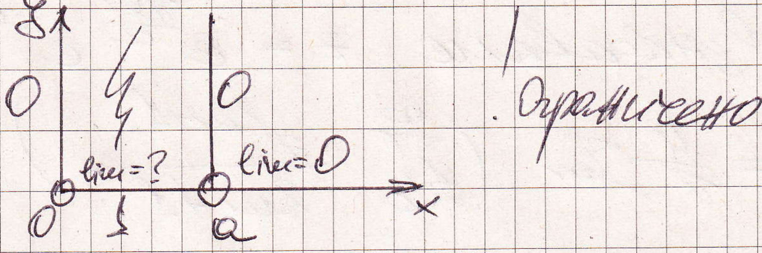
ЧДУ 22.12.14

1) $\Delta U = 0$



$U|_{x=0} = 0$
$U|_{x=a} = 0$
$U|_{y=0} = 1$

$$U(x,y) = X(x)Y(y) \neq 0$$

$$X''(x)Y(y) + Y''(y)X(x) = 0 \quad :/X\cdot Y \Rightarrow$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$$\left.\begin{array}{l} X''(x) + \lambda X(x) = 0 \\ X(0) = 0,\ X(a) = 0 \end{array}\right\} \to \begin{array}{l} \lambda_k = \left(\frac{k\pi}{a}\right)^2 \quad k = 1, 2 \ldots \\ X_k(x) = \sin\frac{k\pi x}{a} \end{array}$$

$$Y''_k(y) \left(\frac{k\pi}{a}\right)^2 Y_k(y) = 0$$

$$Y_{(k)}(y) = A_k e^{\frac{k\pi y}{a}} + B_k e^{-\frac{k\pi y}{a}}$$

Искаме Y_k – ограничено $\Rightarrow A_k = 0$

$$U(x,y) = \sum_{k=1}^{\infty} B_k e^{-\frac{k\pi y}{a}} \sin\frac{k\pi x}{a}$$

$$U|_{y=0} = 1 = \sum_{k=1}^{\infty} B_k \sin\frac{k\pi x}{a}, \text{ т.е.}$$

$$B_k = \frac{2}{a}\int_0^a 1\cdot \sin\frac{k\pi x}{a}\,dx = -\frac{2}{a}\frac{a}{k\pi}\cos\frac{k\pi x}{a}\Big|_0^a =$$

$$\cancel{\ldots} \; -\frac{2}{k\pi}\left((-1)^k - 1\right)$$

$$B_{2m+1} = \frac{4}{\pi(2m+1)}$$

$$U(x,y) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{e^{-\frac{(2m+1)\pi y}{a}} \sin\frac{(2m+1)\pi x}{a}}{(2m+1)}$$

Означаване $z = e^{\frac{\pi x i}{a}} e^{-\frac{\pi y}{a}}$

$$u = \frac{4}{\pi} \operatorname{Im} \left(\sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1} \right) \qquad (\text{от } \mathrm{kA}) \Rightarrow$$

$$\left. \sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1} = \frac{1}{2} \log_0 \frac{1+z}{1-z} \right\} \Rightarrow u = \frac{2}{\pi} \operatorname{Im} \left(\log_0 \frac{1+z}{1-z} \right)$$

$$\log_0 z = \ln |z| + i \arg z \Rightarrow u = \frac{2}{\pi} \arg \frac{1+z}{1-z}$$

$$\arg \frac{1+z}{1-z} = \arg \frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})} = \arg \frac{1-\bar{z}+z-|z|^2}{|1-z|^2}$$

Нека $z = p + iq \Rightarrow$

$$\arg \frac{1-p+iq-p+iq-(p+iq)^2}{(1-p)^2} = \frac{1-(p^2+q^2)+2iq}{(1-p)^2+q^2} =$$

$$= \arg \underbrace{\left(\frac{1-q^2-p^2}{(1-p)^2+q^2} \right)}_{x} + i \underbrace{\frac{2q}{(1-p)^2+q^2}}_{y} \quad \text{т.е.}$$

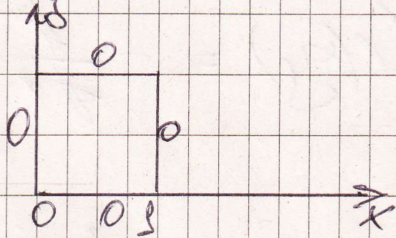
$$\operatorname{arctg} \frac{2q}{(1-p)^2-q^2} = \operatorname{arctg} \frac{2e^{-\frac{\pi y}{a}} \sin \frac{\pi x}{a}}{1-e^{-\frac{2\pi y}{a}}}$$

$$\Rightarrow u = \frac{2}{\pi} \operatorname{Im} \left(\log_0 \frac{1+z}{1-z} \right) = \frac{2}{\pi} \operatorname{arctg} \frac{e^{-\frac{\pi y}{a}} \sin \frac{\pi x}{a}}{e^{-\pi y/a} \left(\frac{e^{\pi y/a} - e^{-\pi y/a}}{2} \right)} =$$

$$\frac{2}{\pi} \operatorname{arctg} \left(\frac{\sin \frac{\pi x}{a}}{\operatorname{sh} \pi y/a} \right) \xrightarrow{a \to \infty} \frac{2}{\pi} \operatorname{arctg} \left(\frac{x}{y} \right)$$

2/ $\Delta U = f(x,y) = xy$

Уравнението на Поасон



$$U(x,y) = \sum_{m,n=1}^{\infty} E_{m,n} \sin m\pi x \sin n\pi y$$

$$\Delta U = -\pi^2 \sum_{m,n=1}^{\infty} E_{m,n} (m^2+n^2) \sin m\pi x \sin n\pi y$$

$$xy = -\pi^2 \sum_{m,n=1}^{\infty} E_{m,n} (m^2+n^2) \sin m\pi x \sin n\pi y$$

$$\sum_{m,n=1}^{\infty} A_{m,n} \sin m\pi x \sin n\pi y \Rightarrow E_{m,n} = \frac{A_{m,n}}{\pi^2(m^2+n^2)} =$$

$$-\frac{2\cdot 2}{\pi^2(m^2+n^2)} \int_0^1 \left(\int_0^1 xy \sin m\pi x \sin n\pi y \, dx \right) dy =$$

$$= -\frac{4}{\pi^2(m^2+n^2)} \left(\int_0^1 x \sin m\pi x \, dx \right) \left(\int_0^1 y \sin n\pi y \, dy \right)$$ (по части)

$$\int x \sin m\pi x \, dx = -\frac{1}{m\pi} \int x \cos m\pi x =$$

$$= -\frac{1}{m\pi} x \cos m\pi x + \frac{1}{m\pi} \int \cos m\pi x \, dx =$$

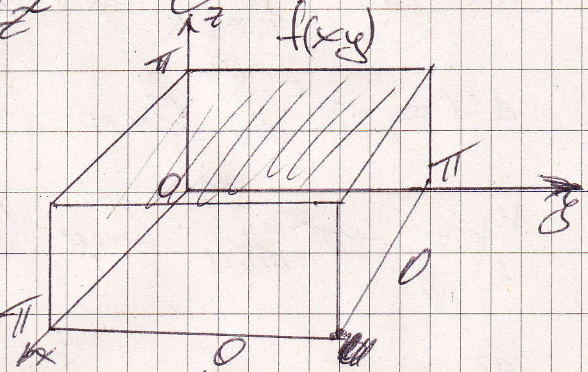
$$-\frac{1}{m\pi} x \cos m\pi x + \frac{1}{m^2\pi^2} \sin m\pi x \Rightarrow$$

$$-\frac{4}{\pi^2(m^2+n^2)} \left(-\frac{1}{m\pi} (-1)^m \right) \left(-\frac{1}{n\pi} (-1)^n \right)$$ т.е.

$$E_{m,n} = -\frac{4}{\pi^4 mn(m^2+n^2)} \Rightarrow$$

$$u(x,y) = -\frac{4}{\pi^4}\sum_{m,n=1}^{\infty}\frac{1}{mn(m^2+n^2)}\sin m\pi x \sin n\pi y$$

3) $$\Delta u = 0 = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$



$u|_{y=0}=0, \quad u|_{y=\pi}=0$

$u|_{z=0}=0, \quad u|_{z=\pi}=f(x,y)$

$u|_{x=0}=0, \quad u|_{x=\pi}=0$

$$u(x,y,z)=X(x)Y(y)Z(z)\neq 0$$

$$X''(x)Y(y)Z(z)+X(x)Y''(y)Z(z)+X(x)Y(y)Z''(z)=0$$

$$u(x,y,z)=X(x)Y(y)Z(z)\neq 0 \quad !!!$$

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$$

$$\frac{X''}{X}+\frac{Y''}{Y}=-\frac{Z''}{Z},$$

$\longrightarrow u|_{x=\pi}=0$

Нехай $\left.\begin{array}{l}\dfrac{X''}{X}=-\lambda \\ \dfrac{Y''}{Y}=-\mu\end{array}\right\} \Rightarrow \dfrac{Z''}{Z}=\mu+\lambda$

$\longrightarrow u|_{y=\pi}=0$

$$X''(x)+\lambda X(x)=0, \quad X(0)=X(\pi)=0$$

$$Y''(y)+\mu Y(y)=0, \quad Y(0)=Y(\pi)=0$$

$$\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \quad X_n(x) = \sin nx$$
$$\mu_m = \left(\frac{m\pi}{\pi}\right)^2 = m^2, \quad Y_m(y) = \sin my$$
$\Rightarrow$

$$Z'' - (m^2+n^2)Z = 0 \text{ т.е. } Z = C_1 e^{\sqrt{m^2+n^2}\,z} + C_2 e^{-\sqrt{m^2+n^2}\,z}$$

$$Z_{m,n} = A_{m,n}\,\mathrm{ch}\sqrt{m^2+n^2}\,z + B_{m,n}\,\mathrm{sh}\sqrt{m^2+n^2}\,z$$

$$u(x,y,z) = \sum_{m,n=1}^{\infty} \sin nx \sin my \left(A_{mn}\,\mathrm{ch}\sqrt{m^2+n^2}\,z + B_{mn}\,\mathrm{sh}\sqrt{m^2+n^2}\,z\right)$$

$$u|_{z=0} = 0 = \sum_{m,n=0}^{\infty} \sin nx \sin my\, A_{m,n} \Rightarrow A_{mn} = 0$$

$$u|_{z=\pi} = f(x,y) = \sum_{m,n=1}^{\infty} \sin nx \sin my\; B_{m,n}\,\mathrm{sh}\sqrt{m^2+n^2}\,\pi, \text{ т.е.}$$

$$B_{m,n} = \frac{1}{\mathrm{sh}\sqrt{m^2+n^2}\,\pi} = \frac{2}{\pi}\cdot\frac{2}{\pi}\int_0^{\pi}\int_0^{\pi} f(x,y)\sin nx \sin my\,dx\,dy$$