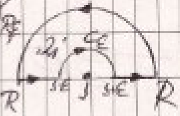
КА 15.01.15

Зад. $I = \text{v.p.} \int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+4)}\,dx$

Нека $f(z) = \frac{1}{(z-1)(z^2+4)}$. Особените точки на $f(z)$ в $\mathbb{C}$ са 1 и $\pm 2i$



По Th за резидуумите

$$\int_{\gamma_{R\varepsilon}} f(z)\,dz = 2\pi i \operatorname{Res}(f, 2i)$$

$$\int_{-R}^{1-\varepsilon} f(x)\,dx + \int_{C_\varepsilon} f(z)\,dz + \int_{1+\varepsilon}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \operatorname{Res}(f, 2i)$$

(1)

Разгл.

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{(R-1)(R^2+4)}\pi R \xrightarrow[R\to\infty]{} 0 \Rightarrow$$

$$\int_{C_R} f(z)\,dz \xrightarrow[R\to\infty]{} 0 \quad \text{за } z \in C_R \quad \begin{array}{l} |z-1| \geq |z|-1 = R-1 > 0 \\ |z^2+4| \geq |z|^2 - 4 = R^2 - 4 > 0 \end{array}$$

В прободена околност на 1 развиваме $f(z)$ в ред на Лоран $\Rightarrow$

$$f(z) = \frac{1}{(z^2+4)} \Big/ z-1 = \frac{a_0 + a_1(z-1) \cdots}{z-1} = a_0 =$$

$\frac{1}{5(z-1)} + \varphi(z)$ където $\varphi(z)$ е холоморфна в околност

на 1 и са ограничена в околност на 1

(от $\frac{1}{z^2+4}\Big|_{z=1}$ а другите от развитието на Лоран са равни на 0)

Нека $|\varphi(z)| \le M \Rightarrow \int\limits_{C_\varepsilon} \frac{1}{(z-1)(z^2+4)}dz = \frac{1}{5}\int\limits_{C_\varepsilon}\frac{1}{z-1}dz + \int\limits_{C_\varepsilon}\varphi(z)dz$

$\left|\int\limits_{C_\varepsilon}\varphi(z)dz\right| \le M\pi\varepsilon \xrightarrow[\varepsilon\to 0]{} 0 \Rightarrow \int\limits_{C_\varepsilon}\varphi(z)dz \xrightarrow[\varepsilon\to 0]{} 0$

$C_\varepsilon : z = 1+\varepsilon e^{it},\ t\in[0,\pi]$

$\int\limits_{C_\varepsilon}\frac{1}{z-1}dz = -\int\limits_{C_\varepsilon}\frac{1}{z-1}dz = -\int\limits_0^\pi\frac{1}{\varepsilon e^{it}}\,d(1+\varepsilon e^{it}) =$

$= -\int\limits_0^\pi\frac{1}{\varepsilon e^{it}}\,i\varepsilon e^{it}dt = -\pi i$. Сл. $\int\limits_{C_\varepsilon} f(z)dz \xrightarrow[\varepsilon\to 0]{} -\frac{\pi i}{5}$

От (1) при $R\to\infty,\ \varepsilon\to 0 \Rightarrow$

$I - \frac{\pi i}{5} = 2\pi i\,Res\,(f, 2i)$

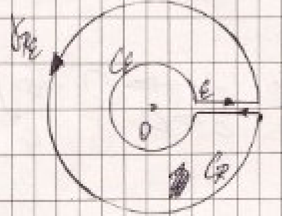
$Res(f, 2i) = Res\left(\frac{\frac{1}{z-1}}{z^2+4}, 2i\right) = \frac{\frac{1}{z-1}\big|_{z=2i}}{(z^2+4)'\big|_{z=2i}} = \frac{1}{4i(2i-1)} = \frac{2i+1}{-20i}$

$I - \frac{\pi i}{5} = 2i\pi\,\frac{2i+1}{-20i} \Rightarrow I - \frac{\pi i}{5} = -\frac{\pi i}{5} - \frac{\pi}{10} \Rightarrow$

$I = -\frac{\pi}{10}$

Интеграли от вида $\int_0^\infty x^\alpha R(x)\ln^m x\,dx$, където $\alpha\in\mathbb{R}$, $m\in\mathbb{N}\cup\{0\}$ и $R(x)$ е рационална ф/ия. Тези интеграли се пресмятат, като интегрираме ф/ията $f(x) = z^\alpha R(z)\log^{m+1} z$, ако $\alpha\in\mathbb{Z}$, или ф/ията $f(z) = e^{\alpha\log z}R(z)\log^m z$, ако $\alpha\notin\mathbb{Z}$ по

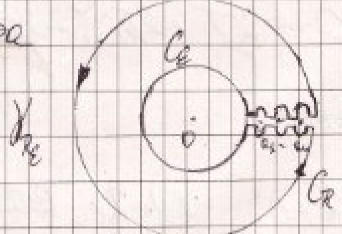
контура



Ако $R(z)$ няма положителни полюси

или

по контура

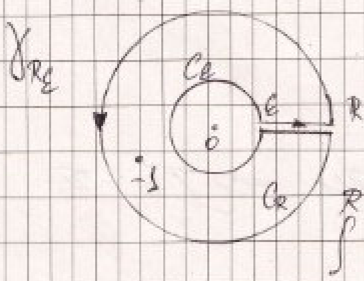


Ако $a_1 \dots a_n$ са всичките положителни полюси на $R(z)$

Тук $\log z = \ln|z| + i\arg z$, $0 \le \arg z \le 2\pi$

(25а) $I = \int_0^\infty \frac{\ln x}{\sqrt{x}\,(x+1)^2}\,dx$. Нека $f(z) = \frac{\log z}{e^{\frac{1}{2}\log z}(z+1)^2}$,

където $\log z = \ln|z| + i\arg z$, $0 \le \arg z \le 2\pi$



$\Rightarrow \int_{\gamma_{R\varepsilon}} f(z)\,dz = 2\pi i\,\mathrm{Res}(f,-1)$

$$\int_\varepsilon^R \frac{\ln|x| + i\cdot 0}{e^{\frac{1}{2}\log x}(x+1)^2}\,dx + \int_{C_R} f(z)\,dz - \int_\varepsilon^R \frac{\ln|x| + i2\pi}{e^{\frac{1}{2}(\ln|x|+i2\pi)}(x+1)^2}\,dx + \int_{C_\varepsilon} f(z)\,dz = 2\pi i\,\mathrm{Res}(f,-1)$$

$$2\int_{\varepsilon}^{R} \frac{\ln x}{\sqrt{x}(x+1)^2}dx + \int_{\varepsilon}^{R}\frac{2\pi i}{\sqrt{x}(x+1)^2}dx + \int_{C_R} f(z)dz + \int_{C_\varepsilon} f(z)dz = 2\pi i \operatorname{Res}(f,-1)$$

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\ln R + 2\pi}{\sqrt{R}(R-1)^2}2\pi R \xrightarrow[R\to\infty]{} 0 \Rightarrow$$

$$\int_{C_R} f(z)dz \xrightarrow[R\to\infty]{} 0$$ от окръжността (оценка)

$$\left|\int_{C_\varepsilon} f(z)dz\right| \le \frac{2\pi - \ln\varepsilon}{\sqrt{\varepsilon}(1-\varepsilon)^2}2\pi\varepsilon \xrightarrow[\varepsilon\to 0]{} 0 \Rightarrow \int_{C_\varepsilon} f(z)dz \xrightarrow[\varepsilon\to 0]{} 0$$

от (1) при $R\to\infty, \varepsilon\to 0 \Rightarrow 2I + 2\pi i J = 2\pi i \operatorname{Res}(f,-1)$ за

$$J = \int_0^{+\infty}\frac{1}{\sqrt{x}(x+1)^2}dx$$

$$\operatorname{Res}(f,-1) = \operatorname{Res}\left(\frac{\log z}{e^{1/2\log z}}\Big/(z+1)^2, -1\right) = \operatorname{Res}\left(\frac{a_0 + a_1(z-1)+\dots}{(z-1)^2}, -1\right)$$

$$= a_1 = \frac{\left(\log z / e^{1/2\log z}\right)'}{1!}\Big/_{z=-1} = \frac{1/z \cdot e^{1/2\log z} - \log z\, e^{1/2\log z}\cdot 1/2z}{e^{(1/2\log z)^2}}$$

$$= \frac{-1 + 1/2\cdot i\pi}{e^{i\pi/2}} = i + \pi/2 \Rightarrow$$

$$I + \pi i J = \pi i (i + \pi/2)$$

Приравняваме реалните части $\Rightarrow I = -\pi$