c| + |f(t_2) - c|$$

$$\varepsilon > 0 \quad T > 0 \quad t_n > T \quad t_n + \delta > T$$

$$\left| \int_0^{t+\delta} y(s) ds - \int_0^{t_n} y(s) ds \right| < 2\varepsilon$$

//

$$\left| \int_{t_n}^{t_n+\delta} y(s) ds \right| = \int_{t_n}^{t_n+\delta} |y(s)| ds > \frac{\varepsilon_0 \delta}{2}$$

✗

$$\begin{cases} \dot{x}(t) = f(x(t)) \geq 0 \\ x(0) = x_0 \end{cases}$$



не по
нагнута

$$x(t) = \int_0^t f(x(s)) ds$$

$t \rightarrow T$
когда
се выпукл
 $f(x(t))$

$$\begin{aligned} \|f(x(t_1)) - f(x(t_2))\| &= \\ &= |t_1 - t_2| f'(x(\theta)) \\ &\quad \theta \in (t_1, t_2) \\ &\quad \text{о.р.} \end{aligned}$$

10.12.2013г.

Лема $\gamma: [0, +\infty) \rightarrow \mathbb{R}$ интегр. по Риману

$$\Rightarrow \left| \int_0^{t_n + \delta} \gamma(s) ds - \int_0^{t_n} \gamma(s) ds \right| < \varepsilon$$

$$\varepsilon > 0 \quad \exists \tau_0 > 0$$

$$\left| \int_0^{\tau} \gamma(s) ds - \int_0^{\infty} \gamma(s) ds \right| < \varepsilon$$

равном. непрерыв.

$$\Rightarrow \forall \varepsilon > 0 \quad \delta > 0 : |t_2 - t_1| < \delta$$

$$\Rightarrow |\gamma(t_2) - \gamma(t_1)| < \varepsilon$$

$$\varepsilon > 0 \quad \tau > 0 \quad t > \tau \Rightarrow |\gamma(t) - 0| < \varepsilon$$

Обратно,

$$\exists \varepsilon_0 > 0 \quad |\gamma(t_n)| \geq \varepsilon_0 \quad t_n \rightarrow \infty$$

$$|t_2 - t_1| < \delta \Rightarrow |\gamma(t_2) - \gamma(t_1)| < \frac{\varepsilon_0}{2}$$

$$t \in (t_n - \delta, t_n + \delta) \quad |\gamma(t) - \gamma(t_n)| < \frac{\varepsilon_0}{2}$$

$$|y(t)| = (y(t) - y(t_n) + y(t_n)) \geq$$

$$|y(t_n)| - (y(t) - y(t_n)) \geq \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2}$$

$\Rightarrow$ знака е един и същ в

интервала

$$\Rightarrow (*) = \left| \int_{t_n}^{t_n+\delta} y(s) ds \right| = \int_{t_n}^{t_n+\delta} |y(s)| ds \geq \int_{t_n}^{t_n+\delta} \frac{\varepsilon_0}{2} ds > 0$$

$$\left| \begin{array}{l} \frac{d}{dt} y(t) = f(y(t)) \\ y(0) = y_0 \end{array} \right.$$

$$\begin{array}{l} y(t) \in S \\ f(y) \geq 0 \\ y \in S \end{array}$$

След интегриране

$$\Rightarrow y(t) = y_0 + \int_0^t f(y(s)) ds \rightarrow \text{обучава,}$$

при $t \rightarrow \infty$,

защото реш. е инвариантно

$$\parallel \int_0^t y'(s) ds$$

$$|f(y(t_2)) - f(y(t_1))| < \varepsilon$$

$$|f(y_2(t)) - f(y_1(t))| = |f'(z)| |y(t_2) - y(t_1)|$$

$$= |f'(z)| |f(y_0)| |t_2 - t_1| \leq C |t_2 - t_1|$$

→ равн. метр.

$$\lim_{t \rightarrow \infty} f(y(t)) = 0$$

⇒ $y(t)$ клоникън точка където се нулира функцията f

Модел за пречиствателна станция

$$\frac{d}{dt} s(t) = u(s^0 - s(t)) - \mu(s(t))x(t)$$

$$\frac{d}{dt} x(t) = x(t)(\mu(s(t)) - \lambda u)$$

$$s(0) = s_0 > 0$$

$$x(0) = x_0 > 0$$

$$u = \frac{\text{колич. вл. вода}}{\text{колич. изл. вода}}$$

x - ф-я на растетса

s^0 - кон. на кр.?

λ - част от орг., които се закрепват

Прегнан

1. $\mu(0) = 0$

$$\mu(s) > 0 \quad \exists a \quad s > 0$$

$$\mu(\bar{s}) = 2u$$

$$\bar{x} = \frac{s^0 - \bar{s}}{k_2}$$

2. $0 < \bar{s} < s^0$

$$\mu(s) = 2u \quad \left| \begin{array}{l} > 0 \quad s > \bar{s} \\ < 0 \quad s < \bar{s} \end{array} \right. \quad \bar{s} - \text{точка}$$

Мам. уоуен

$$\begin{aligned} s(0) &= s_0 \\ x(0) &= x_0 \end{aligned}$$

$$\left| \frac{d}{dt} s(t) = u(s^0 - s(t)) - k \mu(s(t)) \right| x(t)$$

$$\left| \frac{d}{dt} x(t) = x(t) (\mu(s(t)) - 2u) \right|$$

1) $\mu(0) = 0 \quad \mu(s) > 0 \quad \exists a \quad s > 0$

2) $0 < \bar{s} < s^0$

$$\mu(\bar{s}) = 2u$$

$$\bar{x} = \frac{s^0 - \bar{s}}{k_2}$$

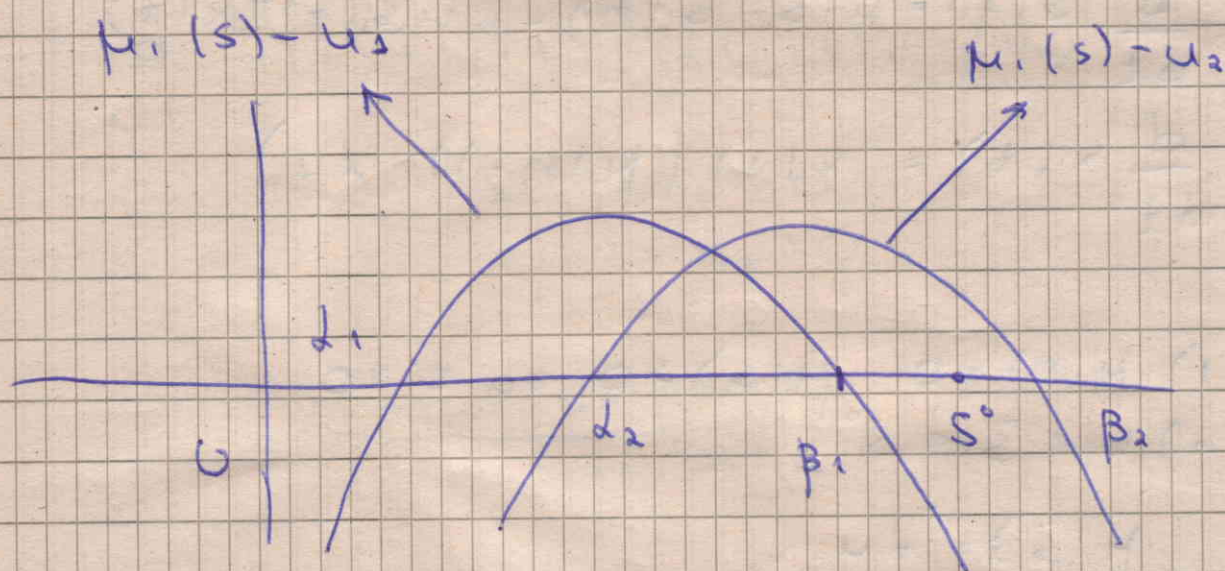
3) $\mu(s) = 2u \quad \left\{ \begin{array}{l} > 0 \quad s > \bar{s} \\ < 0 \quad \bar{s} < s \end{array} \right.$

Моген с повете ат (2 буга) ограничу

$$\begin{aligned} \frac{d}{dt} s(t) &= u(s^* - s(t)) - k_1 \mu_1(s(t)) x_1(t) - k_2 \mu_2(s(t)) x_2(t) & s(0) &= s_0 > 0 \\ \frac{d}{dt} x_1(t) &= x_1(t) (\mu_1(s(t)) - u_1) & x_1(0) &= x_0' > 0 \\ \frac{d}{dt} x_2(t) &= x_2(t) (\mu_2(s(t)) - u_2) & x_2(0) &= x_0'' > 0 \end{aligned}$$

$$\mu_1(0) = \mu_2(0) = 0$$

$$\mu_1(s) > 0 \quad \mu_2(s) > 0 \quad \exists a < s < b$$



първия оцелява при по-ниска
концентрация на храната
=> той ще надделее на втория и
втория ще излезне

$$P(t) = s(t) + k_1 x_1(t) + k_2 x_2(t) - \frac{u s^0}{u_{\min}}$$

$$\frac{d}{dt} P(t) = u(s^0 - s(t)) - k_1 \mu_1(s(t)) x_1(t) - k_2 \mu_2(s(t)) x_2(t) + k_1 x_1(t) (\mu_1(s(t)) - u_1) + k_2 x_2(t) (\mu_2(s(t)) - u_2)$$

$$= u(s^0 - s(t)) - k_1 x_1(t) u_1 - k_2 x_2(t) u_2 \geq (1)$$

$$0 < u_{\min} = \min(u, u_1, u_2)$$

$$0 < u_{\max} = \max(u, u_1, u_2)$$

$$(1) \leq u s^0 - u_{\min} (s(t) + k_1 x_1(t) + k_2 x_2(t))$$

$$= -u_{\min} (s(t) + k_1 x_1(t) + \underbrace{k_2 x_2(t)}_{P(t)} - \frac{u s^0}{u_{\min}})$$

$$\dot{p}(t) \leq -u_{\min} p(t)$$

$$q(t) = s(t) + k_1 x_1(t) + k_2 x_2(t) - \frac{u s^0}{u_{\max}}$$

$$\frac{d}{dt} q(t) = (s^0 - s(t))u - k_1 x_1(t) u_1 - k_2 x_2(t) u_2 \geq -u_{\max} q(t)$$

$$\dot{p}(t) \leq -u_{\min} p(t) \quad | \cdot e^{u_{\min} t}$$

$$e^{u_{\min} t} \frac{d}{dt} p(t) \leq -u_{\min} p(t) e^{u_{\min} t}$$

$$\frac{d}{dt} (p(t) \cdot e^{u_{\min} t}) \leq 0$$

$$p(t) e^{u_{\min} t} \leq p(0) e^{u_{\min} \cdot 0} = p(0)$$

$$p(t) \leq p(0) \cdot e^{-u_{\min} t} \quad // \quad e^{-u_{\min} t} \rightarrow 0$$

$$\frac{u s^0}{u_{\max}} - \varepsilon \leq s(t) + k_1 x_1(t) + k_2 x_2(t)$$

$$\leq \frac{u s^0}{u_{\min}} + \varepsilon$$

$$\Rightarrow p_{\text{em}} \cdot e \quad t \in (0; +\infty)$$

$$\frac{d}{dt} (s(t) - s^0) \stackrel{= \frac{d}{dt} s(t)}{=} \frac{d}{dt} s(t) < u(s^0 - s(t))$$

$$\frac{d}{dt} r(t) < -u r(t) \quad | \cdot e^{ut}$$

$$r(t) = s(t) - s^0$$

$$\frac{d}{dt} (r(t) e^{\mu t}) \leq 0$$

$$r(t) \leq r(0) \cdot e^{-\mu t} \quad // e^{-\mu t} \rightarrow 0$$

$$s(t) \leq s^0 + \epsilon$$

$$B(s^0, s^0 + \epsilon)$$

$$\mu_2(s) - \mu_2 > \eta > 0$$

$$\Rightarrow \frac{d}{dt} x_2(t) \geq \eta x_2(t) \quad \dot{x}$$

17.12.2013г.

μ, X - за параметри

$$\frac{d}{dt} s(t) = \mu (s_{in}^0 - s(t)) - k_1 \mu_1(s(t)) x_1(t) - k_2 \mu_2(s(t)) x_2(t)$$

конко са издана

$$\frac{d}{dt} x_1(t) = (\mu_1(s(t)) - \mu_1) x_1(t)$$

конко са издана

$$\frac{d}{dt} x_2(t) = (\mu_2(s(t)) - \mu_2) x_2(t)$$

2 вида във ванна

$s(t)$ - конко на устр.

$x_1(t)$ - конко бичага 1

$x_2(t)$ - " 2

S° - граница

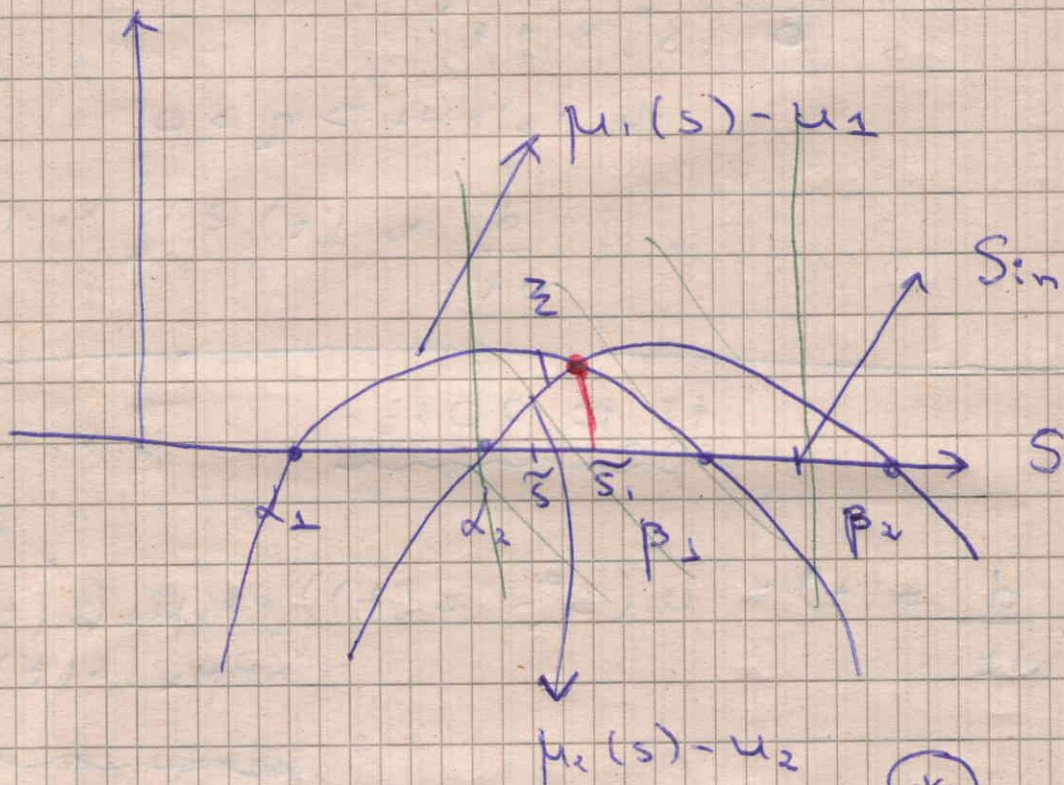
$$s(0) = S_0 > 0$$

$$x_1(0) = x_1^{\circ} > 0$$

$$x_2(0) = x_2^{\circ} > 0$$

$$\mu_1(0) = \mu_2(0) = 0$$

$$\mu_1(s) \geq 0 \quad \mu_2(s) \geq 0$$



$$\frac{u S_{in} - \varepsilon}{u_{max}}$$

$$s(t) + \kappa_1 x_1(t) + \kappa_2 x_2(t)$$

(*)

$$\leq \frac{u S_{in} + \varepsilon}{u_{min}}$$

$$u_{max} = \max(u, u_1, u_2)$$

$$u_{min} = \min(u, u_1, u_2)$$

за $\forall \varepsilon > 0 \exists T > 0$
 $t > T$ е изпълнено
 (от динамика)

$$0 < s(t) < S_{in}$$

Ще док, че $\exists T$ т.е. s се
намира в нево от Δ_2

Док., че за $\forall t > T$ $s(t) \geq \Delta_2$

$$\Delta_2 \leq s(t) \leq S_{in}$$

Лема Барбала

растуща
не е
смен знака

$$x_2(t) = \int_0^t (\mu_2(s(\sigma)) - u_2) d\sigma + x_2^0$$

подитт. ф-я се
нулира

$$\frac{d}{dt} x_1(t) \geq x_1(t) \eta, \eta > 0$$

равном
непр.

$$\mu_1 - u_1 > 0$$

$\Rightarrow$ раш. расте неогр
в този път срво
X

~~d/dt~~

Док. че $\exists$ момент $\tilde{t}$

$$s(\tilde{t}) = \tilde{s}$$

? произв. в този момент

$$\frac{d}{dt} s(\tilde{t}) = u(s_{in} - \tilde{s}) - \kappa_1 x_1(\mu_1(\tilde{s}))$$

$$-k_2 x_2(\tilde{t}) \mu_2(\tilde{s}) =$$

$$= \cancel{k_1 x_1(\tilde{t}) \mu_1(\tilde{s}) - k_2 x_2(\tilde{t}) \mu_2(\tilde{s})}$$

$$= u(\sin - \tilde{s}) - (k_1 x_1(\tilde{t}) + k_2 x_2(\tilde{t})) \cdot$$

$$\left[\frac{k_1 x_1(\tilde{t}) \mu_1(\tilde{s})}{k_1 x_1(\tilde{t}) + k_2 x_2(\tilde{t})} + \frac{k_2 x_2(\tilde{t}) \mu_2(\tilde{s})}{k_1 x_1(\tilde{t}) + k_2 x_2(\tilde{t})} \right]$$

изтъкнала комб. на $\mu_1(\tilde{s})$
 $\mu_2(\tilde{s})$

$\Rightarrow$ тя е $u/4$ голяма

$$\leq u(\sin - \tilde{s}) - (k_1 x_1(\tilde{t}) + k_2 x_2(\tilde{t})) \cdot$$

$$\min(\mu_1(\tilde{s}), \mu_2(\tilde{s}))$$

$$\leq u(\sin - \tilde{s}) - \left(\tilde{s} + \epsilon - \frac{u \sin}{u_{\max}} \right)$$

$$\min(\mu_1(\tilde{s}), \mu_2(\tilde{s}))$$

$$u(\sin - \tilde{s}) - \left(\tilde{s} + \frac{u \sin}{u_{\max}} \right) \cdot$$

$$\min(\mu_1(\tilde{s}), \mu_2(\tilde{s})) < 0$$

$\Rightarrow$ от 0 до $\tilde{s}$
 не може да излезе

$$? \quad x_2 \rightarrow 0$$

II Bug uzreizba

$$\frac{d}{dt} p_n(x_2(t)) - \frac{d}{dt} p_n(x_1(t)) =$$

$$= (\mu_2(s(t)) - u_2) - (\mu_1(s(t)) - u_1)$$

$$\frac{d}{dt} p_n \frac{x_2(t)}{x_1(t)} < -\zeta \quad \zeta > 0$$

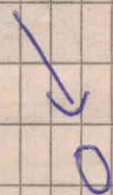
$$\frac{d}{dt} \left(p_n \left(\frac{x_2(t)}{x_1(t)} \right) e^{\zeta t} \right) < 0$$

$$p_n \frac{x_1(t)}{x_1(t)} e^{\zeta t} < p_n \frac{x_2(0)}{x_1(0)} \cdot 1$$

$$\frac{x_2(t)}{x_1(t)} < c e^{-\zeta t}$$

$$x_2(t) = \frac{x_2(t)}{x_1(t)} \cdot x_1(t) <$$

$$c e^{-\zeta t} \cdot 0$$



$$s_1 \rightarrow d_1$$

Ако цялата с-ма $\rightarrow \odot$
разл. с-мата, която
остава

La Salle

мивотни

ишане и продукт, от деля се продукт
хране се със сурт

$$\frac{d}{dt} s(t) = u(s_{in} - s(t)) - k \mu(s(t)) x(t)$$

$$\frac{d}{dt} x(t) = (\mu(s(t)) - 2u) \cdot x(t)$$

$$\frac{d}{dt} p(t) = h \mu(s) x(t) - p u p(t)$$

продукт

$$s(0) = s_0 > 0$$

$$x(0) = x_0 > 0$$

$$p(0) = p_0$$

$$0 < \bar{s} < s_{in} > 0$$

конц.

↓
фиксиране точка

как да мърдаме u , те

ишаната да
отиде в $\bar{s}$.

u -произволна

$$\frac{d}{dt} x(t) = f(x(t), u(t))$$

$$u(x(t))$$

|

да зависи от
фазовата
променлива
затворена
система

$\bar{x}$

от кои точки може да се
стигне до $\bar{x}$?

$$\mu(0) = 0$$

$$\mu(s) > 0 \quad s > 0$$

непр.

$$u = \frac{k \mu(s) x}{\sin - \bar{s}}$$

уедно, че
тази

система

клучи към $\bar{s}$

$$\frac{d}{dt} s(t) = \frac{k \mu(s(t)) x(t)}{\sin - \bar{s}} (\sin - s(t)) -$$

$$- k \mu(s(t)) x(t) =$$

$$= k \mu(s(t)) x(t) \left[\frac{\sin - s(t)}{\sin - \bar{s}} - 1 \right] =$$

$$= - \frac{k \mu(s(t)) x(t)}{\sin - \bar{s}} (s - \bar{s}) = \frac{d}{dt} (s(t) - \bar{s})$$

$$\frac{d}{dt} x(t) = \left(\mu(s(t)) - \frac{2 \cdot k \mu(s)}{\sin - \bar{s}} x(t) \right) x(t)$$

$$= \mu(s(t)) x(t) \left\{ \sin - \bar{s} - 2 k x \right\}$$

$$\frac{d}{dt} P(t) = \frac{\mu(s(t)) x(t)}{\sin - \bar{s}} - \frac{\frac{\sin - \bar{s}}{2k} \cdot \frac{2k}{\sin - \bar{s}} x(t)}{\sin - \bar{s}} - \frac{\beta \cdot k \mu(s(t)) x(t)}{\sin - \bar{s}}$$

$$= \mu(s(t)) x(t) \left[h - \frac{\beta k P(t)}{\sin - \bar{s}} \right]$$

$$\frac{h \sin - h \bar{s} - \beta k P(t)}{\sin - \bar{s}}$$

$$\frac{d}{dt} (s(t) - \bar{s}) = -u(t) (s(t) - \bar{s})$$

$$\frac{d}{dt} (x(t) - \bar{x}) = -2u(t) (x(t) - \bar{x})$$

$$\frac{d}{dt} (P(t) - \bar{P}) = -\beta u(t) (P(t) - \bar{P})$$

$$\dot{z}(t) = \lambda(t) z(t) \quad z(0) = z_0 > 0$$

$$\Rightarrow z(t) = z_0 e^{\int_0^t \lambda(s) ds}$$

$\bar{p}$

$$p(t) = - \frac{\beta \kappa \mu(s(t)) x(t)}{s(t) - \bar{s}} \left[p(t) - \frac{h(s(t) - \bar{s})}{\beta \kappa} \right]$$

$$s(t) - \bar{s} = (s_0 - \bar{s}) e^{-\int_0^t u(s) ds}$$

$$x(t) - \bar{x} = (x_0 - \bar{x}) e^{-\int_0^t u(s) ds}$$

$$p(t) - \bar{p} = (p_0 - \bar{p}) e^{-\int_0^t u(s) ds}$$

also $\int \rightarrow \infty$ o.k. $\int \rightarrow \bar{s}$

$$\begin{aligned} s(t) &= \bar{s} + (s_0 - \bar{s}) e^{-\int_0^t u(s) ds} \\ x(t) &= \bar{x} + (x_0 - \bar{x}) e^{-\int_0^t u(s) ds} \\ p(t) &= \bar{p} + (p_0 - \bar{p}) e^{-\int_0^t u(s) ds} \end{aligned}$$

$$s(t) = \bar{s} (1 - \lambda(t)) \rightarrow \lambda(t) s_0$$

$$\lambda(t) = e^{-\int_0^t u(s) ds}$$

$$\lambda(t) \in [0, 1]$$

отгорт се менат (в момент t)

$$Q(t) = \lambda \mu(s(t)) \cdot x(t)$$

$$\Rightarrow u = \frac{\kappa Q(t)}{\lambda(s(t) - \bar{s})}$$

$$\frac{d}{dt} s(t) = u(s(t) - \bar{s}) - \kappa \mu(s(t)) x(t)$$

$$\frac{d}{dt} x(t) = (\mu(s(t)) - \lambda u) x(t)$$

$$\frac{d}{dt} p(t) = h \mu(s(t)) x(t) - p u p(t)$$

$$\frac{d}{dt} \gamma(t) = -C(s(t) - \bar{s}) Q(t) (\gamma(t) - \gamma^-)(\gamma^+ - \gamma(t))$$

$$u(t) = \lambda \mu(s(t)) x(t)$$

$$u(t) = \gamma(t) Q(t) \quad \text{обратна връзка}$$

стабилизира
системата

квм т. $\bar{s}$

адаптивно обратна връзка

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), x(t)) \\ \dot{y}(t) = g(y(t), x(t)) \end{cases}$$

разширяване
методата
стабилизиране

07.01.2014г.

$\mu(s)$ - текуща
скорост
на
растение

влиза

$$\begin{aligned} \dot{s}(t) &= u(s^i - s(t)) - \kappa \mu(s(t)) x(t) \\ \dot{x}(t) &= (\mu(s(t)) - \lambda u) x(t) \\ \dot{p}(t) &= p_n \mu(s(t)) x(t) - p_u p(t) \end{aligned}$$

p - продукт

u - управление; обратна връзка;

$$s(0) = s_0 > 0$$

$$x(0) = x_0 > 0$$

$$p(0) = p_0 > 0$$

$$u(t) = \frac{\kappa \mu(s(t)) x(t)}{s^i - \bar{s}}$$

$$0 < \bar{s} < s^i$$

равновесна
точка

Прешотване

/
анаеробни
отделя се газ

x - не може да се мери в
реално време

обем на газа, който се отделя

$$y(t) = \lambda \mu(s(t)) x(t)$$

$$0 < \lambda < 1$$

знаем

$$s(t) \blacksquare$$

$s(t)$ се измерва лесно

адаптивно
у-е

$$\dot{y}(t) = -c(s(t) - \bar{s})(M - y(t)) \\ (y(t) - m) y(t)$$

$$0 < \bar{s} < s^i$$

$$y(0) \in (m, M)$$

$$0 < m < M$$

$$u(t) = y(t) \cdot y(t)$$

отвора, с който пускаме
нашето влияние

$$u(t) = \frac{k \mu(s(t)) x(t)}{s^i - \bar{s}} \quad \begin{matrix} \text{не} \\ \text{се} \\ \text{мери} \end{matrix}$$

искаме само измерими и
да стабилизираме
процеса

автономна с-ма-добавяме $\dot{x}(t) = \dots$

Заместваме и в с-мата

$$\dot{s}(t) = \underline{y(t)} \left[x(t) \cdot (s' - s(t)) - \frac{\kappa}{\lambda} \right]$$

$$\dot{x}(t) = y(t) \left[\frac{1}{\lambda} - \alpha \underline{x(t)} x(t) \right]$$

$$\dot{p}(t) = y(t) \left[\frac{h}{\lambda} - \beta x(t) p(t) \right]$$

$$\dot{x}(t) = y(t) \cdot c (s - s(t)) (m - x(t)) (x(t) - m)$$

има единствено реш.

$x \neq 0$, защото там има реш, а то
е единствено

$$x > 0$$

$s, x, p \neq 0$ - съответства реш.

$$\Rightarrow y(t) > 0$$

$$\mu(0) = 0$$

$$\mu(s) > 0 \rightarrow s > 0$$

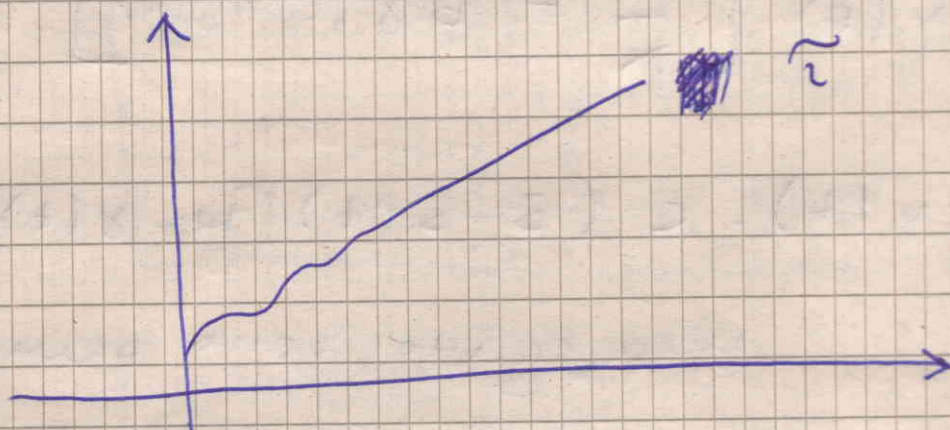
Сменяме скалата на Времето

$$z(t) = \tilde{z} := \int_0^t \gamma(s) ds$$

$$\tau > 0$$

$$z'(t) = \gamma(t) > 0$$

$\Rightarrow z$ има обратна



$$\tau = z(z^{-1}(\tau))$$

$$1 = z'(z^{-1}(\tau)) \cdot \frac{d}{d\tau} z^{-1}(\tau)$$

Взимаме s и правим смяна

$$\tilde{s}(\tau) = s(z^{-1}(\tau))$$

$$\frac{d}{d\tau} \tilde{s}(\tau) = \frac{d}{dt} \tilde{s}(z^{-1}(\tau)) \cdot \frac{d}{d\tau} z^{-1}(\tau)$$

$$= \gamma(z^{-1}(\tau)) \cdot \left[\gamma(z^{-1}(\tau)) \cdot (s - s(z^{-1}(\tau)) - \frac{\kappa}{\gamma}) \right]$$

$$\Rightarrow \frac{d}{d\tau} z^{-1}(\tau) = \frac{1}{\gamma(z^{-1}(\tau))} = \frac{1}{\gamma(z(\tau))}$$

$$\Rightarrow \frac{d}{d\tau} \tilde{s}(\tau) = \left[\gamma(z^{-1}(\tau)) (s - \tilde{s}(\tau) - \frac{\kappa}{\gamma}) \right]$$

$\gamma \rightarrow \tilde{\gamma}$ изглаголо е $\gamma(t)$

$\tilde{\gamma} \rightarrow \gamma$ и т.н.

$$\tilde{x}(t) = x(z^{-1}(\tau))$$

$$\tilde{p}(t) = p(z^{-1}(\tau))$$

$$\tilde{\gamma}(\tau) = \gamma(z^{-1}(\tau))$$

$$\frac{d}{d\tau} \tilde{s}(\tau) = \tilde{s}'(\tau) = \tilde{\gamma}(\tau) (s - \tilde{s}(\tau) - \frac{\kappa}{\gamma})$$

$$\frac{d}{d\tau} \tilde{s}(\tau) = \tilde{s}'(\tau) = \tilde{\gamma}(\tau) (s^i - \tilde{s}(\tau) - \frac{\kappa}{\lambda})$$

$$\frac{d}{d\tau} \tilde{x}(\tau) = \frac{1}{\lambda} - 2\tilde{\gamma}(\tau)\tilde{x}(\tau)$$

$$\frac{d}{d\tau} \tilde{p}(\tau) = \tilde{p}'(\tau) = \frac{h}{\lambda} - p\tilde{\gamma}(\tau)\tilde{p}(\tau)$$

$$\frac{d}{d\tau} \tilde{\gamma}(\tau) = \tilde{\gamma}'(\tau) = c(\bar{s} - \tilde{s}(\tau))(M - \tilde{\gamma}(\tau)) \cdot \textcircled{*} (\tilde{\gamma}(\tau) - m)$$

$$v(\tau) = s^i - \tilde{s}(\tau) \quad \textcircled{*}$$

$$v'(\tau) = -\tilde{s}'(\tau) = \frac{\kappa}{\lambda} - \tilde{\gamma}(\tau)v(\tau)$$

$$\tilde{\gamma}'(\tau) = c(M - \tilde{\gamma}(\tau))(\tilde{\gamma}(\tau) - m) \frac{(v(\tau) - \bar{v})}{v(\tau)}$$

$$\begin{aligned} \bar{s} - \tilde{s}(\tau) &= \bar{s} - s^i + \overbrace{s^i - \tilde{s}(\tau)}^{v(\tau)} = \\ &= v(\tau) - \bar{v} \end{aligned}$$

$$\bar{v} = \bar{s} - s^i$$

→ получаем двумерную систему

Принцип Ла Сале $\tilde{\gamma}$

$$F(v, \tilde{\gamma}) = \int_{\tilde{\gamma}} \frac{z - \bar{v}}{z} dz + \int_{\tilde{\gamma}} \frac{\eta - \bar{\gamma}}{c(H - \eta)(\eta - m)} d\eta$$

!

$$F(v, \tilde{\gamma}) = \int_{\tilde{v}}^v \frac{z - \bar{v}}{z} dz + \int_{\tilde{\gamma}}^{\gamma} \frac{\eta - \bar{\gamma}}{c(H - \eta)(\eta - m)} d\eta$$

$$F'(v, \tilde{\gamma}) =$$

аннулира (*)

$$\bar{\gamma} = \bar{s}$$

$\bar{\gamma}$ -прави 0 производната по s

$$\bar{\gamma} = \frac{\kappa}{\lambda(s^i - \bar{s})} = \frac{\kappa}{\lambda \bar{v}}$$

$$\bar{v} = s^i - \bar{s}$$

$$\dot{F}^*(v, \tilde{\gamma})$$

градиент . дясната страна

(**)

$$\dot{F}^*(v, \tilde{\gamma}) = \left(\frac{v - \bar{v}}{\bar{v}}, \frac{\gamma - \bar{\gamma}}{c(H - \gamma)(\gamma - m)} \right) \cdot \begin{pmatrix} \frac{\partial F}{\partial v} \\ \frac{\partial F}{\partial \gamma} \end{pmatrix}$$

$$\dot{F}(v, \gamma) = \frac{v - \bar{v}}{v} \cdot (\bar{v}\bar{\gamma} - v\gamma) +$$

$$+ \frac{\gamma - \bar{\gamma}}{c(\mu - \gamma)(\gamma - m)} \cdot c(\mu - \gamma)(\gamma - m)(v - \bar{v})$$

$$\dot{F}(v, \gamma) = \frac{v - \bar{v}}{v} \left[\bar{\gamma}\bar{v} - \cancel{\gamma v} + \cancel{\gamma v} - \bar{\gamma}v \right] =$$

$$= \frac{-\bar{\gamma}(v - \bar{v})^2}{v} \quad \text{о.к.:))}$$

От Принципа на La Salle:

компл. на $S = \bar{S}$

$\dot{S} \stackrel{=0}{\neq} 0$ само при $v = \bar{v}$

$$v'(\tilde{t}) = \bar{v}\bar{\gamma} - \tilde{\gamma}(\tilde{t})v(\tilde{t})$$

$$\begin{matrix} v \rightarrow \bar{v} \\ \gamma \rightarrow \bar{\gamma} \end{matrix} \quad :)$$

при $\gamma = \bar{\gamma}$

$$\tilde{x}'(\tilde{t}) = \frac{1}{\gamma} - 2\bar{\gamma}\tilde{x}(\tilde{t})$$

$$\tilde{p}'(\tilde{t}) = \frac{1}{\gamma} - p\bar{\gamma}\tilde{p}(\tilde{t})$$

$$\begin{cases} \tilde{x}'(\tau) = \frac{1}{\gamma} - \lambda \bar{\gamma} \tilde{x}(\tau) \\ \tilde{p}'(\tau) = \frac{1}{\gamma} - \beta \bar{\gamma} \tilde{p}(\tau) \end{cases}$$

$$\tilde{x}(\tau) = e^{-\int_0^\tau \lambda \bar{\gamma} ds} \left(x_0 + \int_0^\tau \frac{1}{\gamma} e^{\int_0^s \lambda \bar{\gamma} ds} ds \right)$$

$$\tilde{x}'(\tau) = -\lambda \bar{\gamma} \tilde{x}(\tau) + \frac{1}{\gamma}$$

$$\tilde{x}'(\tau) = \lambda \bar{\gamma} \left(\frac{1}{\lambda \bar{\gamma}} - \tilde{x}(\tau) \right) =$$

$$\parallel = -\lambda \bar{\gamma} (\tilde{x}(\tau) - c)$$

$$\frac{d}{d\tau} (\tilde{x}(\tau) - c)$$

$$\tilde{x}(\tau) - c = (x_0 - c) \cdot e^{-\lambda \bar{\gamma} \tau}$$

$$\begin{matrix} x \rightarrow c \\ t \rightarrow \infty \end{matrix}$$

$$c = \frac{1}{\lambda \bar{\gamma}}$$

~~---~~ c duon pouceu :))

? Как се учи за изпит ?

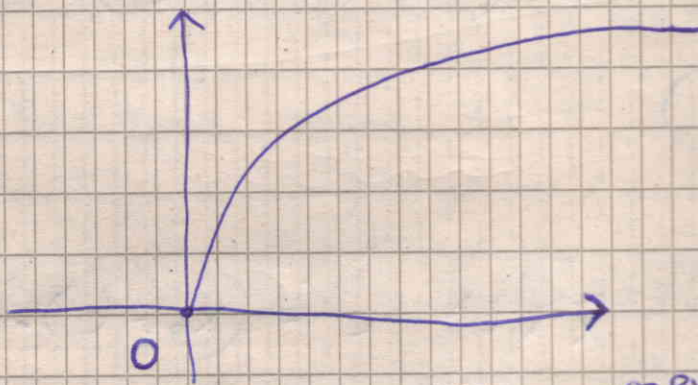
A_0 A_1 дни

$A_0, A_1, \dots, A_m = \bar{A}$

A_1 - познаница $\uparrow$ ден

$$A_1 = A_0 - \underbrace{2A_0}_{\text{забравихме}} + \beta y_1^{1/2} - \text{времето, в което сме учили}$$

$$f(y) = y^2 \quad y \in (0, 1)$$



$$A_2 = A_1 - 2A_1 + \beta y_2^{1/2} - \text{ефективност на ученето}$$

$$A_n = A_{n-1} - 2A_{n-1} + \beta y_n^{1/2}$$

$$y > 0$$

искаме да min времето

$$\sum_{i=1}^3 Y_i \rightarrow \min$$

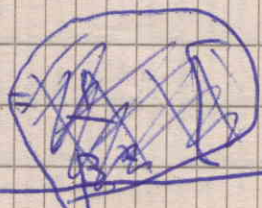
$$Y_1 = \left(\frac{A_1 + 2A_0 - A_0}{\beta} \right)^2$$

$$Y_2 = \left(\frac{A_2 + 2A_1 - A_1}{\beta} \right)^2$$

⋮

$$Y_n = \left(\frac{A_n + 2A_{n-1} - A_{n-1}}{\beta} \right)^2$$

$$Y_1 + Y_2 + \dots + Y_n = \frac{1}{\beta^2} \left[\frac{A_1 + A_0(2-1)}{\beta}^2 + \dots + \frac{A_n + 2A_{n-1} - A_{n-1}}{\beta}^2 \right]$$



$$A_{k+1} = \gamma A_k + \beta^{1/2} Y_{k+1} \quad \gamma = 1-2$$

знаем A_0 и A_n

геним γ^{k+1}

$$\frac{A_{k+1}}{\gamma_{k+1}} = \frac{A_k}{\gamma_k} + \frac{\beta^{1/2} Y_{k+1}}{\gamma_{k+1}} \quad \text{и собираем}$$

$$C B_k = \frac{A_k}{\gamma_k}$$

$$\Rightarrow \cancel{B_k} \cancel{B_k} = \cancel{B_k}$$

$$\Rightarrow C_{k+1} = C_k + \frac{B}{\gamma_{k+1}} \cdot \gamma_{k+1}^{1/2}$$

суммарно

$$\underset{\substack{\uparrow \\ \text{знаем}}}{d} = C_m - C_0 = B \sum_{k=1}^m \frac{\gamma_k^{1/2}}{\gamma_k}$$

14.01.2014г.

$$a = (a_1, \dots, a_n)$$

$$b = (b_1, \dots, b_n)$$

$$\langle a, b \rangle \leq \|a\| \cdot \|b\|$$

$$\left(\sum_{i=1}^m \gamma_i \right)^{1/2} \geq \sum_{i=1}^m \frac{\gamma_i^{1/2}}{\gamma_i} = F, \quad \gamma = \left(\sum_{i=1}^n \frac{1}{\gamma^{2i}} \right)^{1/2}$$



$$i = 1, \dots, m$$

$$\langle a, b \rangle = \|a\| \cdot \|b\|$$

$$a_i = \tilde{r} b_i$$

$$b = \left(\frac{1}{\delta}, \frac{1}{\delta^2}, \dots, \frac{1}{\delta^m} \right)$$

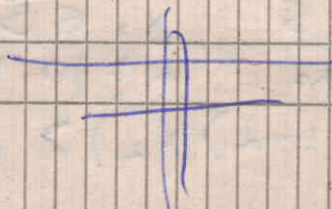
$$a = \left(y_1^{1/2}, \dots, y_m^{1/2} \right)$$

$$y_i^{1/2} = \frac{\tilde{r}}{\delta^i} \Rightarrow y_i = \frac{\tilde{r}^2}{\delta^{2i}}$$

$$\sum_{i=1}^3 \frac{\tilde{r}}{\delta^{2i}} = \pi$$

$$\tilde{r} = \frac{\pi}{\sum_{i=1}^3 \frac{1}{\delta^{2i}}}$$

$$A_{i+1} = \delta A_i + \beta y_i^{1/2} \quad \delta \in (0, 1)$$



$u(c, e)$ - ф-я на полезност
 c - потребление на стоки
 e - изразел :D

$$u'_c > 0$$

$$u'_e > 0$$

$$u(c, e) \rightarrow \max$$

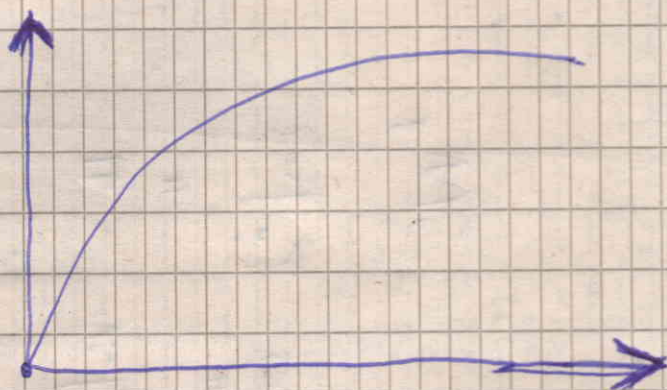
$$u(0, e) = 0$$

$$u(c, 0) = 0$$

$$\lim_{c \rightarrow 0^+} u'_c(c, e) = \infty$$

$$\lim_{e \rightarrow 0^+} u'_e(c, e) = \infty$$

инко вгдъбната



под графиката - изтъкнало

$$f: S \rightarrow \mathbb{R}, S \subseteq \mathbb{R}^n$$

! S - изтъкнало 0

$$x_1 \in S, x_2 \in S, \lambda \in [0, 1]$$

$$x_1 \neq x_2, \lambda \neq \{0, 1\}$$

$$f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda)f(x_2)$$

$\{(x, z) \in \mathbb{R}^{n+1} : z \leq f(x)\}$ подграфика-
изтъкнало $\odot$
при f -вдълб-
ната

$(x_1, z_1) \in \odot$ $f(x_1) \geq z_1$
 $(x_2, z_2) \in \odot$ $f(x_2) \geq z_2$

$\Rightarrow f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$

$\lambda(x_1, z_1) + (1-\lambda)(x_2, z_2) =$
 $= (\lambda x_1 + (1-\lambda)x_2, \lambda z_1 + (1-\lambda)z_2)$

изтъкнало $\odot$

$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$
 $\geq \lambda z_1 + (1-\lambda)z_2$
 $= z$

$\{x \in \mathbb{R}^n : f(x) \geq r\}$

x_1 $f(x_1) \geq r$
 x_2 $f(x_2) \geq r$

$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$
 $\geq \lambda r + (1-\lambda)r$
 $= r$

к-капитал

n-время на
работа

$$u(c, l) \rightarrow \max$$
$$0 \leq c \leq \overset{\text{изплата}}{w(1-l)} + \tau k$$

цено
время

$$l = n + l$$

$$k \rightarrow k + \mu k = (1 + \mu)k$$

цена капитал

$$0 \leq k \leq k_0$$

$$0 \leq l \leq 1$$

$k = k_0$ Влагаем целую
капитал за
повече потребление

$$l \neq 0$$

$$l \neq 1$$

$$c > 0$$

(=)

$$u(c, l) \rightarrow \max$$
$$c \leq w(1-l) + \tau k_0$$

Б.О.

$$\left\{ \begin{array}{l} c = w(1-l) + \tau k_0 \\ u(c, l) \rightarrow \max \end{array} \right.$$

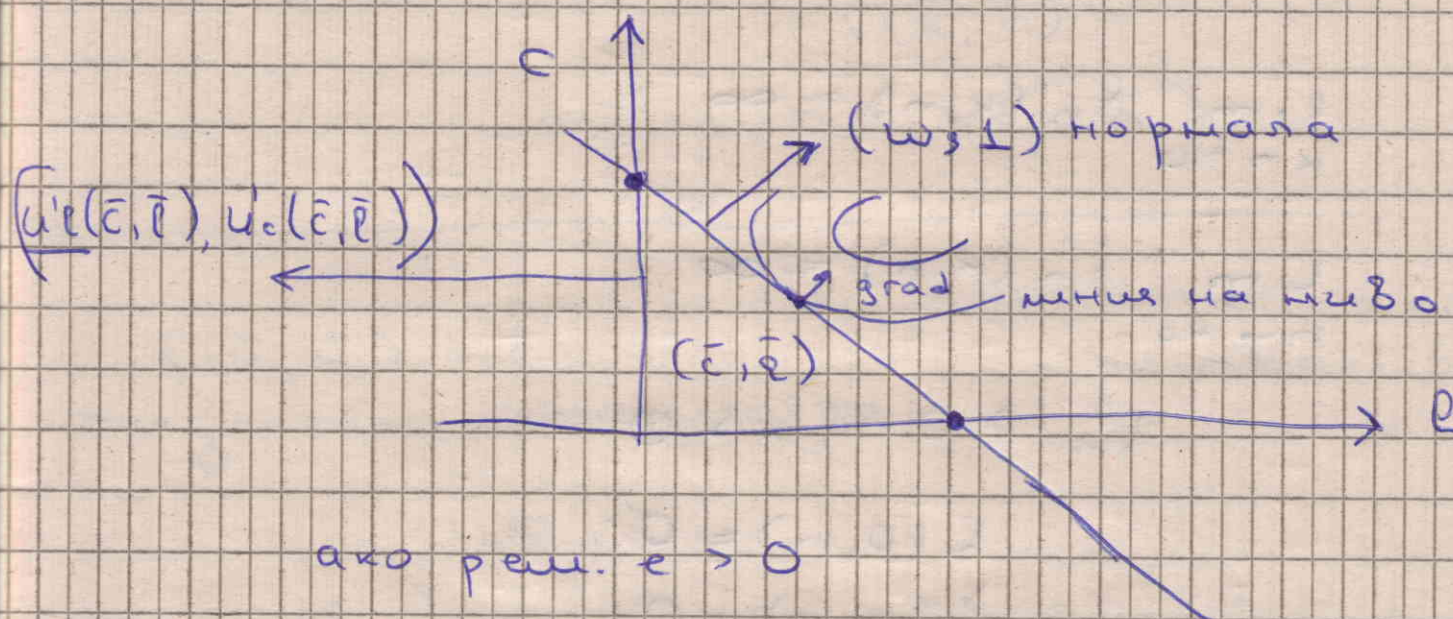
$$L(c, l, \lambda) = u(c, l) + \lambda (w(1-l) + \tau k_0 - c)$$

$$0 = L_c = u'_c(\bar{c}, \bar{l}) - \lambda$$

$$0 = L_l = u'_l(\bar{c}, \bar{l}) - \lambda w$$
$$c + w l = w + \tau k_0$$

$$u'_c(\bar{c}, \bar{e}) = \lambda$$

$$u'_e(\bar{c}, \bar{e}) = \lambda w$$



$$\{x \in \mathbb{R}^n : f(x) \geq \alpha\} \quad \begin{array}{l} \uparrow \\ \text{произвольно} \end{array} \quad \text{изтъкнало}$$

$$\left\{ (c, e) : u(c, e) = u(\bar{c}, \bar{e}) \right\}$$

изтъкнало $\odot$

$$\left| \begin{array}{l} u'_c(\bar{c}, \bar{e}) = \lambda \\ u'_e(\bar{c}, \bar{e}) = \lambda w \end{array} \right. \Rightarrow w u'_c(\bar{c}, \bar{e}) = u'_e(\bar{c}, \bar{e})$$

$$w u'_c(\bar{c}, \bar{e}) = u'_e(\bar{c}, \bar{e})$$

капитал
, труд

За фирма

$$f(k, n) = \omega n - r k \rightarrow \max$$

зарплата цена на капитал

$$\lim_{k \rightarrow +0} f'_k(k, n) = \infty$$

$$\lim_{n \rightarrow +0} f'_n(k, n) = \infty$$

$$f'_k > 0 \quad f'_n > 0$$

$$f(0, n) = 0$$

$$f(k, 0) = 0$$

$$f'_k(\bar{k}, \bar{n}) = r$$

$$f'_n(\bar{k}, \bar{n}) = \omega$$

и добавим $\omega u'_e(\bar{c}, \bar{e}) = u'_e(\bar{c}, \bar{e})$

$$f(\lambda k, \lambda n) = \lambda f(k, n) \quad \lambda > 0$$

гомогенно

напр. $f(k, n) = k^{\frac{1}{3}} n^{\frac{2}{3}}$

$$f(\lambda \bar{k}, \lambda \bar{n}) = \lambda f(\bar{k}, \bar{n})$$

диф. по λ

$$f(\bar{k}, \bar{n}) = f'_k(\lambda \bar{k}, \lambda \bar{n}) \bar{k} + f'_n(\lambda \bar{k}, \lambda \bar{n}) \bar{n}$$

$$\lambda = 1$$

$$\begin{aligned} f'_k(\bar{k}, \bar{n}) \cdot \bar{k} + f'_n(\bar{k}, \bar{n}) \cdot \bar{n} &= \\ &= f(\bar{k}, \bar{n}) = \\ &= r\bar{k} + w\bar{n} = f(\bar{k}, \bar{n}) \end{aligned}$$

$$f(\bar{k}, \bar{n}) = \bar{c}$$

предлагание консумации

равенство на производств

$$\bar{n} + \bar{e} = 1$$

$$\bar{k} = k_0$$

$$\textcircled{1} \quad w u'_c(\bar{c}, \bar{e}) = u'_e(\bar{c}, \bar{e})$$

$$\textcircled{2} \quad f'_k(\bar{k}, \bar{n}) = r$$

$$\textcircled{3} \quad f'_n(\bar{k}, \bar{n}) = w$$

$$\textcircled{4} \quad f(\bar{k}, \bar{n}) = \bar{c}$$

$$\textcircled{5} \quad \bar{n} + \bar{e} = 1$$

$$\textcircled{6} \quad \bar{k} = k_0$$

(*)

③①

$$f'_n(k_0, 1 - \bar{e}) u'_c(f(k_0, 1 - \bar{e}), \bar{e}) = u'_e(f(k_0, 1 - \bar{e}), \bar{e})$$

нашируем $\bar{e}$

K_0 - капитал,
наличия в
обществе

Пример: $u(c, e) = 2c^{\frac{1}{2}} + e$
 $f(k, n) = k^{\frac{1}{3}} n^{\frac{2}{3}}$

$$f'_k(\bar{k}, \bar{n}) = \bar{n}^{\frac{2}{3}} \cdot \frac{1}{3} \bar{k}^{-\frac{2}{3}}$$

$$f'_n(\bar{k}, \bar{n}) = \frac{2}{3} \bar{k}^{\frac{1}{3}} \bar{n}^{-\frac{1}{3}}$$

$$u'_c = c^{-\frac{1}{2}} \quad u'_e = 1$$

$$f(K_0, 1-e) = K_0^{\frac{1}{3}} (1-e)^{\frac{2}{3}}$$

$$\frac{2}{3} K_0^{\frac{1}{3}} (1-e)^{-\frac{1}{3}} \cdot c^{-\frac{1}{2}} \quad (\text{ка})$$

$$\frac{2}{3} K_0^{\frac{1}{3}} (1-e)^{-\frac{1}{3}} \cdot K_0^{-\frac{1}{6}} (1-e)^{-\frac{1}{3}} = 1$$

$$K_0^{\frac{1}{6}} (1-e)^{-\frac{2}{3}} = \frac{3}{2}$$

$$\frac{3}{2 (1-e)^{\frac{2}{3}}} = K_0^{\frac{1}{6}}$$

$$\frac{3}{2 K_0^{\frac{1}{6}}} = (1-e)^{\frac{2}{3}}$$

$$\left(\frac{3}{2 K_0^{\frac{1}{6}}} \right)^{\frac{3}{2}} = 1-e$$

$$e = 1 - \left(\frac{3}{2 K_0^{\frac{1}{6}}} \right)^{\frac{3}{2}}$$

Кога има реш.

$$u(c, \ell) \rightarrow \max$$

↓

$$c = f(k_0, 1 - \ell)$$
$$0 \leq \ell \leq 1$$

Допустими стойности да $\exists c, \ell$
непразно огр. зате. $\odot$
ако $u(c, \ell) \rightarrow$ непрек. :))

① има решение

$$u(f(k_0, 1 - \ell), \ell) \rightarrow \max$$
$$\ell \in (0, 1)$$

Диференц.

$$-u'_c(f(k_0, 1 - \bar{\ell}), \bar{\ell}) \cdot f'_n(k_0, 1 - \bar{\ell})$$
$$+ u'_\ell(f(k_0, 1 - \bar{\ell}), \bar{\ell}) = 0$$

? единствено

Доп., че има 2 решения ℓ_1 и ℓ_2

$$c_1 = f(k_0, 1 - \ell_1)$$

$$c_2 = f(k_0, 1 - \ell_2)$$

$\parallel$

$$\Rightarrow u(c_1, \ell_1) = u(c_2, \ell_2) = \max u(c, \ell)$$

$$\frac{c_1 + c_2}{2} = \frac{1}{2} c_1 + \frac{1}{2} c_2 = \frac{1}{2} f(k_0, 1 - \ell_1)$$

$$+ \frac{1}{2} f(k_0, 1 - \ell_2)$$

или
вдлъбнатата

$$f(k_0, \frac{1}{2}(1-l_1) + \frac{1}{2}(1-l_2)) =$$

$$= f(k_0, 1 - (\frac{1}{2}l_1 + \frac{1}{2}l_2))$$

$$\Rightarrow \frac{c_1 + c_2}{2} \text{ год.}$$

$$u\left(\frac{c_1 + c_2}{2}, \frac{l_1 + l_2}{2}\right) > \frac{1}{2}u(c_1, l_1) +$$

$$+ \frac{1}{2}u(c_2, l_2) = M$$

max
↓

X

$\Rightarrow$ реш. е единствено

Държавата $\rightarrow$ данъци

$$u(c, e) \rightarrow \max_c$$

държавата
ко е
обложила

$$\text{приход} \quad \tau_w w n + \underbrace{\tau_k k_0}_{\text{капитал}} \tau_u > 0$$

$$\tau_w > 0$$

д. приход държавата

$$u(c, e) \rightarrow \max$$

$$c \leq (1 - \tilde{t}_w)w(1 - e) + (1 - \tilde{t}_k)r_k$$

$$f(k, n) - wn - rk \rightarrow \max$$

$$c + g = f(\bar{k}, \bar{n})$$

$$1 = \bar{n} + \bar{e}$$

$$\bar{k} = k_0$$



$$(1 - \tilde{t}_w)w u'_c(\bar{c}, \bar{e}) = u'_e(\bar{c}, \bar{e})$$

$$\begin{aligned} & (1 - \tilde{t}_w) f'_n(k_0, 1 - \bar{e}) u'_c(f(k_0, 1 - \bar{e}) - \\ & - \tilde{t}_w f'_n(k_0, 1 - \bar{e})(1 - \bar{e}) - \tilde{t}_k(f'_k(k_0, 1 - \bar{e})k_0, \bar{e})) \\ & = u'_c(\text{same argument}) \end{aligned}$$

21.01.2014г.

едномерна система

Тема + Пример (с д-во)

изтъкнала

без биопроцес с добавяне (адаптивно) управление

Принцип La Salle - без д-во

биопроцес с 2 популации (без)

Казино 2 игри

I

II

Игра 5% - 1000 лв.
95% - 0 лв.

5% - 5000 лв.
95% - 200 лв.

$$E(R_i) = \sum_{i=1}^n p_i a_i - \text{стойност}$$

Игра
отакване
случайна
величина

Вероятност

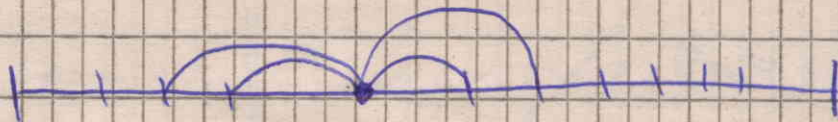
$$E(R_1) = \frac{5}{100} \cdot 1000 + \frac{95}{100} \cdot 0 = 50$$

$$E(R_2) = 5000 \cdot \frac{5}{100} + \frac{95}{100} \cdot 200 =$$

$$= 250 + 190 = 440 \quad 60$$

Риск - Б средно квадратично отклонение
дотерия

$$\sigma^2 = E[(R - E(R))^2]$$



$E(R)$ - изтъкната комбинация

$$\sigma^2 = E[(R_i - E(R_i))^2]$$

$$E(R_i) = 50$$

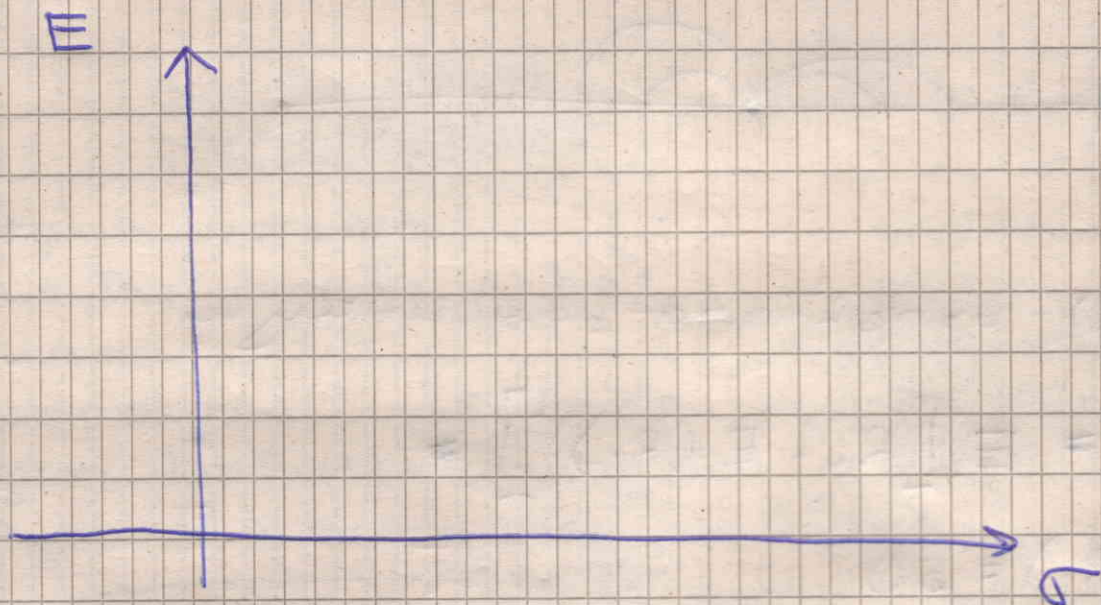
$$\begin{array}{r} 95 \\ 25 \\ \hline 475 \\ 190 \end{array}$$

$$\sigma_1^2 = \frac{5}{100} (1000 - 50)^2 + \frac{95}{100} (0 - 50)^2 = 2375$$

$$= \frac{5}{100} \left(\frac{2}{950} \right)^2 + \frac{95}{100} \cdot 2500 =$$

$$= \frac{475 + 2375}{10} = \frac{2850}{10} = 285$$

Ана $\sigma_1 = 212,48$



Доходност на една акция

$$R = \frac{\text{продажна} \\ \text{покупна цена} + \text{дивидент} - \text{покупна} \\ \text{(в момента)} \quad \text{цена}}{\text{покупна цена}}$$

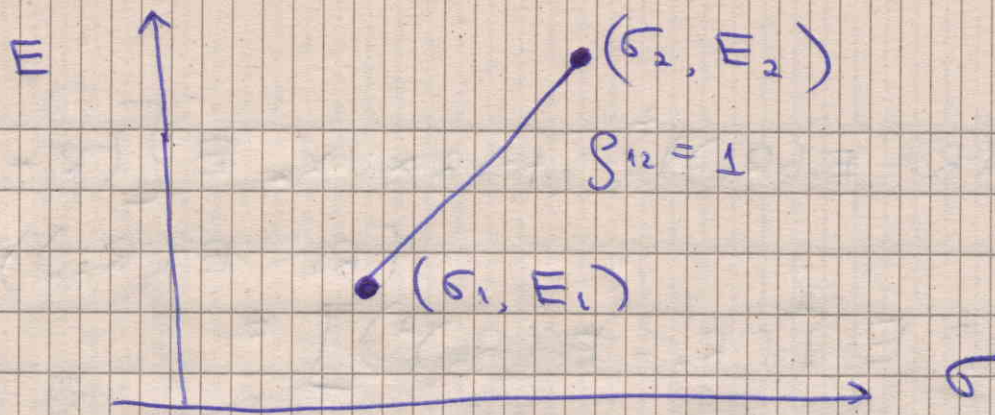


време

$$R \rightarrow \left(\begin{array}{cc} \sigma_R, E(R) \\ \downarrow \quad \downarrow \\ \text{риск} \quad \text{оакваана} \\ \quad \quad \text{доходност} \end{array} \right)$$

средна σ - т $\rightarrow$ средна

$$\frac{R_1 + R_2 + \dots + R_K}{K}$$



само 2 акции

Портфейл от 2 акции - случайна величина

$$R = x_1 R_1 + x_2 R_2$$

$$x_1 + x_2 = 1 \quad x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_1 = \frac{1}{3} \quad \frac{1}{3} \text{ от парите в I акция}$$

$$\frac{2}{3} \rightarrow \text{всё в II акция}$$

$$E(R) = x_1 E(R_1) + x_2 E(R_2)$$

отаква не

$$\sigma_R^2 = E \left[\left(x_1 R_1 + x_2 R_2 - x_1 E(R_1) - x_2 E(R_2) \right)^2 \right]$$

$$= E \left[\left(x_1 (R_1 - E(R_1)) + x_2 (R_2 - E(R_2)) \right)^2 \right]$$

$$= E \left\{ x_1^2 (R_1 - E(R_1))^2 + 2x_1 x_2 (R_1 - E(R_1)) (R_2 - E(R_2)) + x_2^2 (R_2 - E(R_2))^2 \right\}$$

$$\begin{aligned}
 &= x_1^2 E[(R_1 - E(R_1))^2] + 2x_1 x_2 E[(R_1 - E(R_1))(R_2 - E(R_2))] + \\
 &\quad + x_2^2 E[(R_2 - E(R_2))^2] = \\
 &= x_1^2 \sigma_1^2 + 2x_1 x_2 \sigma_1 \sigma_2 \rho_{12} + x_2^2 \sigma_2^2
 \end{aligned}$$

$$\rho_{12} = \frac{E[(R_1 - E(R_1))(R_2 - E(R_2))]}{\sigma_1 \sigma_2}$$

1
коэффициент на корреляция
 $\rho \in [-1, 1]$ (доказва се)

Ако $\rho_{12} = 1$

$$\rightarrow \sigma_R = x\sigma_1 + (1-x)\sigma_2$$

$$(\sigma_R, E(R)) = (x\sigma_1 + (1-x)\sigma_2, xE_1 + (1-x)E_2)$$

$$x \in [0, 1]$$

отсечка :))

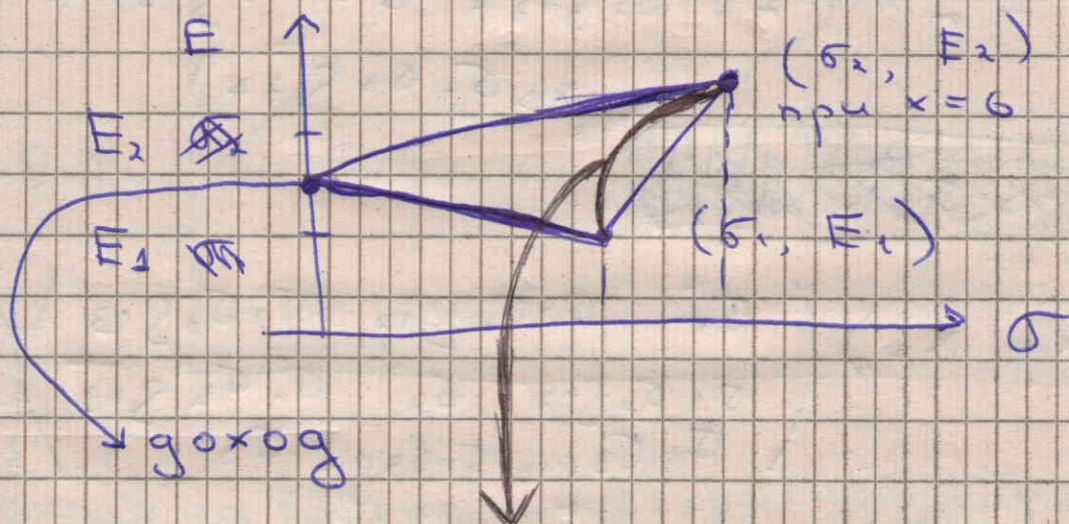
$$\rho_{1,2} = -1$$

$$\sigma_R = |x\sigma_1 - (1-x)\sigma_2|$$

$$x\sigma_1 - (1-x)\sigma_2 = 0$$

$$x\sigma_1 - \sigma_2 + x\sigma_2 = 0$$

$$\bar{x} = \frac{\sigma_2}{\sigma_1 + \sigma_2} \quad ! \text{ винаги може 0-ев риск}$$



$$-1 < \rho_{12} < 1 \text{ строго}$$

с к величини

$$R = \sum_{i=1}^k x_i R_i$$

$$x_i \geq 0$$

$$\sum_{i=1}^k x_i = 1$$

$$E(R) = \sum_{i=1}^k x_i E(R_i)$$

$x_1(R_1 - E(R_1)) + x_2(R_2 - E(R_2)) + x_3(R_3 - E(R_3))$
 $x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + x_3^2 \sigma_3^2 + 2x_1x_2\rho_{12}\sigma_1\sigma_2 + 2x_1x_3\rho_{13}\sigma_1\sigma_3 + 2x_2x_3\rho_{23}\sigma_2\sigma_3$

с 3 величини:

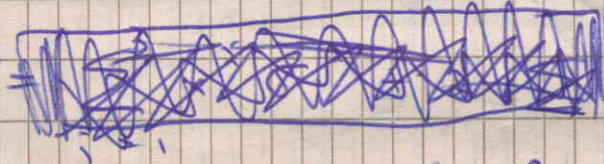
$$\sigma_R^2 = E \left[(x_1 R_1 + x_2 R_2 + x_3 R_3 - x_1 E(R_1) - x_2 E(R_2) - x_3 E(R_3)) \right]$$

$$\sigma_R^2 = \sum_{i=1}^k x_i^2 \sigma_i^2 + 2 \sum_{i < j} x_i x_j \rho_{ij} \sigma_i \sigma_j$$

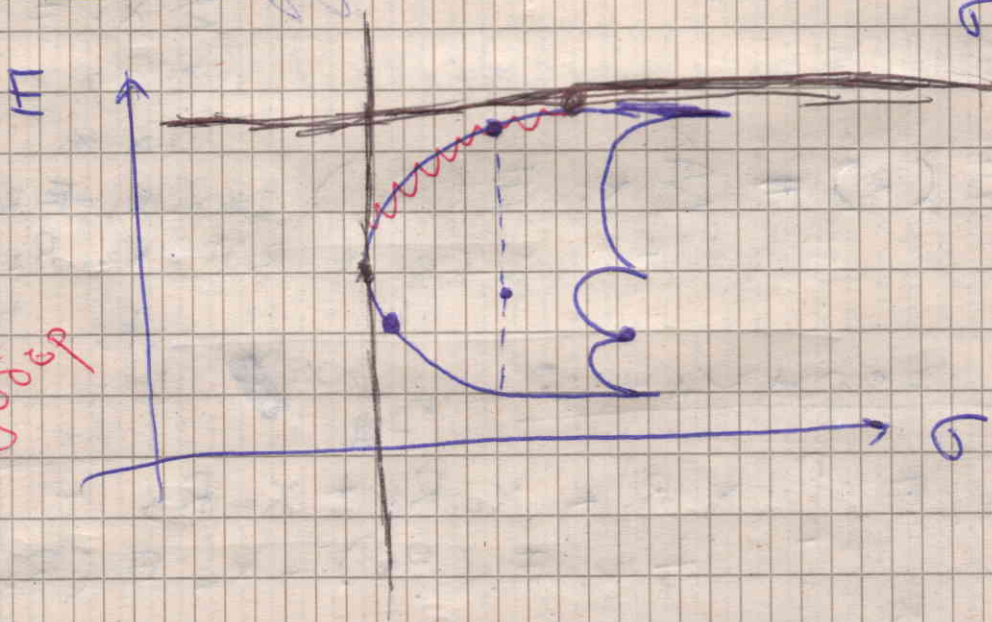
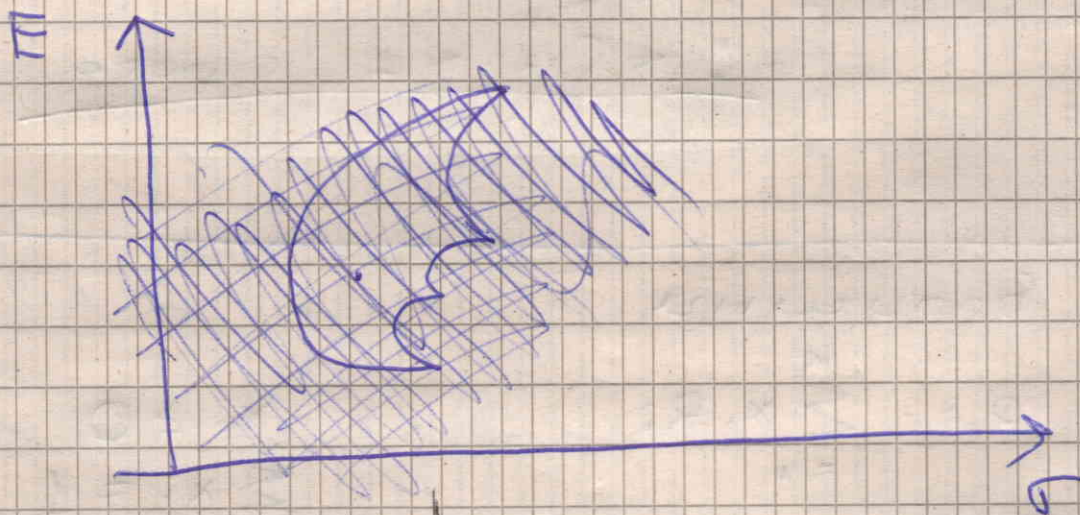
$$\sigma_R^2 = (x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + x_3^2 \sigma_3^2) +$$

$$+ 2x_1(x_2 \sigma_1 \sigma_2 \rho_{12} + x_3 \sigma_1 \sigma_3 \rho_{13}) +$$

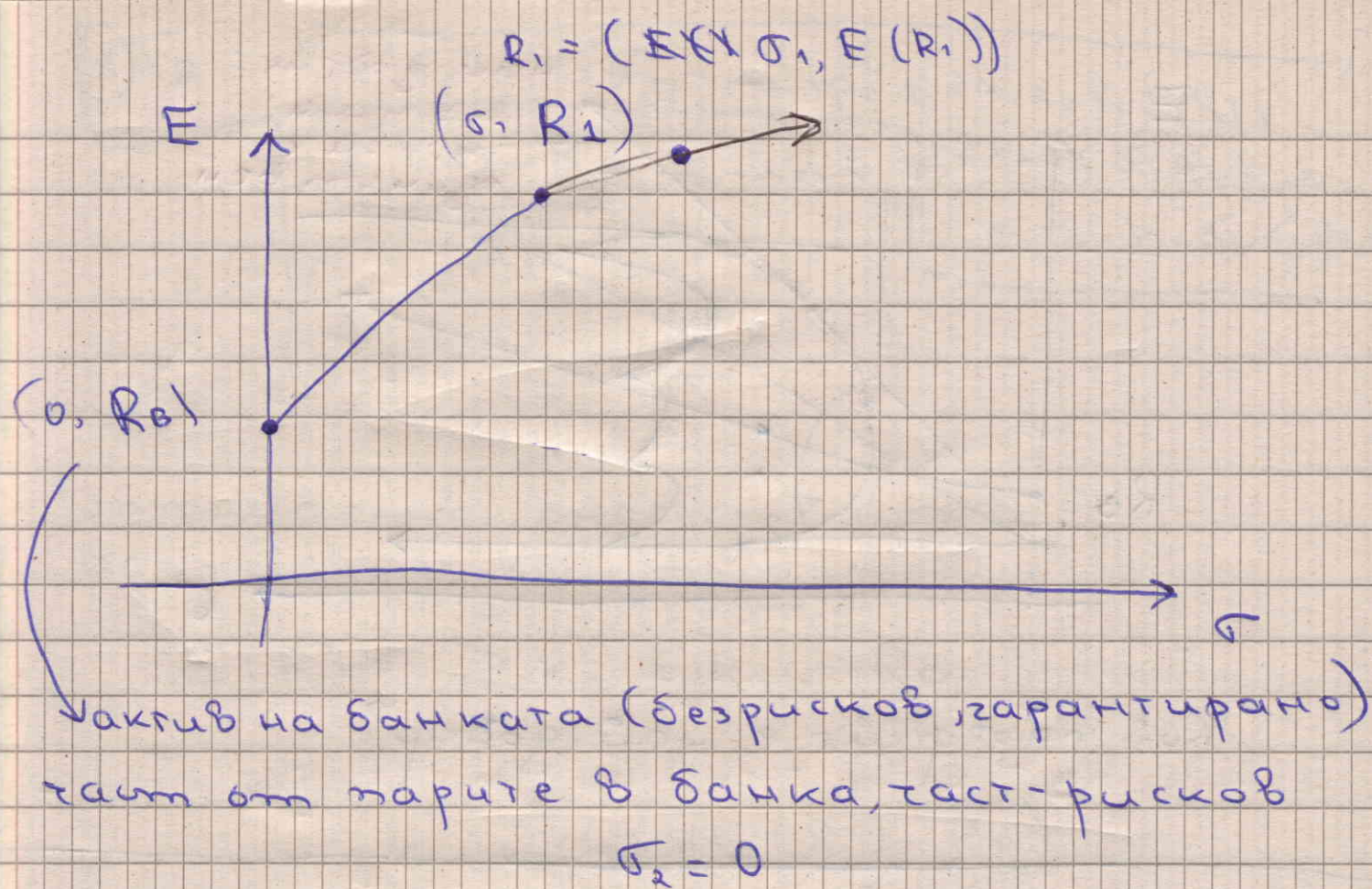
$$+ x_2 x_3 \sigma_2 \sigma_3 \rho_{23})$$



$$= (x_1, x_2, x_3) \begin{pmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho_{12} & \sigma_1 \sigma_3 \rho_{13} \\ \sigma_1 \sigma_2 \rho_{12} & \sigma_2^2 & \sigma_2 \sigma_3 \rho_{23} \\ \sigma_1 \sigma_3 \rho_{13} & \sigma_2 \sigma_3 \rho_{23} & \sigma_3^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



красив
по фрейму
м/у макс
и мин
на годер



$$\sigma_R = x \sigma_1$$

$$x = 0 + \sigma_1 \cdot \lambda$$

$$y = R_B + (R_1 - R_B) \cdot \lambda$$

$$E(R) = R_B + \frac{E(R_1) - R_B}{\sigma_1} \sigma_R$$

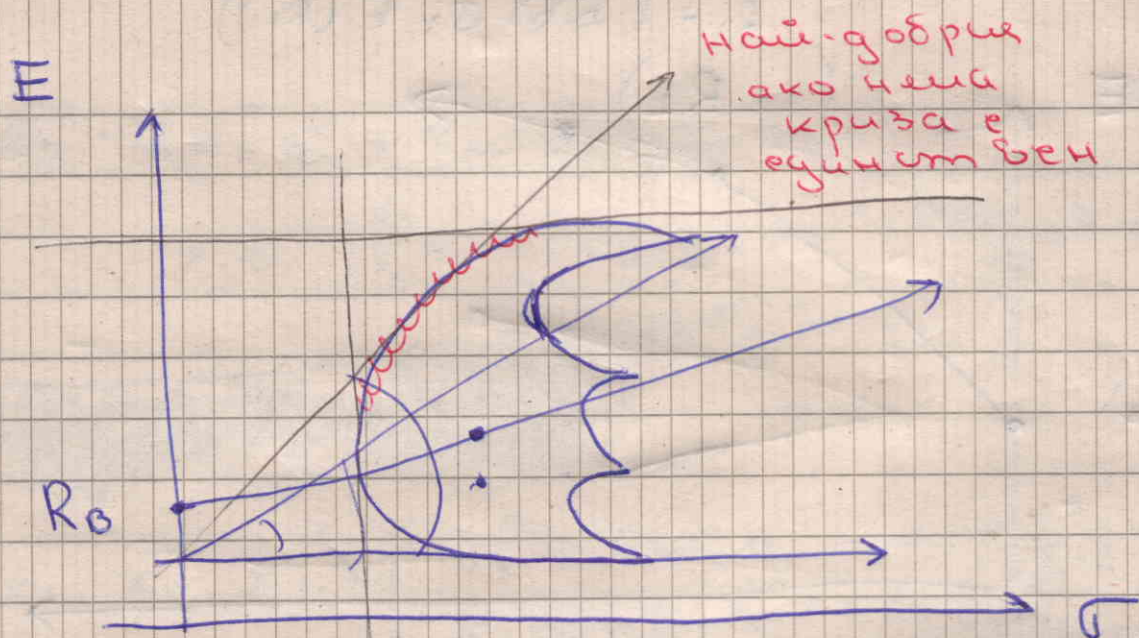
Разно. нбта

~~Е(Р) = x E(R₁) + (1-x) R_B~~

$$E(R) = x E(R_1) + (1-x) R_B$$

Нека $x > 1$

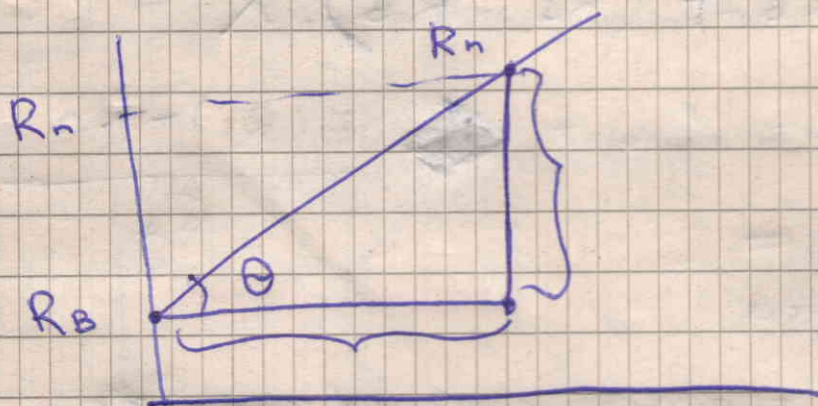
$$= R_B + x (E(R_1) - R_B)$$



Кой портфейл е най-добър?

Нека активът на банката е 3

Търсим портфейл с най-голям възв.



$$\operatorname{tg} \theta = \frac{E(R_m) - R_0}{\sigma_R} \rightarrow \max$$

$$x_i \geq 0 \quad \sum_{i=1}^m x_i = 1$$

$$\sigma_R = \sqrt{x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + x_3^2 \sigma_3^2 + 2x_1 x_2 \sigma_1 \sigma_2 \rho_{12} + 2x_1 x_3 \sigma_1 \sigma_3 \rho_{13} + 2x_2 x_3 \sigma_2 \sigma_3 \rho_{23}}$$

$$\frac{E(R) - R_D}{\sigma_R} \rightarrow \max$$

$$x_i \geq 0 \quad \sum_{i=1}^n x_i = 1$$

? оптималният портфейл при
разрешение към продажби

$$E(R) = \sum_{i=1}^n x_i E(R_i) - R_D =$$

$$= \sum_{i=1}^n x_i (E(R_i) - R_D)$$

за n=3

Търсене на условна ф.з.
Диф по $\forall x_i$:

$$\Rightarrow E(R) = x_1 (E(R_1) - R_D) + x_2 (E(R_2) - R_D) + x_3 (E(R_3) - R_D)$$

Диф:

$$\frac{E(R) - R_D}{\sigma_R} + \frac{1}{2} \frac{(E(R) - R_D)}{\sigma_R^3} \cdot 2 \left[x_1 \sigma_1^2 + x_2 \sigma_1 \sigma_2 \rho_{12} + x_3 \sigma_1 \sigma_3 \rho_{13} \right]$$

$$E(R_i) - R_0 = \frac{E(R) - R_0}{\sigma_R^2} (x_1 \sigma_1^2 + x_2 \sigma_1 \sigma_2 \rho_{12} + x_3 \sigma_1 \sigma_3 \rho_{13})$$

$$\lambda x_1 = z_1$$

$$\lambda x_2 = z_2$$

$$\lambda x_3 = z_3$$

→ I ред матрица

$$E(R_i) - R_0 = z_1 \sigma_1^2 + z_2 \sigma_1 \sigma_2 \rho_{12} + z_3 \sigma_1 \sigma_3 \rho_{13}$$

Едно и също диф. по x_1, x_2, x_3

Диф. по x_2

$$E(R_2) - R_0 = z_1 \sigma_1 \sigma_2 \rho_{12} + z_2 \sigma_2^2 + z_3 \sigma_2 \sigma_3 \rho_{23}$$

$$E(R_3) - R_0 = z_1 \sigma_1 \sigma_3 \rho_{13} + z_2 \sigma_2 \sigma_3 \rho_{23} + z_3 \sigma_3^2$$

има z_1, z_2, z_3

$$z_1 + z_2 + z_3 = \lambda x_1 + \lambda x_2 + \lambda x_3 = \lambda (x_1 + x_2 + x_3)$$

$$x_1 = \frac{z_1}{\lambda}$$

329.
10

$$E(R_1) = 14\%$$

$$\sigma_1 = 6\%$$

$$E(R_2) = 8\%$$

$$\sigma_2 = 3\%$$

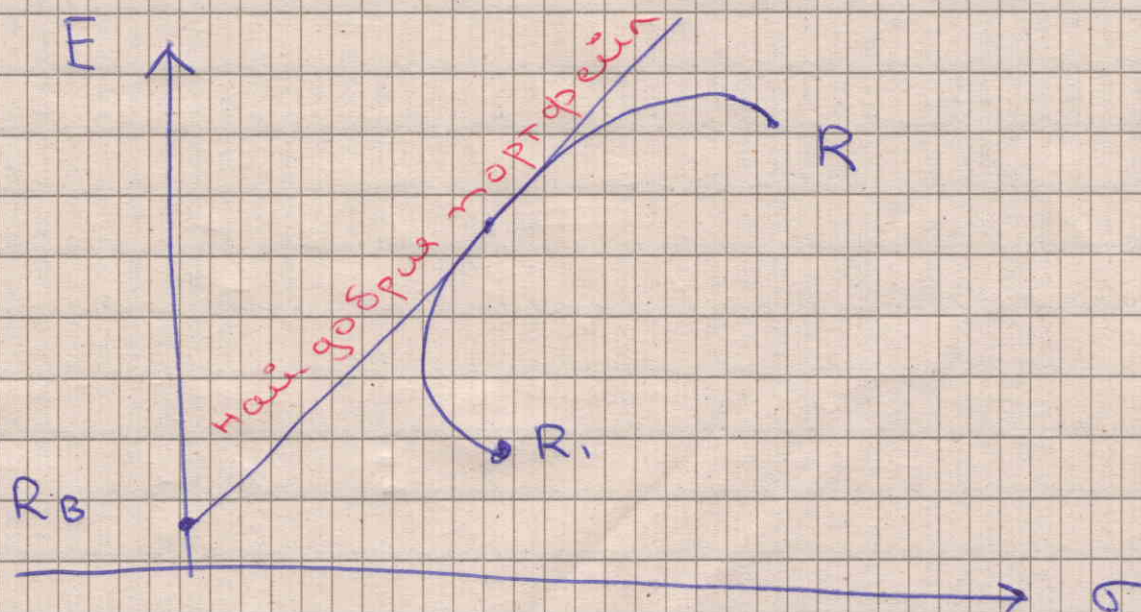
$$E(R_3) = 20\%$$

$$\sigma_3 = 15\%$$

Ако $x_i < 0$ - продажени и купени
групи

$$x_1 R_1 + x_2 R_2$$

2 актива
индекс



? при какви условия едното x е 0
(не инвестиране вcurities)
за 2 актива

$$E(R_1) - R_0 = z_1 \sigma_1^2 + z_2 \sigma_1 \sigma_2 \rho_{12}$$

$$E(R_2) - R_0 = z_1 \sigma_1 \sigma_2 \rho_{12} + z_2 \sigma_2^2$$

$$z_1 \sigma_1^2 + z_2 \sigma_1 \sigma_2 \rho_{12}$$

$$z_1 \sigma_1 \sigma_2 \rho_{12} + z_2 \sigma_2^2$$

$$\rho_{12} = \frac{E(R_2) - R_B}{\sigma_1 \sigma_2}$$

$$E(R_2) - R_B = \sigma_1^2$$

Модел на Маклувіз?