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Вариационни принципи

Принцип на най-малкото действие на Хамилтон

1) $A[q] = \int_{t_0}^{t_1} L(q, \dot{q})\, dt$ - Екстремалите са у-ята на Ойлер-Лагранж

2) $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$, $q = (q_1 \dots q_n)$

Пример 1) $L = \sqrt{1+\dot{q}^2}$

$\frac{d}{dt}\left(\frac{\dot{q}}{\sqrt{1+\dot{q}^2}}\right) = 0 \Rightarrow \dot{q} = c$, $q(t) = ct + D$

L - лагранжиан, $L = L(x, y, u, u_x, u_y)$
$p = u_x$, $q = u_y$

$A[u] = \iint_D L\, dx\, dy$, $\frac{\partial L}{\partial u} = \frac{\partial}{\partial x}\frac{\partial L}{\partial p} + \frac{\partial}{\partial y}\frac{\partial L}{\partial q}$

Пример 2) $L = \frac{1}{2}(p^2 + q^2)$

$A[u] = \frac{1}{2}\iint_D (u_x^2 + u_y^2)\, dx\, dy$

$0 = \frac{\partial}{\partial x} p + \frac{\partial}{\partial y} q = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u$ - Лаплас

Пример 3 $\quad A[u] = \frac{1}{2}\iint\limits_D [(u_t)^2 - a^2 u_x^2] dx dt$

$$\frac{\partial L}{\partial t} - \frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x},$$

$$0 = u_{tt} - a^2 u_{xx} \rightarrow u_{tt} = a^2 u_{xx} \quad \text{ЛЕ} \ \tfrac{1}{4}$$

Пример 4
$$A[u] = \iint \left(\frac{1}{2} u_x u_t + u_x^3 - \frac{1}{2} u_{xx}^2\right) dx\,dt$$

Полагаме $g(\varepsilon) = A[u - \varepsilon\omega] =$

$$= \iint \left[\frac{1}{2}(u_x + \varepsilon\omega_x)(u_t + \varepsilon\omega_t) + (u_x + \varepsilon\omega_x)^3 - \frac{1}{2}(u_{xx} + \varepsilon\omega_{xx})^2\right] dx\,dt$$

$$\left.\frac{dg(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0,$$

$$0 = \iint \left[\frac{1}{2}(u_x\omega_t + \omega_x u_t) + 3u_x^2\omega_x - u_{xx}\omega_{xx}\right] dx\,dt$$

$$= \iint \left[-u_{xt} - 3(u_x^2)_x - (u_{xx})_{xx}\right] \omega\, dx\,dt$$

Понеже ω е произволен полином $\neq 0$,
полагаме $v = u_x$
$v_t + 6vv_x + u_{xxx} = 0 \quad$ KdV
у-е на плоските вълни

Основна лема на вариационното смятане

Лема $0 = \int_{t_0}^{t_1} A(t) B(t)\, dt$ за $\forall B(t) \Rightarrow A(t) = 0$

Нека допуснем че $\exists t_1 : A(t_1) \neq 0$

$\exists$ околност на $t_1 : (\alpha, \beta)$, $A(t) > 0$

$$0 = \int_\alpha^\beta A(t) B(t) dt = A(t_1) B(t_1) > 0 \Rightarrow \lightning$$

$$A[u] = \frac{u_1^2}{2I_1} + \frac{u_2^2}{2I_2} + \frac{u_3^2}{2I_3}, \quad u_j = \frac{\omega_j}{I_j}$$

Дефинираме скобката на Поасон така

$$\{u_1 u_2\} = -u_3, \quad \{u_1 u_3\} = u_2, \quad \{u_2 u_3\} = -u_1$$

$$\dot{u}_1 = \{u_1 H\} = \frac{I_2 - I_3}{I_2 I_3} u_2 u_3, \quad \dot{u}_2 = \{u_2 H\} = \frac{I_3 - I_1}{I_3 I_1} u_3 u_1$$

$$\dot{u}_3 = \{u_3 H\} = \frac{I_1 - I_2}{I_1 I_2} u_1 u_2, \quad C = u_1^2 + u_2^2 + u_3^2$$

Вариационна производна

$$F[u] \quad \frac{\delta F}{\delta u}, \quad F[u + \varepsilon \omega]|_{\varepsilon = 0} = \int \frac{\delta F}{\delta u} v dx$$

$$H[u] = \int \left(\frac{u_x^2}{2} + u^3 \right) dx, \quad \{S, P\} = \int \frac{\delta S}{\delta u} \frac{\partial}{\partial x} \frac{\delta P}{\delta u} dx \Rightarrow$$

$u_t = \frac{\partial}{\partial x} \frac{\delta H}{\delta u}$ където $\{S, P\}$ е скобка на Поасон

$$I_1 = \int u dx$$

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Задача $u_{tt} = a^2 u_{xx}$, $x \in (0, L)$, $t > 0$

$$\begin{cases} u(x,0) = x = \varphi \\ u_t(x,0) = 1 = \psi \end{cases} \qquad \varphi'(0) = 1 \neq 0, \ \varphi'(L) = 1 \neq 0$$

$u(x,t) = X(x)T(t) \neq 0$, $\dfrac{\ddot{T}}{a^2 T} = \dfrac{X''}{X} = -\lambda$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases} \qquad \begin{matrix} \lambda_0 = 0 \\ X_0 = 1 \end{matrix}$$

$\lambda_k = \left(\dfrac{k\pi}{L}\right)^2$, $X_k = \cos\dfrac{k\pi x}{L}$, $\ddot{T}_k + a^2 \dfrac{k^2\pi^2}{L^2} T_k = 0$

$T_k = A_k \cos\dfrac{kat\pi}{L} + B_k \sin\dfrac{kat\pi}{L}$, $\ddot{T}_0 = 0$

$T_0 = At + B$, $u(x,t) = X_0 T_0 + \sum\limits_1^\infty X_k(x) T_k(t)$

$u(x,t) = A_0 t + B_0 + \sum\limits_1^\infty \cos\dfrac{k\pi x}{L}\left(A_k \cos\dfrac{k\pi a t}{L} + B_k \sin\dfrac{k\pi a t}{L}\right)$

$u_t = A_0 + \sum\limits_1^\infty \cos\dfrac{k\pi x}{L}\left(A_k\left(-\dfrac{k\pi a}{L}\right)\sin\dfrac{k\pi a t}{L} + B_k \dfrac{k\pi a}{L}\cos\dfrac{k\pi a t}{L}\right)$

$u_t\Big|_{t=0} = 1 = A_0 + \sum\limits_1^\infty \cos\dfrac{k\pi x}{L}\dfrac{k\pi a}{L} B_k$

$\left\{1, \cos\dfrac{k\pi x}{L}\right\} \Rightarrow A_0 = 1$, $B_k = 0$

$u(x,t) = t + B_0 + \sum\limits_1^\infty A_k \cos\dfrac{k\pi x}{L} \cdot \cos\dfrac{k\pi a t}{L}$

$u(x,0) = B_0 + \sum\limits_1^\infty A_k \cos\dfrac{k\pi x}{L} = x$

$x = C_0 + \sum C_k \cos\dfrac{k\pi x}{L}$, $C_0 = \dfrac{1}{L}\int\limits_0^L x\,dx$

$B_0 = \dfrac{1}{L}\int\limits_0^L x\,dx = \dfrac{x^2}{2L}\Big|_0^L = \dfrac{L}{2}$; $C_k = \dfrac{2}{L}\int\limits_0^L x\cos\dfrac{k\pi x}{L}\,dx$

$$A_k = \frac{2}{L}\int_0^L x\cos\frac{k\pi x}{L}\,dx = \frac{L}{k\pi}\frac{2}{L}\int_0^L x\,d\sin\frac{k\pi}{L}x =$$

$$= \frac{2L}{k\pi L}\, x\sin\frac{k\pi x}{L}\Big|_0^L - \frac{L}{k\pi}\frac{2}{L}\int_0^L \sin\frac{k\pi x}{L}\,dx$$

$$A_k = \frac{L^2}{k\pi^2}\cdot\frac{2}{L}\cos\frac{k\pi x}{L}\Big|_0^L \Rightarrow A_{2k+1} = \frac{-4}{L\pi^2}\frac{1}{(2k\pi)^2}$$

$$u(x,t) = t + \frac{L}{2} - \frac{4}{\pi^2 L^2}\sum_{1}^{\infty}\frac{1}{(2k+1)^2}\cos\frac{(2k\pi)}{L}x\cos\frac{(2k\pi)\pi a t}{L}$$