# Geometric Algorithms

- range search
- quad and kd trees
- intersection search
- VLSI rules check

References:

Algorithms in C (2nd edition), Chapters 26-27

http://www.cs.princeton.edu/introalgsds/73range

http://www.cs.princeton.edu/introalgsds/74intersection

## Overview

Types of data. Points, lines, planes, polygons, circles, ...
This lecture. Sets of N objects.

Geometric problems extend to higher dimensions.

- Good algorithms also extend to higher dimensions.
- Curse of dimensionality.

Basic problems.

- Range searching.
- Nearest neighbor.
- Finding intersections of geometric objects.

- ▸ range search
- ▸ quad and kd trees
- ▸ intersection search
- ▸ VLSI rules check

# 1D Range Search

Extension to symbol-table ADT with comparable keys.

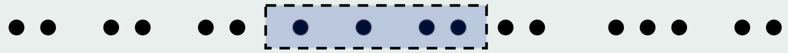
- Insert key-value pair.
- Search for key k.
- How many records have keys between $k_1$ and $k_2$?
- Iterate over all records with keys between $k_1$ and $k_2$.

Application: database queries.

Geometric intuition.

- Keys are point on a line.
- How many points in a given interval?

| | |
|---|---|
| insert B | B |
| insert D | B D |
| insert A | A B D |
| insert I | A B D I |
| insert H | A B D H I |
| insert F | A B D F H I |
| insert P | A B D F H I P |
| count G to K | 2 |
| search G to K | H I |



# 1D Range search: implementations

Range search. How many records have keys between $k_1$ and $k_2$?

Ordered array. Slow insert, binary search for $k_1$ and $k_2$ to find range.

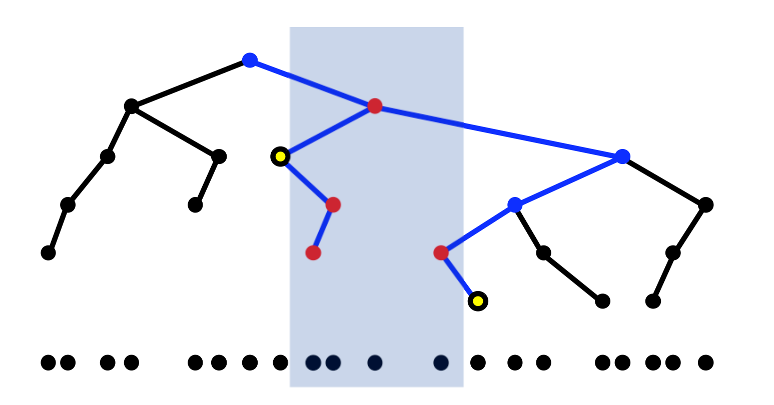
Hash table. No reasonable algorithm (key order lost in hash).

BST. In each node x, maintain number of nodes in tree rooted at x.

Search for smallest element ≥ $k_1$ and largest element ≤ $k_2$.

| | insert | count | range |
|---|---|---|---|
| ordered array | N | log N | R + log N |
| hash table | 1 | N | N |
| BST | log N | log N | R + log N |

N = # records
R = # records that match



● nodes examined
● within interval
● not touched

# 2D Orthogonal Range Search

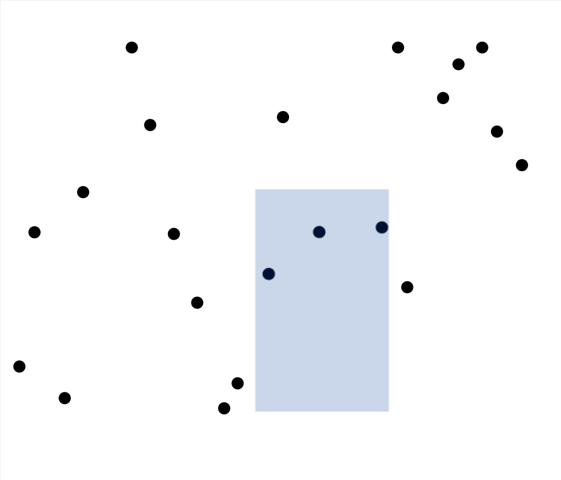
Extension to symbol-table ADT with 2D keys.

- Insert a 2D key.
- Search for a 2D key.
- Range search: find all keys that lie in a 2D range?
- Range count: how many keys lie in a 2D range?

Applications: networking, circuit design, databases.

Geometric interpretation.

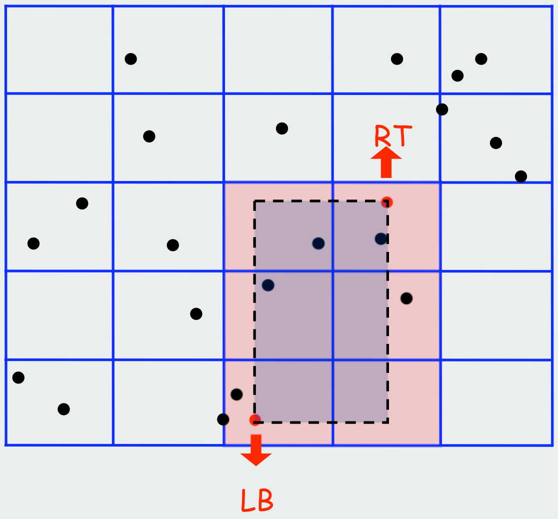
- Keys are point in the plane
- Find all points in a given h-v rectangle



# 2D Orthogonal range Search: Grid implementation

Grid implementation. [Sedgewick 3.18]

- Divide space into M-by-M grid of squares.
- Create linked list for each square.
- Use 2D array to directly access relevant square.
- Insert: insert (x, y) into corresponding grid square.
- Range search: examine only those grid squares that could have points in the rectangle.



# 2D Orthogonal Range Search: Grid Implementation Costs

Space-time tradeoff.

- Space: $M^2 + N$.
- Time: $1 + N / M^2$ per grid cell examined on average.

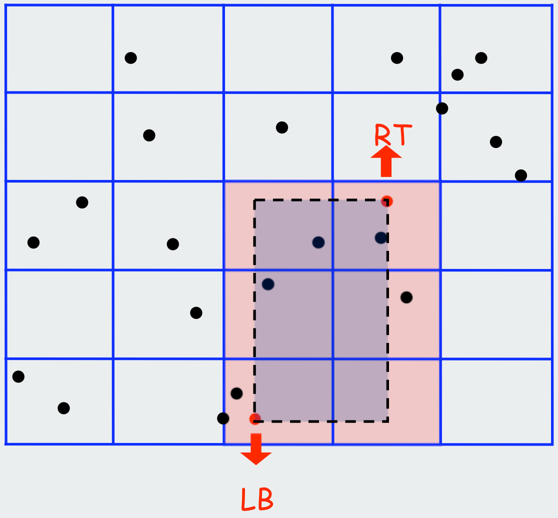
Choose grid square size to tune performance.

- Too small: wastes space.
- Too large: too many points per grid square.
- Rule of thumb: $\sqrt{N}$ by $\sqrt{N}$ grid.

Running time. [if points are evenly distributed]

- Initialize: O(N).
- Insert: O(1).
- Range: O(1) per point in range.

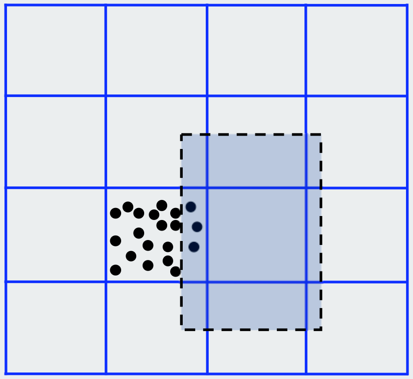
$M \approx \sqrt{N}$



# Clustering

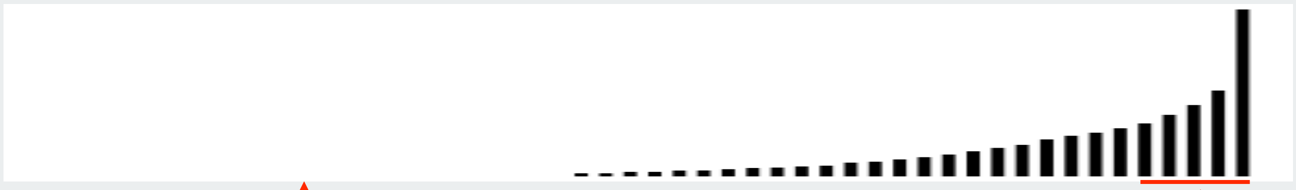
Grid implementation. Fast, simple solution for well-distributed points.

Problem. Clustering is a well-known phenomenon in geometric data.



Ex: USA map data.

13,000 points, 1000 grid squares.



half the squares are empty

half the points are in 10% of the squares

Lists are too long, even though average length is short.

Need data structure that gracefully adapts to data.

- range search
- **quad and kd trees**
- intersection search
- VLSI rules check

# Space Partitioning Trees

Use a tree to represent a recursive subdivision of d-dimensional space.

BSP tree. Recursively divide space into two regions.

Quadtree. Recursively divide plane into four quadrants.

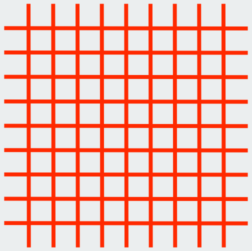
Octree. Recursively divide 3D space into eight octants.

kD tree. Recursively divide k-dimensional space into two half-spaces.

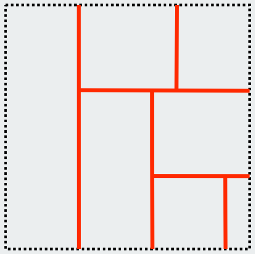
[possible but much more complicated to define Voronoi-based structures]

Applications.

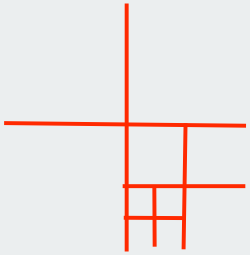
- Ray tracing.
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.



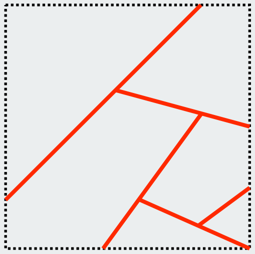
Grid



kD tree



Quadtree



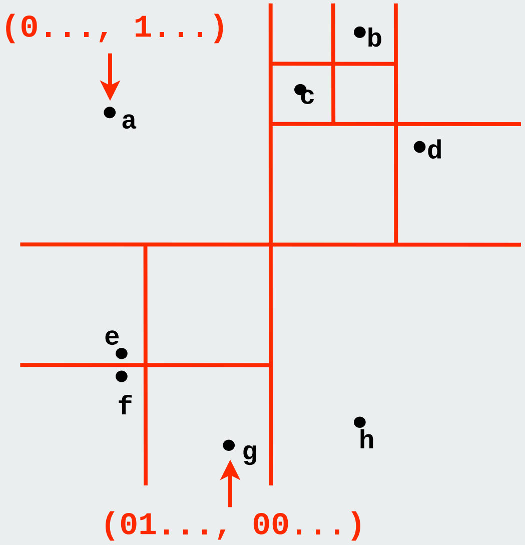
BSP tree

# Quadtree

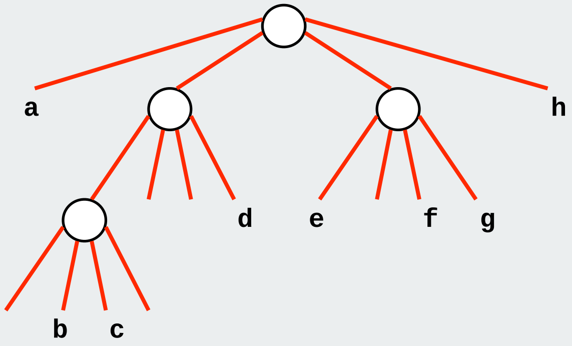
Recursively partition plane into 4 quadrants.

Implementation: 4-way tree.

actually a trie
partitioning on bits of coordinates



```
public class QuadTree
{
   private Quad quad;
   private Value value;
   private QuadTree NW, NE, SW, SE;
}
```

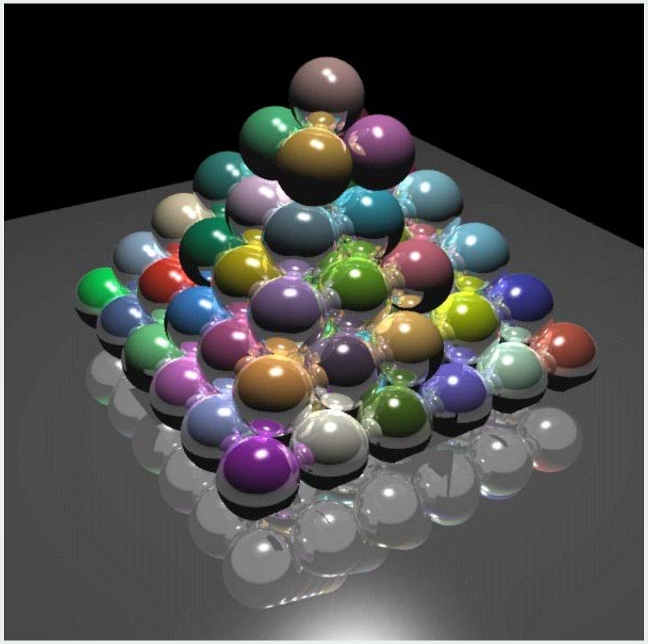


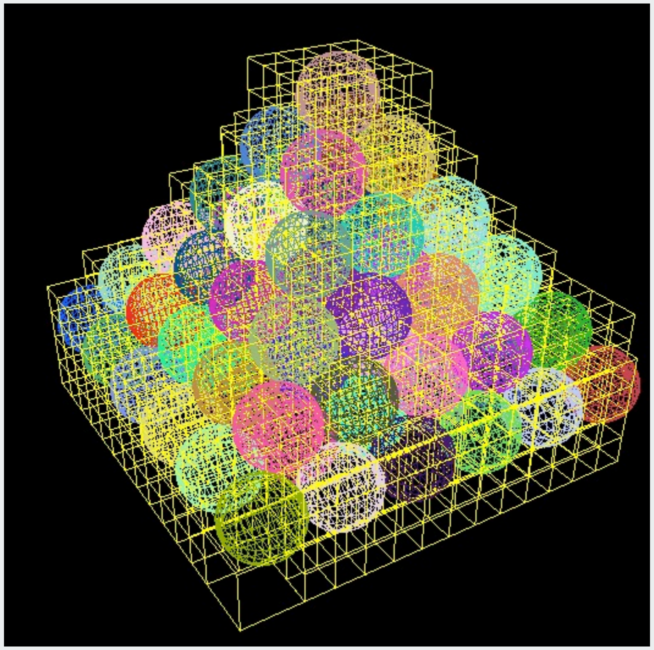
Primary reason to choose quad trees over grid methods:
good performance in the presence of clustering

# Curse of Dimensionality

Range search / nearest neighbor in k dimensions?
Main application. Multi-dimensional databases.

3D space. Octrees: recursively divide 3D space into 8 octants.
100D space. Centrees: recursively divide into $2^{100}$ centrants???





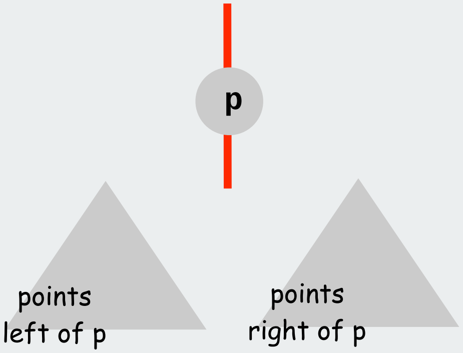
Raytracing with octrees
http://graphics.cs.ucdavis.edu/~gregorsk/graphics/275.html

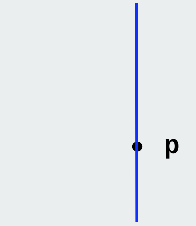
# 2D Trees

Recursively partition plane into 2 halfplanes.

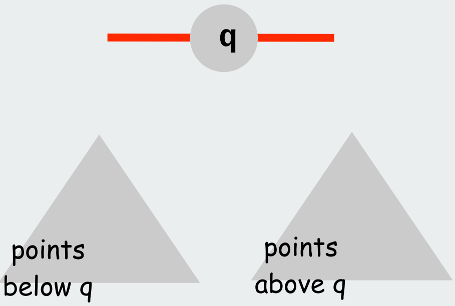
Implementation: BST, but alternate using x and y coordinates as key.

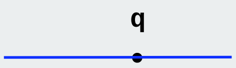
- Search gives rectangle containing point.
- Insert further subdivides the plane.



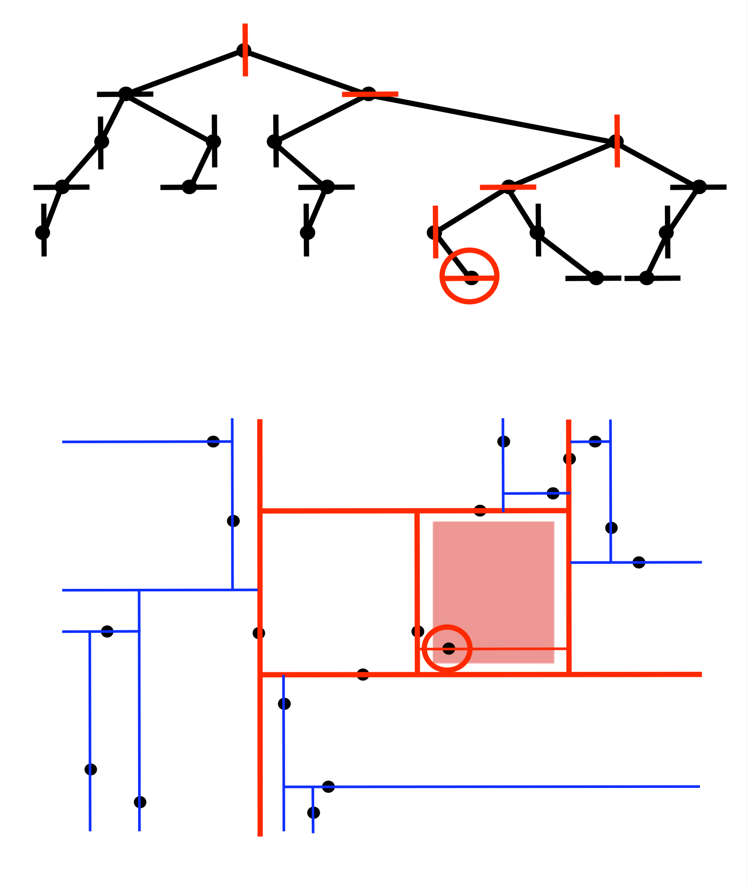


even levels





odd levels



# Near Neighbor Search

Useful extension to symbol-table ADT for records with metric keys.

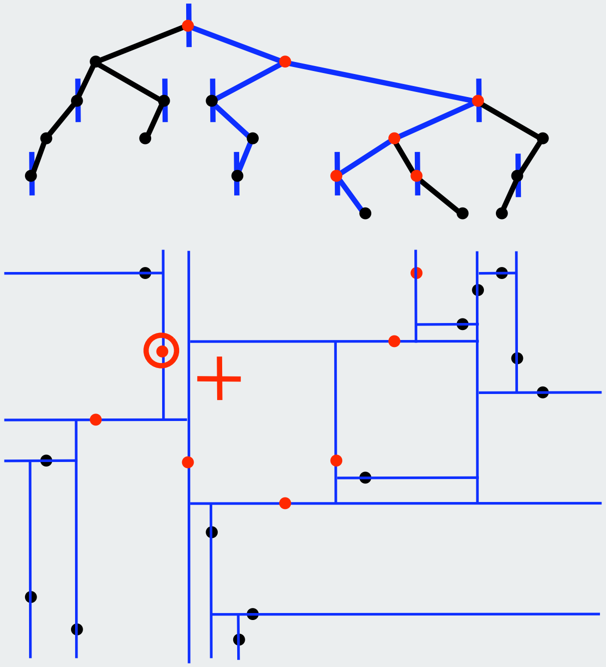
- Insert a k dimensional point.
- Near neighbor search: given a point p, which point in data structure is nearest to p?

Need concept of distance, not just ordering.

kD trees provide fast, elegant solution.

- Recursively search subtrees that could have near neighbor (may search both).
- O(log N) ?

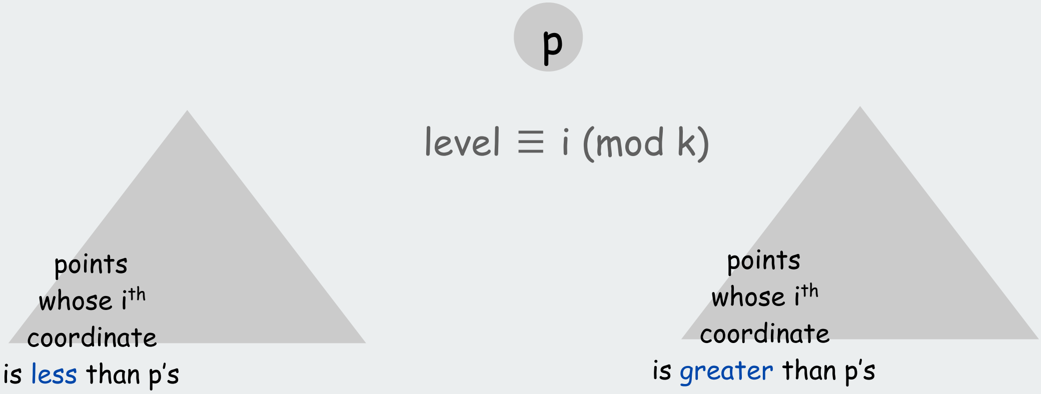
Yes, in practice (but not proven)



# kD Trees

kD tree. Recursively partition k-dimensional space into 2 halfspaces.

Implementation: BST, but cycle through dimensions ala 2D trees.

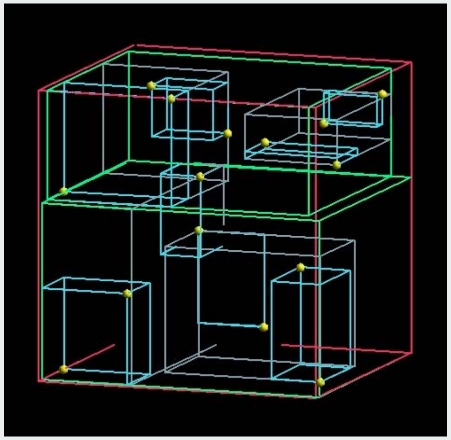


p

level ≡ i (mod k)

points whose $i^{th}$ coordinate is less than p's

points whose $i^{th}$ coordinate is greater than p's



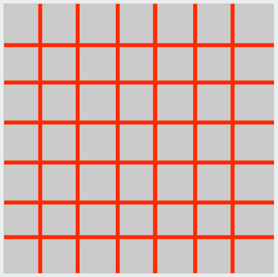
Efficient, simple data structure for processing k-dimensional data.

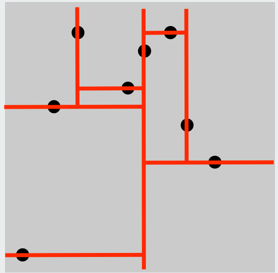
- adapts well to clustered data.
- adapts well to high dimensional data.
- widely used.
- discovered by an undergrad in an algorithms class!

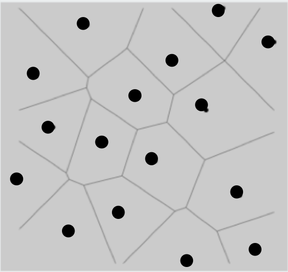
# Summary

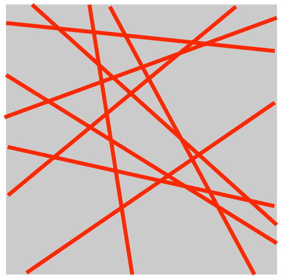
Basis of many geometric algorithms: search in a planar subdivision.

| | grid | 2D tree | Voronoi diagram | intersecting lines |
|---|---|---|---|---|
| basis | √N h-v lines | N points | N points | √N lines |
| representation | 2D array of N lists | N-node BST | N-node multilist | ~N-node BST |
| cells | ~N squares | N rectangles | N polygons | ~N triangles |
| search cost | 1 | log N | log N | log N |
| extend to kD? | too many cells | easy | cells too complicated | use (k-1)D hyperplane |







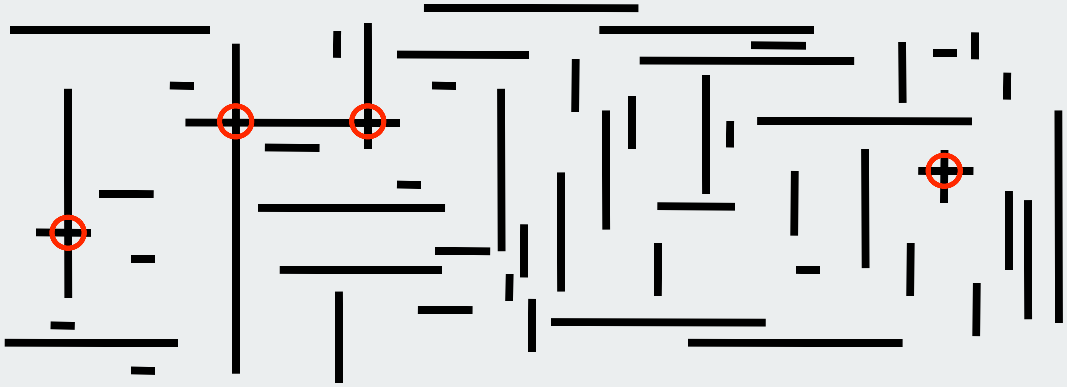


- range search
- quad and kd trees
- **intersection search**
- VLSI rules check

# Search for intersections

Problem. Find all intersecting pairs among set of N geometric objects.
Applications. CAD, games, movies, virtual reality.

Simple version: 2D, all objects are horizontal or vertical line segments.

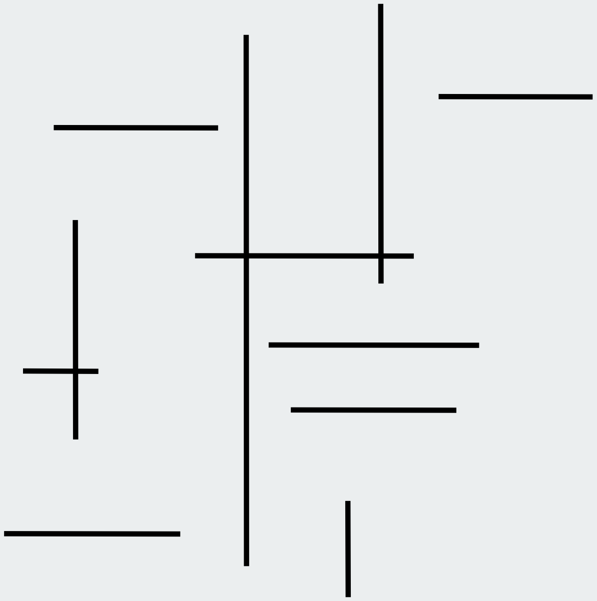


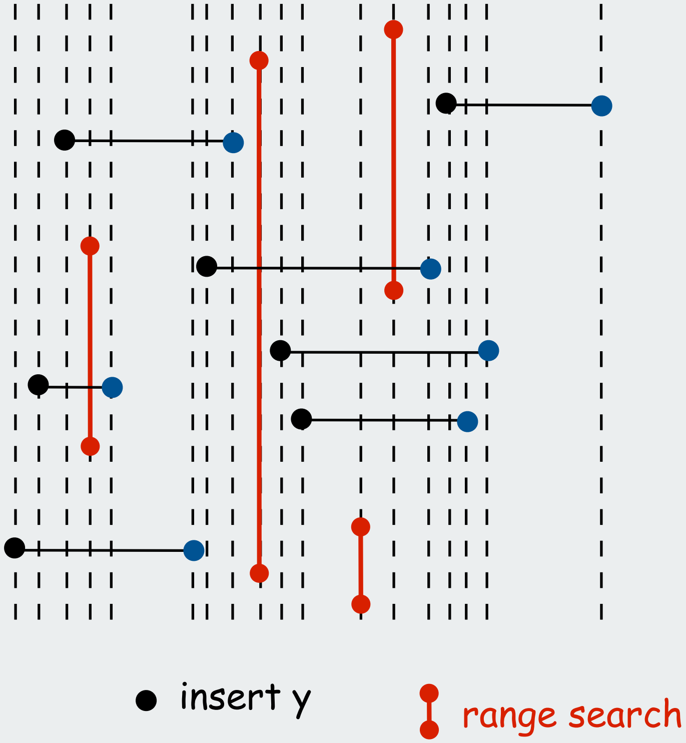
Brute force. Test all $\Theta(N^2)$ pairs of line segments for intersection.
Sweep line. Efficient solution extends to 3D and general objects.

# Orthogonal segment intersection search:  Sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- left endpoint of h-segment:  insert y coordinate into ST.
- right endpoint of h-segment:  remove y coordinate from ST.
- v-segment:  range search for interval of y endpoints.





# Orthogonal segment intersection: Sweep-line algorithm

Reduces 2D orthogonal segment intersection search to 1D range search!

Running time of sweep line algorithm.

| | |
|---|---|
| • Put x-coordinates on a PQ (or sort). | O(N log N) |
| • Insert y-coordinate into SET. | O(N log N) |
| • Delete y-coordinate from SET. | O(N log N) |
| • Range search. | O(R + N log N) |

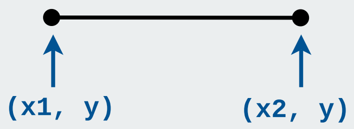
N = # line segments
R = # intersections

Efficiency relies on judicious use of data structures.

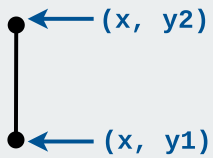
# Immutable H-V segment ADT

```
public final class SegmentHV implements Comparable<SegmentHV>
{
   public final int x1, y1;
   public final int x2, y2;

   public SegmentHV(int x1, int y1, int x2, int y2)
   {  ...  }
   public boolean isHorizontal()
   {  ...  }
   public boolean isVertical()
   {  ...  }
   public int compareTo(SegmentHV b)
   {  ...  }
   public String toString()
   {  ...  }
}
```

compare by x-coordinate; break ties by y-coordinate



horizontal segment



vertical segment

# Sweep-line event

```
public class Event implements Comparable<Event>
{
   private int time;
   private SegmentHV segment;

   public Event(int time, SegmentHV segment)
   {
      this.time    = time;
      this.segment = segment;
   }

   public int compareTo(Event b)
   {
      return a.time - b.time;
   }
}
```

# Sweep-line algorithm:  Initialize events

```
MinPQ<Event> pq = new MinPQ<Event>();

for (int i = 0; i < N; i++)
{
   if (segments[i].isVertical())
   {
      Event e = new Event(segments[i].x1, segments[i]);
      pq.insert(e);
   }
   else if (segments[i].isHorizontal())
   {
      Event e1 = new Event(segments[i].x1, segments[i]);
      Event e2 = new Event(segments[i].x2, segments[i]);
      pq.insert(e1);
      pq.insert(e2);
   }
}
```

initialize PQ

vertical segment

horizontal segment

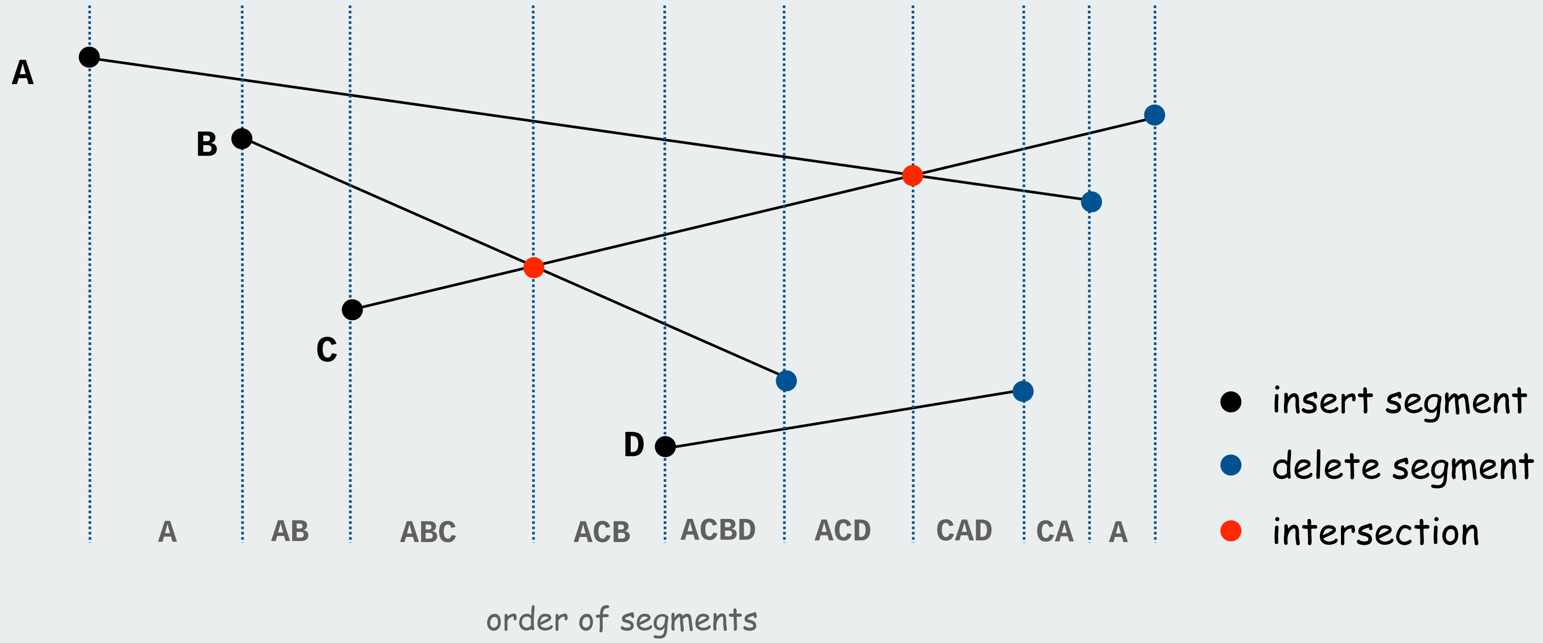
# Sweep-line algorithm:  Simulate the sweep line

```
int INF = Integer.MAX_VALUE;

SET<SegmentHV> set = new SET<SegmentHV>();

while (!pq.isEmpty())
{
   Event e = pq.delMin();
   int sweep = e.time;
   SegmentHV segment = e.segment;

   if (segment.isVertical())
   {
      SegmentHV seg1, seg2;
      seg1 = new SegmentHV(-INF, segment.y1, -INF, segment.y1);
      seg2 = new SegmentHV(+INF, segment.y2, +INF, segment.y2);
      for (SegmentHV seg : set.range(seg1, seg2))
          System.out.println(segment + " intersects " + seg);
   }


   else if (sweep == segment.x1) set.add(segment);
   else if (sweep == segment.x2) set.remove(segment);
}
```

# General line segment intersection search

## Extend sweep-line algorithm

- Maintain order of segments that intersect sweep line by y-coordinate.
- Intersections can only occur between adjacent segments.
- Add/delete line segment ⇒ one new pair of adjacent segments.
- Intersection ⇒ swap adjacent segments.



# Line Segment Intersection: Implementation

Efficient implementation of sweep line algorithm.

- Maintain PQ of important x-coordinates: endpoints and intersections.
- Maintain SET of segments intersecting sweep line, sorted by y.
- O(R log N + N log N).

↑ to support "next largest" and "next smallest" queries

Implementation issues.

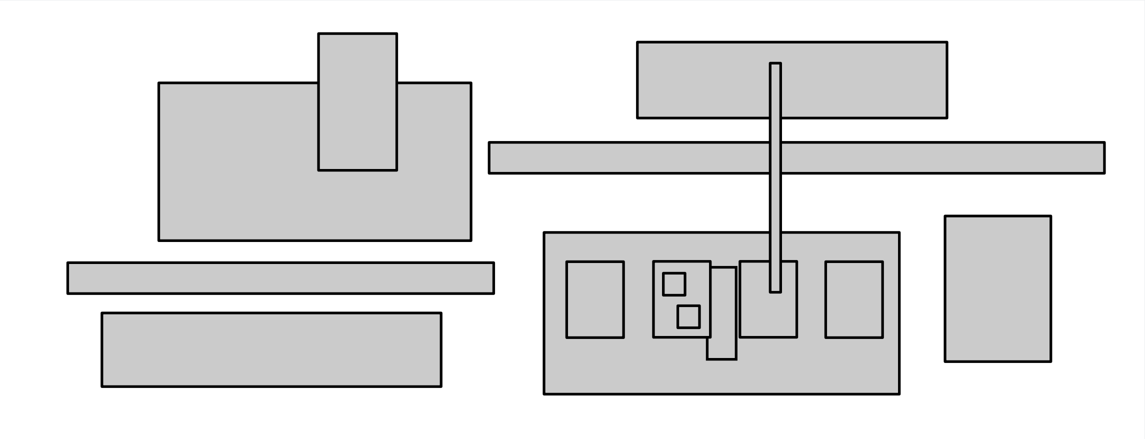
- Degeneracy.
- Floating point precision.
- Use PQ, not presort (intersection events are unknown ahead of time).

- range search
- quad and kd trees
- intersection search
- **VLSI rules check**

# Algorithms and Moore's Law

Rectangle intersection search. Find all intersections among h-v rectangles.

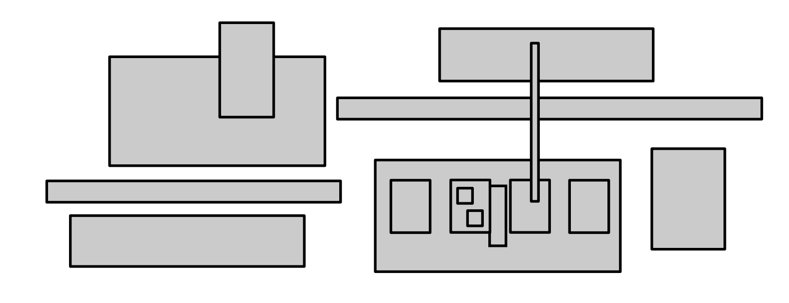
Application. Design-rule checking in VLSI circuits.



# Algorithms and Moore's Law

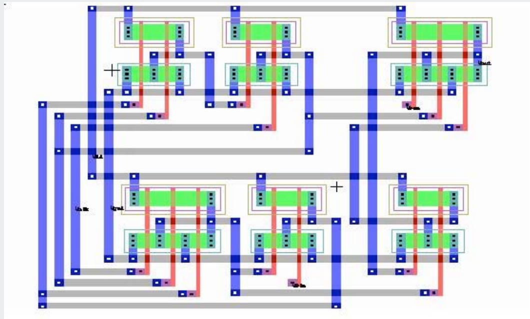
Early 1970s: microprocessor design became a geometric problem.

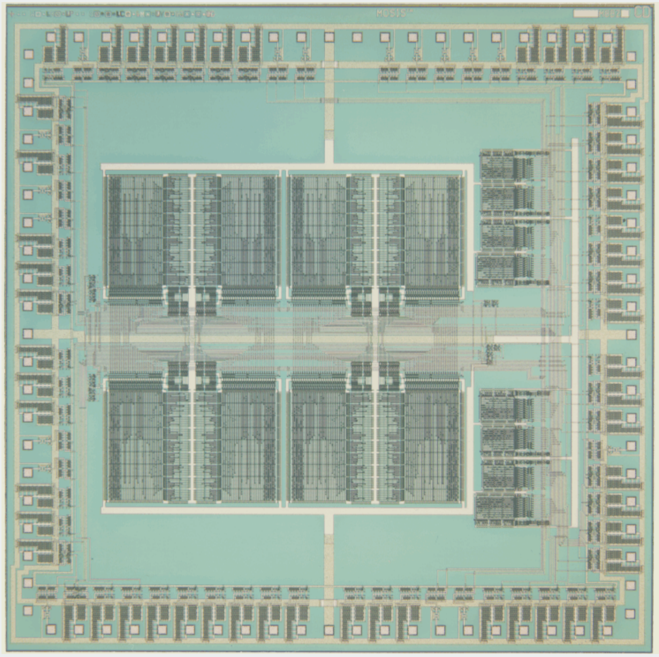
- Very Large Scale Integration (VLSI).
- Computer-Aided Design (CAD).



Design-rule checking:

- certain wires cannot intersect
- certain spacing needed between different types of wires
- debugging = rectangle intersection search





# Algorithms and Moore's Law

"Moore's Law." Processing power doubles every 18 months.

- 197x: need to check N rectangles.
- 197(x+1.5): need to check 2N rectangles on a 2x-faster computer.

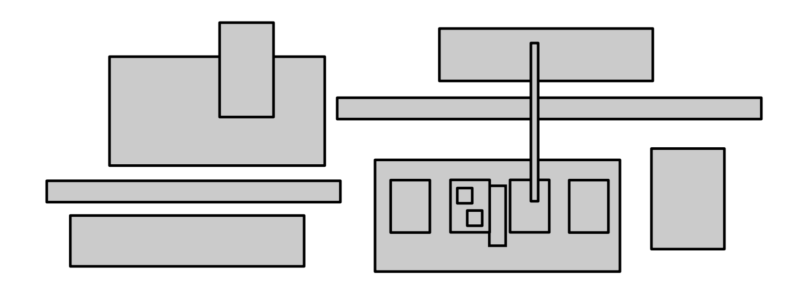
Bootstrapping: we get to use the faster computer for bigger circuits

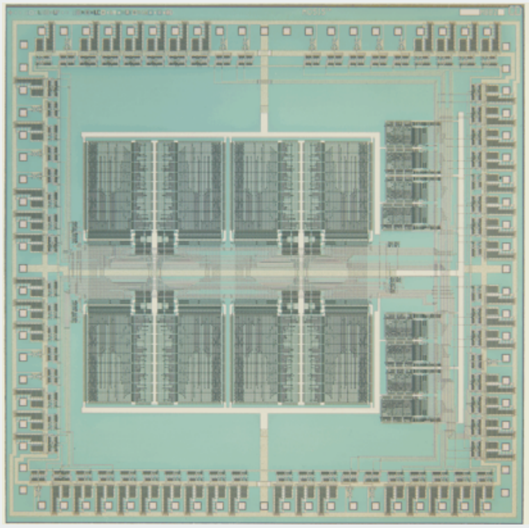
But bootstrapping is not enough if using a quadratic algorithm

- 197x: takes M days.
- 197(x+1.5): takes (4M)/2 = 2M days. (!)

quadratic algorithm

2x-faster computer



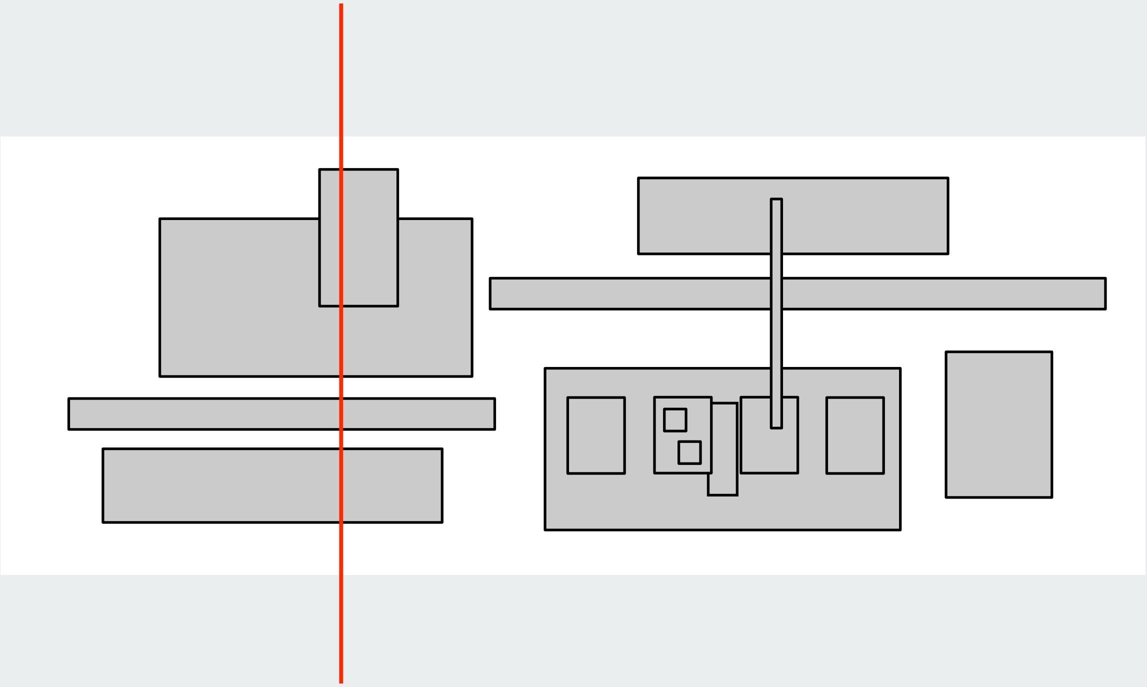


O(N log N) CAD algorithms are necessary to sustain Moore's Law.

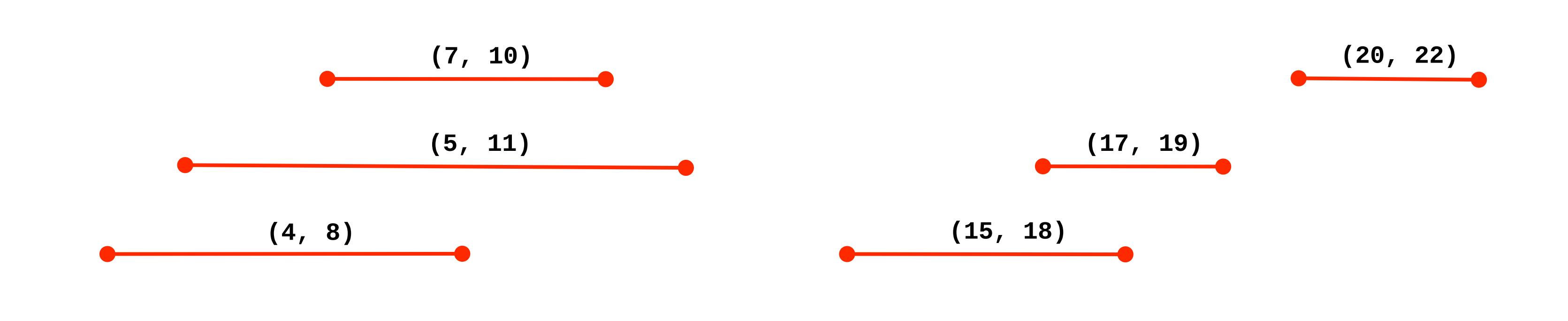
# Rectangle intersection search

Move a vertical "sweep line" from left to right.

- Sweep line: sort rectangles by x-coordinate and process in this order, stopping on left and right endpoints.
- Maintain set of intervals intersecting sweep line.
- Key operation: given a new interval, does it intersect one in the set?



# Interval Search Trees

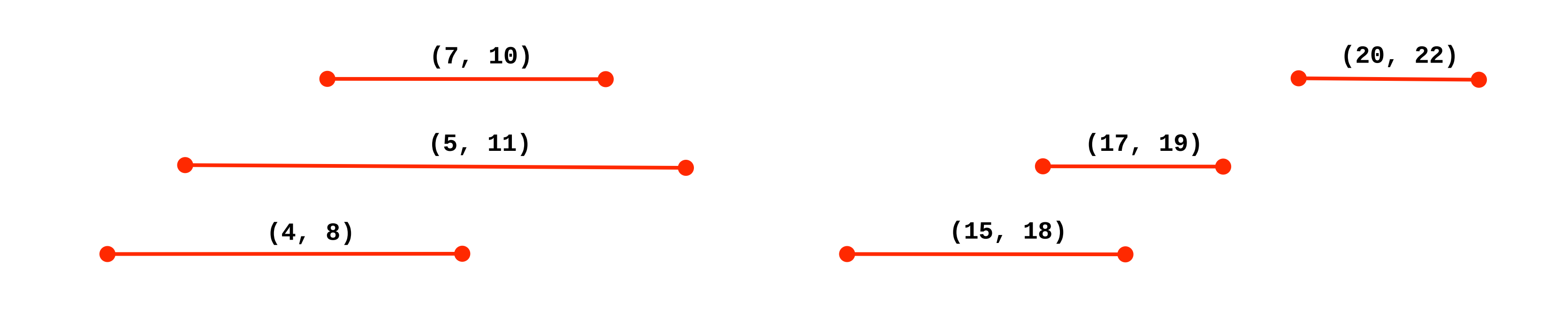


Support following operations.

- Insert an interval `(lo, hi)`.
- Delete the interval `(lo, hi)`.
- Search for an interval that intersects `(lo, hi)`.

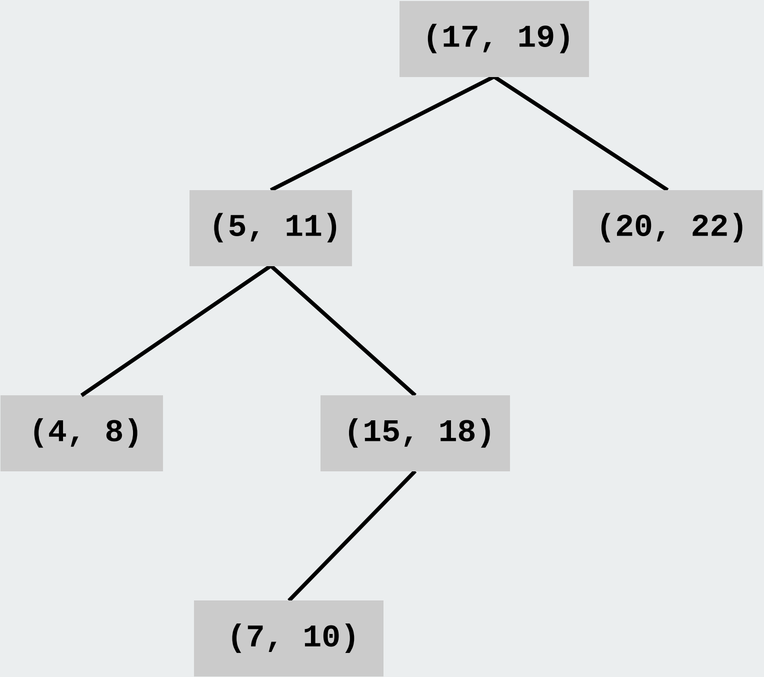
Non-degeneracy assumption. No intervals have the same x-coordinate.

# Interval Search Trees

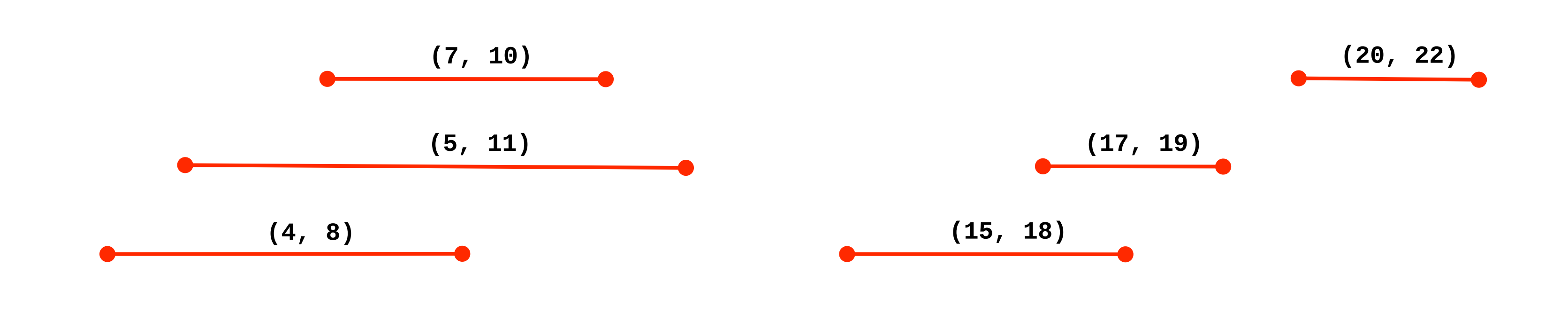


Interval tree implementation with BST.

- Each BST node stores one interval.
- use `lo` endpoint as BST key.

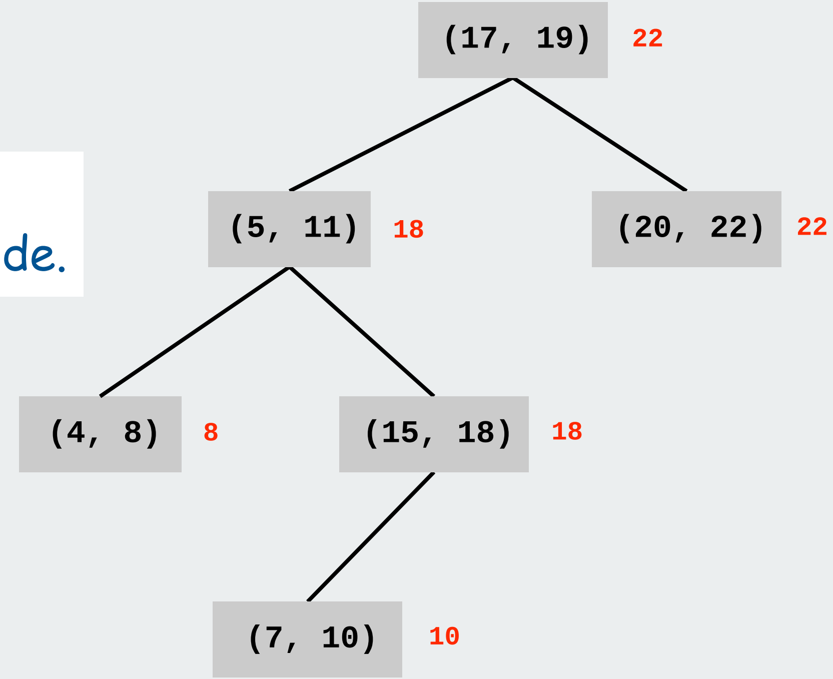


# Interval Search Trees



Interval tree implementation with BST.

- Each BST node stores one interval.
- BST nodes sorted on `lo` endpoint.
- Additional info: store and maintain max endpoint in subtree rooted at node.



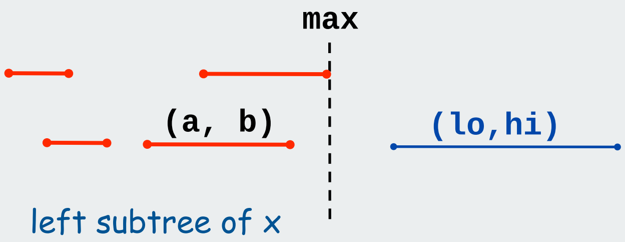
# Finding an intersecting interval

Search for an interval that intersects `(lo, hi)`.

```
Node x = root;
while (x != null)
{
   if (x.interval.intersects(lo, hi)) return x.interval;
   else if (x.left == null)  x = x.right;
   else if (x.left.max < lo) x = x.right;
   else                      x = x.left;
}
return null;
```

Case 1. If search goes right, then either

- there is an intersection in right subtree
- there are no intersections in either subtree.



Pf. Suppose no intersection in right.

- `(x.left == null)` ⇒ trivial.
- `(x.left.max < lo)` ⇒ for any interval `(a, b)` in left subtree of x, we have `b` ≤ `max` < `lo`.

defn of max

reason for going right

# Finding an intersecting interval

Search for an interval that intersects `(lo, hi)`.

```
Node x = root;
while (x != null)
{
   if (x.interval.intersects(lo, hi)) return x.interval;
   else if (x.left == null)  x = x.right;
   else if (x.left.max < lo) x = x.right;
   else                      x = x.left;
}
return null;
```

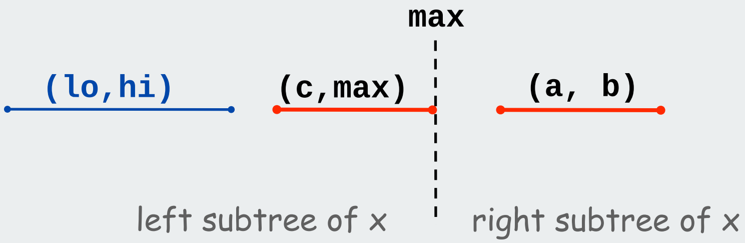
Case 2. If search goes left, then either

- there is an intersection in left subtree
- there are no intersections in either subtree.

Pf. Suppose no intersection in left. Then for any interval `(a, b)` in right subtree, $a \geq c > hi$ ⇒ no intersection in right.

intervals sorted by left endpoint

no intersection in left subtree



# Interval Search Tree: *Analysis*

**Implementation.** Use a red-black tree to guarantee performance.

can maintain auxiliary information using log N extra work per op

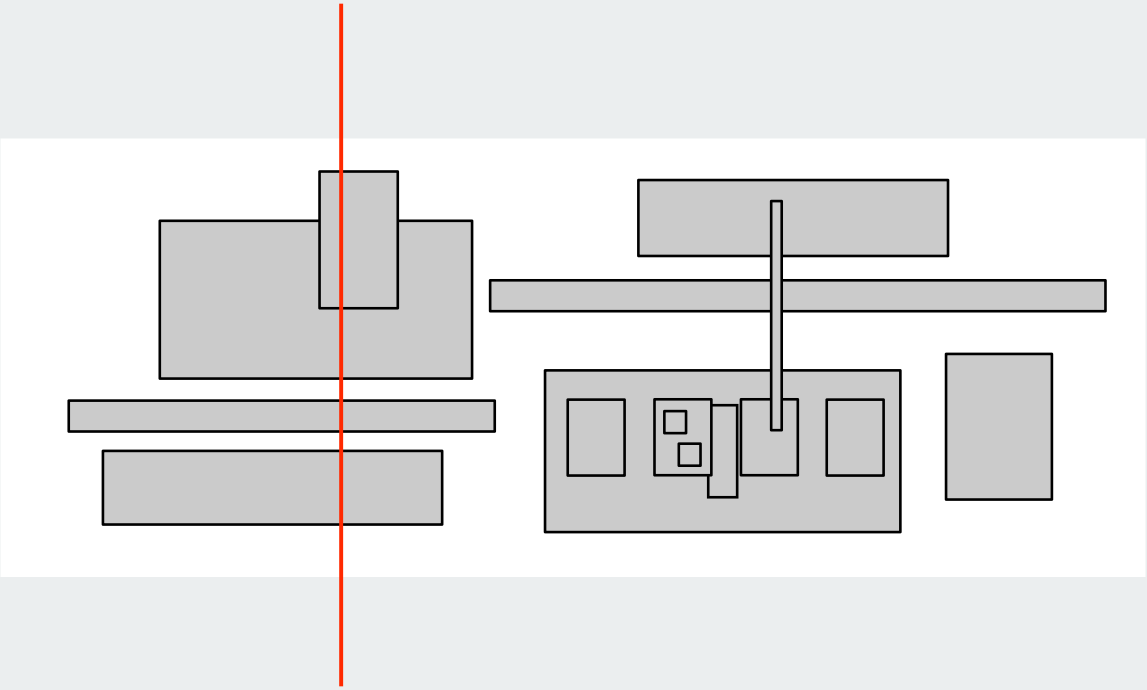
| Operation | Worst case |
|---|---|
| insert interval | log N |
| delete interval | log N |
| find an interval that intersects (lo, hi) | log N |
| find all intervals that intersect (lo, hi) | R log N |

N = # intervals
R = # intersections

# Rectangle intersection sweep-line algorithm: Review

Move a vertical "sweep line" from left to right.

- Sweep line: sort rectangles by x-coordinates and process in this order.
- Store set of rectangles that intersect the sweep line in an interval search tree (using y-interval of rectangle).
- Left side: interval search for y-interval of rectangle, insert y-interval.
- Right side: delete y-interval.



# VLSI Rules checking: Sweep-line algorithm (summary)

Reduces 2D orthogonal rectangle intersection search to 1D interval search!

## Running time of sweep line algorithm.

| | |
|---|---|
| Sort by x-coordinate. | O(N log N) |
| Insert y-interval into ST. | O(N log N) |
| Delete y-interval from ST. | O(N log N) |
| Interval search. | O(R log N) |

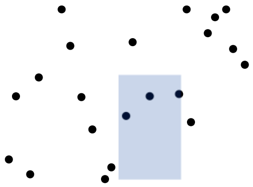
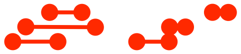
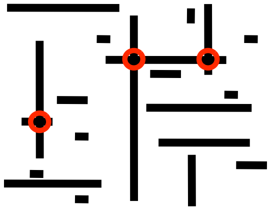
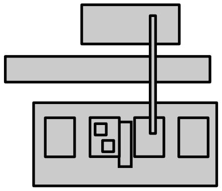
N = # line segments
R = # intersections

Efficiency relies on judicious extension of BST.

## Bottom line.

Linearithmic algorithm enables design-rules checking for huge problems

# Geometric search summary: Algorithms of the day

| | | |
|---|---|---|
| 1D range search | | BST |
| kD range search |  | kD tree |
| 1D interval intersection search |  | interval tree |
| 2D orthogonal line intersection search |  | sweep line reduces to 1D range search |
| 2D orthogonal rectangle intersection search |  | sweep line reduces to 1D interval intersection search |