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CHAPTER

TWO

TWO-DIMENSIONAL TRANSFORMATIONS

2-1 INTRODUCTION

We begin our study of the fundamentals of the mathematics underlying computer graphics by considering the representation and transformation of points and lines. Points and the lines which join them, along with an appropriate drawing algorithm, are used to represent objects or to display information graphically. The ability to transform these points and lines is basic to computer graphics. When visualizing an object, it may be desirable to scale, rotate, translate, distort or develop a perspective view of the object. All of these transformations can be accomplished using the mathematical techniques discussed in this and the next chapter.

2-2 REPRESENTATION OF POINTS

A point is represented in two dimensions by its coordinates. These two values are specified as the elements of a 1-row, 2-column matrix:

$$\begin{bmatrix} x & y \end{bmatrix}$$

In three dimensions a 1×3 matrix

$$\begin{bmatrix} x & y & z \end{bmatrix}$$

is used. Alternately, a point is represented by a 2-row, 1-column matrix

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

in two dimensions or by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

in three dimensions. Row matrices like

$$[\,x \quad y\,]$$

or column matrices like

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

are frequently called position vectors. In this book a row matrix formulation of the position vectors is used.

A series of points, each of which is a position vector relative to some coordinate system, is stored in a computer as a matrix or array of numbers. The position of these points is controlled by manipulating the matrix which defines the points. Lines are drawn between the points to generate lines, curves or pictures.

2–3 TRANSFORMATIONS AND MATRICES

Matrix elements can represent various quantities, such as a number store, a network or the coefficients of a set of equations. The rules of matrix algebra define allowable operations on these matrices (see Appendix B). Many physical problems lead to a matrix formulation. For models of physical systems, the problem is formulated as: given the matrices $[\,A\,]$ and $[\,B\,]$ find the solution matrix $[\,T\,]$, i.e., $[\,A\,][\,T\,] = [\,B\,]$. In this case the solution is $[\,T\,] = [\,A\,]^{-1}[\,B\,]$, where $[\,A\,]^{-1}$ is the inverse of the square matrix $[\,A\,]$ (see Ref. 2–1).

An alternate interpretation is to treat the matrix $[\,T\,]$ as a geometric operator. Here matrix multiplication is used to perform a geometrical transformation on a set of points represented by the position vectors contained in $[\,A\,]$. The matrices $[\,A\,]$ and $[\,T\,]$ are assumed known. It is required to determine the elements of the matrix $[\,B\,]$. The interpretation of the matrix $[\,T\,]$ as a geometrical operator is the foundation of mathematical transformations useful in computer graphics.

2–4 TRANSFORMATION OF POINTS

Consider the results of the multiplication of a matrix $[\,x \quad y\,]$ containing the coordinates of a point P and a general 2×2 transformation matrix:

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\,(ax+cy) \quad (bx+dy)\,] = [\,x^* \quad y^*\,] \qquad (2-1)$$

This mathematical notation means that the initial coordinates x and y are transformed to x^* and y^*, where $x^* = (ax+cy)$ and $y^* = (bx+dy)$.† We are interested

† See Appendix B for the details of matrix multiplication.

in the implications of considering x^* and y^* as the transformed coordinates of the point P. We begin by investigating several special cases.

Consider the case where $a = d = 1$ and $c = b = 0$. The transformation matrix $[\,T\,]$ then reduces to the identity matrix. Thus,

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\,x \quad y\,] = [\,x^* \quad y^*\,] \qquad (2-2)$$

and no change in the coordinates of the point P occurs. Since in matrix algebra multiplying by the identity matrix is equivalent to multiplying by 1 in ordinary algebra, this result is expected.

Next consider $d = 1$, $b = c = 0$, i.e.,

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = [\,ax \quad y\,] = [\,x^* \quad y^*\,] \qquad (2-3)$$

which, since $x^* = ax$, produces a scale change in the x component of the position vector. The effect of this transformation is shown in Fig. 2–1a. Now consider $b = c = 0$, i.e.,

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = [\,ax \quad dy\,] = [\,x^* \quad y^*\,] \qquad (2-4)$$

This yields a scaling of both the x and y coordinates of the original position vector P, as shown in Fig. 2–1b. If $a \neq d$, then the scalings are not equal. If $a = d > 1$, then a pure enlargement or scaling of the coordinates of P occurs. If $0 < a = d < 1$, then a compression of the coordinates of P occurs.

If a and/or d are negative, reflections through an axis or plane occur. To see this, consider $b = c = 0$, $d = 1$ and $a = -1$. Then

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [\,-x \quad y\,] = [\,x^* \quad y^*\,] \qquad (2-5)$$

and a reflection through the y-axis results, as shown in Fig. 2–1c. If $b = c = 0$, $a = 1$, and $d = -1$, then a reflection through the x-axis occurs. If $b = c = 0$, $a = d < 0$, then a reflection through the origin occurs. This is shown in Fig. 2–1d, with $a = -1$, $d = -1$. Note that both reflection and scaling of the coordinates involve only the diagonal terms of the transformation matrix.

Now consider the effects of the off-diagonal terms. First consider $a = d = 1$ and $c = 0$. Thus,

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = [\,x \quad (bx+y)\,] = [\,x^* \quad y^*\,] \qquad (2-6)$$

Note that the x coordinate of the point P is unchanged, while y^* depends linearly on the original coordinates. This effect is called shear, as shown in Fig. 2–1e.

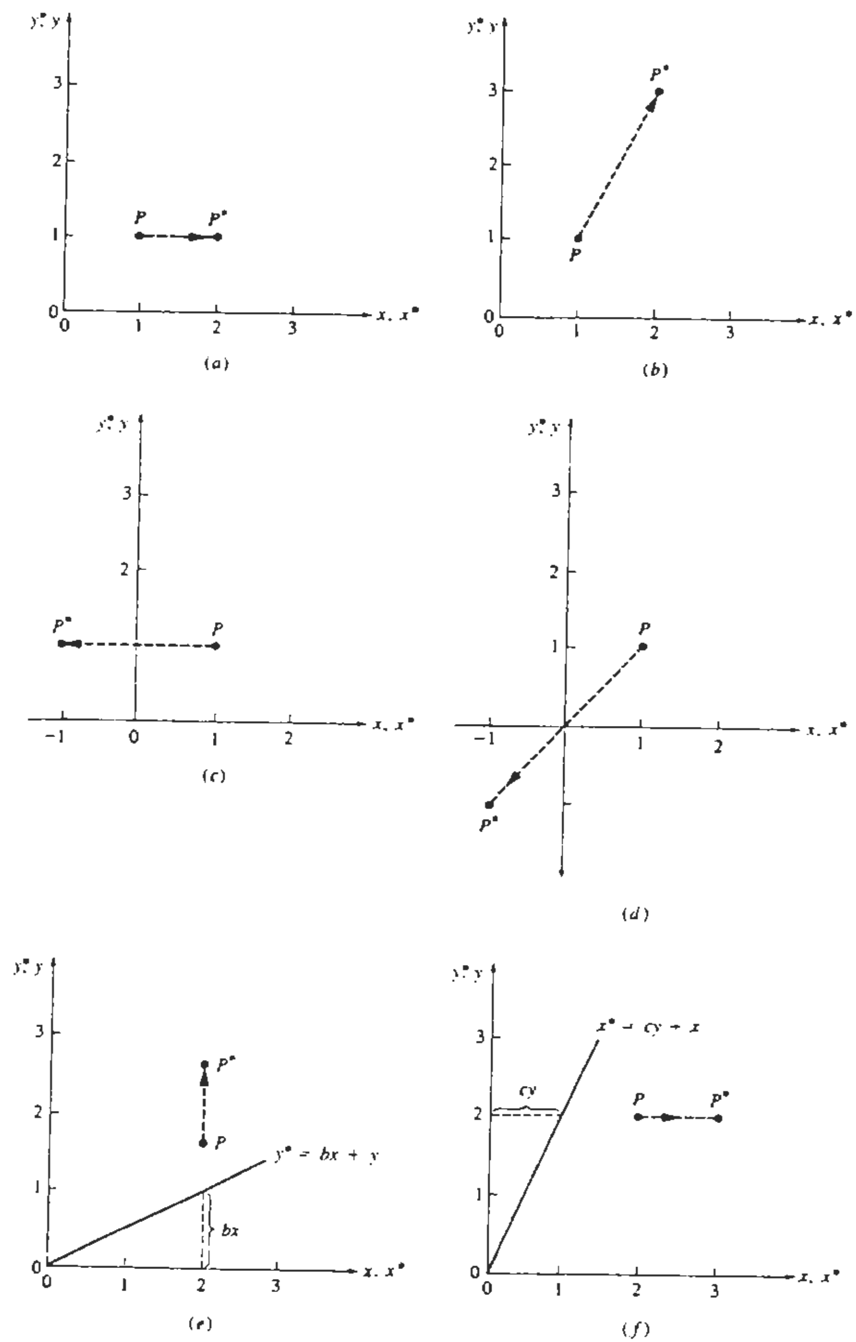

Figure 2–1 Transformation of points.

Similarly, when $a = d = 1$, $b = 0$, the transformation produces shear proportional to the y coordinate, as shown in Fig. 2-1f. Thus, we see that the off-diagonal terms produce a shearing effect on the coordinates of the position vector for P.

Before completing our discussion of the transformation of points, consider the effect of the general 2 × 2 transformation given by Eq. (2–1) when applied to the origin, i.e.,

$$[\,x \quad y\,]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\,(ax+cy) \quad (bx+dy)\,] = [\,x^* \quad y^*\,]$$

or for the origin,

$$[\,0 \quad 0\,]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\,0 \quad 0\,] = [\,x^* \quad y^*\,]$$

Here we see that the origin is invariant under a general 2×2 transformation. This is a limitation which will be overcome by the use of homogeneous coordinates.

2–5 TRANSFORMATION OF STRAIGHT LINES

A straight line can be defined by two position vectors which specify the coordinates of its end points. The position and orientation of the line joining these two points can be changed by operating on these two position vectors. The actual operation of drawing a line between two points depends on the display device used. Here, we consider only the mathematical operations on the position vectors of the end points.

A straight line between two points A and B in a two-dimensional plane is drawn in Fig. 2–2. The position vectors of points A and B are $[\,A\,] = [\,0 \quad 1\,]$ and $[\,B\,] = [\,2 \quad 3\,]$, respectively. Now consider the transformation matrix

$$[\,T\,] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \tag{2-7}$$

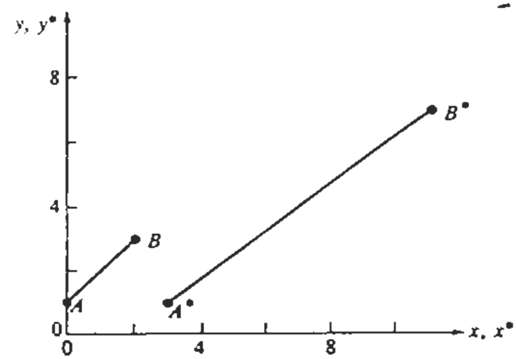

Figure 2–2 Transformation of straight lines.

which we recall from our previous discussion produces a shearing effect. Transforming the position vectors for A and B using $[\,T\,]$ produces new transformed position vectors A^* and B^* given by

$$[\,A\,][\,T\,] = [\,0 \quad 1\,]\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [\,3 \quad 1\,] = [\,A^*\,] \qquad (2-8)$$

and

$$[\,B\,][\,T\,] = [\,2 \quad 3\,]\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [\,11 \quad 7\,] = [\,B^*\,] \qquad (2-9)$$

Thus, the resulting coordinates for A^* are $x^* = 3$ and $y^* = 1$. Similarly, B^* is a new point with coordinates $x^* = 11$ and $y^* = 7$. More compactly the line AB may be represented by the 2×2 matrix

$$[\,L\,] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

Matrix multiplication by $[\,T\,]$ then yields

$$[\,L\,][\,T\,] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 11 & 7 \end{bmatrix} = [\,L^*\,] \qquad (2-10)$$

where the components of $[\,L^*\,]$ represent the transformed position vectors $[\,A^*\,]$ and $[\,B^*\,]$. The transformation of A to A^* and B to B^* is shown in Fig. 2–2. The initial axes are x, y and the transformed axes are x^*y^*. Figure 2–2 shows that the shearing transformation $[\,T\,]$ increased the length of the line and changed its orientation.

2–6 MIDPOINT TRANSFORMATION

Figure 2–2 shows that the 2×2 transformation matrix (see Eq. 2–7) transforms the straight line $y = x + 1$, between points A and B, into another straight line $y = (3/4)x - 5/4$, between A^* and B^*. In fact a 2×2 matrix transforms any straight line into a second straight line. Points on the second line have a one-to-one correspondence with points on the first line. We have already shown this to be true for the end points of the line. To further confirm this we consider the transformation of the midpoint of the straight line between A and B. Letting

$$[\,A\,] = [\,x_1 \quad y_1\,] \qquad [\,B\,] = [\,x_2 \quad y_2\,] \qquad \text{and} \qquad [\,T\,] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and transforming both end points simultaneously yields

$$\begin{bmatrix} A \\ B \end{bmatrix}[\,T\,] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix} \qquad (2-11)$$

Hence, the end points of the transformed line A^*B^* are

$$[\,A^*\,] = [\,ax_1 + cy_1 \quad bx_1 + dy_1\,] = [\,x_1^* \quad y_1^*\,]$$

$$[\,B^*\,] = [\,ax_2 + cy_2 \quad bx_2 + dy_2\,] = [\,x_2^* \quad y_2^*\,] \qquad (2-12)$$

The midpoint of the transformed line A^*B^* calculated from the transformed end points is

$$\begin{aligned}[\,x_m^* \quad y_m^*\,] &= \left[\frac{x_1^* + x_2^*}{2} \quad \frac{y_1^* + y_2^*}{2}\right] \\ &= \left[\frac{(ax_1 + cy_1) + (ax_2 + cy_2)}{2} \quad \frac{(bx_1 + dy_1) + (bx_2 + dy_2)}{2}\right] \\ &= \left[a\frac{(x_1 + x_2)}{2} + c\frac{(y_1 + y_2)}{2} \quad b\frac{(x_1 + x_2)}{2} + d\frac{(y_1 + y_2)}{2}\right]\end{aligned} \qquad (2-13)$$

Returning to the original line AB the midpoint is

$$[\,x_m \quad y_m\,] = \left[\frac{x_1 + x_2}{2} \quad \frac{y_1 + y_2}{2}\right] \qquad (2-14)$$

Using $[\,T\,]$ the transformation of the midpoint of AB is

$$\begin{aligned}[\,x_m \quad y_m\,][\,T\,] &= \left[\frac{x_1 + x_2}{2} \quad \frac{y_1 + y_2}{2}\right]\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \left[a\frac{(x_1 + x_2)}{2} + c\frac{(y_1 + y_2)}{2} \quad b\frac{(x_1 + x_2)}{2} + d\frac{(y_1 + y_2)}{2}\right]\end{aligned} \qquad (2-15)$$

Comparing Eqs. (2–13) and (2–15) shows that they are identical. Consequently, the midpoint of the line AB transforms into the midpoint of the line A^*B^*. This process can be applied recursively to segments of the divided line. Thus, a one-to-one correspondence between points on the line AB and A^*B^* is assured.

Example 2–1 Midpoint of a Line

Consider the line AB shown in Fig. 2–2. The position vectors of the end points are

$$[\,A\,] = [\,0 \quad 1\,] \qquad [\,B\,] = [\,2 \quad 3\,]$$

The transformation

$$[\,T\,] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

yields the position vectors of the end points of the transformed line A^*B^* as

$$\begin{bmatrix} A \\ B \end{bmatrix}[\,T\,] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 11 & 7 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix}$$

The midpoint of A^*B^* is

$$[\, x_m^* \quad y_m^* \,] = \left[\, \frac{3+11}{2} \quad \frac{1+7}{2} \,\right] = [\, 7 \quad 4 \,]$$

The midpoint of the original untransformed line AB is

$$[\, x_m \quad y_m \,] = \left[\, \frac{0+2}{2} \quad \frac{1+3}{2} \,\right] = [\, 1 \quad 2 \,]$$

Transforming this midpoint yields

$$[\, x_m \quad y_m \,][\, T \,] = [\, 1 \quad 2 \,]\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [\, 7 \quad 4 \,] = [\, x_m^* \quad y_m^* \,]$$

which is the same as our previous result.

For computer graphics applications these results show that any straight line can be transformed into any other straight line in any position by simply transforming its end points and redrawing the line between the end points.

2-7 TRANSFORMATION OF PARALLEL LINES

When a 2×2 matrix is used to transform a pair of parallel lines, the result is a second pair of parallel lines. To see this, consider a line between $[\, A \,] = [\, x_1 \quad y_1 \,]$ and $[\, B \,] = [\, x_2 \quad y_2 \,]$ and a line parallel to AB between E and F. To show that these lines and any transformation of them are parallel, examine the slopes of AB, EF, A^*B^* and E^*F^*. Since they are parallel, the slope of both AB and EF is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{2-16}$$

Transforming the end points of AB using a general 2×2 transformation yields the end points of A^*B^*:

$$\begin{bmatrix} A \\ B \end{bmatrix}[\, T \,] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} x_1^* & y_1^* \\ x_2^* & y_2^* \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix} \tag{2-17}$$

Using the transformed end points, the slope of A^*B^* is then

$$m^* = \frac{(bx_2 + dy_2) - (bx_1 + dy_1)}{(ax_2 + cy_2) - (ax_1 + cy_1)} = \frac{b(x_2 - x_1) + d(y_2 - y_1)}{a(x_2 - x_1) + c(y_2 - y_1)}$$

or

$$m^* = \frac{b + d\dfrac{(y_2 - y_1)}{(x_2 - x_1)}}{a + c\dfrac{(y_2 - y_1)}{(x_2 - x_1)}} = \frac{b + dm}{a + cm} \tag{2-18}$$

Since the slope m^* is independent of x_1, x_2, y_1 and y_2, and since m, a, b, c and d are the same for EF and AB, it follows that m^* is the same for both E^*F^* and A^*B^*. Thus, parallel lines remain parallel after transformation. This means that parallelograms transform into other parallelograms when operated on by a general 2×2 transformation matrix. These simple results begin to show the power of using matrix multiplication to produce graphical effects.

2-8 TRANSFORMATION OF INTERSECTING LINES

When a general 2×2 matrix is used to transform a pair of intersecting straight lines, the result is also a pair of intersecting straight lines. To see this consider a pair of lines, e.g., the dashed lines in Fig. 2-3, represented by

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

Reformulating these equations in matrix notation yields

$$[\, x \quad y \,]\begin{bmatrix} -m_1 & -m_2 \\ 1 & 1 \end{bmatrix} = [\, b_1 \quad b_2 \,]$$

or

$$[\, X \,][\, M \,] = [\, B \,] \tag{2-19}$$

If a solution to this pair of equations exists, then the lines intersect. If not, then they are parallel. A solution can be obtained by matrix inversion. Specifically,

$$[\, X_i \,] = [\, x_i \quad y_i \,] = [\, B \,][\, M \,]^{-1} \tag{2-20}$$

The inverse of $[\, M \,]$ is

$$[\, M \,]^{-1} = \begin{bmatrix} \dfrac{1}{m_2 - m_1} & \dfrac{m_2}{m_2 - m_1} \\ \dfrac{-1}{m_2 - m_1} & \dfrac{-m_1}{m_2 - m_1} \end{bmatrix} \tag{2-21}$$

since $[\, M \,][\, M \,]^{-1} = [\, I \,]$, the identity matrix. Hence, the intersection of the two lines is

$$[\, X_i \,] = [\, x_i \quad y_i \,] = [\, b_1 \quad b_2 \,]\begin{bmatrix} \dfrac{1}{m_2 - m_1} & \dfrac{m_2}{m_2 - m_1} \\ \dfrac{-1}{m_2 - m_1} & \dfrac{-m_1}{m_2 - m_1} \end{bmatrix}$$

$$[\, X_i \,] = [\, x_i \quad y_i \,] = \left[\, \frac{b_1 - b_2}{m_2 - m_1} \quad \frac{b_1 m_2 - b_2 m_1}{m_2 - m_1} \,\right] \tag{2-22}$$

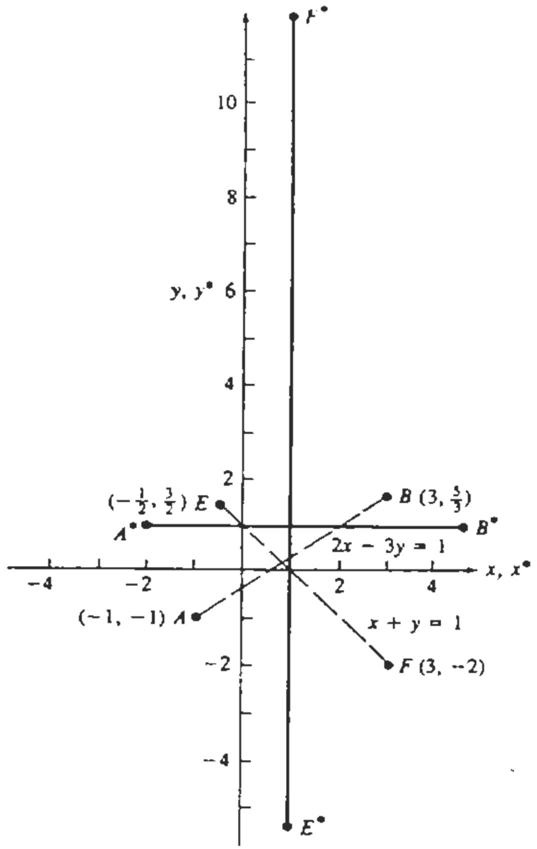

Figure 2–3 Transformation of intersecting lines.

If these two lines are now transformed using a general 2 × 2 transformation matrix given by

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then they have the form

$$y^* = m_1^* x^* + b_1^*$$
$$y^* = m_2^* x^* + b_2^*$$

It is relatively easy to show that

$$m_i^* = \frac{b + dm_i}{a + cm_i} \qquad (2-23)$$

and

$$b_i^* = b_i(d - cm_i^*) = b_i \frac{ad - bc}{a + cm_i} \qquad i = 1, 2 \qquad (2-24)$$

The intersection of the transformed lines is obtained in the same manner as that for the untransformed lines. Thus,

$$[X_i^*] = [x_i^* \quad y_i^*] = \begin{bmatrix} \dfrac{b_1^* - b_2^*}{m_2^* - m_1^*} & \dfrac{b_1^* m_2^* - b_2^* m_1^*}{m_2^* - m_1^*} \end{bmatrix}$$

Rewriting the components of the intersection point using Eqs. (2–23) and (2–24) yields

$$[X_i^*] = [x_i^* \quad y_i^*] = \begin{bmatrix} \dfrac{a(b_1 - b_2) + c(b_1 m_2 - b_2 m_1)}{m_2 - m_1} & \dfrac{b(b_1 - b_2) + d(b_1 m_2 - b_2 m_1)}{m_2 - m_1} \end{bmatrix} \qquad (2-25)$$

Returning now to the untransformed intersection point $[x_i \quad y_i]$ and applying the same general 2 × 2 transformation we have

$$[x_i^* \quad y_i^*] = [x_i \quad y_i][T] = \begin{bmatrix} \dfrac{b_1 - b_2}{m_2 - m_1} & \dfrac{b_1 m_2 - b_2 m_1}{m_2 - m_1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{a(b_1 - b_2) + c(b_1 m_2 - b_2 m_1)}{m_2 - m_1} & \dfrac{b(b_1 - b_2) + d(b_1 m_2 - b_2 m_1)}{m_2 - m_1} \end{bmatrix} \qquad (2-26)$$

Comparing Eqs. (2–25) and (2–26) shows that they are identical. Consequently, the intersection point transforms into the intersection point.

Example 2–2 Intersecting Lines

Consider the two dashed lines AB and EF shown in Fig. 2–3 with end points

$$[A] = [-1 \quad -1] \qquad [B] = [3 \quad 5/3]$$

and

$$[E] = [-1/2 \quad 3/2] \qquad [F] = [3 \quad -2]$$

The equation of the line AB is $-(2/3)x + y = -(1/3)$ and of the line EF, $x + y = 1$. In matrix notation the pair of lines is represented by

$$[x \quad y]\begin{bmatrix} -2/3 & 1 \\ 1 & 1 \end{bmatrix} = [-1/3 \quad 1]$$

Using matrix inversion (see Eq. 2–21) the intersection of these lines is

$$[x_i \quad y_i] = [-1/3 \quad 1]\begin{bmatrix} -3/5 & -3/5 \\ 3/5 & 2/5 \end{bmatrix} = [4/5 \quad 1/5]$$

Now consider the transformation of these lines using

$$[T] = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$$

The resulting lines are shown as A^*B^* and E^*F^* in Fig. 2-3. In matrix form the equations of the transformed lines are

$$[x^* \quad y^*]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [1 \quad 1]$$

with intersection point at $[x_i^* \quad y_i^*] = [1 \quad 1]$.

Transforming the intersection point of the untransformed lines yields

$$[x_i^* \quad y_i^*] = [x_i \quad y_i][T]$$

$$= [4/5 \quad 1/5]\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} = [1 \quad 1]$$

which is identical to the intersection point of the transformed lines.

Examination of Fig. 2-3 and Ex. 2-2 shows that the original pair of untransformed dashed lines AB and EF are *not* perpendicular. However, the transformed solid lines A^*B^* and E^*F^* *are* perpendicular. Thus, the transformation $[T]$ changed a pair of intersecting nonperpendicular lines into a pair of intersecting perpendicular lines. By implication, $[T]^{-1}$, the inverse of the transformation, changes a pair of intersecting perpendicular lines into a pair of intersecting nonperpendicular lines. This effect can have disastrous geometrical consequences. It is thus of considerable interest to determine under what conditions perpendicular lines transform into perpendicular lines. We will return to this question in Sec. 2-14 when a little more background has been presented.

Additional examination of Fig. 2-3 and Ex. 2-2 shows that the transformation $[T]$ involved a rotation, a reflection and a scaling. Let's consider each of these effects individually.

2-9 ROTATION

Consider the plane triangle ABC shown in Fig. 2-4. The triangle ABC is rotated through 90° about the origin in a counterclockwise sense by the transformation

$$[T] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

If we use a 3 × 2 matrix containing the x and y coordinates of the triangle's vertices, then

$$\begin{bmatrix} 3 & -1 \\ 4 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 4 \\ -1 & 2 \end{bmatrix}$$

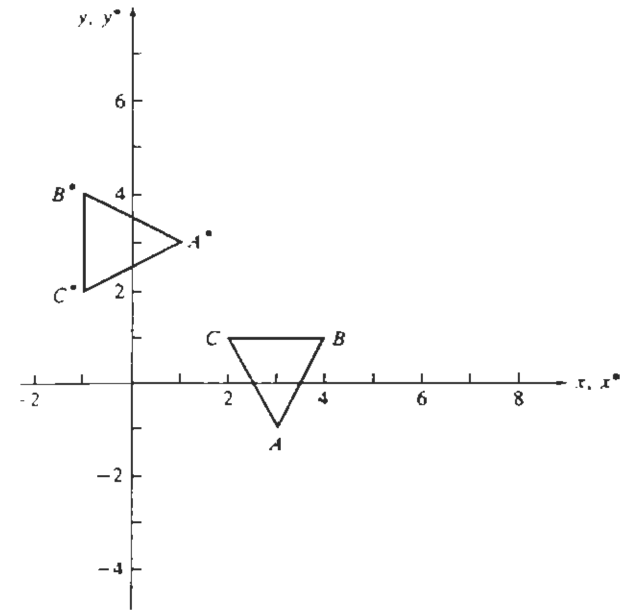

Figure 2-4 Rotation.

which produces the triangle $A^*B^*C^*$. A 180° rotation about the origin is obtained by using the transformation

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and a 270° rotation about the origin by using

$$[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Of course, the identity matrix

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

corresponds to a rotation about the origin of either 0° or 360°. Note that neither scaling nor reflection has occurred in these examples.

These example transformations produce specific rotations about the origin: 0°, 90°, 180°, 270°. What about rotation about the origin by an arbitrary angle θ? To obtain this result consider the position vector from the origin to the point P shown in Fig. 2-5. The length of the vector is r at an angle ϕ to the x-axis. The position vector P is rotated about the origin by the angle θ to P^*.

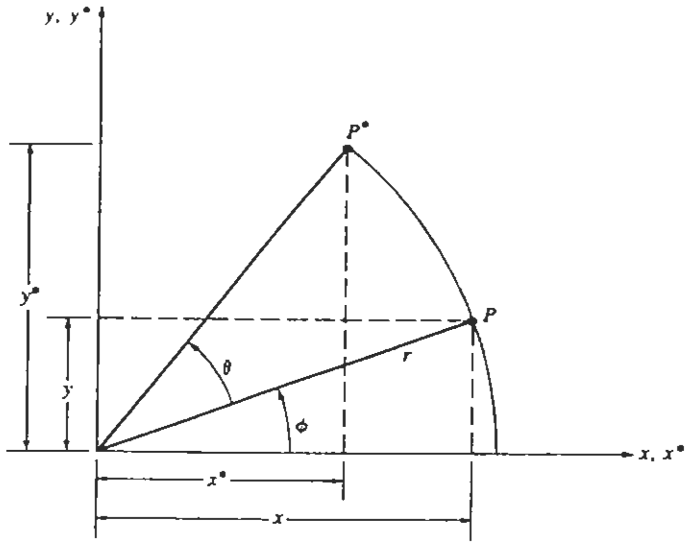

Figure 2-5 Rotation of a position vector.

Writing the position vectors for P and P^* we have

$$P = [x \quad y] = [r\cos\phi \quad r\sin\phi]$$

and
$$P^* = [x^* \quad y^*] = [r\cos(\phi+\theta) \quad r\sin(\phi+\theta)]$$

Using the sum of the angles formulas† allows writing P^* as

$$P^* = [x^* \quad y^*] = [r(\cos\phi\cos\theta - \sin\phi\sin\theta) \quad r(\cos\phi\sin\theta + \sin\phi\cos\theta)]$$

Using the definitions of x and y allows rewriting P^* as

$$P^* = [x^* \quad y^*] = [x\cos\theta - y\sin\theta \quad x\sin\theta + y\cos\theta]$$

Thus, the transformed point has components

$$x^* = x\cos\theta - y\sin\theta \qquad (2-27a)$$
$$y^* = x\sin\theta + y\cos\theta \qquad (2-27b)$$

In matrix form

$$[X^*] = [X][T] = [x^* \quad y^*]$$
$$= [x \quad y]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (2-28)$$

† $\cos(\phi \pm \theta) = \cos\phi\cos\theta \mp \sin\phi\sin\theta$
$\sin(\phi \pm \theta) = \cos\phi\sin\theta \pm \sin\phi\cos\theta$

Thus, the transformation for a general rotation about the origin by an arbitrary angle θ is

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (2-29)$$

Rotations are positive counterclockwise about the origin, as shown in Fig. 2-5.

Evaluation of the determinant of the general rotation matrix yields

$$\det[T] = \cos^2\theta + \sin^2\theta = 1 \qquad (2-30)$$

In general, transformations with a determinant identically equal to +1 yield pure rotations.

Suppose now that we wish to rotate the point P^* back to P, i.e., perform the inverse transformation. The required rotation angle is obviously $-\theta$. From Eq. (2-29) the required transformation matrix is

$$[T]^{-1} = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad (2-31)$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. $[T]^{-1}$ is a formal way of writing the inverse of $[T]$. We can show that $[T]^{-1}$ is the inverse of $[T]$ by recalling that the product of a matrix and its inverse yields the identity matrix. Here,

$$[T][T]^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

where $[I]$ is the identity matrix.

Examining Eqs. (2-29) and (2-31) reveals another interesting and useful result. Recall that the transpose of a matrix is obtained by interchanging its rows and columns. Forming the transpose of $[T]$, i.e., $[T]^T$, and comparing it with $[T]^{-1}$ shows that

$$[T]^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = [T]^{-1} \qquad (2-32)$$

The inverse of the general rotation matrix $[T]$ is its transpose. Since formally determining the inverse of a matrix is more computationally expensive than determining its transpose, Eq. (2-32) is an important and useful result. In general, the inverse of any pure rotation matrix, i.e., one with a determinant identically equal to +1, is its transpose.†

† Such matrices are said to be orthogonal.

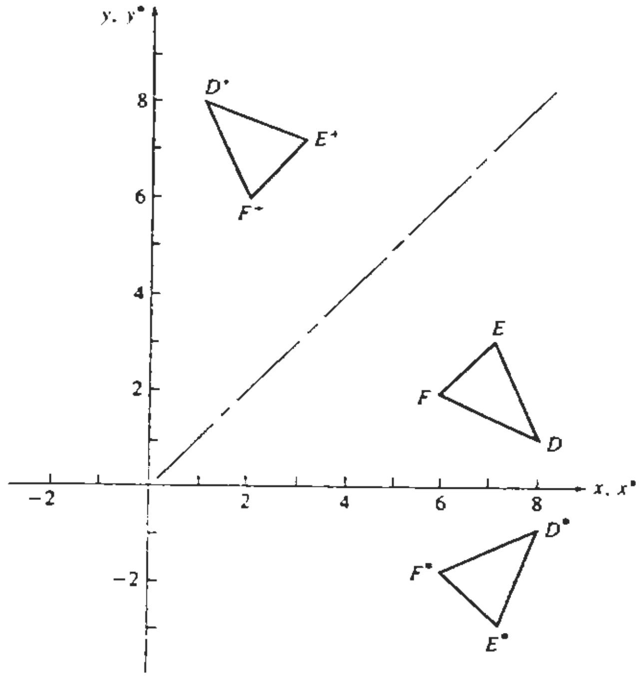

Figure 2–6 Reflection.

2–10 REFLECTION

Whereas a pure two-dimensional rotation in the xy plane occurs entirely in the two-dimensional plane about an axis normal to the xy plane, a reflection is a 180° rotation out into three space and back into two space about an axis in the xy plane. Two reflections of the triangle DEF are shown in Fig. 2–6. A reflection about $y = 0$, the x-axis, is obtained by using

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (2-33)$$

In this case the new vertices $D^*E^*F^*$ for the triangle are given by

$$\begin{bmatrix} 8 & 1 \\ 7 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ 7 & -3 \\ 6 & -2 \end{bmatrix}$$

Similarly reflection about $x = 0$, the y-axis, is given by

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (2-34)$$

A reflection about the line $y = x$ occurs for

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (2-35)$$

The transformed, new vertices $D^+E^+F^+$ are given by

$$\begin{bmatrix} 8 & 1 \\ 7 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 3 & 7 \\ 2 & 6 \end{bmatrix}$$

Similarly, a reflection about the line $y = -x$ is given by

$$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad (2-36)$$

Each of these reflection matrices has a determinant that is identically -1. In general, if the determinant of a transformation matrix is identically -1, then the transformation produces a pure reflection.

If two pure reflection transformations about lines passing through the origin are applied successively, the result is a pure rotation about the origin. To see this, consider the following example.

Example 2–3 Reflection and Rotation

Consider the triangle ABC shown in Fig. 2-7, first reflected about the x axis (see Eq. 2–33) and then about the line $y = -x$ (see Eq. 2-36). Specifically, the result of the reflection about the x-axis is

$$[X^*] = [X][T_1] = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & -2 \\ 4 & -3 \end{bmatrix}$$

Reflecting the triangle $A^*B^*C^*$ about the line $y = -x$ yields

$$[X^+] = [X^*][T_2] = \begin{bmatrix} 4 & -1 \\ 5 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -4 \end{bmatrix}$$

Rotation about the origin by an angle $\theta = 270°$ (see Eq. 2-29) yields the identical result, i.e.,

$$[X^+] = [X][T_3] = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -4 \end{bmatrix}$$

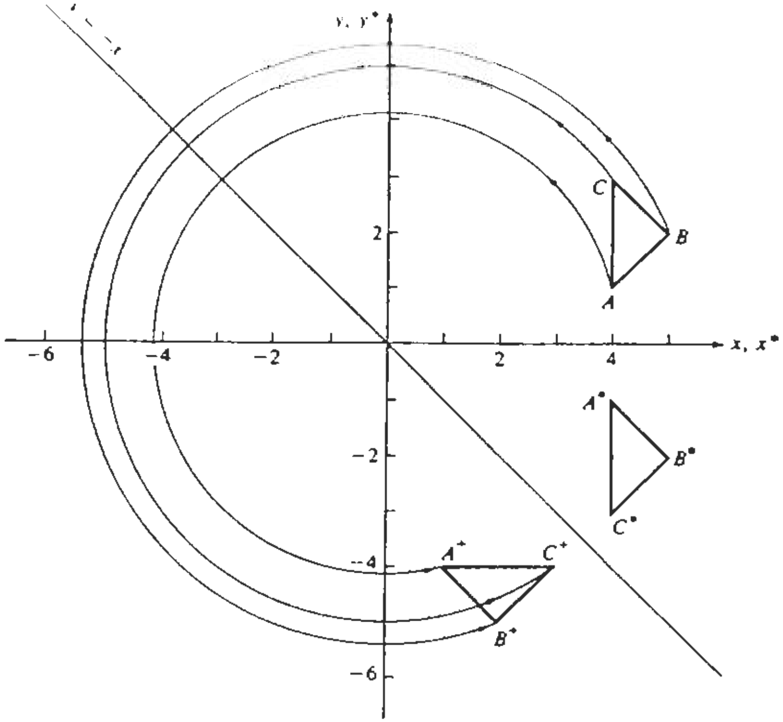

Figure 2–7 Combined reflections yield rotations.

Note that the reflection matrices given above in Eqs. (2-33) and (2-36) are orthogonal; i.e., the transpose is also the inverse. For example,

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^{-1}$$

2–11 SCALING

Recalling our discussion of the transformation of points, we see that scaling is controlled by the magnitude of the two terms on the primary diagonal of the matrix. If the matrix

$$[T] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

is used as an operator on the vertices of a triangle, a '2-times' enlargement, or uniform scaling, occurs about the origin. If the magnitudes are unequal, a distortion occurs. These effects are shown in Fig. 2-8. Triangle ABC is transformed by

$$[T] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

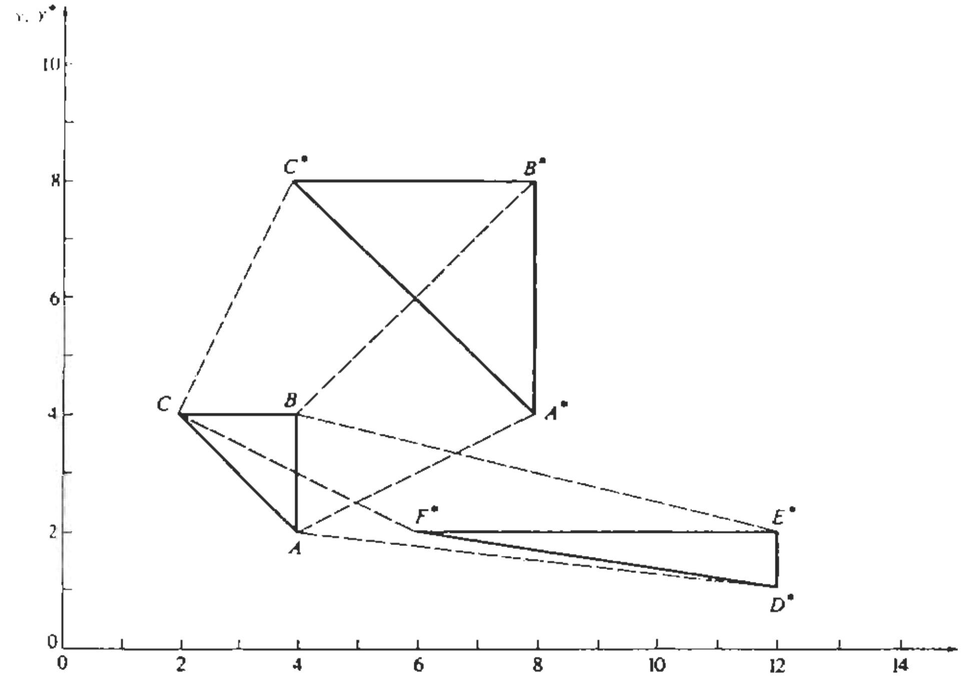

Figure 2–8 Uniform and nonuniform scaling or distortion.

to yield $A^*B^*C^*$, where a uniform scaling occurs. Transforming triangle ABC by

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix}$$

to $D^*E^*F^*$ shows distortion due to the nonuniform scale factors.

In general, if

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2-37}$$

with $a = d$, $b = c = 0$, a uniform scaling occurs; and if $a \neq d$, $b = c = 0$, a nonuniform scaling occurs. For a uniform scaling, if $a = d > 1$, a uniform expansion occurs; i.e., the figure gets larger. If $a = d < 1$, then a uniform compression occurs; i.e., the figure gets smaller. Nonuniform expansions and compressions occur, depending on whether a and d are individually > 1 or < 1.

Figure 2-8 also reveals what at first glance is an apparent translation of the transformed triangles. This apparent translation is easily understood if we recall that the ***position vectors***, not the ***points***, are scaled with respect to the origin.

To see this more clearly examine the transformation of ABC to $D^*E^*F^*$ more closely. Specifically,

$$[X^*] = [X][T] = \begin{bmatrix} 4 & 2 \\ 4 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 12 \\ 1 & 12 \end{bmatrix}$$

Note that each of the x components of the *position vectors* of DEF is increased by a scale factor of 3 and the y components of the *position vectors* by a scale factor of 2.

To obtain a pure scaling without apparent translation, the centroid of the figure must be at the origin. This effect is shown in Fig. 2-9, where the triangle ABC with the centroid coordinates (1/3 the base and 1/3 the height) at the origin is scaled by a factor of 2. Specifically,

$$[X^*] = [X][T] = \begin{bmatrix} -1 & -1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 4 & -2 \\ -2 & 4 \end{bmatrix}$$

2-12 COMBINED TRANSFORMATIONS

The power of the matrix methods described in the previous sections is clear. By performing matrix operations on the position vectors which define the vertices, the shape and position of the surface can be controlled. However, a desired orientation may require more than one transformation. Since matrix multiplication is noncommutative, the order of application of the transformations is important.

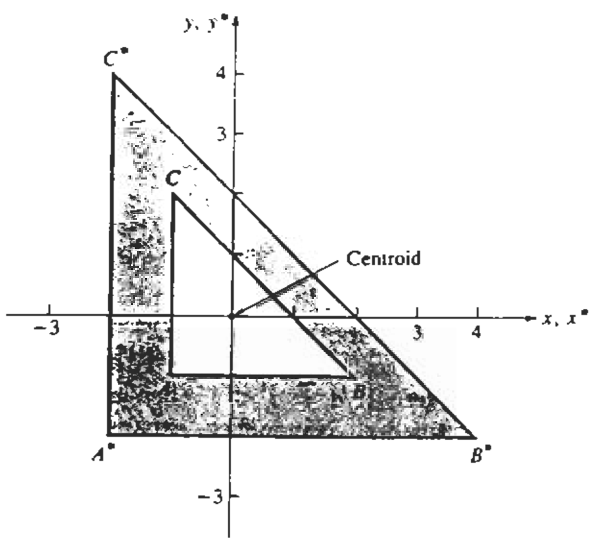

Figure 2-9 Uniform scaling without apparent translation.

In order to illustrate the effect of noncommutative matrix multiplication, consider the operations of rotation and reflection on the position vector $[x \quad y]$. If a 90° rotation, $[T_1]$, is followed by reflection through the line $y = -x$, $[T_2]$, these two consecutive transformations give

$$[X'] = [X][T_1] = [x \quad y] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [-y \quad x]$$

and then

$$[X^*] = [X'][T_2] = [-y \quad x] \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = [-x \quad y]$$

On the other hand, if reflection is followed by rotation, the results given by

$$[X'] = [X][T_2] = [x \quad y] \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = [-y \quad -x]$$

and

$$[X^*] = [X'][T_1] = [-y \quad -x] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [x \quad -y]$$

are obtained. The results are different, confirming that the order of application of matrix transformations is important.

Another important point is illustrated by the above results and by the example given below. Above, the individual transformation matrices were successively applied to the successively obtained position vectors, e.g.,

$$[x \quad y][T_1] \rightarrow [x' \quad y']$$

and

$$[x' \quad y'][T_2] \rightarrow [x^* \quad y^*]$$

In the example below the individual transformations are first combined or concatenated and then the *concatenated* transformation is applied to the original position vector, e.g., $[T_1][T_2] \rightarrow [T_3]$ and $[x \quad y][T_3] \rightarrow [x^* \quad y^*]$.

Example 2-4 Combined Two-Dimensional Transformations

Consider the triangle ABC shown in Fig. 2-10. The two transformations are a +90° rotation about the origin:

$$[T_1] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and a reflection through the line $y = -x$

$$[T_2] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

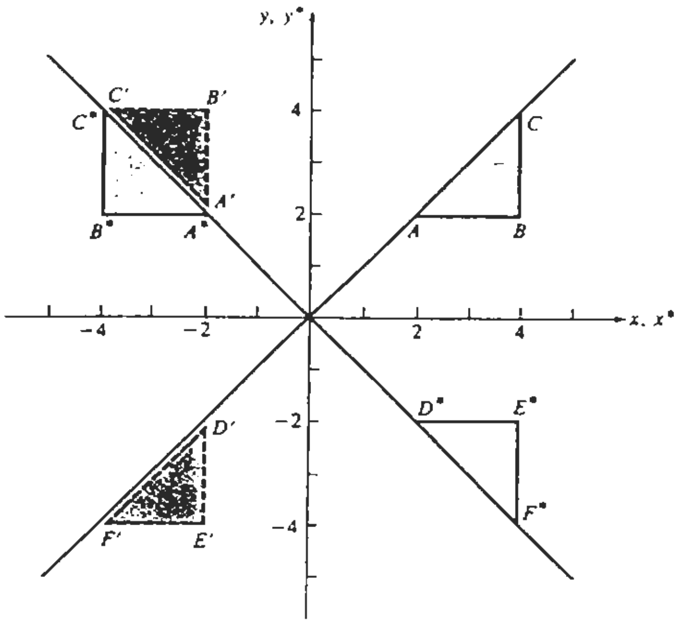

Figure 2-10 Combined two-dimensional transformations.

The effect of the combined transformation $[T_3] = [T_1][T_2]$ on the triangle ABC is

$$[X^*] = [X][T_1][T_2] = [X][T_3]$$

or

$$\begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & 2 \\ -4 & 4 \end{bmatrix}$$

The final result is shown as $A^*B^*C^*$ and the intermediate result as $A'B'C'$ in Fig. 2-10.

Reversing the order of application of the transformations yields

$$[X^*] = [X][T_2][T_1] = [X][T_4]$$

or

$$\begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \\ 4 & -4 \end{bmatrix}$$

The final result is shown as $D^*E^*F^*$ and the intermediate result as $D'E'F'$ in Fig. 2-10.

The results are different, again confirming that the order of application of the transformations is important. Note also that $\det[T_3] = -1$ and $\det[T_4] = -1$, indicating that both results can be obtained by a single reflection. $A^*B^*C^*$ can be obtained from ABC by reflection through the y-axis (see $[T_3]$ and Eq. 2-34). $D^*E^*F^*$ can be obtained from ABC by reflection through the x-axis (see $[T_4]$ and Eq. 2-33).

2 13 TRANSFORMATION OF THE UNIT SQUARE

So far we have concentrated on the behavior of points and lines to determine the effect of simple matrix transformations. However, the matrix is correctly considered to operate on *every* point in the plane. As has been shown, the only point that remains invariant under a 2 × 2 matrix transformation is the origin. All other points within the plane are transformed. This transformation may be interpreted as a stretching of the original plane and coordinate system into a new shape. More formally, we say that the transformation causes a mapping from one coordinate space into a second.

Consider a square-grid network consisting of unit squares in the xy plane as shown in Fig. 2-11. The four position vectors of a unit square with one corner at the origin of the coordinate system are

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{matrix} \text{origin of the coordinates} - A \\ \text{unit point on the } x\text{-axis} - B \\ \text{outer corner} - C \\ \text{unit point on the } y\text{-axis} - D \end{matrix}$$

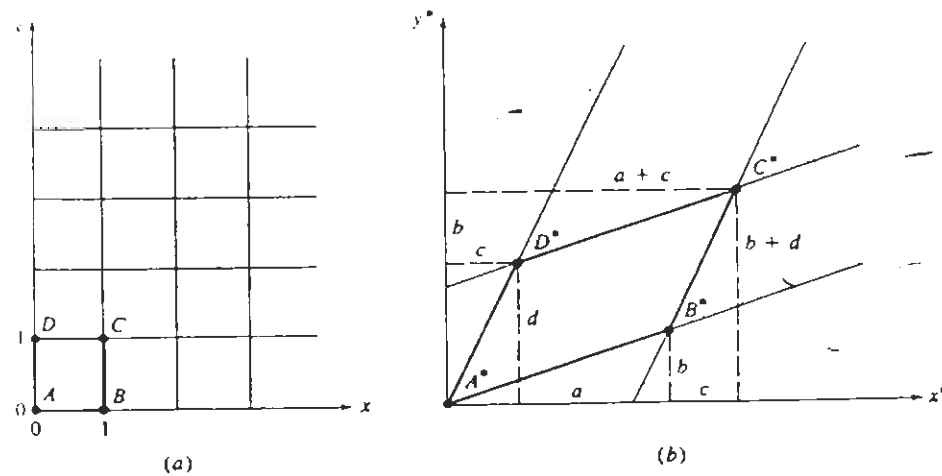

Figure 2-11 General transformation of unit square. (a) Before transformation; (b) after transformation.

This unit square is shown in Fig. 2-11a. Application of a general 2×2 matrix transformation to the unit square yields

$$\begin{matrix} A \\ B \\ C \\ D \end{matrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \\ a+c & b+d \\ c & d \end{bmatrix} \begin{matrix} A^* \\ B^* \\ C^* \\ D^* \end{matrix} \qquad (2-38)$$

The results of this transformation are shown in Fig. 2-11b. First notice from Eq. (2-38) that the origin is not affected by the transformation, i.e., $[A] = [A^*] = [0 \quad 0]$. Further, notice that the coordinates of B^* are equal to the first row in the general transformation matrix, and the coordinates of D^* are equal to the second row in the general transformation matrix. Thus, once the coordinates of B^* and D^* (the transformed unit vectors $[1 \quad 0]$ and $[0 \quad 1]$, respectively) are known, the general transformation matrix is determined. Since the sides of the unit square are originally parallel, and since we have previously shown that parallel lines transform into parallel lines, the transformed figure is a parallelogram.

The effect of the terms a, b, c and d in the 2×2 matrix can be identified separately. The terms b and c cause a shearing (see Sec. 2-4) of the initial square in the y and x directions, respectively, as can be seen in Fig. 2-11b. The terms a and d act as scale factors, as noted earlier. Thus, the general 2×2 matrix produces a combination of shearing and scaling.

It is also possible to easily determine the area of $A^*B^*C^*D^*$, the parallelogram shown in Fig. 2-11b. The area within the parallelogram can be calculated as follows:

$$A_p = (a+c)(b+d) - \frac{1}{2}(ab) - \frac{1}{2}(cd) - \frac{c}{2}(b+b+d) - \frac{b}{2}(c+a+c)$$

which yields

$$A_p = ad - bc = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad (2-39)$$

It can be shown that the area of any parallelogram A_p, formed by transforming a square, is a function of the transformation matrix determinant and is related to the area of the initial square A_s by the simple relationship

$$A_p = A_s(ad - bc) = A_s \det[T] \qquad (2-40)$$

In fact, since the area of a general figure is the sum of unit squares, the area of any transformed figure A_t is related to the area of the initial figure A_i by

$$A_t = A_i(ad - bc) \qquad (2-41)$$

This is a useful technique for determining the areas of arbitrary shapes.

Example 2-5 Area Scaling

The triangle ABC with position vectors $[1 \quad 0]$, $[0 \quad 1]$ and $[-1 \quad 0]$, is transformed by

$$[T] = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$$

to create a second triangle $A^*B^*C^*$ as shown in Fig. 2-12.

The area of the triangle ABC is

$$A_i = \frac{1}{2}(base)(height) = \frac{1}{2}(2)(1) = 1$$

Using Eq. (2-41) the area of the transformed triangle $A^*B^*C^*$ is

$$A_t = A_i(ad - bc) = 1(6+2) = 8$$

Now the vertices of the transformed triangle $A^*B^*C^*$ are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \\ -3 & -2 \end{bmatrix}$$

Calculating the area from the transformed vertices yields

$$A_t = \frac{1}{2}(base)(height) = \frac{1}{2}(4)(4) = 8$$

which confirms the previous result.

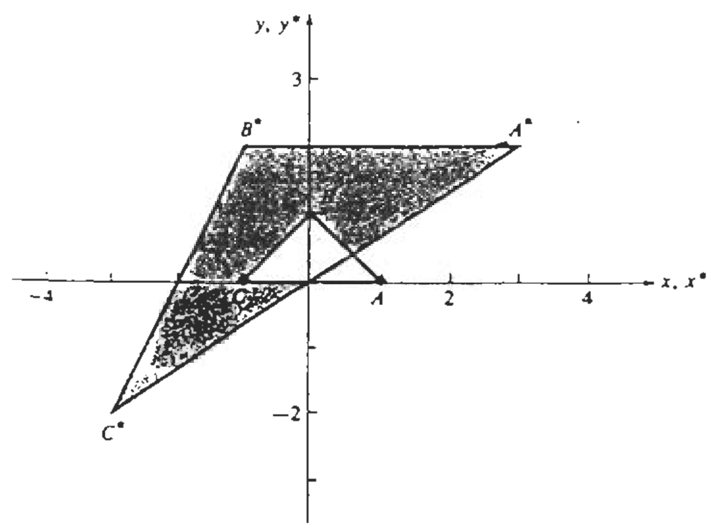

Figure 2-12 Area scaling.

2-14 SOLID BODY TRANSFORMATIONS

We now return to the question posed in Sec. 2-8, i.e., when do perpendicular lines transform as perpendicular lines? First consider the somewhat more general question of when is the angle between intersecting lines preserved?

Recall that the dot or scalar product of two vectors is

$$\bar{V}_1 \cdot \bar{V}_2 = V_{1x}V_{2x} + V_{1y}V_{2y} = |\bar{V}_1||\bar{V}_2|\cos\theta \qquad (2-42)$$

and the cross product of two vectors confined to the two-dimensional xy plane is

$$\bar{V}_1 \times \bar{V}_2 = (V_{1x}V_{2y} - V_{2x}V_{1y})\bar{k} = |\bar{V}_1||\bar{V}_2|\bar{k}\sin\theta \qquad (2-43)$$

where the subscripts x, y refer to the x and y components of the vector, θ is the acute angle between the vectors and $\bar{k}$ is the unit vector perpendicular to the xy plane.

Transforming $\bar{V}_1$ and $\bar{V}_2$ using a general 2×2 transformation yields

$$\begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} [T] = \begin{bmatrix} V_{1x} & V_{1y} \\ V_{2x} & V_{2y} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} aV_{1x} + cV_{1y} & bV_{1x} + dV_{1y} \\ aV_{2x} + cV_{2y} & bV_{2x} + dV_{2y} \end{bmatrix} = \begin{bmatrix} \bar{V}_1^* \\ \bar{V}_2^* \end{bmatrix} \qquad (2-44)$$

The cross product of $\bar{V}_1^*$ and $\bar{V}_2^*$ is

$$\bar{V}_1^* \times \bar{V}_2^* = (ad - cb)(V_{1x}V_{2y} - V_{2x}V_{1y})\bar{k} = |\bar{V}_1^*||\bar{V}_2^*|\bar{k}\sin\theta \qquad (2-45)$$

Similarly the scalar product is

$$\bar{V}_1^* \cdot \bar{V}_2^* = (a^2 + b^2)V_{1x}V_{2x} + (c^2 + d^2)V_{1y}V_{2y} + (ac + bd)(V_{1x}V_{2y} + V_{1y}V_{2x})$$
$$= |\bar{V}_1^*||\bar{V}_2^*|\cos\theta \qquad (2-46)$$

Requiring that the magnitude of the vectors, as well as the angle between them, remains unchanged, comparing Eqs. (2-42) and (2-46) and Eqs. (2-43) and (2-45) and equating coefficients of like terms yields

$$a^2 + b^2 = 1 \qquad (2-47a)$$
$$c^2 + d^2 = 1 \qquad (2-47b)$$
$$ac + bd = 0 \qquad (2-47c)$$

$$ad - bc = +1 \qquad (2-48)$$

Equations (2-47a, b, c) correspond to the conditions that a matrix be orthogonal, i.e.,

$$[T][T]^{-1} = [T][T]^T = [I]$$

or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equation (2-48) requires that the determinant of the transformation matrix be +1.

Thus, the angles between intersecting lines are preserved by pure rotation. Since reflective transformations are also orthogonal with a determinant of -1, these results are easily extended. In this case the magnitude of the vectors is preserved, but the angle between the transformed vectors is technically $2\pi - \theta$. Hence, the angle is technically not preserved. Still, perpendicular lines transform as perpendicular lines. Since $\sin(2\pi - \theta) = -\sin\theta$, $ad - bc = -1$. Pure rotations and reflections are called rigid body transformations. In addition, a few minutes' thought or experimentation reveals that uniform scalings also preserve the angle between intersecting lines but not the magnitudes of the transformed vectors.†

2-15 TRANSLATIONS AND HOMOGENEOUS COORDINATES

A number of transformations governed by the general 2×2 transformation matrix, e.g., rotation, reflection, scaling, shearing etc., were discussed in the previous sections. As noted previously, the origin of the coordinate system is invariant with respect to all of these transformations. However, it is necessary to be able to modify the position of the origin, i.e., to transform every point in the two-dimensional plane. This can be accomplished by translating the origin or any other point in the two-dimensional plane, i.e.,

$$x^* = ax + cy + m$$
$$y^* = bx + dy + n$$

Unfortunately, it is not possible to introduce the constants of translation m, n into the general 2×2 transformation matrix; there is no room!

This difficulty can be overcome by introducing homogeneous coordinates. The homogeneous coordinates of a nonhomogeneous position vector $[x \quad y]$ are $[x' \quad y' \quad h]$ where $x = x'/h$ and $y = y'/h$ and h is any real number. Note that $h = 0$ has special meaning. One set of homogeneous coordinates is always of the form $[x \quad y \quad 1]$. We choose this form to represent the position vector $[x \quad y]$ in the physical xy plane. All other homogeneous coordinates are of the form $[hx \quad hy \quad h]$. There is no unique homogeneous coordinate representation, e.g., $[6 \quad 4 \quad 2]$, $[12 \quad 8 \quad 4]$, $[3 \quad 2 \quad 1]$ all represent the physical point $(3, 2)$.

The general transformation matrix is now 3×3. Specifically,

$$[T] = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{bmatrix} \qquad (2-49)$$

†Since an orthogonal matrix preserves *both* the angle between the vectors and their magnitudes, the uniform scaling transformation matrix is *not* orthogonal.

where the elements a, b, c, d of the upper left 2×2 submatrix have exactly the same effects revealed by our previous discussions. m, n are the translation factors in the x and y directions, respectively. The pure two-dimensional translation matrix is

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix} = [x+m \quad y+n \quad 1] \qquad (2-50)$$

Notice that now every point in the two-dimensional plane, even the origin ($x = y = 0$), can be transformed.

2–16 ROTATION ABOUT AN ARBITRARY POINT

Previously we have considered rotations as occurring about the origin. Homogeneous coordinates provide a mechanism for accomplishing rotations about points other than the origin. In general, a rotation about an arbitrary point can be accomplished by first translating the point to the origin, performing the required rotation, and then translating the result back to the original center of rotation. Thus, rotation of the position vector $[x \quad y \quad 1]$ about the point m, n through an arbitrary angle can be accomplished by

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & -n & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix} \qquad (2-51)$$

By carrying out the two interior matrix products we can write

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ \left\{\begin{matrix} -m(\cos\theta - 1) \\ +n\sin\theta \end{matrix}\right\} & \left\{\begin{matrix} -n(\cos\theta - 1) \\ -m\sin\theta \end{matrix}\right\} & 1 \end{bmatrix} \qquad (2-52)$$

An example illustrates this result.

Example 2–6 Rotation About an Arbitrary Point.

Suppose the center of an object is at $[4 \quad 3]$ and it is desired to rotate the object 90° counterclockwise about its center. Using the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

causes a rotation about the origin, not the object center. The necessary procedure is to first translate the object so that the desired center of rotation is at the origin by using the translation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}$$

Next apply the rotation matrix, and finally translate the results of the rotation back to the original center by means of the inverse translation matrix. The entire operation

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

can be combined into one matrix operation by concatenating the transformation matrices, i.e.,

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 7 & -1 & 1 \end{bmatrix}$$

2–17 REFLECTION THROUGH AN ARBITRARY LINE

Previously (see Sec. 2–10) reflection through lines that passed through the origin was discussed. Occasionally reflection of an object through a line that does not pass through the origin is required. This can be accomplished using a procedure similar to that for rotation about an arbitrary point. Specifically,

Translate the line and the object so that the line passes through the origin.

Rotate the line and the object about the origin until the line is coincident with one of the coordinate axes.

Reflect through the coordinate axis.

Apply the inverse rotation about the origin.

Translate back to the original location.

In matrix notation the resulting concatenated matrix is

$$[T] = [T'][R][R'][R]^{-1}[T']^{-1} \qquad (2-53)$$

where

T' is the translation matrix
R is the rotation matrix about the origin
R' is the reflection matrix

The translations, rotations and reflections are also applied to the figure to be transformed. An example is given below.

Example 2–7 Reflection Through an Arbitrary Line

Consider the line L and the triangle ABC shown in Fig. 2–13a. The equation of the line L is

$$y = \frac{1}{2}(x + 4)$$

The position vectors $[2\ \ 4\ \ 1]$, $[4\ \ 6\ \ 1]$ and $[2\ \ 6\ \ 1]$ describe the vertices of the triangle ABC.

The line L will pass through the origin by translating it -2 units in the y direction. The resulting line can be made coincident with the x-axis by rotating it by $-\tan^{-1}(\frac{1}{2}) = -26.57°$ about the origin. Equation (2–33) is then used to reflect the triangle through the x-axis. The transformed position vectors of the triangle are then rotated and translated back to the original orientation. The combined transformation is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ -8/5 & 16/5 & 1 \end{bmatrix}$$

and the transformed position vectors for the triangle $A^*B^*C^*$ are

$$\begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 1 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ -8/5 & 16/5 & 1 \end{bmatrix} = \begin{bmatrix} 14/5 & 12/5 & 1 \\ 28/5 & 14/5 & 1 \\ 22/5 & 6/5 & 1 \end{bmatrix}$$

as shown in Fig. 2–13a. Figures 2–13b through 2–13e show the various steps in the transformation.

2–18 PROJECTION – A GEOMETRIC INTERPRETATION OF HOMOGENEOUS COORDINATES

The general 3×3 transformation matrix for two-dimensional homogeneous coordinates can be subdivided into four parts:

$$[T] = \begin{bmatrix} a & b & \vdots & p \\ c & d & \vdots & q \\ \cdots & \cdots & \cdots & \cdots \\ m & n & \vdots & s \end{bmatrix} \qquad (2-54)$$

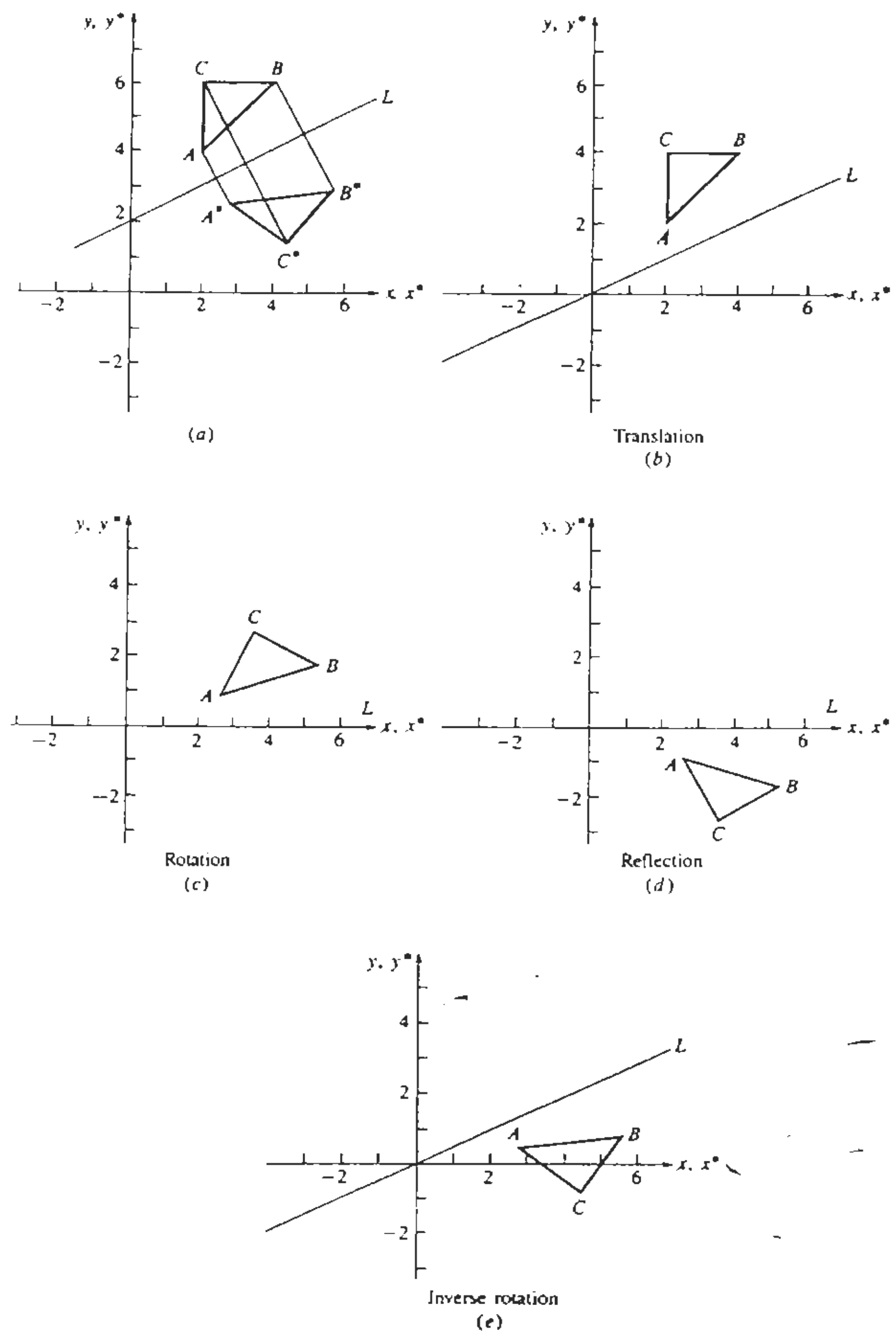

Figure 2–13 Reflection through an arbitrary line. (a) Original and final position; (b) translate line through origin; (c) rotate line to x-axis; (d) reflect about x-axis; (e) undo rotation; (a) undo translation.

Recall that a, b, c and d produce scaling, rotation, reflection and shearing; and m and n produce translation. In the previous two sections $p = q = 0$ and $s = 1$. Suppose p and q are not zero. What are the effects? A geometric interpretation is useful.

When $p = q = 0$ and $s = 1$, the homogeneous coordinate of the transformed position vectors is always $h = 1$. Geometrically this result is interpreted as confining the transformation to the $h = 1$ physical plane.

To show the effect of $p \neq 0$, $q \neq 0$ in the third column in the general 3×3 transformation matrix, consider the following:

$$[X \quad Y \quad h] = [hx \quad hy \quad h] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [x \quad y \quad (px + qy + 1)] \qquad (2-55)$$

Here $X = hx$, $Y = hy$ and $h = px + qy + 1$. The transformed position vector expressed in homogeneous coordinates now lies in a plane in three-dimensional space defined by $h = px + qy + 1$. This transformation is shown in Fig. 2-14, where the line AB in the physical ($h = 1$) plane is transformed to the line CD in the $h \neq 1$ plane, i.e., $pX + qY - h + 1 = 0$.

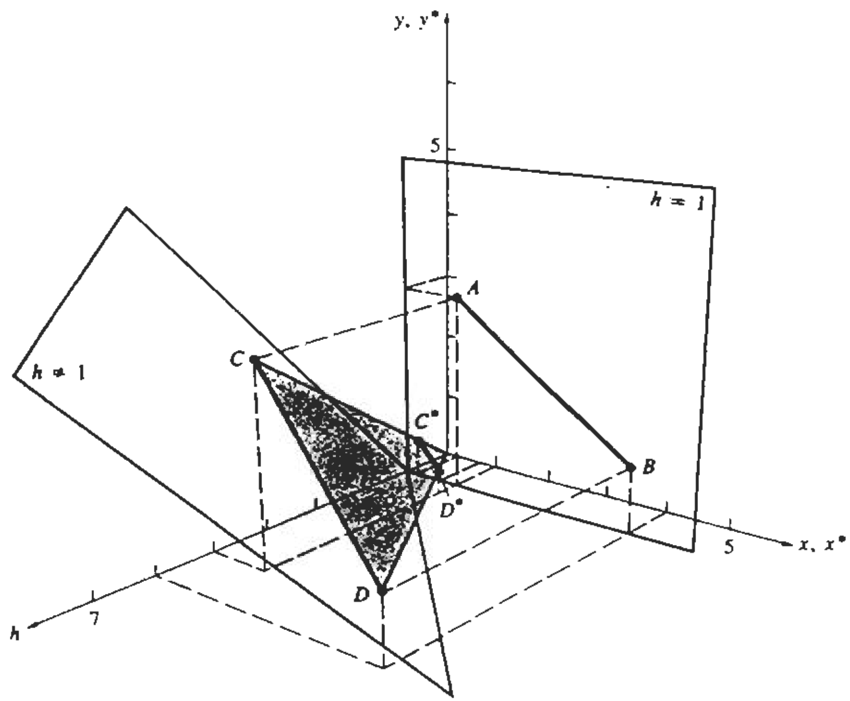

Figure 2–14 Transformation from the physical ($h = 1$) plane into the $h \neq 1$ plane and projection from the $h \neq 1$ plane back into the physical plane.

However, the results of interest are those in the physical plane corresponding to $h = 1$. These results can be obtained by geometrically projecting CD from the $h \neq 1$ plane back onto the $h = 1$ plane using a pencil of rays through the origin. From Fig. 2-14, using similar triangles,

$$x^* = \frac{X}{h} \qquad y^* = \frac{Y}{h}$$

or in homogeneous coordinates

$$[x^* \quad y^* \quad 1] = \left[\frac{X}{h} \quad \frac{Y}{h} \quad 1\right]$$

Now, normalizing Eq. (2-55) by dividing through by the homogeneous coordinate value h yields

$$[x^* \quad y^* \quad 1] = \left[\frac{X}{h} \quad \frac{Y}{h} \quad 1\right] = \left[\frac{x}{px + qy + 1} \quad \frac{y}{px + qy + 1} \quad 1\right] \qquad (2-56)$$

or

$$x^* = \frac{X}{h} = \frac{x}{px + qy + 1} \qquad (2-57a)$$

$$y^* = \frac{Y}{h} = \frac{y}{px + qy + 1} \qquad (2-57b)$$

The details are given in the example below.

Example 2–8 Projection in Homogeneous Coordinates

For the line AB in Fig. 2-14 we have, with $p = q = 1$, $[A] = [1 \quad 3 \quad 1]$ and $[B] = [4 \quad 1 \quad 1]$,

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}[T] = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 1 & 6 \end{bmatrix}$$

Thus, $[C] = [1 \quad 3 \quad 5]$ and $[D] = [4 \quad 1 \quad 6]$ in the plane $h = x + y + 1$. Projecting back onto the $h = 1$ physical plane by dividing through by the homogeneous coordinate factor yields the two-dimensional transformed points

$$[C^*] = [1 \quad 3 \quad 5] = [1/5 \quad 3/5 \quad 1]$$

$$[D^*] = [4 \quad 1 \quad 6] = [2/3 \quad 1/6 \quad 1]$$

The result is shown in Fig. 2-14.

2-19 OVERALL SCALING

The remaining unexplained element in the general 3 × 3 transformation matrix (see Eq. 2-54), s, produces overall scaling; i.e., all components of the position vector are equally scaled. To show this, consider the transformation

$$[X \quad Y \quad h] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix} = [x \quad y \quad s] \qquad (2-58)$$

Here, $X = x$, $Y = y$ and $h = s$. After normalizing, this yields

$$X^* = \frac{x}{s} \quad \text{and} \quad Y^* = \frac{y}{s}$$

Thus, the transformation is $[x \quad y \quad 1][T] = [\frac{x}{s} \quad \frac{y}{s} \quad 1]$, a uniform scaling of the position vector. If $s < 1$, then an expansion occurs; and if $s > 1$, a compression occurs.

Note that this is also a transformation out of the $h = 1$ plane. Here, $h = s =$ constant. Hence, the $h \neq 1$ plane is parallel to the $h = 1$ plane. A geometric interpretation of this effect is shown in Fig. 2-15. If $s < 1$, then the $h =$ constant plane lies between the $h = 1$ and $h = 0$ planes. Consequently, when the transformed line AB is projected back onto the $h = 1$ plane to A^*B^*, it becomes larger. Similarly, if $s > 1$, then the $h =$ constant plane lies beyond the $h = 1$ plane along the h-axis. When the transformed line CD is projected back onto the $h = 1$ plane to C^*D^*, it becomes smaller.

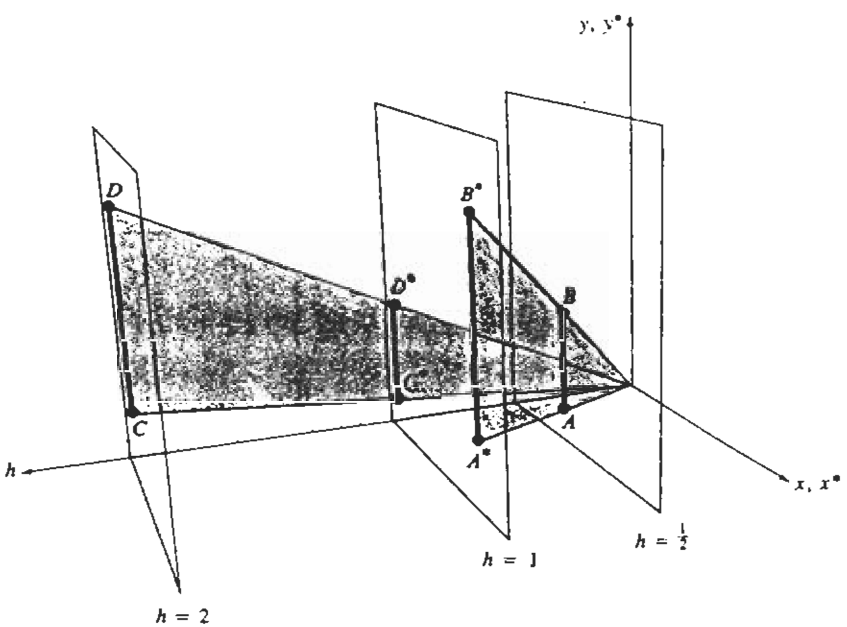

Figure 2-15 A geometric interpretation of overall scaling.

2-20 POINTS AT INFINITY

Homogeneous coordinates provide a convenient and efficient technique for mapping a set of points from one coordinate system into a corresponding set in an alternate coordinate system. Frequently, an infinite range in one coordinate system is mapped into a finite range in an alternate coordinate system. Unless the mappings are carefully chosen, parallel lines may not map into parallel lines. However, intersection points map into intersection points. This property is used to determine the homogeneous coordinate representation of a point at infinity.

We begin by considering the pair of intersecting lines given by

$$x + y = 1$$
$$2x - 3y = 0$$

which have an intersection point at $x = 3/5$, $y = 2/5$. Writing the equations as $x + y - 1 = 0$ and $2x - 3y = 0$ and casting them in matrix form yields

$$[x \quad y \quad 1]\begin{bmatrix} 1 & 2 \\ 1 & -3 \\ -1 & 0 \end{bmatrix} = [0 \quad 0]$$

or

$$[X][M'] = [R]$$

If $[M']$ were square, the intersection could be obtained by matrix inversion. This can be accomplished by slightly rewriting the system of original equations. Specifically,

$$x + y - 1 = 0$$
$$2x - 3y = 0$$
$$1 = 1$$

In matrix form this is

$$[X][M] = [R]$$

i.e.,

$$[x \quad y \quad 1]\begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \\ -1 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 1]$$

The inverse of this square matrix is†

$$[M]^{-1} = \begin{bmatrix} 3/5 & 2/5 & 0 \\ 1/5 & -1/5 & 0 \\ 3/5 & 2/5 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix}$$

† Matrix inversion techniques are discussed in Ref. 2-1 or any good linear algebra book.

Multiplying both sides of the equation by $[M]^{-1}$ and noting that $[M][M]^{-1} = [I]$, the identity matrix, yields

$$[x \quad y \quad 1] = \frac{1}{5}[0 \quad 0 \quad 1]\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix} = [3/5 \quad 2/5 \quad 1]$$

Thus, the intersection point is again $x = 3/5$ and $y = 2/5$.

Now consider two parallel lines defined by

$$x + y = 1$$
$$x + y = 0$$

By definition, in Euclidean (common) geometric space, the intersection point of this pair of parallel lines occurs at infinity. Proceeding, as above, to calculate the intersection point of these lines leads to the matrix formulation

$$[x \quad y \quad 1]\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 1]$$

However, even though the matrix is square it does not have an inverse, since two rows are identical. The matrix is said to be singular. Another alternate formulation is possible which does have an invertible matrix. This is obtained by rewriting the system of equations as

$$x + y - 1 = 0$$
$$x + y = 0$$
$$x = x$$

In matrix form this is

$$[x \quad y \quad 1]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = [0 \quad 0 \quad x]$$

Here, the matrix is not singular; the inverse exists and is

$$[M]^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Multiplying both sides of the equation by the inverse yields

$$[x \quad y \quad 1] = [0 \quad 0 \quad x]\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = [x \quad -x \quad 0] = x[1 \quad -1 \quad 0]$$

The resulting homogeneous coordinates $[1 \quad -1 \quad 0]$ represent the 'point of intersection' for the two parallel lines, i.e., a point at infinity. Specifically it represents the point at infinity in the direction $[1 \quad -1]$ in the two-dimensional plane. In general, the two-dimensional homogeneous vector $[a \quad b \quad 0]$ represents the point at infinity on the line $ay - bx = 0$. Some examples are:

$[1 \quad 0 \quad 0]$ on the positive x-axis
$[-1 \quad 0 \quad 0]$ on the negative x-axis
$[0 \quad 1 \quad 0]$ on the positive y-axis
$[0 \quad -1 \quad 0]$ on the negative y-axis
$[1 \quad 1 \quad 0]$ along the line $y = x$ in the direction $[1 \quad 1]$

The fact that a vector with the homogeneous component $h = 0$ does indeed represent a point at infinity can also be illustrated by the limiting process shown in Table 2–1. Consider the line $y^* = (3/4)x^*$ and the point $[X \quad Y \quad h] = [4 \quad 3 \quad 1]$. Recalling that a unique representation of a position vector does not exist in homogeneous coordinates, the point $[4 \quad 3 \quad 1]$ is represented in homogeneous coordinates in all the ways shown in Table 2–1. Note that in Table 2–1 as $h \to 0$, the ratio of y^*/x^* remains at 3/4, as is required by the governing equation. Further, note that successive pairs of (x^*, y^*) all of which fall on the line $y^* = (3/4)x^*$, become closer to infinity. Thus, in the limit as $h \to 0$, the point at infinity is given by $[X \quad Y \quad h] = [4 \quad 3 \quad 0]$ in homogeneous coordinates.

By recalling Fig. 2–15, a geometrical interpretation of the limiting process as $h \to 0$ is also easily illustrated. Consider a line of unit length from $x = 0$, $y = 0$ in the direction $[1 \quad 0]$, in the plane $h = s$ $(s < 1)$. As $s \to 0$ the projection of this line back onto the $h = 1$ physical plane by a pencil of rays through the origin becomes of infinite length. Consequently, the end point of the line must represent the point at infinity on the x-axis.

Table 2–1 Homogeneous Coordinates for the Point [4 3]

h	x^*	y^*	X	Y
1	4	3	4	3
1/2	8	6	4	3
1/3	12	9	4	3
.				
.				
.				
1/10	40	30	4	3
.				
.				
.				
1/100	400	300	4	3
.				
.				
.				

2-21 TRANSFORMATION CONVENTIONS

Various conventions are used to represent data and to perform transformations with matrix multiplication. Extreme care is necessary in defining the problem and interpreting the results. For example, before performing a rotation the following decisions must be made:

Are the position vectors (vertices) to be rotated defined relative to a right-hand coordinate or a left-hand coordinate system?

Is the object or the coordinate system being rotated?

How are positive and negative rotations defined?

Are the position vectors stored as a row matrix or as a column matrix?

About what line, or axis, is rotation to occur?

In this text a right-hand coordinate system is used, the object is rotated in a fixed coordinate system, positive rotation is defined using the right-hand rule, i.e., clockwise about an axis as seen by an observer at the origin looking outward along the positive axis, and position vectors are represented as row matrices.

Equation (2-29) gives the transformation for positive rotation about the origin or about the z-axis. Since position vectors are represented as row matrices, the transformation matrix appears *after* the data or position vector matrix. This is a post-multiplication transformation. Using homogeneous coordinates for positive rotation by an angle θ of an object about the origin (z-axis) using a post-multiplication transformation gives

$$[X^*] = [X][R]$$

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2-59)$$

If we choose to represent the position vectors in homogeneous coordinates as a column matrix, then the same rotation is performed using

$$[X^*] = [R]^{-1}[X]$$

$$\begin{bmatrix} x^* \\ y^* \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad (2-60)$$

Equation (2-60) is called a premultiplication transformation because the transformation matrix appears *before* the column position vector or data matrix. Notice that the 3×3 matrix in Eq. (2-60) is also the transpose of the 3×3 matrix in Eq. (2-59). That is, the rows and columns have been interchanged.

To rotate the coordinate system and keep the position vectors fixed, simply replace θ with $-\theta$ in Eq. (2-59). Recall that $\sin\theta = -\sin(-\theta)$ and $\cos\theta = \cos(-\theta)$. Equation (2-59) is then

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2-61)$$

Notice that the 3×3 matrix is again the inverse and also the transpose of that in Eq. (2-59).

If the coordinate system is rotated *and* a left-hand coordinate system used, then the replacement of θ with $-\theta$ is made *twice* and Eq. (2-59) is again valid, assuming a post-multiplication transformation is used on a row data matrix.

Note that, as shown in Fig. 2-16, a counterclockwise rotation of the vertices which represent an object is identical to a clockwise rotation of the coordinate axes for a fixed object. Again, no change occurs in the 3×3 transformation

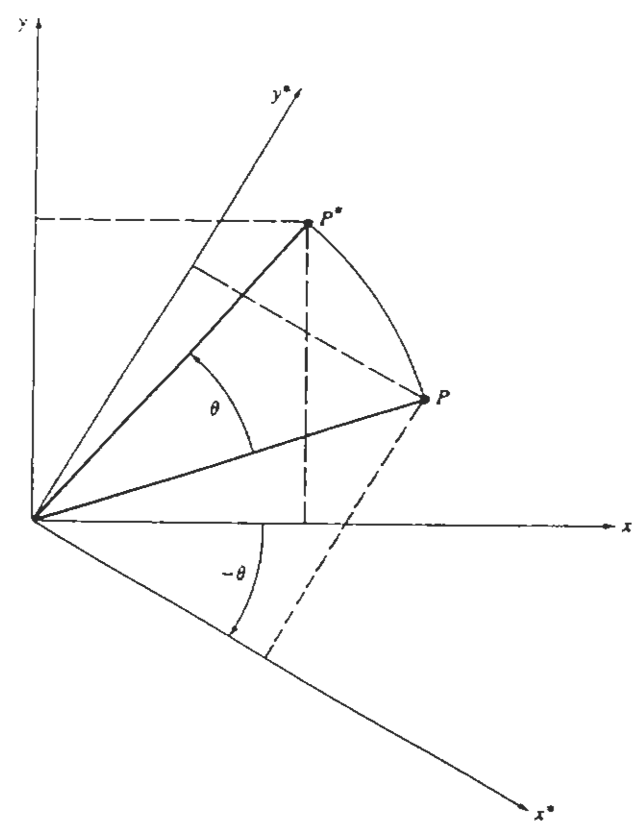

Figure 2-16 Equivalence of position vector and coordinate system rotation.

matrix *if* no other options are modified. These few examples show how careful we must be when performing matrix transformations.

2–22 REFERENCES

2-1 Fox, L., *An Introduction to Numerical Linear Algebra*, Oxford University Press, London, 1964.

2-2 Forrest, A. R., "Co-ordinates, Transformations, and Visualization Techniques," CAD Group Document No. 23, Cambridge University, June 1969.

CHAPTER

THREE

THREE-DIMENSIONAL TRANSFORMATIONS AND PROJECTIONS

3–1 INTRODUCTION

The ability to represent or display a three-dimensional object is fundamental to the understanding of the shape of that object. Furthermore, the ability to rotate, translate, and project views of that object is also, in many cases, fundamental to the understanding of its shape. This is easily demonstrated by picking up a relatively complex unfamiliar object. Immediately it is rotated, held at arm's length, moved up and down, back and forth, etc., in order to obtain an understanding of its shape. To do this with a computer we must extend our previous two-dimensional analysis to three dimensions. Based on our previous experience, we immediately introduce homogeneous coordinates. Hence, a point in three-dimensional space $[x \quad y \quad z]$ is represented by a four-dimensional position vector

$$[x' \quad y' \quad z' \quad h] = [x \quad y \quad z \quad 1][T]$$

where $[T]$ is some transformation matrix. Again, the transformation from homogeneous coordinates to ordinary coordinates is given by

$$[x^* \quad y^* \quad z^* \quad 1] = \begin{bmatrix} \frac{x'}{h} & \frac{y'}{h} & \frac{z'}{h} & 1 \end{bmatrix} \qquad (3-1)$$

The generalized 4×4 transformation matrix for three-dimensional homogeneous coordinates is

$$[T] = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix} \qquad (3-2)$$

matrix *if* no other options are modified. These few examples show how careful we must be when performing matrix transformations.

2–22 REFERENCES

2-1 Fox, L., *An Introduction to Numerical Linear Algebra*, Oxford University Press, London, 1964.
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$$[\,x' \quad y' \quad z' \quad h\,] = [\,x \quad y \quad z \quad 1\,][\,T\,]$$

where $[\,T\,]$ is some transformation matrix. Again, the transformation from homogeneous coordinates to ordinary coordinates is given by

$$[\,x^* \quad y^* \quad z^* \quad 1\,] = \begin{bmatrix} \dfrac{x'}{h} & \dfrac{y'}{h} & \dfrac{z'}{h} & 1 \end{bmatrix} \qquad (3-1)$$

The generalized 4×4 transformation matrix for three-dimensional homogeneous coordinates is

$$[\,T\,] = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix} \qquad (3-2)$$

The 4 × 4 transformation matrix in Eq. (3–2) can be partitioned into four separate sections:

$$\left[\begin{array}{ccc:c} & 3 \times 3 & & 3 \\ & & & \times \\ & & & 1 \\ \hdashline & 1 \times 3 & & 1 \times 1 \end{array}\right]$$

The upper left 3 × 3 submatrix produces a linear transformation† in the form of scaling, shearing, reflection and rotation. The 1 × 3 lower left submatrix produces translation, and the upper right 3 × 1 submatrix produces a perspective transformation. The final lower right-hand 1 × 1 submatrix produces overall scaling. The total transformation obtained after operating on a homogeneous position vector with this 4 × 4 matrix and obtaining the ordinary coordinate is called a bilinear transformation.‡ In general, this transformation yields a combination of shearing, local scaling, rotation, reflection, translation, perspective and overall scaling.

3–2 THREE-DIMENSIONAL SCALING

The diagonal terms of the general 4 × 4 transformation produce local and overall scaling. To illustrate this, consider the transformation

$$[X][T] = [x \quad y \quad z \quad 1]\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [ax \quad ey \quad jz \quad 1] = [x^* \quad y^* \quad z^* \quad 1] \qquad (3-3)$$

which shows the local scaling effect. An example follows.

Example 3–1 Local Scaling

Consider the rectangular parallelepiped (RPP) shown in Fig. 3–1a with homogeneous position vectors:

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

†A linear transformation is one which transforms an initial linear combination of vectors into the same linear combination of transformed vectors.

‡A bilinear transformation results from two sequential linear transformations.

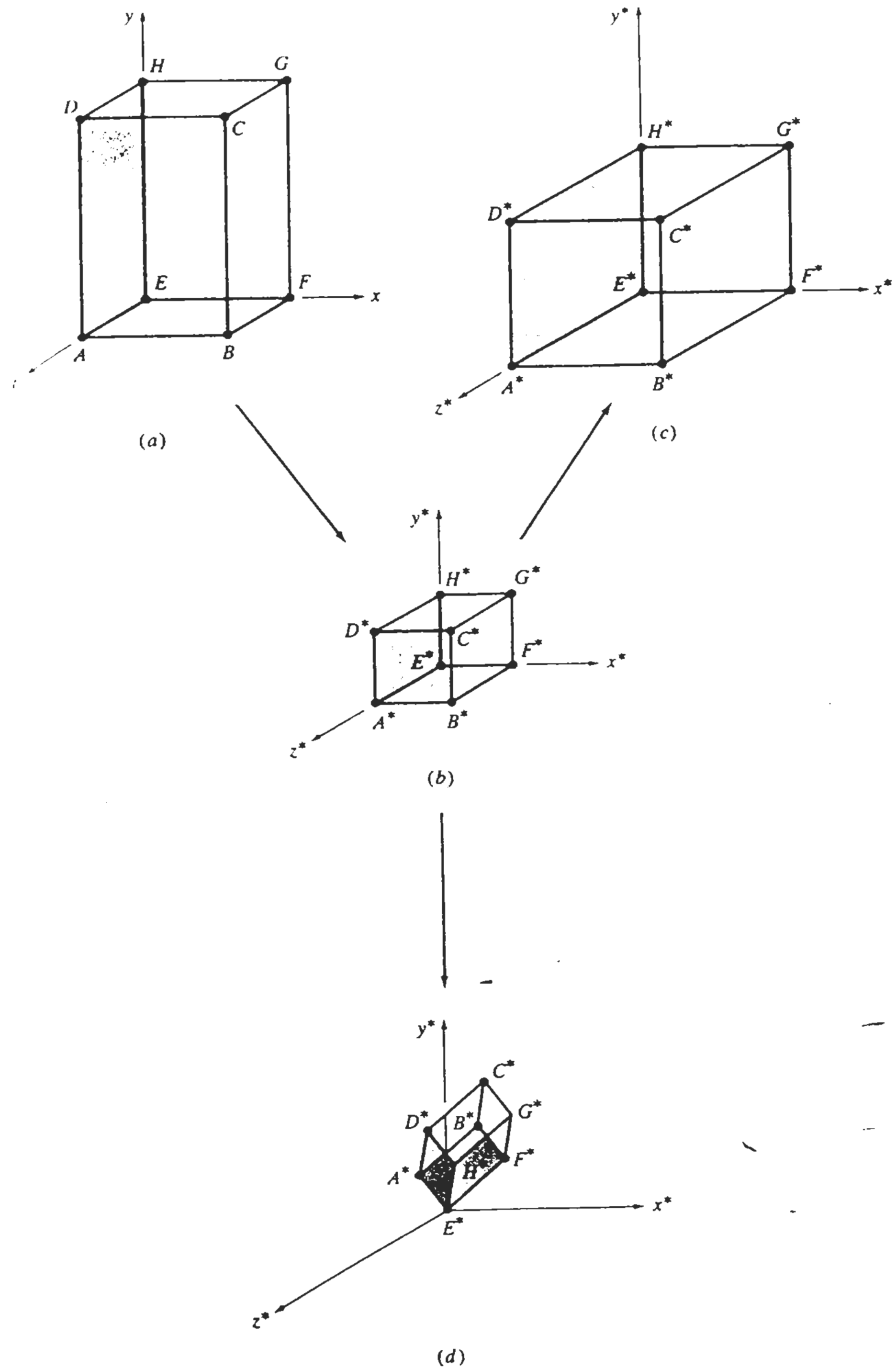


Figure 3–1 Three-dimensional scale transformations.

Locally scaling the RPP to yield a unit cube requires scale factors of 1/2, 1/3, 1 along the x, y, z axes, respectively. The local scaling transformation is

$$[T] = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The resulting cube has homogeneous position vectors

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Notice that the homogeneous coordinate factor h is unity for each of the transformed position vectors. The result is shown in Fig. 3–1b.

Overall scaling is obtained by using the fourth diagonal element, i.e.,

$$[X][T] = [x \quad y \quad z \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} = [x' \quad y' \quad z' \quad s] \qquad (3-4)$$

The ordinary or physical coordinates are

$$[x^* \quad y^* \quad z^* \quad 1] = \left[\frac{x'}{s} \quad \frac{y'}{s} \quad \frac{z'}{s} \quad 1\right]$$

Again, an example illustrates the effect.

Example 3–2 Overall Scaling

Uniformly scaling the unit cube shown in Fig. 3–1b by a factor of two (doubling the size) requires the transformation (see Eq. 3–4)

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

The resulting RPP has homogeneous position vectors given by

$$[X'] = [X^*][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0.5 \\ 1 & 0 & 1 & 0.5 \\ 1 & 1 & 1 & 0.5 \\ 0 & 1 & 1 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0.5 \\ 1 & 1 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 \end{bmatrix}$$

Notice that the homogeneous coordinate factor for each of the transformed position vectors is $h = 0.5$. Thus, to obtain the ordinary or physical coordinates each position vector must be divided by h. The result, shown in Fig. 3–1c, is

$$[X^*] = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

Notice here, as in the two-dimensional overall scaling transformation, that the homogeneous coordinate factor is not unity. By analogy with the previous discussion (see Sec. 2–18) this represents transformation out of the physical $h = 1$ volume into another volume in 4-space. The transformed physical coordinates are obtained by projecting back into the physical $h = 1$ volume through the center of the 4-space coordinate system. Again, if $s < 1$, a uniform expansion of the position vectors occurs. If $s > 1$, a uniform compression of the position vectors occurs.

The same effect can be obtained by means of equal local scalings. In this case the transformation matrix is

$$[T] = \begin{bmatrix} 1/s & 0 & 0 & 0 \\ 0 & 1/s & 0 & 0 \\ 0 & 0 & 1/s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that here the homogeneous coordinate factor is unity, i.e., $h = 1$. Thus, the entire transformation takes place in the $h = 1$ physical volume.

3–3 THREE-DIMENSIONAL SHEARING

The off-diagonal terms in the upper left 3×3 submatrix of the generalized 4×4 transformation matrix produce shear in three dimensions, i.e.,

$$[X][T] = [x \quad y \quad z \quad 1] \begin{bmatrix} 1 & b & c & 0 \\ d & 1 & f & 0 \\ g & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [x + yd + gz \quad bx + y + iz \quad cx + fy + z \quad 1] \qquad (3-5)$$

An example clarifies these results.

Example 3–3 Shearing

Again consider the unit cube shown in Fig. 3–1b. Applying the shearing transformation

$$[T] = \begin{bmatrix} 1 & -0.85 & 0.25 & 0 \\ -0.75 & 1 & 0.7 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

yields

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.85 & 0.25 & 0 \\ -0.75 & 1 & 0.7 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 1 & 1 & 1 \\ 1.5 & 0.15 & 1.25 & 1 \\ 0.75 & 1.15 & 1.95 & 1 \\ -0.25 & 2 & 1.7 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -0.85 & 0.25 & 1 \\ 0.25 & 0.15 & 0.95 & 1 \\ -0.75 & 1 & 0.7 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–1d. Notice that in all three examples the origin is unaffected by the transformation.

3–4 THREE-DIMENSIONAL ROTATION

Before considering three-dimensional rotation about an arbitrary axis, we examine rotation about each of the coordinate axes. For rotation about the x-axis, the x coordinates of the position vectors do not change. In effect, the rotation occurs in planes perpendicular to the x-axis. Similarly, rotation about the y- and z-axes occurs in planes perpendicular to the y- and z-axes, respectively. The transformation of the position vectors in each of these planes is governed by the general two-dimensional rotation matrix given in Eq. (2–29). Recalling that matrix, and again noting that for rotation about the x-axis the x coordinate of the transformed position vectors does not change, allows writing down the 4×4 homogeneous coordinate transformation by the angle θ as

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-6)$$

Rotation is assumed positive in a right-hand sense, i.e., clockwise as one looks outward from the origin in the positive direction along the rotation axis.† The block shown in Fig. 3–2b is the result of a $-90°$ rotation about the x-axis of the block shown in Fig. 3–2a.

In a similar manner, the transformation matrix for rotation by an angle ψ about the z-axis is

$$[T] = \begin{bmatrix} \cos\psi & \sin\psi & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-7)$$

For rotation by an angle ϕ about the y-axis, the transformation is

$$[T] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-8)$$

Note that in Eq. 3–8 the signs of the sine terms are reversed from those of Eqs. (3–6) and (3–7). This is required to maintain the positive right-hand rule convention.

Examining Eqs. (3–6) to (3–8) shows that the determinant of each transformation matrix is +1 as required for pure rotation. An example more fully illustrates these results.

†The right-hand rule for rotation is stated as follows: align the thumb of the right hand with the positive direction of the rotation axis. The natural curl of the fingers gives the positive rotation direction.

Example 3–4 Rotation

Consider the rectangular parallelepiped shown in Fig. 3–2a. The matrix of position vectors $[X]$ is

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ A \\ \\ \\ \\ \\ \\ \end{matrix}$$

Here, the row labeled A in the position matrix $[X]$ corresponds to the point A in Fig. 3-2. For rotation by $\theta = -90°$ about the x-axis Eq. (3–6) yields

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying the transformation gives the new position vectors

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 \\ 3 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{matrix} \\ \\ A^* \\ \\ \\ \\ \\ \\ \end{matrix}$$

Notice that the x components of $[X]$ and $[X^*]$ are identical as required. The result is shown in Fig. 3–2b.

For rotation by $\phi = 90°$ about the y-axis, Eq. (3–7) yields

$$[T'] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

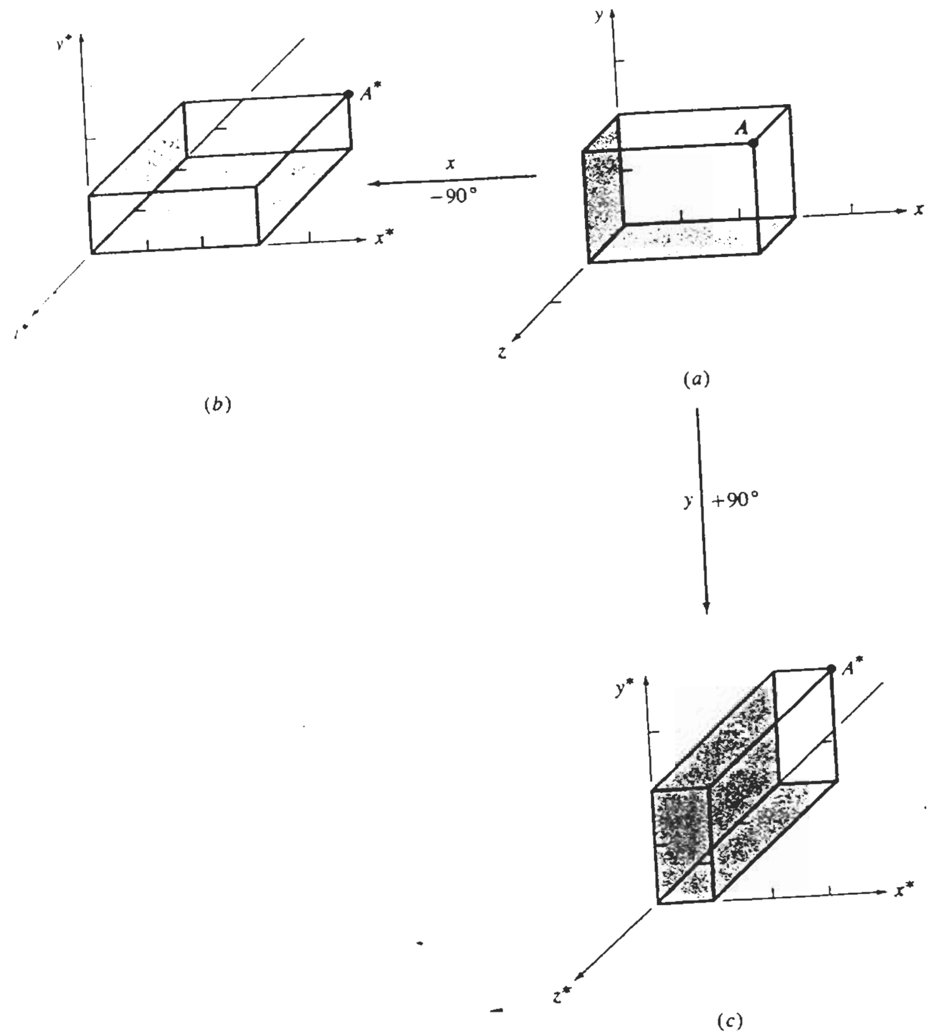


Figure 3–2 Three-dimensional rotations.

Again applying the transformation to the original block yields the new position vectors, i.e.,

$$[X^{*\prime}] = [X][T'] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & -3 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 2 & -3 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} A^*$$

Notice that in this case the y components of $[X]$ and $[X^{*\prime}]$ are identical. The result is shown in Fig. 3-2c.

Since three-dimensional rotations are obtained using matrix multiplication, they are noncommutative; i.e., the order of multiplication affects the final result (see Sec. 2-12). In order to show this, consider a rotation about the x-axis followed by an equal rotation about the y-axis. Using Eqs. (3-6) and (3-8) with $\theta = \phi$, we have

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ \sin^2\theta & \cos\theta & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & -\sin\theta & \cos^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-9)$$

On the other hand, the reverse operation, i.e., a rotation about the y-axis followed by an equal rotation about the x-axis with $\theta = \phi$, yields

$$[T] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin^2\theta & -\cos\theta\sin\theta & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta\sin\theta & \cos^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-10)$$

Comparison of the right-hand sides of Eqs. (3-9) and (3-10) shows that they are not the same. The fact that three-dimensional rotations are noncommutative must be kept in mind when more than one rotation is to be made.

The result of transformation of the object in Fig. 3-3a consisting of two 90° rotations using the matrix product given in Eq. (3-9) is shown dashed in Figs. 3-3c and 3-3d. When the opposite order of rotation as specified by Eq. (3-10) is used, the solid figures shown in Figs. 3-3b and 3-3d graphically demonstrate that different results are obtained by changing the order of rotation. A numerical example further illustrates this concept.

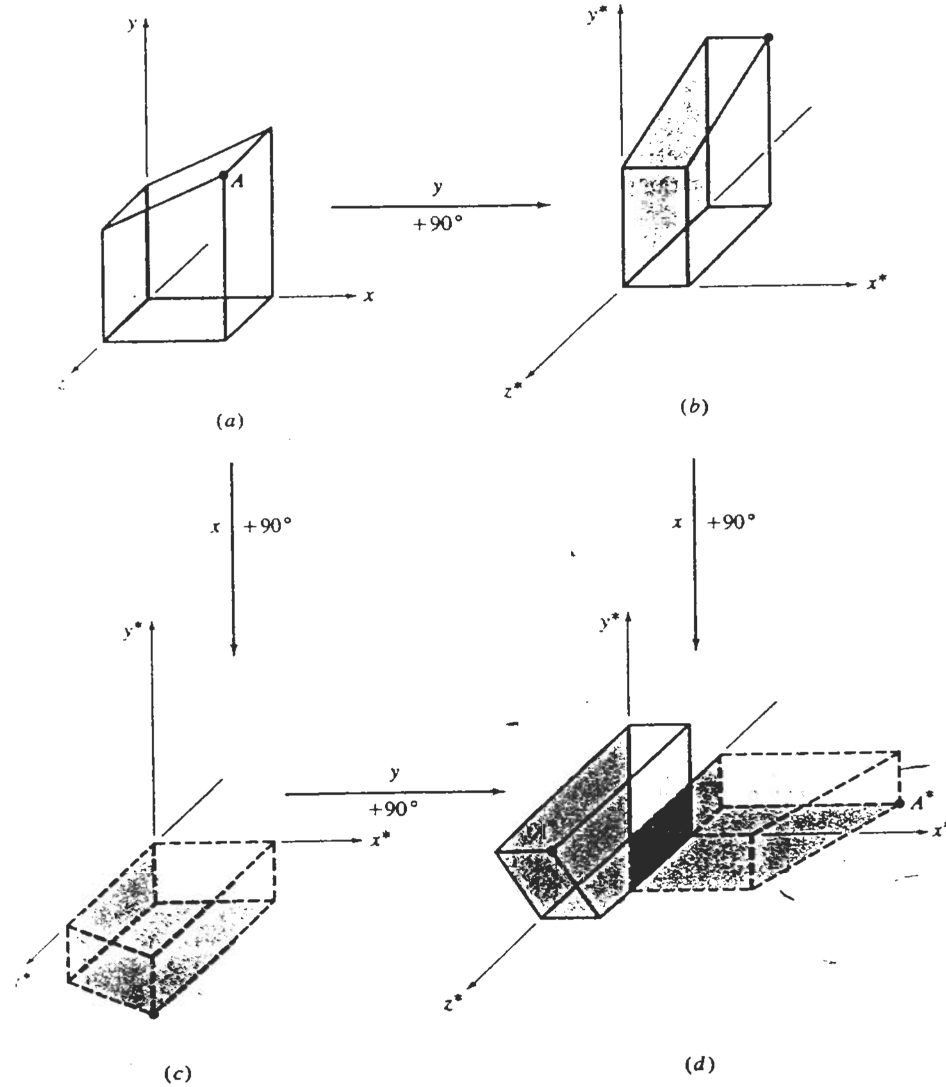


Figure 3–3 Three-dimensional rotations are noncommutative.

Example 3–5 Combined Rotations

The object in Fig. 3–3a has position vectors

$$[X]=\begin{bmatrix}0&0&1&1\\2&0&1&1\\2&3&1&1\\0&2&1&1\\0&0&0&1\\2&0&0&1\\2&3&0&1\\0&2&0&1\end{bmatrix} A$$

The concatenated matrix for a rotation about the x-axis by $\theta = 90°$ followed by a rotation about the y-axis by $\phi = 90°$ is given by Eq. (3–9) as

$$[T]=\begin{bmatrix}0&0&-1&0\\1&0&0&0\\0&-1&0&0\\0&0&0&1\end{bmatrix}$$

The transformed position vectors are

$$[X^*]=[X][T]=\begin{bmatrix}0&0&1&1\\2&0&1&1\\2&3&1&1\\0&2&1&1\\0&0&0&1\\2&0&0&1\\2&3&0&1\\0&2&0&1\end{bmatrix}\begin{bmatrix}0&0&-1&0\\1&0&0&0\\0&-1&0&0\\0&0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&-1&0&1\\0&-1&-2&1\\3&-1&-2&1\\2&-1&0&1\\0&0&0&1\\0&0&-2&1\\3&0&-2&1\\2&0&0&1\end{bmatrix} A^*$$

The transformed object is shown dashed in Fig. 3–3d.

The concatenated matrix for a rotation about the y-axis by $\phi = 90°$ followed by a rotation about the x-axis by $\theta = 90°$ is given by Eq. (3–10) as

$$[T']=\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$$

Here, the resulting transformed position vectors are

$$[X^{*\prime}]=[X][T']=\begin{bmatrix}0&0&1&1\\2&0&1&1\\2&3&1&1\\0&2&1&1\\0&0&0&1\\2&0&0&1\\2&3&0&1\\0&2&0&1\end{bmatrix}\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&1\\1&2&0&1\\1&2&3&1\\1&0&2&1\\0&0&0&1\\0&2&0&1\\0&2&3&1\\0&0&2&1\end{bmatrix} A^{*\prime}$$

The transformed object is shown by solid lines in Fig. 3–3d.

Comparing the two numerical results also clearly shows that the orientation of the transformed objects is considerably different. Hence, the order of matrix multiplication is important.

3-5 THREE DIMENSIONAL REFLECTION

Some orientations of a three-dimensional object cannot be obtained using pure rotations; they require reflections. In three dimensions, reflections occur through a plane. By analogy with the previous discussion of two-dimensional reflection (see Sec. 2–10), three-dimensional reflection through a plane is equivalent to rotation about an axis in three-dimensional space out into four-dimensional space and back into the original three-dimensional space. For a pure reflection the determinant of the reflection matrix is identically −1.

In a reflection through the xy plane, only the z coordinate values of the object's position vectors change. In fact, they are reversed in sign. Thus, the transformation matrix for a reflection through the xy plane is

$$[T]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix} \qquad (3-11)$$

The reflection of a unit cube through the xy plane is shown in Fig. 3–4. For a reflection through the yz plane,

$$[T]=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \qquad (3-12)$$

and for a reflection through the xz plane,

$$[T]=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \qquad (3-13)$$

A numerical example further illustrates these results.

Example 3–6 Reflection

The block $ABCDEFGH$ shown in Fig. 3–4 has position vectors

$$[X] = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & -2 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

The transformation matrix for reflection through the xy plane is given by Eq. (3–11). After reflection the transformed position vectors are

$$[X^*] = [X][T] = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & -2 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

The result $A^*B^*C^*D^*E^*F^*G^*H^*$ is shown in Fig. 3–4.

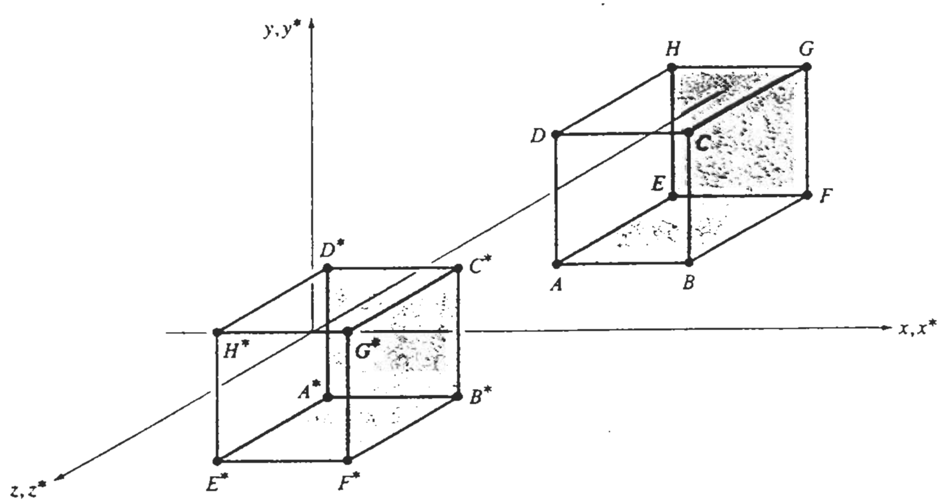


Figure 3–4 Three-dimensional reflection through the xy plane.

3–6 THREE DIMENSIONAL TRANSLATION

The three-dimensional translation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix} \qquad (3-14)$$

The translated homogeneous coordinates are obtained by writing

$$[x' \quad y' \quad z' \quad h] = [x \quad y \quad z \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix}$$

When expanded this yields

$$[x' \quad y' \quad z' \quad h] = [(x+l) \quad (y+m) \quad (z+n) \quad 1] \qquad (3-15)$$

It follows that the transformed physical coordinates are

$$x^* = x + l$$
$$y^* = y + m$$
$$z^* = z + n$$

3–7 MULTIPLE TRANSFORMATIONS

Successive transformations can be combined or concatenated into a single 4 × 4 transformation that yields the same result. Since matrix multiplication is noncommutative, the order of application is important (in general $[A][B] \neq [B][A]$). The proper order is determined by the position of the individual transformation matrix relative to the position vector matrix. The matrix nearest the position vector matrix generates the first individual transformation, and the farthest, the last individual transformation. Mathematically this is expressed as

$$[X][T] = [X][T_1][T_2][T_3][T_4]\cdots$$

where $$[T] = [T_1][T_2][T_3][T_4]\cdots$$

and the $[T_i]$ are any combination of scaling, shearing, rotation, reflection, translation, perspective and projective matrices. Since perspective transformations distort geometric objects (see Sec. 3–15) and projective transformations result in lost information (see Sec. 3–12), if these matrices are included, they must occur next to last and last in the order, respectively.

The example below explicitly illustrates this concept.

Example 3–7 Multiple Transformations

Consider the effect of a translation in the x, y, z directions by -1, -1, -1, respectively, followed successively by a $+30°$ rotation about the x-axis, and a $+45°$ rotation about the y-axis on the homogeneous coordinate position vector $[3 \quad 2 \quad 1 \quad 1]$.

First derive the concatenated transformation matrix. Using Eqs. (3–14), (3–6) and (3–8) yields

$$[T] = [Tr][R_x][R_y]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta & 0 \\ \begin{matrix} l\cos\phi \\ +m\sin\phi\sin\theta \\ +n\sin\phi\cos\theta \end{matrix} & \begin{matrix} m\cos\theta \\ -n\sin\theta \end{matrix} & \begin{matrix} -l\sin\phi \\ +m\cos\phi\sin\theta \\ +n\cos\phi\cos\theta \end{matrix} & 1 \end{bmatrix} \tag{3-16}$$

where θ, ϕ are the rotation angles about the x- and y-axes, respectively; and l, m, n are the translation factors in the x, y, z directions, respectively.

For a general position vector we have

$$[X][T] = [x \quad y \quad z \quad 1] \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta & 0 \\ \begin{matrix} l\cos\phi \\ +m\sin\phi\sin\theta \\ +n\sin\phi\cos\theta \end{matrix} & \begin{matrix} m\cos\theta \\ -n\sin\theta \end{matrix} & \begin{matrix} -l\sin\phi \\ +m\cos\phi\sin\theta \\ +n\cos\phi\cos\theta \end{matrix} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} (x+l)\cos\phi \\ +(y+m)\sin\phi\sin\theta \\ +(z+n)\sin\phi\cos\theta \end{matrix} & \begin{matrix} (y+m)\cos\theta \\ -(z+n)\sin\theta \end{matrix} & \begin{matrix} -(x+l)\sin\phi \\ +(y+m)\cos\phi\sin\theta \\ +(z+n)\cos\phi\cos\theta \end{matrix} & 1 \end{bmatrix}$$

For specific values of $\theta = +30°$, $\phi = +45°$, $l = -1$, $m = -1$, $n = -1$ the transformed position vector is $[3 \quad 2 \quad 1 \quad 1]$.

$$[X][T] = [3 \quad 2 \quad 1 \quad 1] \begin{bmatrix} 0.707 & 0 & -0.707 & 0 \\ 0.354 & 0.866 & 0.354 & 0 \\ 0.612 & -0.5 & 0.612 & 0 \\ -1.673 & -0.366 & -0.259 & 1 \end{bmatrix}$$

$$[X][T] = [1.768 \quad 0.866 \quad -1.061 \quad 1]$$

To confirm that the concatenated matrix yields the same result as individually applied matrices consider

$$[X'] = [X][Tr]$$

$$= [3 \quad 2 \quad 1 \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$= [2 \quad 1 \quad 0 \quad 1]$$

$$[X''] = [X'][R_x] = [2 \quad 1 \quad 0 \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [2 \quad 0.866 \quad 0.5 \quad 1]$$

$$[X'''] = [X''][R_y] = [2 \quad 0.866 \quad .5 \quad 1] \begin{bmatrix} 0.707 & 0 & -0.707 & 0 \\ 0 & 1 & 0 & 0 \\ 0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X'''] = [1.768 \quad 0.866 \quad -1.061 \quad 1]$$

which confirms our previous result.

3–8 ROTATIONS ABOUT AN AXIS PARALLEL TO A COORDINATE AXIS

The transformations given in Eqs. (3–6) to (3–8) cause rotation about the x, y and z coordinate axes. Often it is necessary to rotate an object about an axis other than these. Here, the special case of an axis that is parallel to one of the x, y or z coordinate axes is considered. Figure 3–5 shows a body with a local axis system $x'y'z'$ parallel to the fixed global axis system xyz. Rotation of the body about any of the individual x', y' or z' local axes is accomplished using the following procedure:

Translate the body until the local axis is coincident with the coordinate axis in the same direction.

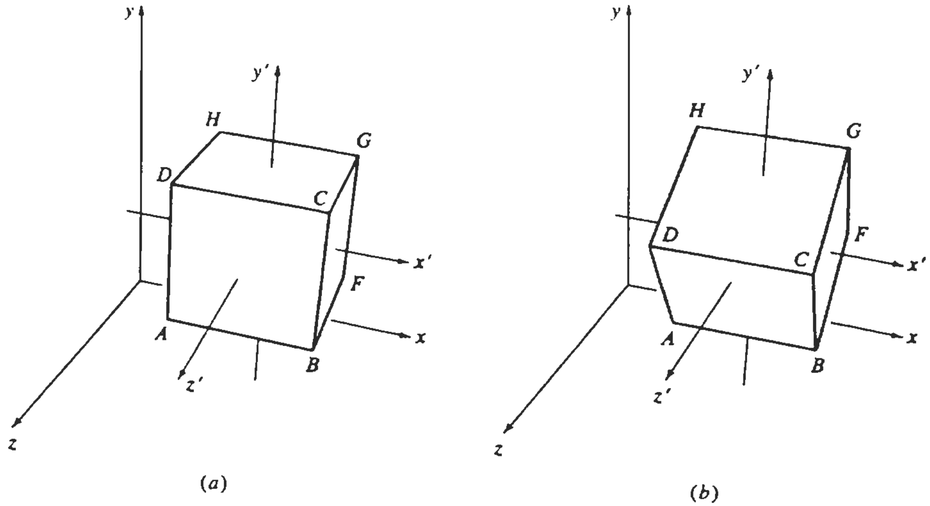


Figure 3–5 Rotation about an axis parallel to one of the coordinate axes.

Rotate about the specified axis.

Translate the transformed body back to its original position.

Mathematically

$$[X^*] = [X][T_r][R_x][T_r]^{-1}$$

where

- $[X^*]$ represents the transformed body
- $[X]$ is the untransformed body
- $[T_r]$ is the translation matrix
- $[R_x]$ is the appropriate rotation matrix
- $[T_r]^{-1}$ is the inverse of the translation matrix

An illustrative example is given below.

Example 3–8 Single Relative Rotation

Consider the block in Fig. 3-5a defined by the position vectors

$$[X] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{matrix}$$

relative to the global xyz-axis system. Let's rotate the block $\theta = +30^\circ$ about the local x'-axis passing through the centroid of the block. The origin of the local axis system is assumed to be the centroid of the block.

The centroid of the block is $[x_c \;\; y_c \;\; z_c \;\; 1] = [3/2 \;\; 3/2 \;\; 3/2 \;\; 1]$. The rotation is accomplished by

$$[X^*] = [X][T_r][R][T_r]^{-1}$$

where

$$[T_r] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -y_c & -z_c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3/2 & -3/2 & 1 \end{bmatrix}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$[T_r]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y_c & z_c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3/2 & 3/2 & 1 \end{bmatrix}$$

The first matrix $[T_r]$ translates the block parallel to the $x = 0$ plane until the x'-axis is coincident with the x-axis. The second matrix $[R_x]$ performs the required rotation about the x-axis, and the third matrix $[T_r]^{-1}$ translates the x'-axis and hence the rotated block back to its original position. Concatenating these three matrices yields

$$[T] = [T_r][R_x][T_r]^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & y_c(1-\cos\theta) + z_c\sin\theta & z_c(1-\cos\theta) - y_c\sin\theta & 1 \end{bmatrix}$$

After substituting numerical values the transformed coordinates are

$$[X'] = [X][T] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0.951 & -0.549 & 1 \end{bmatrix}$$

$$[X'] = \begin{bmatrix} 1 & 0.817 & 1.683 & 1 \\ 2 & 0.817 & 1.683 & 1 \\ 2 & 1.683 & 2.183 & 1 \\ 1 & 1.683 & 2.183 & 1 \\ 1 & 1.317 & 0.817 & 1 \\ 2 & 1.317 & 0.817 & 1 \\ 2 & 2.183 & 1.317 & 1 \\ 1 & 2.183 & 1.317 & 1 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{matrix}$$

The result is shown in Fig. 3-5b.

In the previous example only rotation about a single axis parallel to one of the coordinate axes was required. Thus, it was only necessary to make the rotation axis coincident with the corresponding coordinate axis. If multiple rotations in a local axis system parallel to the global axis system are required, then the origin of the local axis system must be made coincident with that of the global axis system. Specifically, the rotations can be accomplished with the following procedure:

Translate the origin of the local axis system to make it coincident with that of the global coordinate system.

Perform the required rotations.

Translate the local axis system back to its original position.

The example below illustrates this procedure.

Example 3–9 Multiple Relative Rotations

Again consider the block shown on Fig. 3-5a. To rotate the block $\phi = -45°$ about the y'-axis, followed by a rotation of $\theta = +30°$ about the x'-axis, requires that the origin of the $x'y'z'$-axis system be made coincident with the origin of the xyz-axis system, the rotations performed and then the result translated back to the original position.

The combined transformation is

$$[X'] = [X][T] = [X][Tr][R_y][R_x][Tr]^{-1}$$

Specifically,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_c & -y_c & -z_c & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_c & y_c & z_c & 1 \end{bmatrix}$$

where ϕ and θ represent the rotation angle about the y'- and x'-axes respectively. Concatenating these matrices yields

$$[T] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & -\sin\phi\cos\theta & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ \sin\phi & -\cos\phi\sin\theta & \cos\phi\cos\theta & 0 \\ \begin{matrix} x_c(1-\cos\phi) \\ -z_c\sin\phi \end{matrix} & \begin{matrix} -x_c\sin\phi\sin\theta \\ +y_c(1-\cos\theta) \\ +z_c\cos\phi\sin\theta \end{matrix} & \begin{matrix} x_c\sin\phi\cos\theta \\ -y_c\sin\theta \\ +z_c(1-\cos\phi\cos\theta) \end{matrix} & 1 \end{bmatrix} \tag{3-17}$$

The transformed position vectors are then

$$[X'] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & -0.354 & 0.612 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ -0.707 & -0.354 & 0.612 & 0 \\ 1.5 & 1.262 & -1.087 & 1 \end{bmatrix}$$

$$[X'] = \begin{bmatrix} 0.793 & 1.067 & 1.25 & 1 \\ 1.5 & 0.713 & 1.862 & 1 \\ 1.5 & 1.579 & 2.362 & 1 \\ 0.793 & 1.933 & 1.75 & 1 \\ 1.5 & 1.421 & 0.638 & 1 \\ 2.207 & 1.067 & 1.25 & 1 \\ 2.207 & 1.933 & 1.75 & 1 \\ 1.5 & 2.287 & 1.138 & 1 \end{bmatrix}$$

The result is shown in Fig. 3-6.

3–9 ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

The general case of rotation about an arbitrary axis in space frequently occur e.g., in robotics, animation, and simulation. Following the previous discussio rotation about an arbitrary axis in space is accomplished with a procedure usi translations and simple rotations about the coordinate axes. Since the techniq for rotation about a coordinate axis is known, the underlying procedural idea to make the arbitrary rotation axis coincident with one of the coordinate axe

Assume an arbitrary axis in space passing through the point $(x_0, y_0, z$ with direction cosines (c_x, c_y, c_z). Rotation about this axis by some angle δ accomplished using the following procedure:

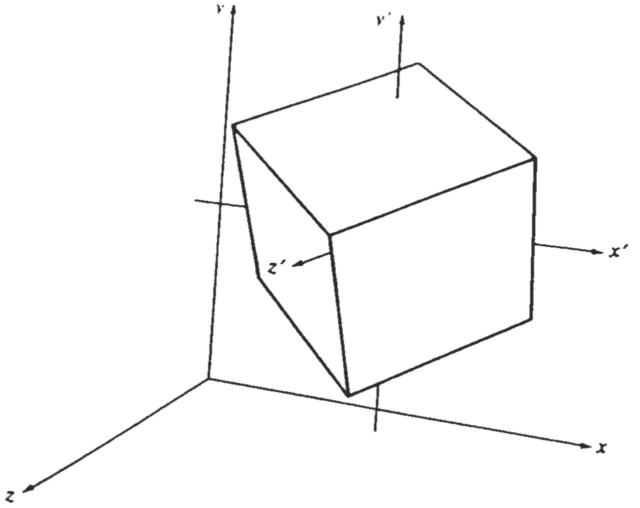


Figure 3–6 Multiple rotations about a local axis system.

Translate so that the point (x_0, y_0, z_0) is at the origin of the coordinate system.

Perform appropriate rotations to make the axis of rotation coincident with the z-axis.[†]

Rotate about the z-axis by the angle δ.

Perform the inverse of the combined rotation transformation.

Perform the inverse of the translation.

In general, making an arbitrary axis passing through the origin coincident with one of the coordinate axes requires two successive rotations about the other two coordinate axes. To make the arbitrary rotation axis coincident with the z-axis, first rotate about the x-axis and then about the y-axis. To determine the rotation angle, α, about the x-axis used to place the arbitrary axis in the xz plane, first project the unit vector along the axis onto the yz plane as shown in Fig. 3–7a. The y and z components of the projected vector are c_y and c_z, the direction cosines of the unit vector along the arbitrary axis. From Fig. 3–7a we have that

$$d = \sqrt{c_y^2 + c_z^2} \tag{3-18}$$

and

$$\cos\alpha = \frac{c_z}{d} \qquad \sin\alpha = \frac{c_y}{d} \tag{3-19}$$

[†]The choice of the z-axis is arbitrary.

After rotation about the x-axis into the xz plane, the z component of the unit vector is d, and the x component is c_x, the direction cosine in the x direction as shown in Fig. 3–7b. The length of the unit vector is, of course, 1. Thus, the rotation angle β about the y-axis required to make the arbitrary axis coincident with the z-axis is

$$\cos\beta = d \qquad \sin\beta = c_x \tag{3-20}$$

The complete transformation is then

$$[\,M\,] = [\,T\,][\,R_x\,][\,R_y\,][\,R_\delta\,][\,R_y\,]^{-1}[\,R_x\,]^{-1}[\,T\,]^{-1} \tag{3-21}$$

where the required translation matrix is

$$[\,T\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_0 & -y_0 & -z_0 & 1 \end{bmatrix} \tag{3-22}$$

the transformation matrix for rotation about the x-axis is

$$[\,R_x\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_z/d & c_y/d & 0 \\ 0 & -c_y/d & c_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3-23}$$

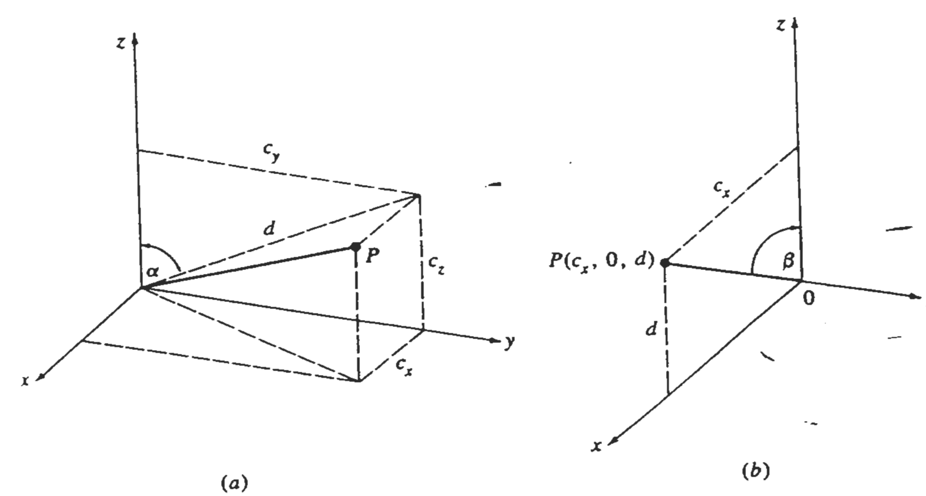


Figure 3–7 Rotations required to make the unit vector OP coincident with the z-axis. (a) Rotation about x; (b) rotation about y.

and about the y-axis

$$[\,R_y\,] = \begin{bmatrix} \cos(-\beta) & 0 & -\sin(-\beta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\beta) & 0 & \cos(-\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d & 0 & c_x & 0 \\ 0 & 1 & 0 & 0 \\ -c_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-24)$$

Finally, the rotation about the arbitrary axis is given by a z-axis rotation matrix

$$[\,R_\delta\,] = \begin{bmatrix} \cos\delta & \sin\delta & 0 & 0 \\ -\sin\delta & \cos\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-25)$$

In practice, the angles α and β are not explicitly calculated. The elements of the rotation matrices $[\,R_x\,]$ and $[\,R_y\,]$ in Eq. (3–21) are obtained from Eqs. (3–18) to (3–20) at the expense of two divisions and a square root calculation. Although developed with the arbitrary axis in the first quadrant, these results are applicable in all quadrants.

If the direction cosines of the arbitrary axis are not known, they can be obtained knowing a second point on the axis (x_1, y_1, z_1) by normalizing the vector from the first to the second point. Specifically, the vector along the axis from (x_0, y_0, z_0) to (x_1, y_1, z_1) is

$$[\,V\,] = [\,(x_1 - x_0) \quad (y_1 - y_0) \quad (z_1 - z_0)\,]$$

Normalized, it yields the direction cosines

$$[\,c_x \quad c_y \quad c_z\,] = \frac{[\,(x_1 - x_0) \quad (y_1 - y_0) \quad (z_1 - z_0)\,]}{[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{\frac{1}{2}}} \qquad (3-26)$$

An example more fully illustrates the procedure.

Example 3–10 Rotation About an Arbitrary Axis

Consider the cube with one corner removed shown in Fig. 3–8a. Position vectors for the vertices are

$$[\,X\,] = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 1 & 2 & 1 \\ 3 & 1.5 & 2 & 1 \\ 2.5 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 2 & 1.5 & 1 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \\ J \end{matrix}$$

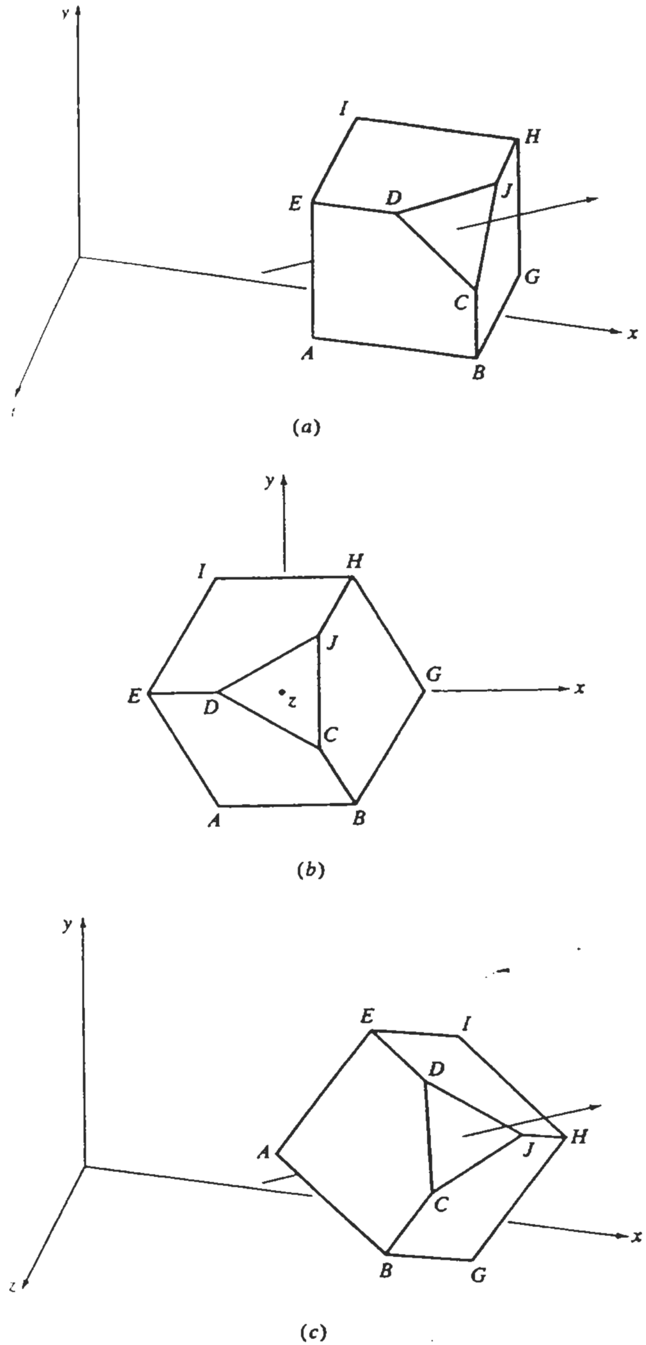


Figure 3–8 Rotation about an arbitrary axis.

The cube is to be rotated by $-45°$ about a local axis passing through the point F and the diagonally opposite corner. The axis is directed from F to the opposite corner and passes through the center of the corner face.

First, determine the direction cosines of the rotation axis. Observing that the corner cut off by the triangle CDJ also lies on the axis, Eq. (3–26) yields

$$[\,c_x \quad c_y \quad c_z\,] = \frac{[\,(3-2) \quad (2-1) \quad (2-1)\,]}{((3-2)^2+(2-1)^2+(2-1)^2)^{\frac{1}{2}}}$$
$$= [\,1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3}\,]$$

Using Eqs. (3–18) to (3–20) yields

$$d = \sqrt{(1/\sqrt{3})^2 + (1/\sqrt{3})^2} = \sqrt{2/3}$$

and

$$\alpha = \cos^{-1}(1/\sqrt{3}\big/\sqrt{2/3}) = \cos^{-1}(1/\sqrt{2}) = 45°$$
$$\beta = \cos^{-1}(\sqrt{2/3}) = 35.26°$$

Since the point F lies on the rotation axis, the translation matrix is

$$[\,T\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & -1 & 1 \end{bmatrix}$$

The rotation matrices to make the arbitrary axis coincident with the z-axis are then

$$[\,R_x\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$[\,R_y\,] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$[\,R_x\,]^{-1}$, $[\,R_y\,]^{-1}$, and $[\,T\,]^{-1}$ are obtained by substituting $-\alpha$, $-\beta$ and (x_0, y_0, z_0) for α, β and $(-x_0, -y_0, -z_0)$, respectively, in Eqs. (3–22) to (3–24).

Concatenating $[\,T\,][\,R_x\,][\,R_y\,]$ yields

$$[\,M\,] = [\,T\,][\,R_x\,][\,R_y\,] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -2/\sqrt{6} & 0 & -4/\sqrt{3} & 1 \end{bmatrix}$$

The transformed intermediate position vectors are

$$[\,X\,][\,M\,] = \begin{bmatrix} -0.408 & -0.707 & 0.577 & 1 \\ 0.408 & -0.707 & 1.155 & 1 \\ 0.204 & -0.354 & 1.443 & 1 \\ -0.408 & 0 & 1.443 & 1 \\ -0.816 & 0 & 1.155 & 1 \\ 0 & 0 & 0 & 1 \\ 0.816 & 0 & 0.577 & 1 \\ 0.408 & 0.707 & 1.155 & 1 \\ -0.408 & 0.707 & 0.577 & 1 \\ 0.204 & 0.354 & 1.443 & 1 \end{bmatrix}$$

This intermediate result is shown in Fig. 3–8b. Notice that point F is at $(0,0,0)$.

The rotation about the arbitrary axis is now given by the equivalent rotation about the z-axis. Hence (see Eq. 3–7)

$$[\,R_\delta\,] = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed object is returned to its 'original' position in space, using

$$[\,M\,]^{-1} = [\,R_y\,]^{-1}[\,R_x\,]^{-1}[\,T\,]^{-1} = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

This result can be obtained either by concatenating the inverses of the individual component matrices of $[\,M\,]$ or by formally taking the inverse of $[\,M\,]$. Incidentally, notice that $[\,R_x\,][\,R_y\,]$ is a pure rotation. The upper left 3×3 submatrix of $[\,M\,]^{-1}$ is just the transpose of the upper left 3×3 submatrix of $[\,M\,]$.

The resulting position vectors are

$$[\,X\,][\,M\,][\,R_\delta\,][\,M\,]^{-1} = \begin{bmatrix} 1.689 & 1.506 & 1.805 & 1 \\ 2.494 & 1.195 & 2.311 & 1 \\ 2.747 & 1.598 & 2.155 & 1 \\ 2.598 & 2.155 & 1.747 & 1 \\ 2.195 & 2.311 & 1.494 & 1 \\ 2 & 1 & 1 & 1 \\ 2.805 & 0.689 & 1.506 & 1 \\ 3.311 & 1.494 & 1.195 & 1 \\ 2.506 & 1.805 & 0.689 & 1 \\ 3.155 & 1.747 & 1.598 & 1 \end{bmatrix}$$

where

$$[M][R_\delta][M]^{-1} = \begin{bmatrix} 0.805 & -0.311 & 0.506 & 0 \\ 0.506 & 0.805 & -0.311 & 0 \\ -0.311 & 0.506 & 0.805 & 0 \\ 0.195 & 0.311 & -0.506 & 1 \end{bmatrix}$$

The transformed object is shown in Fig. 3-8c.

3-10 REFLECTION THROUGH AN ARBITRARY PLANE

The transformations given in Eqs. (3-11) to (3-13) cause reflection through the $x = 0$, $y = 0$, $z = 0$ coordinate planes, respectively. Often it is necessary to reflect an object through a plane other than one of these. Again, this can be accomplished using a procedure incorporating the previously defined simple transformations. One possible procedure is:

Translate a known point P, that lies in the reflection plane, to the origin of the coordinate system.

Rotate the normal vector to the reflection plane at the origin until it is coincident with the $+z$-axis (see Sec. 3-9, Eqs. 3-23 and 3-24); this makes the reflection plane the $z = 0$ coordinate plane.

After also applying the above transformations to the object, reflect the object through the $z = 0$ coordinate plane (see Eq. 3-11).

Perform the inverse transformations to those given above to achieve the desired result.

The general transformation is then

$$[M] = [T][R_x][R_y][Rflt_z][R_y]^{-1}[R_x]^{-1}[T]^{-1}$$

where the matrices $[T]$, $[R_x]$, $[R_y]$ are given by Eqs. (3-22) to (3-24), respectively. $(x_0, y_0, z_0) = (P_x, P_y, P_z)$, the components of point P in the reflection plane; and (c_x, c_y, c_z) are the direction cosines of the normal to the reflection plane.†

An example more fully illustrates the procedure.

Example 3-11 Reflection

Again consider the cube with one corner removed shown in Fig. 3-8a. Reflect the cube through the plane containing the triangle CDJ.

†If the equation of the reflection plane $ax + by + cz + d = 0$ is known, then the unit normal to the plane is

$$[\hat{n}] = [c_x \quad c_y \quad c_z] = \frac{[a \quad b \quad c]}{\sqrt{a^2 + b^2 + c^2}}$$

See Ref. 3-1 for more details.

Recalling the position vectors for the cube, and choosing to translate the point C to the origin, yields the translation matrix

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -3/2 & -2 & 1 \end{bmatrix}$$

The normal to the reflection plane is obtained using the position vectors C, D, J (see Ref. 3-1). Specifically, taking the cross product of the vectors CJ and CD prior to translation yields

$$\begin{aligned} n &= ([J] - [C]) \times ([D] - [C]) \\ &= [(3-3) \quad (2-1.5) \quad (1.5-2)] \times [(2.5-3) \quad (2-1.5) \quad (2-2)] \\ &= [0 \quad 1/2 \quad -1/2] \times [-1/2 \quad 1/2 \quad 0] \\ &= [1/4 \quad 1/4 \quad 1/4] \end{aligned}$$

Normalizing yields

$$\hat{n} = [1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3}]$$

Using Eqs. (3-19) and (3-20) gives

$$d = \sqrt{n_y^2 + n_z^2} = \sqrt{(1/\sqrt{3})^2 + (1/\sqrt{3})^2} = \sqrt{2/3}$$

and

$$\alpha = 45^\circ \qquad \beta = 35.26^\circ$$

The rotation matrices to make the normal at C coincide with the z-axis are (see Eqs. 3-23 and 3-24)

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_y] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrices $[R_x]^{-1}$, $[R_y]^{-1}$ and $[T]^{-1}$ are obtained by substituting $-\alpha$, $-\beta$, and $[x_0 \quad y_0 \quad z_0] = [C]$ into Eqs. (3-22) to (3-24).

Concatenating $[T]$, $[R_x]$ and $[R_y]$ yields

$$[M] = [T][R_x][R_y] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -5/2\sqrt{6} & 1/2\sqrt{2} & -13/2\sqrt{3} & 1 \end{bmatrix}$$

The transformed intermediate position vectors are

$$[X][M] = \begin{bmatrix} -0.612 & -0.354 & -0.876 & 1 \\ 0.204 & -0.354 & -0.287 & 1 \\ 0 & 0 & 0 & 1 \\ -0.612 & 0.354 & 0 & 1 \\ -1.021 & 0.354 & -0.287 & 1 \\ -0.204 & 0.354 & -1.443 & 1 \\ 0.612 & 0.354 & -0.876 & 1 \\ 0.204 & 1.061 & -0.287 & 1 \\ -0.612 & 1.061 & -0.876 & 1 \\ 0 & 0.707 & 0 & 1 \end{bmatrix}$$

This intermediate result is shown in Fig. 3-9b. Notice that the point C is at the origin and the z-axis points out of the page.

Reflection through the arbitrary plane is now given by reflection through the $z = 0$ plane. Hence (see Eq. 3-11)

$$[Rflt] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Returning the transformed object to its 'original' position in space requires

$$[M]^{-1} = [R_y]^{-1}[R_x]^{-1}[T]^{-1} = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 3 & 3/2 & 2 & 1 \end{bmatrix}$$

The resulting position vectors are

$$[X][M][Rflt][M]^{-1} = \begin{bmatrix} 3 & 2 & 3 & 1 \\ 10/3 & 4/3 & 7/3 & 1 \\ 3 & 3/2 & 2 & 1 \\ 5/2 & 2 & 2 & 1 \\ 7/3 & 7/3 & 7/3 & 1 \\ 11/3 & 8/3 & 8/3 & 1 \\ 4 & 2 & 2 & 1 \\ 10/3 & 7/3 & 4/3 & 1 \\ 3 & 3 & 2 & 1 \\ 3 & 2 & 3/2 & 1 \end{bmatrix}$$

where

$$[M][Rflt][M]^{-1} = \begin{bmatrix} 1/3 & -2/3 & -2/3 & 0 \\ -2/3 & 1/3 & -2/3 & 0 \\ -2/3 & -2/3 & 1/3 & 0 \\ 13/3 & 13/3 & 13/3 & 1 \end{bmatrix}$$

The transformed object is shown in Fig. 3-9c.

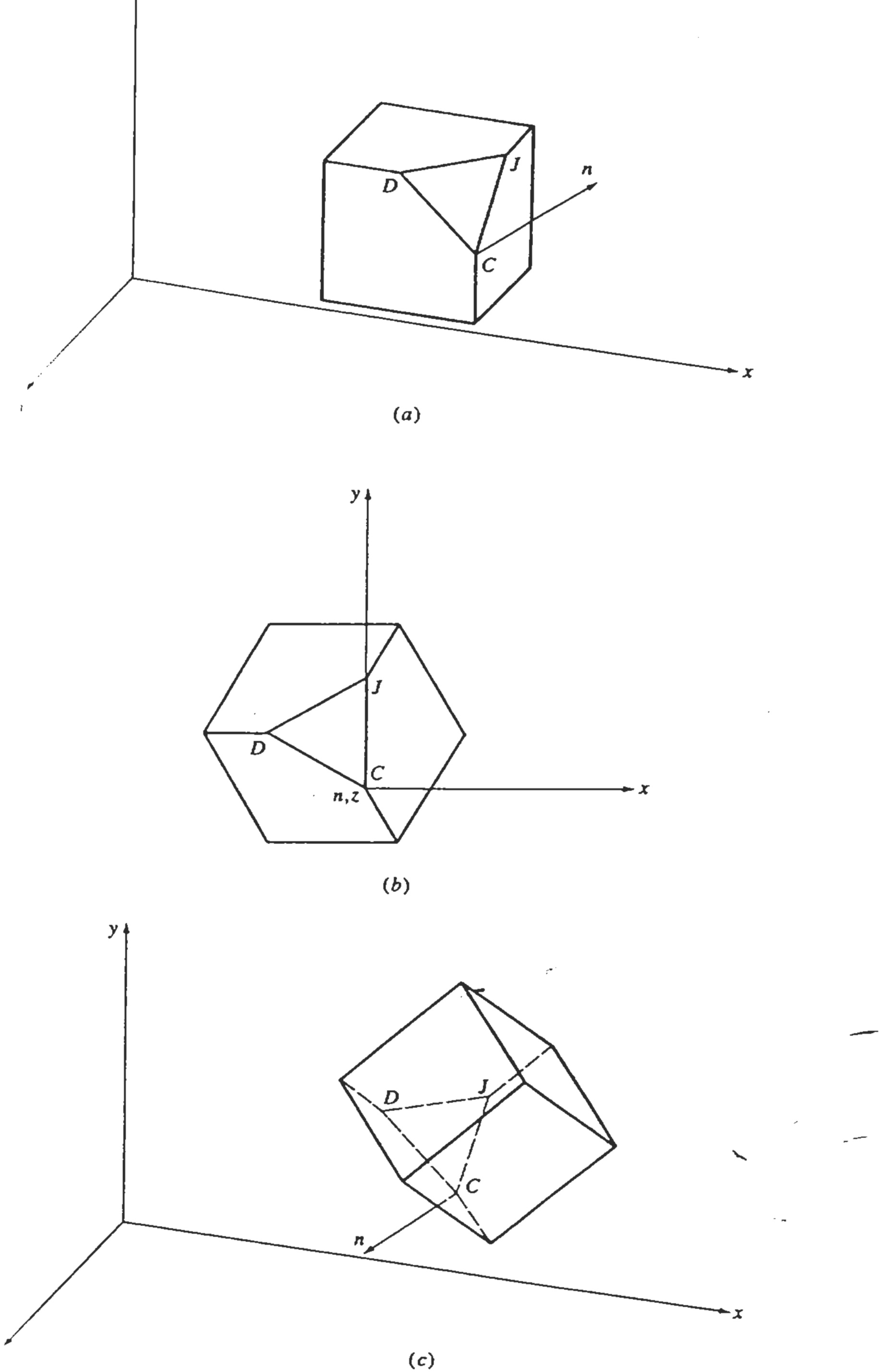


Figure 3-9 Reflection through an arbitrary plane.

As this and the previous section show, complex manipulative transformations can easily be constructed using procedures involving simple single-action transformations. This is the recommended approach. Generally it is less error prone and is computationally more efficient than a direct mathematical approach.

3–11 AFFINE AND PERSPECTIVE GEOMETRY

Geometric theorems have been developed for both perspective and affine geometry. The theorems of affine geometry are identical to those for Euclidean geometry. In both affine and Euclidean geometry parallelism is an important concept. In perspective geometry, lines are generally nonparallel.

An affine transformation is a combination of linear transformations, e.g., rotation followed by translation. For an affine transformation, the last column in the general 4×4 transformation matrix is $[0 \quad 0 \quad 0 \quad 1]^T$. Otherwise, as shown in Sec. 3–15 below, the transformed homogeneous coordinate h is not unity; and there is not a one-to-one correspondence between the affine transformation and the 4×4 matrix operator. Affine transformations form a useful subset of bilinear transformations, since the product of two affine transformations is also affine. This allows the general transformation of a set of points relative to an arbitrary coordinate system while maintaining a value of unity for the homogeneous coordinate h.

Since Euclidean geometry has been taught in schools for many years, drawing and sketching techniques based on Euclidean geometry have become standard methods for graphical communication. Although perspective views are often used by artists and architects to yield more realistic pictures, because of the difficulty of manual construction they are seldom used in technical work. However, with the use of homogeneous coordinates to define an object, both affine and perspective transformations are obtained with equal ease.

Both affine and perspective transformations are three-dimensional, i.e., they are transformations from one three space to another three space. However, viewing the results on a two-dimensional surface requires a projection from three space to two space. The result is called a plane geometric projection. Figure 3–10 illustrates the hierarchy of plane geometric projections. The projection matrix from three space to two space always contains a column of zeros. Consequently the determinant of a projective transformation is always zero.

Plane geometric projections of objects are formed by the intersection of lines called projectors with a plane called the projection plane. Projectors are lines from an arbitrary point called the center of projection, through each point in an object. If the center of projection is located at a finite point in three space, the result is a perspective projection. If the center of projection is located at infinity, all the projectors are parallel and the result is a parallel projection. Plane geometric projections provide the basis for descriptive geometry. Nonplanar and nongeometric projections are also useful; e.g., they are used extensively in cartography.

In developing the various transformations shown in Fig. 3–10 two alternate approaches can be used. The first assumes that the center of projection or eye point is fixed and that the plane of projection is perpendicular to each projector as shown in Fig. 3–11a. The object is manipulated to obtain any required view. The second assumes that the object is fixed, that the center of projection is free to move anywhere in three space, and that the plane of projection is not necessarily perpendicular to the viewing direction. An example is shown in Fig. 3–11b. Both approaches are *mathematically equivalent*.

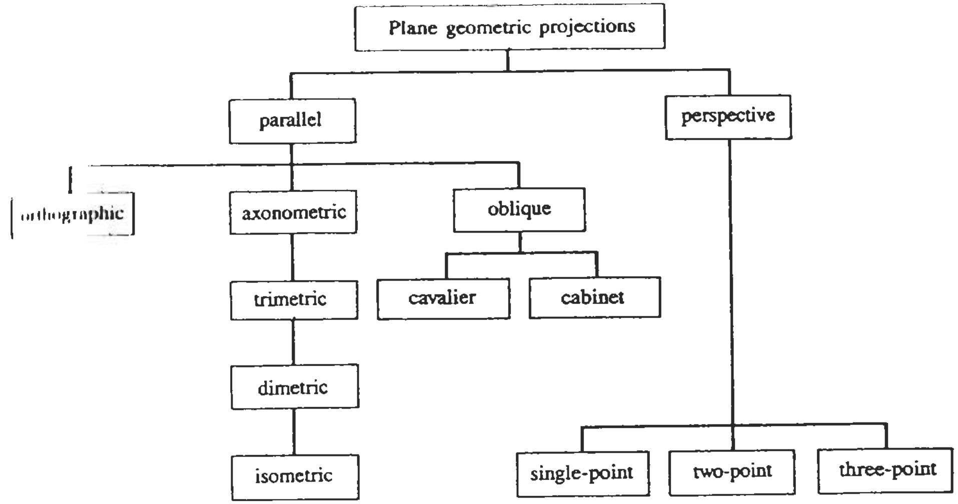


Figure 3–10 Hierarchy of plane geometric projections.

By analogy the first approach is similar to the actions of a human observer when asked to describe a small object, e.g., a book. The object is picked up, rotated and translated in order to view all sides and aspects of the object. The center of projection is fixed and the object is manipulated. The second approach is similar to the actions of the human observer when asked to describe a large object, e.g., an automobile. The observer walks around the object to view the various sides, climbs up on a ladder to view the top, and kneels down to look at the bottom. Here the object is fixed, and the center of projection and eye point are moved.

When designing or viewing an object on a computer graphics display the location of the eye is typically fixed and the plane of projection, i.e., the face of the CRT, is typically perpendicular to the viewing direction. Hence, the first approach is generally more appropriate. However, if the graphics display is used to simulate the motion of a vehicle or of an observer moving through a computer generated model, as is the case for vehicle simulators, or for an observer strolling through an architectured model, then the second approach is more appropriate.

A fixed center of projection, movable object approach is used in this book. The fixed object, movable center of projection approach is nicely developed by Carlbom and Paciorek (Ref. 3–2).

We begin our discussion of plane geometric projections (see Fig. 3–10) by first considering the parallel projections.

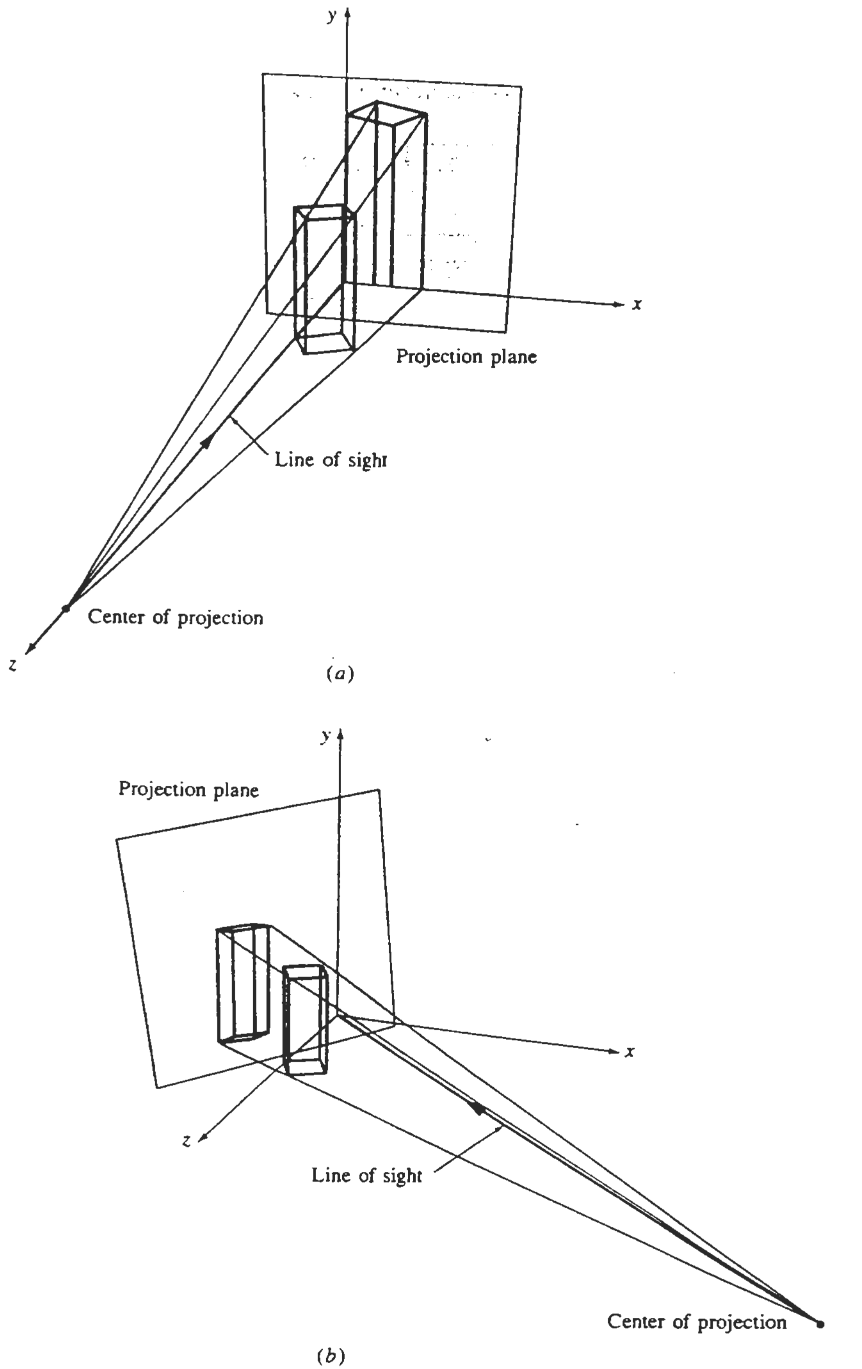


Figure 3–11 Plane projections. (a) Center of projection fixed; (b) object fixed.

3-12 ORTHOGRAPHIC PROJECTIONS

The simplest of the parallel projections is the orthographic projection, commonly used for engineering drawings. They accurately show the correct or 'true' size and shape of a single plane face of an object. Orthographic projections are projections onto one of the coordinate planes $x = 0$, $y = 0$ or $z = 0$. The matrix for projection onto the $z = 0$ plane is

$$[P_z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-27)$$

Notice that the third column (the z column) is all zeros. Consequently, the effect of the transformation is to set the z coordinate of a position vector to zero.

Similarly, the matrices for projection onto the $x = 0$ and $y = 0$ planes are

$$[P_x] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-28)$$

and

$$[P_y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-29)$$

Orthographic projections of the object in Fig. 3–12a onto the $x = 0$, $y = 0$ and $z = 0$ planes from centers of projection at infinity on the $+x$-, $+y$- and $+z$-axes are shown in Figs. 3–12b, 3–12c and 3–12d respectively.

A single orthographic projection does not provide sufficient information to visually and practically reconstruct the shape of an object. Consequently, multiple orthographic projections are necessary. These multiview orthographic projections are by convention† arranged as shown in Fig. 3–13. The front, right side and top views are obtained by projection onto the $z = 0$, $x = 0$ and $y = 0$ planes from centers of projection at infinity on the $+z$-, $+x$- and $+y$-axes. The rear, left side and bottom view projections are obtained by projection onto the $z = 0$, $x = 0$, $y = 0$ planes from centers of projection at infinity on the $-z$-, $-x$- and y-axes. The coordinate axes are not normally shown on the views.

As shown in Fig. 3–13, by convention hidden lines are shown dashed. All six views are normally not required to adequately convey the shape of an object. The front, top and right side views are most frequently used. Even when all six views are not used, the ones that are used appear in the locations shown. The

†This is the convention used in the United States.

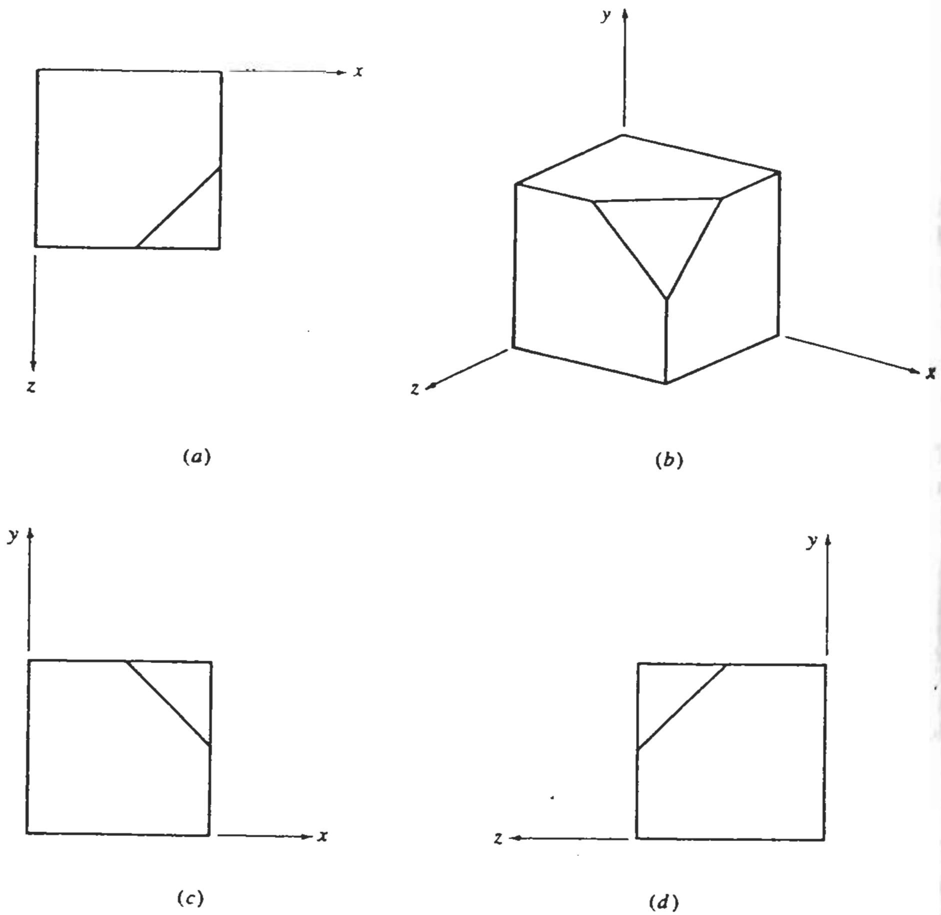


Figure 3–12 Orthographic projections onto (b) $y = 0$, (c) $z = 0$ and (d) $x = 0$ planes.

front and side views are sometimes called the front and side elevations. The top view is sometimes called the plan view.

It is interesting and important to note that all six views can be obtained by combinations of reflection, rotation and translation, followed by projection onto the $z = 0$ plane from a center of projection at infinity on the $+z$-axis. For example, the rear view is obtained by reflection through the $z = 0$ plane, followed by projection onto the $z = 0$ plane. Similarly, the left side view is obtained by rotation about the y-axis by $+90°$, followed by projection onto the $z = 0$ plane.

For objects with planes that are not parallel to one of the coordinate planes, the standard orthographic views do not show the correct or true shape of these planes. Auxiliary views are used for this purpose. An auxiliary view is formed by rotating and translating the object so that the normal to the auxiliary plane

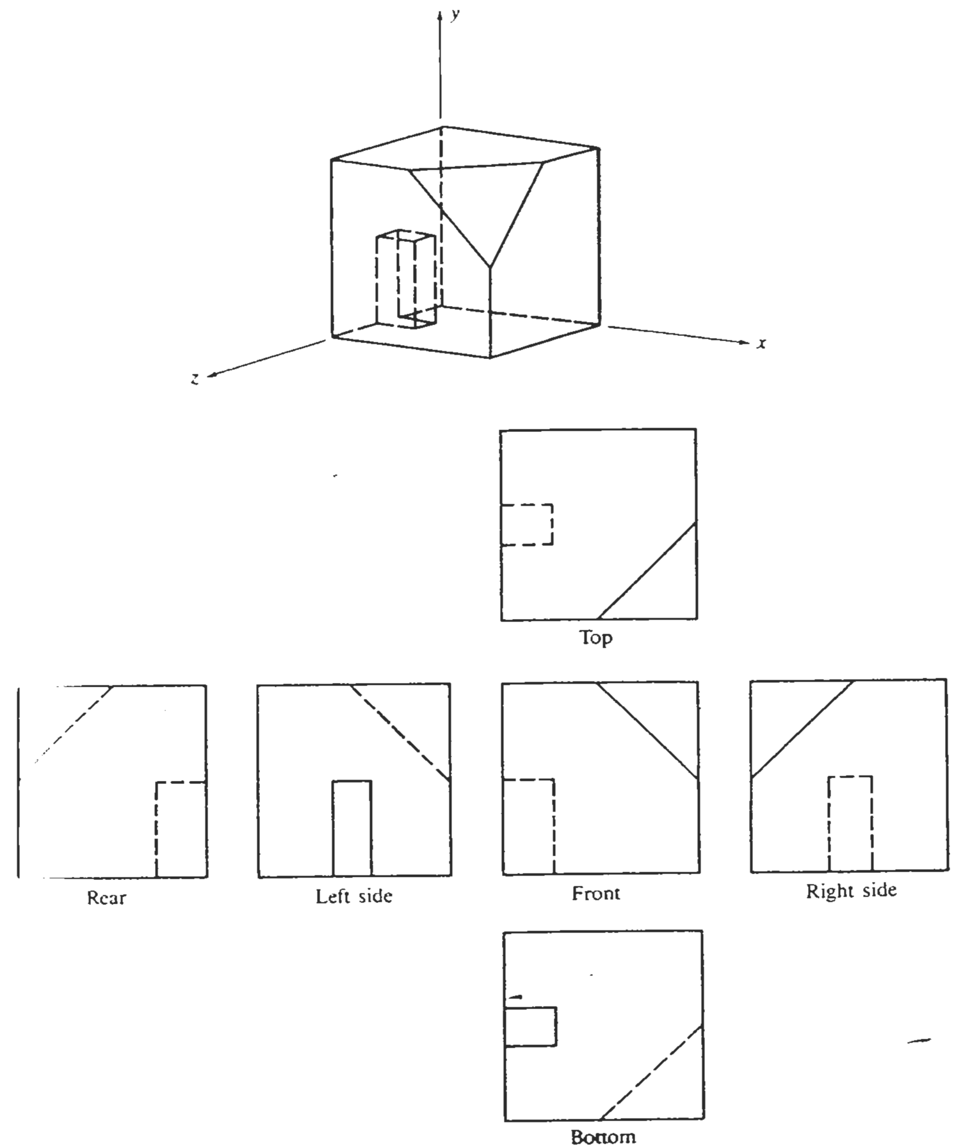


Figure 3–13 Multiview orthographic projection.

is coincident with one of the coordinate axes (see Sec. 3-9). The result is then projected onto the coordinate plane perpendicular to that axis. Figure 3–14c shows an auxiliary plane illustrating the true shape of the triangular corner of the block shown in Fig. 3–13.

An example more fully illustrates these constructions.

Example 3–12 Auxiliary View

Develop an auxiliary view showing the true shape of the triangular corner for the object shown in Fig. 3–14a. The position vectors for the object are

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \\ 0 & 0 & 0.6 & 1 \\ 0.25 & 0 & 0.6 & 1 \\ 0.25 & 0.5 & 0.6 & 1 \\ 0 & 0.5 & 0.6 & 1 \\ 0 & 0 & 0.4 & 1 \\ 0.25 & 0 & 0.4 & 1 \\ 0.25 & 0.5 & 0.4 & 1 \\ 0 & 0.5 & 0.4 & 1 \end{bmatrix}$$

The vertex numbers shown in Fig. 3–14 correspond to the rows in the position vector matrix $[X]$.

The unit outward normal to the triangular face has direction cosines

$$[\,c_x \quad c_y \quad c_z\,] = [\,1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3}\,]$$

and passes through the origin and the point $[\,0.83333 \quad 0.83333 \quad 0.83333\,]$. Recalling the results of Sec. 3–9 and Ex. 3–10, the normal is made coincident with the z-axis by rotation about the x-axis by an angle

$$\alpha = \cos^{-1}(c_z/d) = \cos^{-1}(1/\sqrt{2}) = +45^\circ$$

followed by rotation about the y-axis by an angle

$$\beta = \cos^{-1}(d) = \cos^{-1}(2/\sqrt{6}) = +35.26^\circ$$

Here, the concatenated transformation matrix is

$$[T] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors are

$$[X'] = [X][T] = \begin{bmatrix} -0.408 & -0.707 & 0.577 & 1 \\ 0.408 & -0.707 & 1.155 & 1 \\ 0.204 & -0.354 & 1.443 & 1 \\ -0.408 & 0 & 1.443 & 1 \\ -0.816 & 0 & 1.155 & 1 \\ 0 & 0 & 0 & 1 \\ 0.816 & 0 & 0.577 & 1 \\ 0.408 & 0.707 & 1.155 & 1 \\ -0.408 & 0.707 & 0.577 & 1 \\ 0.204 & 0.354 & 1.443 & 1 \\ -0.245 & -0.424 & 0.354 & 1 \\ -0.041 & -0.424 & 0.491 & 1 \\ -0.245 & -0.071 & 0.779 & 1 \\ -0.449 & -0.071 & 0.635 & 1 \\ -0.163 & -0.283 & 0.231 & 1 \\ 0.041 & -0.283 & 0.375 & 1 \\ -0.163 & 0.071 & 0.664 & 1 \\ -0.367 & 0.071 & 0.52 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–14b. The auxiliary view is created by projecting this intermediate result onto the $z = 0$ plane using Eq. (3–27), i.e.,

$$[P_z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrices $[T]$ and $[P_z]$ are concatenated to yield

$$[T'] = [T][P_z] = \begin{bmatrix} 2/\sqrt{6} & 0 & 0 & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{3} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice the column of zeros. The auxiliary view is then created by

$$[X''] = [X][T']$$

$[X'']$ is the same as $[X']$ except that the third column is all zeros, i.e., the effect of the projection is to neglect the z coordinate. The result is shown in Fig. 3–14c. Hidden lines are shown solid. Notice that the true shape of the triangle, which is equilateral, is shown.

For complex objects it is frequently necessary to show details of the interior. This is accomplished using a sectional view. A sectional view is constructed by passing a plane, called the section or 'cutting' plane, through the object,

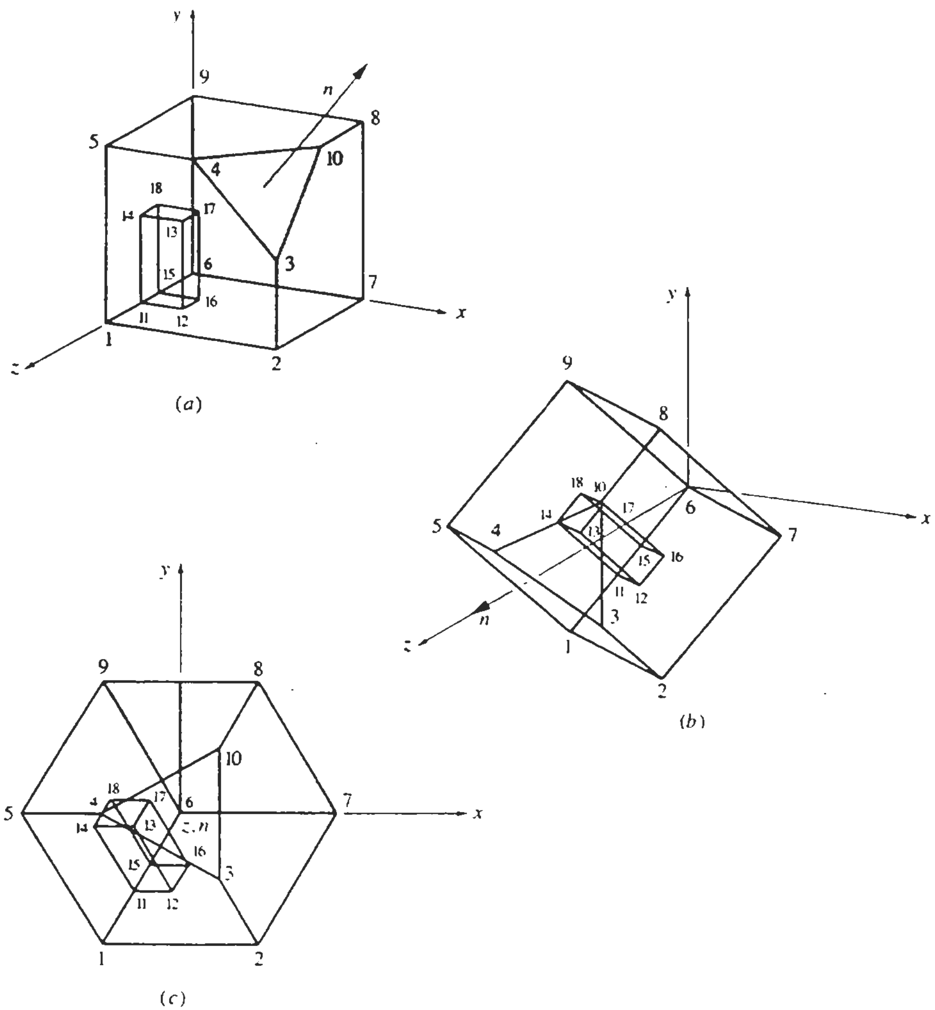


Figure 3–14 Development of an auxiliary view. (a) Trimetric view; (b) normal coincident with the z-axis; (c) projected onto the $z = 0$ plane.

removing the part of the object on one side of the plane and projecting the remainder onto the section plane. Again, a sectional view can be constructed by making the normal to the section plane coincident with one of the coordinate axes (see Sec. 3–9), clipping the object to one side of the section plane (see Ref. 3–1), and finally projecting the result onto the coordinate plane perpendicular to the axis.

Figure 3–15 shows a section plane passing through the notch on the left side of the object of Fig. 3–13. The arrows are used to show the section plane and the viewing direction.

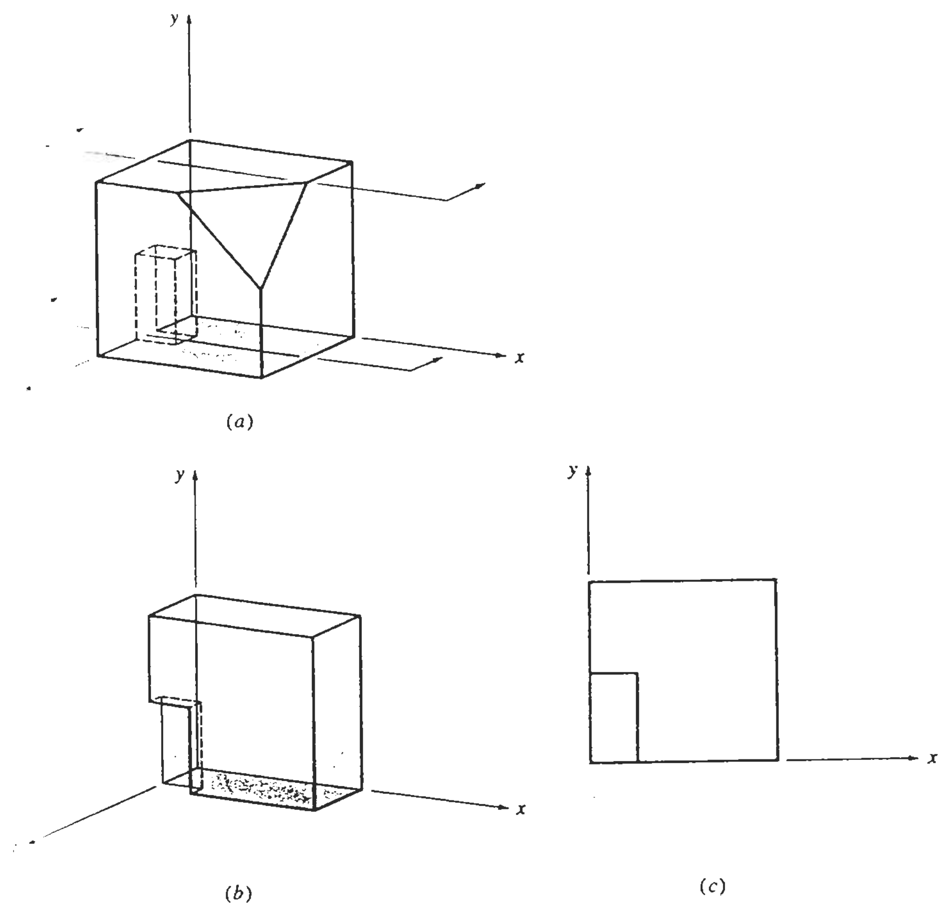


Figure 3–15 Development of a sectional view. (a) Complete object; (b) portion between the section plane and the center of projection removed; (c) projected onto the $z = 0$ plane.

3–13 AXONOMETRIC PROJECTIONS

A single orthographic projection fails to illustrate the general three-dimensional shape of an object. Axonometric projections overcome this limitation. An axonometric projection is constructed by manipulating the object, using rotations and translations, such that at least three adjacent faces are shown.† The result is then projected from a center of projection at infinity onto one of the coordinate planes, usually the $z = 0$ plane. Unless a face is parallel to the plane of projection, an axonometric projection does not show its true shape. However,

†The minimal number of faces occurs for simple cuboidal objects such as are used in most illustrations in this chapter.

the relative lengths of originally parallel lines remain constant, i.e., parallel lines are equally foreshortened. The foreshortening factor is the ratio of the projected length of a line to its true length. There are three axonometric projections of interest: trimetric, dimetric, and isometric, as shown in Fig. 3-10. The trimetric projection is the least restrictive and the isometric projection the most restrictive. In fact, as shown below, an isometric projection is a special case of a dimetric projection, and a dimetric projection is a special case of a trimetric projection.

A trimetric projection is formed by arbitrary rotations, in arbitrary order, about any or all of the coordinate axes, followed by parallel projection onto the $z = 0$ plane. Most of the illustrations in this book are trimetric projections. Figure 3-16 shows several different trimetric projections. Each projection was formed by first rotating about the y-axis and then about the x-axis, followed by parallel projection onto the $z = 0$ plane.

The foreshortening ratios for each projected principal axis (x, y and z) are all different in a general trimetric projection. Here, a principal axis is used in the sense of an axis or edge of the object originally parallel to one of the x, y or z coordinate axes. The wide variety of trimetric projections precludes giving a general equation for these ratios. However, for any specific trimetric projection, the foreshortening ratios are obtained by applying the concatenated transformation matrix to the unit vectors along the principal axes. Specifically,

$$[U][T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} [T]$$

$$= \begin{bmatrix} x_x^* & y_x^* & 0 & 1 \\ x_y^* & y_y^* & 0 & 1 \\ x_z^* & y_z^* & 0 & 1 \end{bmatrix} \qquad (3-30)$$

where $[U]$ is the matrix of unit vectors along the untransformed x, y and z axes, respectively, and $[T]$ is the concatenated trimetric projection matrix. The foreshortening factors along the projected principal axes are then

$$f_x = \sqrt{x_x^{*2} + y_x^{*2}} \qquad (3-31a)$$

$$f_y = \sqrt{x_y^{*2} + y_y^{*2}} \qquad (3-31b)$$

$$f_z = \sqrt{x_z^{*2} + y_z^{*2}} \qquad (3-31c)$$

Example 3-13 provides the details of a trimetric projection.

Example 3-13 Trimetric Projection

Consider the center illustration of Fig. 3-16 formed by a $\phi = 30°$ rotation about the y-axis, followed by a $\theta = 45°$ rotation about the x-axis, and then

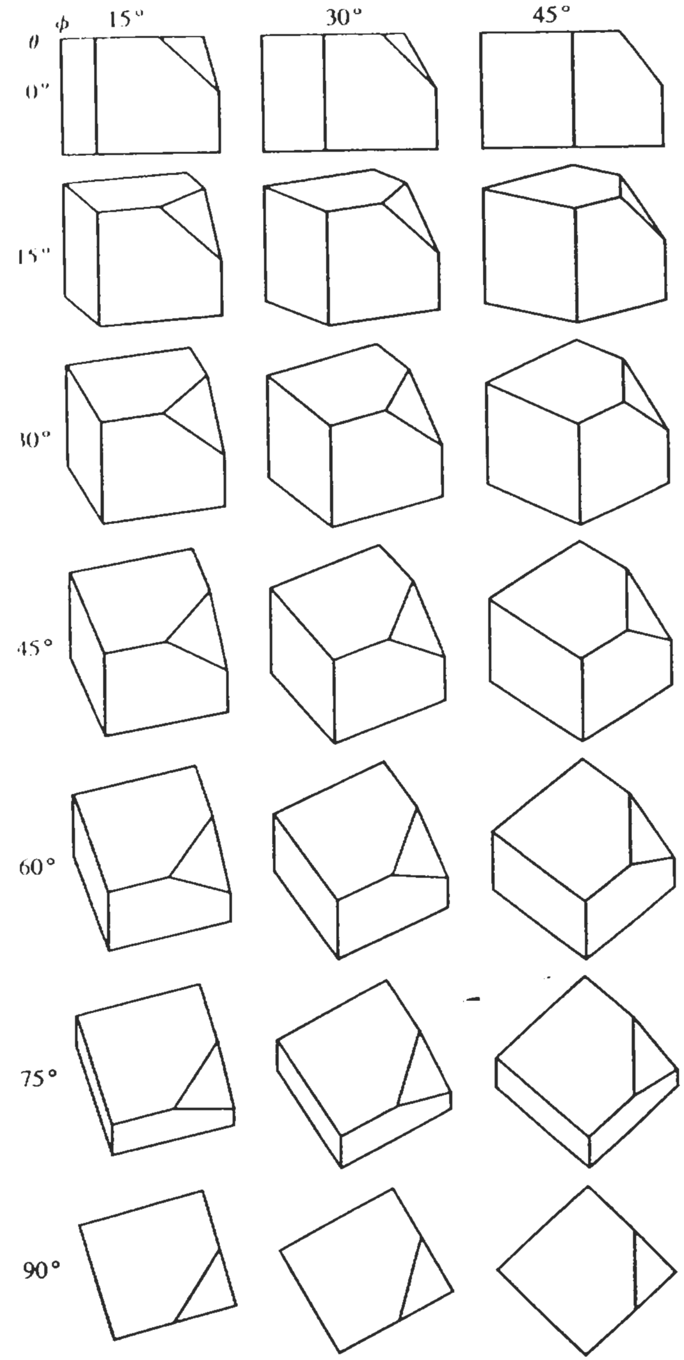


Figure 3-16 Trimetric projections.

parallel projection onto the $z = 0$ plane. The position vectors for the cube with one corner removed are

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix}$$

The concatenated trimetric projection is (see Eqs. 3-8, 3-6, and 3-27)

$$[T] = [R_y][R_x][P_z]$$

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & \sqrt{2}/4 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 & 0 \\ 1/2 & -\sqrt{6}/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the transformed position vectors are

$$[X^*] = [X][T] = \begin{bmatrix} 0.5 & -0.612 & 0 & 1 \\ 1.366 & -0.259 & 0 & 1 \\ 1.366 & 0.095 & 0 & 1 \\ 0.933 & 0.272 & 0 & 1 \\ 0.5 & 0.095 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.866 & 0.354 & 0 & 1 \\ 0.866 & 1.061 & 0 & 1 \\ 0 & 0.707 & 0 & 1 \\ 1.116 & 0.754 & 0 & 1 \end{bmatrix}$$

The foreshortening ratios are

$$[U][T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & \sqrt{2}/4 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 & 0 \\ 1/2 & -\sqrt{6}/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & \sqrt{2}/4 & 0 & 1 \\ 0 & \sqrt{2}/2 & 0 & 1 \\ 1/2 & -\sqrt{6}/4 & 0 & 1 \end{bmatrix}$$

and

$$f_x = \sqrt{(\sqrt{3}/2)^2 + (\sqrt{2}/4)^2} = 0.935$$

$$f_y = \sqrt{2}/2 = 0.707$$

$$f_z = \sqrt{(1/2)^2 + (-\sqrt{6}/4)^2} = 0.791$$

A dimetric projection is a trimetric projection with two of the three foreshortening factors equal; the third is arbitrary. A dimetric projection is constructed by a rotation about the y-axis through an angle ϕ followed by rotation about the x-axis through an angle θ and projection from a center of projection at infinity onto the $z = 0$ plane. The specific rotation angles are as yet unknown. Using Eqs. (3-8), (3-6) and (3-27), the resulting transformation is

$$[T] = [R_y][R_x][P_z]$$

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Concatenation yields

$$[T] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-32)$$

The unit vectors on the x, y, and z principal axes transform to

$$[U^*] = [U][T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[U^*] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 1 \\ 0 & \cos\theta & 0 & 1 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 1 \end{bmatrix} \qquad (3-33)$$

The square of the length of the original unit vector along the x-axis, i.e., the square of the foreshortening factor, is now

$$f_x^2 = (x_x^{*2} + y_x^{*2}) = \cos^2\phi + \sin^2\phi\sin^2\theta \qquad (3-34)$$

Similarly, the squares of the lengths of the original unit vectors along the y-

and z-axes are given by

$$f_y^2 = (x_y^{*2} + y_y^{*2}) = \cos^2\theta \tag{3-35}$$

$$f_z^2 = (x_z^{*2} + y_z^{*2}) = \sin^2\phi + \cos^2\phi\sin^2\theta \tag{3-36}$$

Equating the foreshortening factors along the x and y principal axes† yields one equation in the two unknown rotation angles ϕ and θ. Specifically,

$$\cos^2\phi + \sin^2\phi\sin^2\theta = \cos^2\theta$$

Using the identities $\cos^2\phi = 1 - \sin^2\phi$ and $\cos^2\theta = 1 - \sin^2\theta$ yields

$$\sin^2\phi = \frac{\sin^2\theta}{1 - \sin^2\theta} \tag{3-37}$$

A second relation between ϕ and θ is obtained by choosing the foreshortening factor along the z principal axis f_z. Combining Eqs. (3-36) and (3-37) using $\cos^2\phi = 1 - \sin^2\phi$ yields

$$2\sin^2\theta - 2\sin^4\theta - (1 - \sin^2\theta)f_z^2 = 0$$

or

$$2\sin^4\theta - (2 + f_z^2)\sin^2\theta + f_z^2 = 0 \tag{3-38}$$

After letting $u = \sin^2\theta$, solution yields

$$\sin^2\theta = f_z^2/2,\ 1$$

Since the $\sin^2\theta = 1$ solution yields an infinite result when substituted into Eq. (3-37), it is discarded. Hence,

$$\theta = \sin^{-1}\left(\pm f_z/\sqrt{2}\right) \tag{3-39}$$

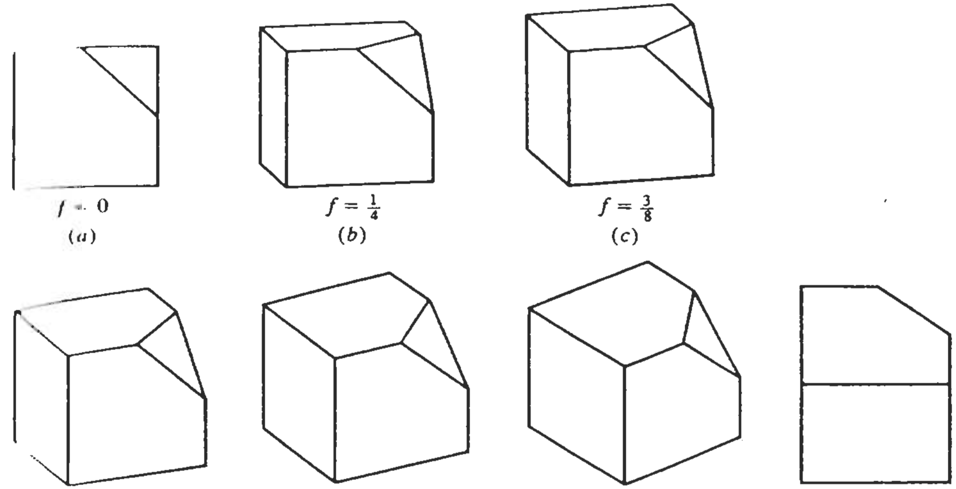
Substituting into Eq. (3-37) yields

$$\phi = \sin^{-1}\left(\pm f_z/\sqrt{2 - f_z^2}\right) \tag{3-40}$$

This result shows that the range of foreshortening factors is $0 \le f_z \le 1$.‡ Further, note that each foreshortening factor f_z yields four possible dimetric projections.

Figure 3-17 shows dimetric projections for various foreshortening factors. For each foreshortening factor, the dimetric projection corresponding to a positive rotation about the y-axis followed by a positive rotation about the x-axis was chosen.

† Any two of the three principal axes could have been used.
‡ Negative foreshortening factors are not sensible.



$f = 0$ (a) $f = \frac{1}{4}$ (b) $f = \frac{3}{8}$ (c) $f = \frac{1}{2}$ (d) $f = \frac{5}{8}$ (e) $f = \frac{3}{4}$ (f) $f = 1$ (g)

Figure 3-17 Dimetric projections for various foreshortening factors. (a) 0; (b) 1/4; (c) 3/8; (d) 1/2; (e) 5/8; (f) 3/4; (g) 1.

Figure 3-18 shows the four possible dimetric projections for a foreshortening factor of 5/8.

An example illustrates specific results.

Example 3-14 Dimetric Projections

For the cube with the corner cut off, determine the dimetric projection for a foreshortening factor along the z-axis of 1/2.

From Eq. (3-39)

$$\theta = \sin^{-1}\left(\pm f_z/\sqrt{2}\right) = \sin^{-1}\left(\pm 1/2\sqrt{2}\right) = \sin^{-1}(\pm 0.35355) = \pm 20.705^\circ$$

From Eq. (3-40)

$$\phi = \sin^{-1}\left(\pm f_z/\sqrt{2 - f_z^2}\right) = \sin^{-1}\left(\pm 1/2/\sqrt{7/4}\right) = \sin^{-1}(\pm 0.378) = \pm 22.208^\circ$$

Choosing $\phi = +22.208^\circ$ and $\theta = +20.705^\circ$, Eq. (3-32) yields the dimetric

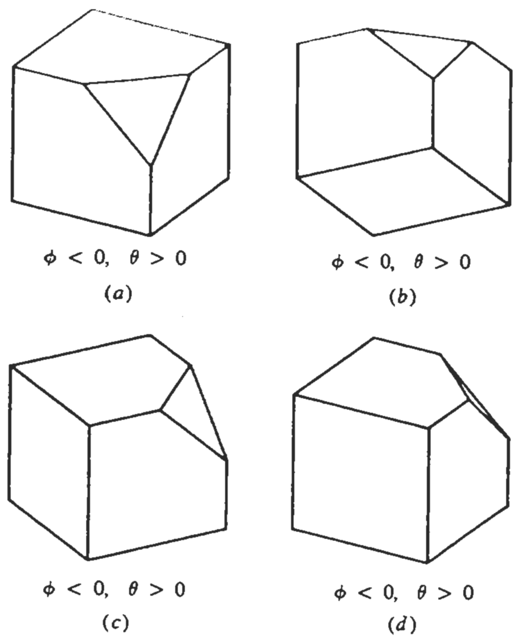


Figure 3–18 Four possible dimetric projections for a foreshortening factor of 5/8 and rotation angles $\phi = \pm 29.52°$, $\theta = \pm 26.23°$. (a) $\phi = -29.52°$, $\theta = +26.23°$; (b) $\phi = -29.52°$, $\theta = -26.23°$; (c) $\phi = +29.52°$, $\theta = +26.23°$; (d) $\phi = +29.52°$, $\theta = -26.23°$.

projection matrix

$$[\,T\,] = \begin{bmatrix} 0.926 & 0.134 & 0 & 0 \\ 0 & 0.935 & 0 & 0 \\ 0.378 & -0.327 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recalling the position vectors for the cube with the corner cut off $[\,X\,]$ (see Ex. 3-13), the transformed position vectors are

$$[\,X^*\,] = [\,X\,][\,T\,] = \begin{bmatrix} 0.378 & -0.327 & 0 & 1 \\ 1.304 & -0.194 & 0 & 1 \\ 1.304 & 0.274 & 0 & 1 \\ 0.841 & 0.675 & 0 & 1 \\ 0.378 & 0.608 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.926 & 0.134 & 0 & 1 \\ 0.926 & 1.069 & 0 & 1 \\ 0 & 0.935 & 0 & 1 \\ 1.115 & 0.905 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–17d.

A dimetric projection allows two of the three transformed principal axes to be measured with the same scale factor. Measurements along the third transformed principal axis require a different scale factor. If accurate scaling of the dimensions of the projected object is required, this can lead to both confusion and error. An isometric projection eliminates this problem.

In an isometric projection all three foreshortening factors are equal. Recalling Eqs. (3–34) to (3–36) and equating Eqs. (3–34) and (3–35) again yields Eq. (3–37), i.e.,

$$\sin^2\phi = \frac{\sin^2\theta}{1-\sin^2\theta} \qquad (3-37)$$

Equating Eqs. (3–35) and (3–36) yields

$$\sin^2\phi = \frac{1-2\sin^2\theta}{1-\sin^2\theta} \qquad (3-41)$$

From Eqs. (3–37) and (3–41) it follows that $\sin^2\theta = 1/3$ or $\sin\theta = \pm\sqrt{1/3}$ and $\theta = \pm 35.26°$. Then

$$\sin^2\phi = \frac{1/3}{1-1/3} = 1/2$$

and $\phi = \pm 45°$. Again note that there are four possible isometric projections. These are shown in Fig. 3–19. The foreshortening factor for an isometric projection is (see Eq. 3–35)

$$f = \sqrt{\cos^2\theta} = \sqrt{2/3} = 0.8165$$

In fact, an isometric projection is a special case of a dimetric projection with $f = 0.8165$.

The angle that the projected x-axis makes with the horizontal is important in manual construction of isometric projections. Transforming the unit vector along the x-axis using the isometric projection matrix yields

$$[\,U_x^*\,] = [\,1 \quad 0 \quad 0 \quad 1\,]\begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [\,\cos\phi \quad \sin\phi\sin\theta \quad 0 \quad 1\,]$$

The angle between the projected x-axis and the horizontal is then

$$\tan\alpha = \frac{y_x^*}{x_x^*} = \frac{\sin\phi\sin\theta}{\cos\phi} = \pm\sin\theta \qquad (3-42)$$

since $\sin\phi = \cos\phi$ for $\phi = 45°$. Alpha is then

$$\alpha = \tan^{-1}(\pm\sin 35.26439°) = \pm 30°$$

A plastic right triangle with included angles of 30° and 60° is a commonly used tool for manually constructing isometric projections. An example illustrates the details.

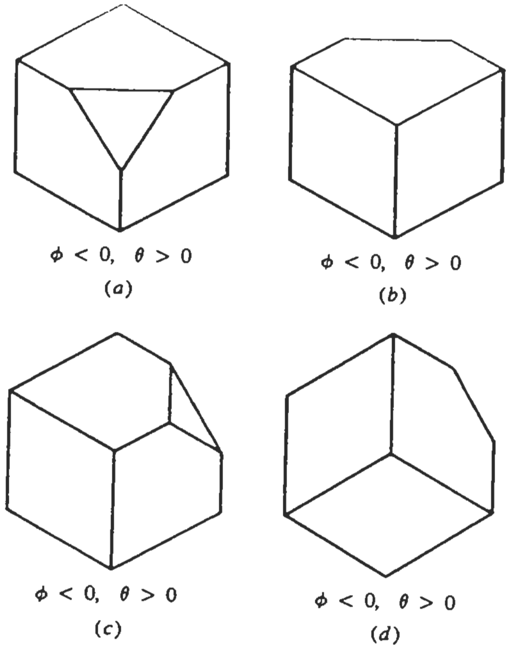


Figure 3–19 Four possible isometric projections with rotation angles $\phi = \pm 45°$, $\theta = \pm 35.26°$. (a) $\phi = -45°$, $\theta = +35.26°$; (b) $\phi = -45°$, $\theta = -35.26°$; (c) $\phi = +45°$, $\theta = +35.26°$; (d) $\phi = +45°$, $\theta = -35.26°$.

Example 3–15 Isometric Projection

Again considering the cube with the corner cut off (see Ex. 3–13), determine the isometric projection for $\phi = -45°$ and $\theta = +35.26439°$. From Eq. (3–32) the isometric projection transformation is

$$[T] = \begin{bmatrix} 0.707 & -0.408 & 0 & 0 \\ 0 & 0.816 & 0 & 0 \\ -0.707 & -0.408 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recalling the position vectors $[X]$, the transformed position vectors are

$$[X^*] = [X][T] = \begin{bmatrix} -0.707 & -0.408 & 0 & 1 \\ 0 & -0.816 & 0 & 1 \\ 0 & -0.408 & 0 & 1 \\ -0.354 & 0.204 & 0 & 1 \\ -0.707 & 0.408 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.707 & -0.408 & 0 & 1 \\ 0.707 & 0.408 & 0 & 1 \\ 0 & 0.816 & 0 & 1 \\ 0.354 & 0.204 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–19a.

3-14 OBLIQUE PROJECTIONS

In contrast to the orthographic and axonometric projections for which the projectors are perpendicular to the plane of projection, an oblique projection is formed by parallel projectors from a center of projection at infinity that intersect the plane of projection at an oblique angle. The general scheme is shown in Fig. 3–20.

Oblique projections illustrate the general three-dimensional shape of the object. However, only faces of the object parallel to the plane of projection are shown at their true size and shape, i.e., angles and lengths are preserved for these faces only. In fact, the oblique projection of these faces is equivalent to an orthographic front view. Faces not parallel to the plane of projection are distorted.

Two oblique projections, cavalier and cabinet, are of particular interest. A cavalier projection is obtained when the angle between the oblique projectors and the plane of projection is 45°. In a cavalier projection the foreshortening factors for all three principal directions are equal. The resulting figure appears too thick. A cabinet projection is used to 'correct' this deficiency.

An oblique projection for which the foreshortening factor for edges perpendicular to the plane of projection is one-half is called a cabinet projection. As is shown below, for a cabinet projection the angle between the projectors and the plane of projection is $\cot^{-1}(1/2) = 63.43°$.

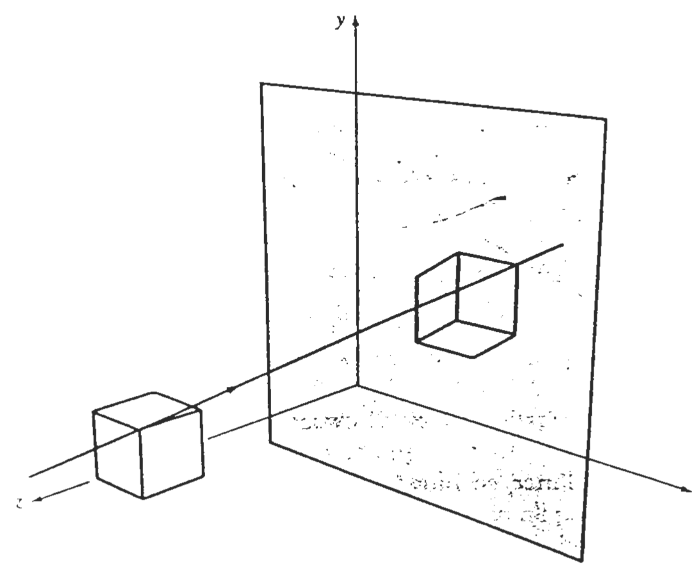


Figure 3–20 Oblique projection.

To develop the transformation matrix for an oblique projection, consider the unit vector $[\,0 \quad 0 \quad 1\,]$ along the z-axis shown in Fig. 3–21. For an orthographic or axonometric projection onto the $z = 0$ plane the vector PO gives the direction of projection. For an oblique projection, the projectors make an angle with the plane of projection. Typical oblique projectors, P_1O and PP_2, are shown in Fig. 3–21. P_1O and PP_2 make an angle β with the plane of projection $z = 0$. Note that all possible projectors through P or O making an angle β with the $z = 0$ plane lie on the surface of a cone with apex at P or O. Thus, there are an infinite number of oblique projections for a given angle β.

The projector P_1O can be obtained from PO by translating the point P to the point P_1 at $[\,-a \quad -b \quad 1\,]$. In the two-dimensional plane through P perpendicular to the z-axis, the 3×3 transformation matrix is

$$[\,T'\,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{bmatrix}$$

In three dimensions this two-dimensional translation is equivalent to a shearing of the vector PO in the x and y directions. The required transformation to accomplish this is

$$[\,T''\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a & -b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

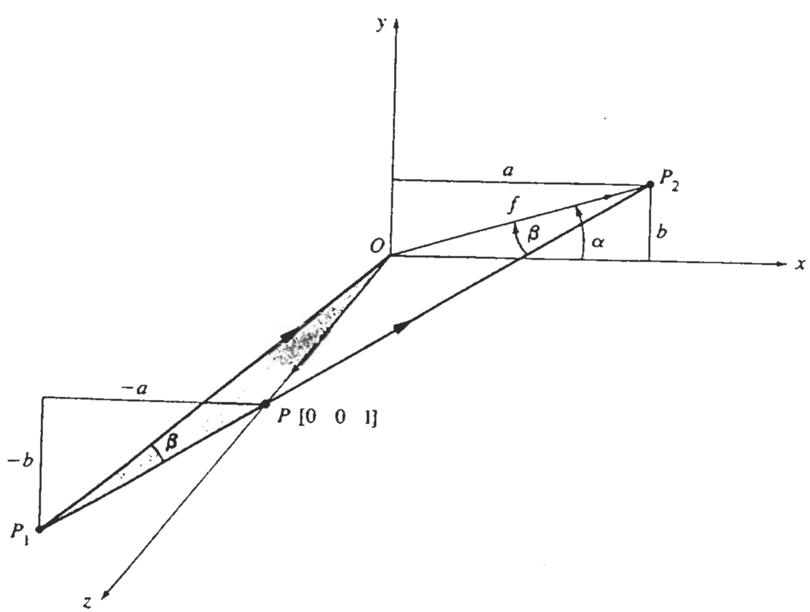


Figure 3–21 Direction of the oblique projection matrix.

Projection onto the $z = 0$ plane yields

$$[\,T\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a & -b & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From Fig. 3–21

$$a = f\cos\alpha$$
$$b = f\sin\alpha$$

where f is the projected length of the z-axis unit vector, i.e., the foreshortening factor, and α is the angle between the horizontal and the projected z-axis. Figure 3–21 also shows that β, the angle between the oblique projectors and the plane of projection, is

$$\beta = \cot^{-1}(f) \qquad (3-43)$$

Thus, the transformation for an oblique projection is

$$[\,T\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-44)$$

If $f = 0$, $\beta = 90°$, then an orthographic projection results. If $f = 1$, the edges perpendicular to the projection plane are not foreshortened. This is the condition for a cavalier projection. From Eq. (3–43)

$$\beta = \cot^{-1}(1) = 45°$$

For a cavalier projection, notice that α is still a free parameter. Figure 3–22 shows cavalier projections for several values of α. Commonly used values of α are 30° and 45°. Values of $180° - \alpha$ are also acceptable.

A cabinet projection is obtained when the foreshortening factor $f = 1/2$. Here

$$\beta = \cot^{-1}(1/2) = 63.435°$$

Again, as shown in Fig. 3–23, α is variable. Common values are 30° and 45°. Values of $180° - \alpha$ are also acceptable. Figure 3–24 shows oblique projections for foreshortening factors $f = 1$, 7/8, 3/4, 5/8, 1/2, with $\alpha = 45°$.

Because one face is shown in its true shape, oblique projections are particularly suited for illustration of objects with circular or otherwise curved faces. Faces with these characteristics should be parallel to the plane of projection to avoid unwanted distortions. Similarly, as in all parallel projections, objects with one dimension significantly larger than the others suffer significant distortion unless the long dimension is parallel to the projection plane. These effects are illustrated in Fig. 3–25.

A detailed example is given below.

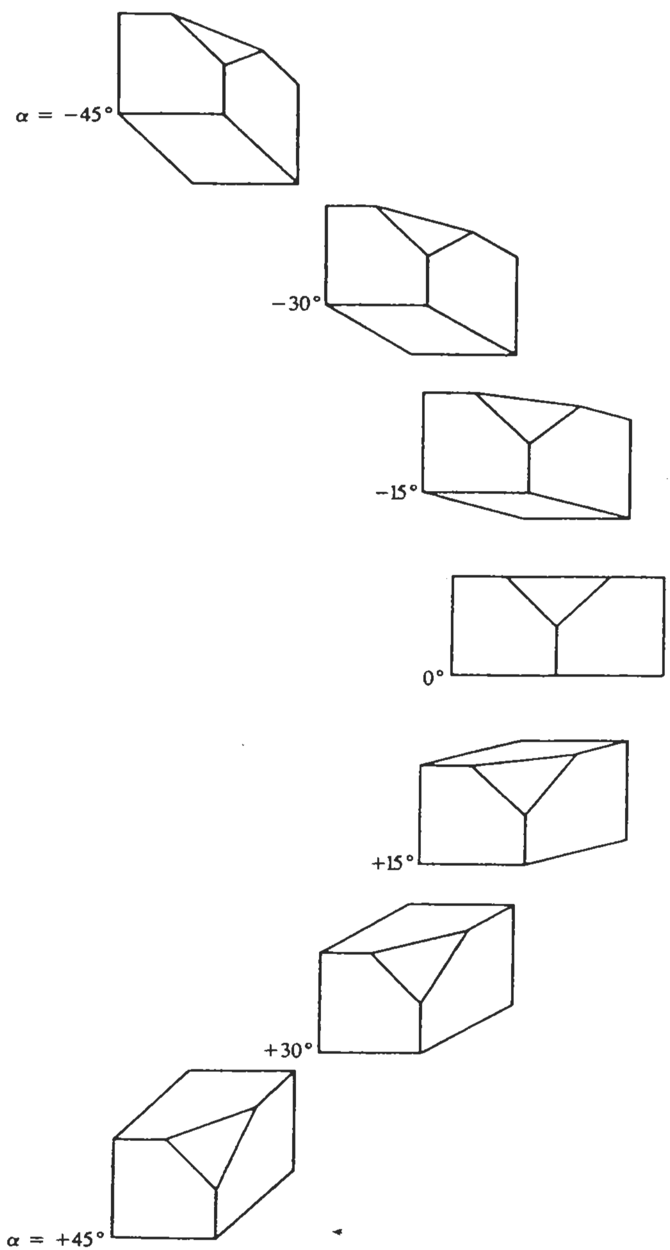


Figure 3–22 Cavalier projections. Top to bottom, $\alpha = -45°$ to $+45°$ at 15° intervals with $\beta = 45°$.

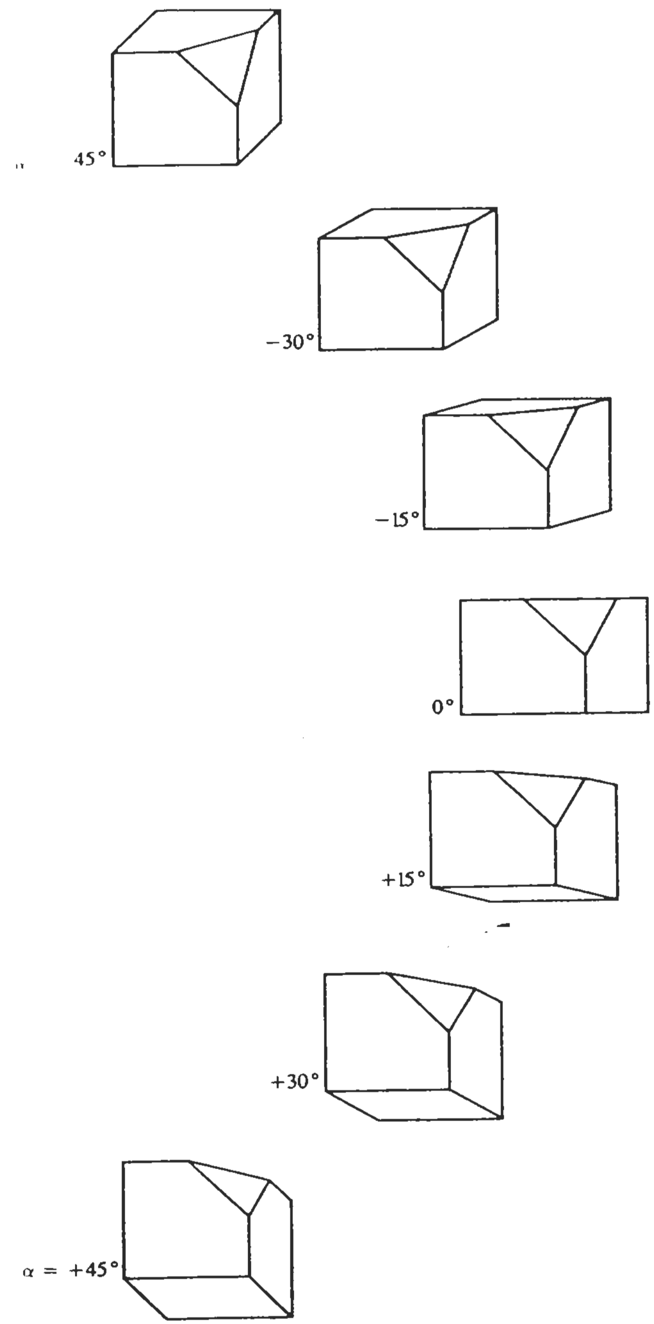


Figure 3–23 Cabinet projections. Top to bottom, $\alpha = -45°$ to $+45°$ at 15° intervals with $f = 0.5$.

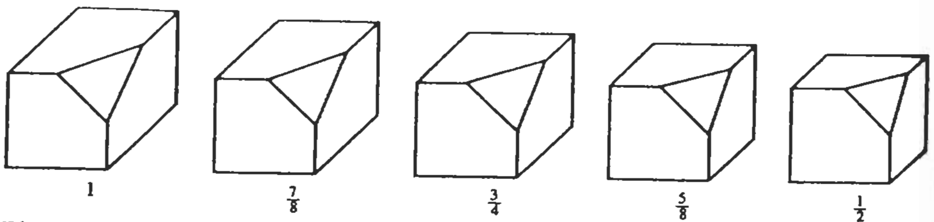


Figure 3-24 Oblique projections. Left to right, $f = 1$, 7/8, 3/4, 5/8, 1/2, with $\alpha = 45°$.

Example 3-16 Oblique Projections

Develop cavalier and cabinet projections for the cube with one corner cut off (see Ex. 3-13).

Recalling that a cavalier projection is an oblique projection with $\beta = 45°$, i.e., a foreshortening factor $f = 1$, and choosing a horizontal inclination angle $\alpha = 30°$, Eq. (3-44) yields the transformation matrix

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.866 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recalling the position vectors for the cube with the corner cut off $[X]$ (see Ex. 3-13), the transformed position vectors are

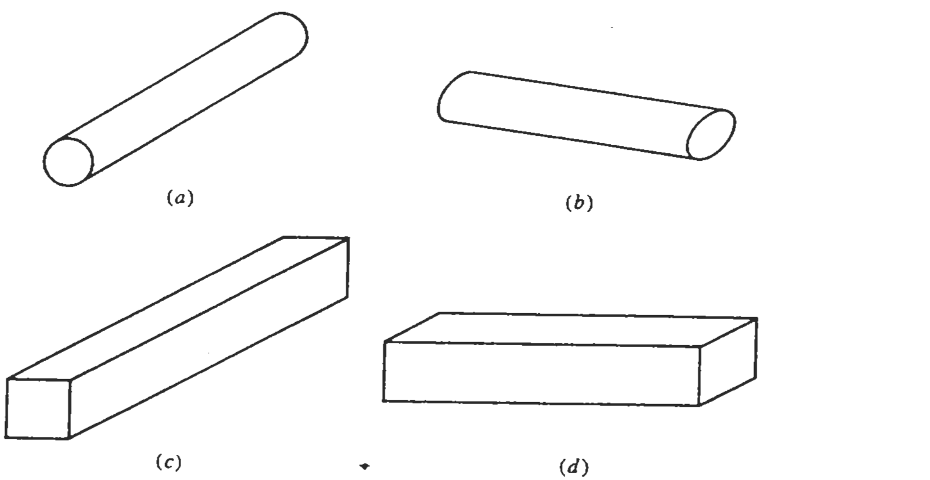


Figure 3-25 Distortion in oblique projections, $f = 5/8, \alpha = 45°$. (a) Circular face parallel to projection plane; (b) circular face perpendicular to projection plane; (c) long dimension perpendicular to projection plane; (d) long dimension parallel to projection plane.

$$[X^*] = [X][T] = \begin{bmatrix} -0.866 & -0.5 & 0 & 1 \\ 0.134 & -0.5 & 0 & 1 \\ 0.134 & 0 & 0 & 1 \\ -0.366 & 0.5 & 0 & 1 \\ -0.866 & 0.5 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0.567 & 0.75 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3-22.

Turning now to the cabinet projection, and recalling that the foreshortening factor is 1/2, Eq. (3-43) yields

$$\beta = \cot^{-1}(1/2) = \tan^{-1}(2) = 63.435°$$

Again choosing $\alpha = 30°$ Eq. (3-44) becomes

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.433 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors for the cabinet projection of the cube with the corner cut off are

$$[X^*] = [X][T] = \begin{bmatrix} -0.433 & -0.25 & 0 & 1 \\ 0.567 & -0.25 & 0 & 1 \\ 0.567 & 0.25 & 0 & 1 \\ 0.067 & 0.75 & 0 & 1 \\ -0.433 & 0.75 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0.783 & 0.875 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3-23.

Notice that for both the cavalier and cabinet projections the triangular corner is *not* shown either true size or true shape because it is *not* parallel to the plane of projection ($z = 0$).

3-15 PERSPECTIVE TRANSFORMATIONS

When any of the first three elements of the fourth column of the general 4×4 homogeneous coordinate transformation matrix is nonzero, a perspective transformation results. As previously mentioned (see Sec. 3-11), a perspective transformation is a transformation from one three space to another three space. In contrast to the parallel transformations previously discussed, in perspective transformations parallel lines converge, object size is reduced with increasing distance from the center of projection, and nonuniform foreshortening of lines in the object as a function of orientation and distance of the object from the center of projection occurs. All of these effects aid the depth perception of the human visual system, but the shape of the object is not preserved.

A single-point perspective transformation is given by

$$[x \quad y \quad z \quad 1]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&r\\0&0&0&1\end{bmatrix} = [x \quad y \quad z \quad rz+1] \tag{3-45}$$

Here $h = rz + 1 \neq 1$. The ordinary coordinates are obtained by dividing through by h, to yield

$$[x^* \quad y^* \quad z^* \quad 1] = \left[\frac{x}{rz+1} \quad \frac{y}{rz+1} \quad \frac{z}{rz+1} \quad 1\right] \tag{3-46}$$

A perspective projection onto some two-dimensional viewing plane is obtained by concatenating an orthographic projection with the perspective transformation. For example, a perspective projection onto the $z = 0$ plane is given by

$$[T] = [P_r][P_z]$$
$$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&r\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&r\\0&0&0&1\end{bmatrix} \tag{3-47}$$

and

$$[x \quad y \quad z \quad 1]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&r\\0&0&0&1\end{bmatrix} = [x \quad y \quad 0 \quad rz+1] \tag{3-48}$$

The ordinary coordinates are

$$[x^* \quad y^* \quad z^* \quad 1] = \left[\frac{x}{rz+1} \quad \frac{y}{rz+1} \quad 0 \quad 1\right] \tag{3-49}$$

To show that Eq. (3–47) produces a perspective projection onto the $z = 0$ plane, consider Fig. 3–26, which illustrates the geometry for a perspective projection of the three-dimensional point P onto a $z = z^* = 0$ plane at P^* from a center of projection at z_c on the z-axis. The coordinates of the projected point P^* are obtained using similar triangles. From Fig. 3–26

$$\frac{x^*}{z_c} = \frac{x}{z_c - z}$$

or

$$x^* = \frac{x}{1 - \frac{z}{z_c}}$$

and

$$\frac{y^*}{\sqrt{x^{*2} + z_c^2}} = \frac{y}{\sqrt{x^2 + (z_c - z)^2}}$$

or

$$y^* = \frac{y}{1 - \frac{z}{z_c}}$$

z^* is, of course, zero.

Letting $r = -1/z_c$ yields results identical to those obtained using Eq. (3–47). Thus, Eq. (3–47) produces a perspective projection onto the $z = 0$ plane from a center of projection at $(-1/r)$ on the z-axis. Notice that as z_c approaches infinity, r approaches zero and an axonometric projection onto the $z = 0$ plane results. Further, notice that for points in the plane of projection, i.e., $z = 0$, the perspective transformation has no effect. Also note that the origin ($x = y = z = 0$) is unaffected. Consequently, if the plane of projection ($z = 0$) passes through an object, then that section of the object is shown at true size and true shape. All other parts of the object are distorted.

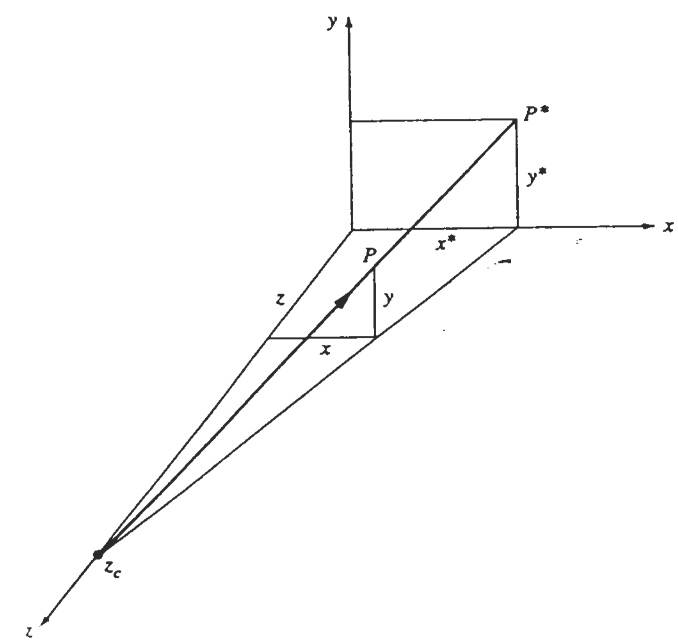


Figure 3–26 Perspective projection of a point.

To help understand the effects of a perspective transformation consider Fig. 3-27. Figure 3-27 shows the perspective projection onto the $z = 0$ plane of the line AB originally parallel to the z-axis, into the line A^*B^* in the $z = 0$ plane, from a center of projection at $-1/r$ on the z-axis. The transformation can be considered in two steps (see Eq. 3-47). First, the perspective transformation of the line AB yields the three-dimensional transformed line $A'B'$ (see Fig. 3-27 below). Subsequent orthographic projection of the line $A'B'$ in three-dimensional perspective space onto the $z = 0$ plane from a center of projection at infinity on the z-axis yields the line A^*B^*.

Examination of Fig. 3-27 shows that the line $A'B'$ intersects the $z = 0$ plane at the same point as the line AB. It also intersects the z-axis at $z = +1/r$. Effectively then, the perspective transformation (see Eqs. 3-45 and 3-46) has transformed the intersection point at infinity of the line AB parallel to the z-axis and the z-axis itself into the finite point at $z = 1/r$ on the z-axis. This point is called the vanishing point.† Notice that the vanishing point lies an equal distance on the opposite side of the plane of projection from the center of projection, e.g., if $z = 0$ is the projection plane and the center of projection is at $z = -1/r$, the vanishing point is at $z = +1/r$.

To confirm this observation consider the perspective transformation of the point at infinity on the $+z$-axis, i.e.,

$$[0 \quad 0 \quad 1 \quad 0]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 1 \quad r] \qquad (3-50)$$

The point $[x^* \quad y^* \quad z^* \quad 1] = [0 \quad 0 \quad 1/r \quad 1]$, corresponding to the transformed point at infinity on the positive z-axis, is now a finite point on the positive z-axis. This means that the entire semi-infinite positive space $(0 \le z \le \infty)$ is transformed to the finite positive half space $0 \le z^* \le 1/r$. Further, all lines originally parallel to the z-axis now pass through the point $[0 \quad 0 \quad 1/r \quad 1]$, the vanishing point.

Before presenting some illustrative examples the single-point perspective transformations with centers of projection and vanishing points on the x- and y-axes are given for completeness. The single-point perspective transformation

$$[x \quad y \quad z \quad 1]\begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad z \quad (px+1)] \qquad (3-51)$$

with ordinary coordinates

$$[x^* \quad y^* \quad z^* \quad 1] = \left[\frac{x}{px+1} \quad \frac{y}{px+1} \quad \frac{z}{px+1} \quad 1\right] \qquad (3-52)$$

†Intuitively the vanishing point is that point in the 'distance' to which parallel lines 'appear' to converge and 'vanish'. A practical example is a long straight railroad track.

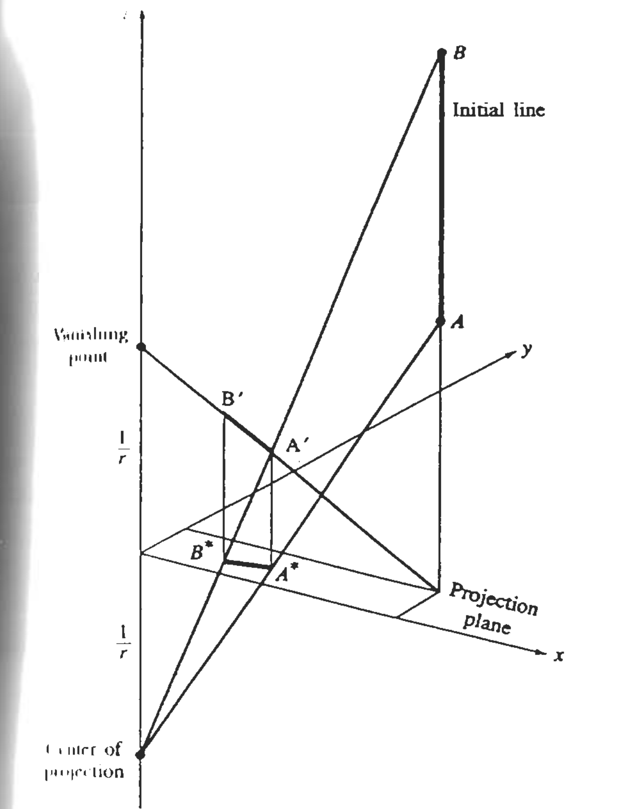


Figure 3–27 Projection of a line parallel to the z-axis.

has a center of projection at $[-1/p \quad 0 \quad 0 \quad 1]$ and a vanishing point located on the x-axis at $[1/p \quad 0 \quad 0 \quad 1]$.

The single-point perspective transformation

$$[x \quad y \quad z \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad z \quad (qy+1)] \qquad (3-53)$$

with ordinary coordinates

$$[x^* \quad y^* \quad z^* \quad 1] = \left[\frac{x}{qy+1} \quad \frac{y}{qy+1} \quad \frac{z}{qy+1} \quad 1\right] \qquad (3-54)$$

has a center of projection at $[0 \quad -1/q \quad 0 \quad 1]$ and a vanishing point located on the y-axis at $[0 \quad 1/q \quad 0 \quad 1]$.

Example 3–17 Perspective Transformation of a Line Parallel to the z-Axis

Consider the line segment AB in Fig. 3–27 parallel to the z-axis with end points $A[3\ \ 2\ \ 4\ \ 1]$ and $B[3\ \ 2\ \ 8\ \ 1]$. Perform a perspective projection onto the $z = 0$ plane from a center of projection at $z_c = -2$. The perspective transformation of AB to $A'B'$ with $r = 0.5$ is

$$\begin{matrix} A \\ B \end{matrix}\begin{bmatrix} 3 & 2 & 4 & 1 \\ 3 & 2 & 8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 & 3 \\ 3 & 2 & 8 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.667 & 1.333 & 1 \\ 0.6 & 0.4 & 1.6 & 1 \end{bmatrix}\begin{matrix} A' \\ B' \end{matrix}$$

The parametric equation of the line segment $A'B'$ is

$$P(t) = [\,A'\,] + [\,B' - A'\,]t \quad 0 \le t \le 1$$

or $\quad P(t) = [1\ \ 0.667\ \ 1.333\ \ 1] + [-0.4\ \ -0.267\ \ 0.267\ \ 0]t$

Intersection of this line with the $x = 0$, $y = 0$ and $z = 0$ planes yields

$$\begin{aligned} x(t) &= 0 = 1 - 0.4t & &\rightarrow & t &= 2.50 \\ y(t) &= 0 = 0.667 - 0.267t & &\rightarrow & t &= 2.50 \\ z(t) &= 0 = 1.333 + 0.267t & &\rightarrow & t &= -5.0 \end{aligned}$$

Substituting $t = 2.5$ into the parametric equation of the line $A'B'$ yields

$$z(2.5) = 1.333 + (0.267)(2.5) = 2.0$$

which represents the intersection of the line $A'B'$ with the z-axis at $z = +1/r$, the vanishing point. Now substituting $t = -5.0$ into the x and y component equations yields the intersection with the $z = 0$ plane, i.e.,

$$\begin{aligned} x(-5.0) &= 1 - (0.4)(-5.0) = 3.0 \\ y(-5.0) &= 0.667 - (0.267)(-5.0) = 2.0 \end{aligned}$$

which is the same as the intersection of the line AB with the $z = 0$ plane.

Projection of line $A'B'$ into the line A^*B^* in the $z = 0$ plane is given by

$$\begin{matrix} A' \\ B' \end{matrix}\begin{bmatrix} 1 & 0.667 & 1.333 & 1 \\ 0.6 & 0.4 & 1.6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.667 & 0 & 1 \\ 0.6 & 0.4 & 0 & 1 \end{bmatrix}\begin{matrix} A^* \\ B^* \end{matrix}$$

An example using a simple cube is given below.

Example 3–18 Single-Point Perspective Transformation of a Cube

Perform a perspective projection onto the $z = 0$ plane of the unit cube shown in Fig. 3–28a from a center of projection at $z_c = 10$ on the z-axis.

The single-point perspective factor r is

$$r = -1/z_c = -1/10 = -0.1$$

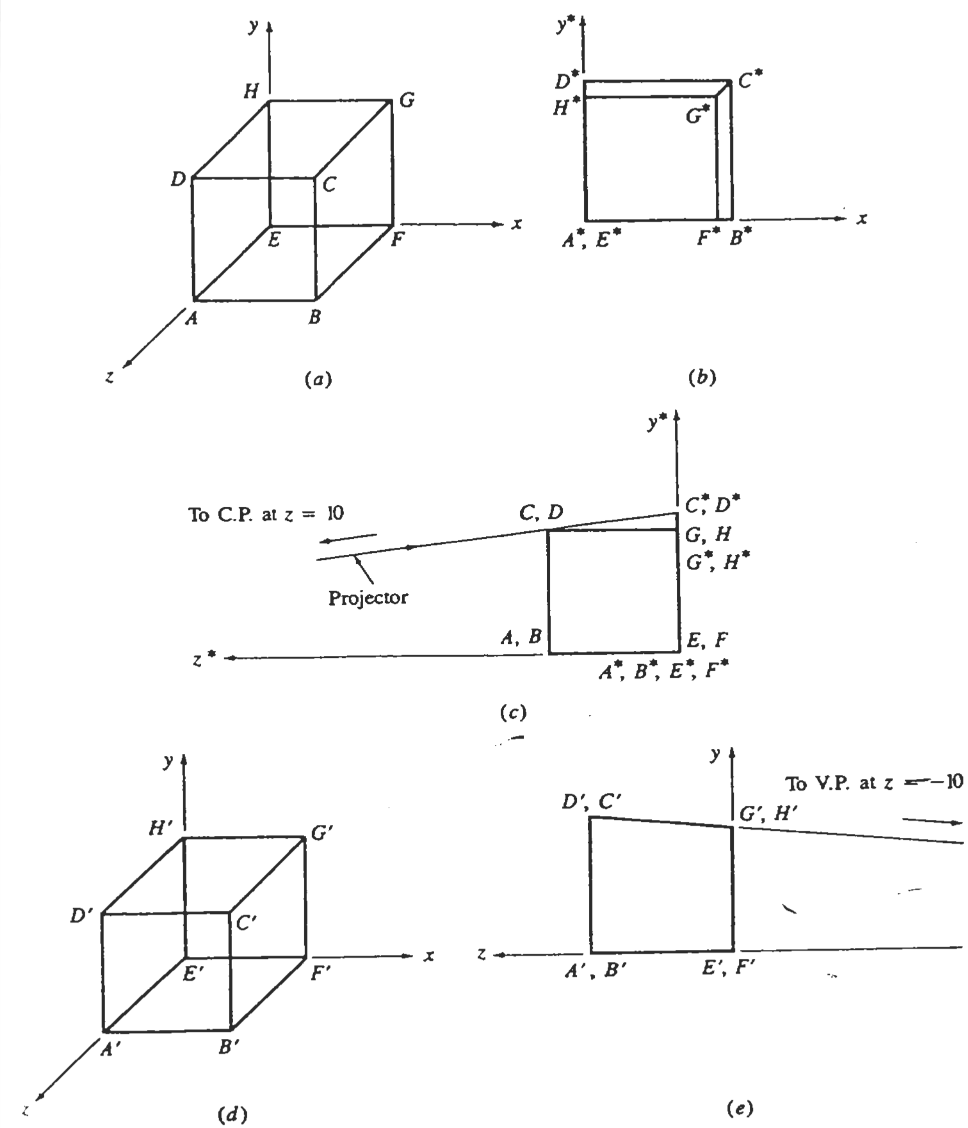


Figure 3–28 Single-point perspective projection of a unit cube.

From Eq. (3-48) the transformation is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0.9 \\ 1 & 0 & 0 & 0.9 \\ 1 & 1 & 0 & 0.9 \\ 0 & 1 & 0 & 0.9 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1.11 & 0 & 0 & 1 \\ 1.11 & 1.11 & 0 & 1 \\ 0 & 1.11 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–28b. Notice that since the center of projection is on the positive z-axis the front face of the cube $ABCD$ projects larger than the back face. Figure 3–28c, which is a parallel projection of the original cube onto the $x = 0$ plane, shows why.

Notice also that because the vanishing point lies on the z-axis the line C^*G^* in Fig. 3–28b passes through the origin.

An alternate and equivalent approach to that above is to first perform the perspective transformation to obtain a distorted object in three space and then to orthographically project the result onto some plane. The distorted object is obtained by

$$[X'] = [X][P_r] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0.9 \\ 1 & 0 & 1 & 0.9 \\ 1 & 1 & 1 & 0.9 \\ 0 & 1 & 1 & 0.9 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1.11 & 1 \\ 1.11 & 0 & 1.11 & 1 \\ 1.11 & 1.11 & 1.11 & 1 \\ 0 & 1.11 & 1.11 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The result is shown using an oblique projection in Fig. 3–28d. Notice that the 'front' face ($A'B'C'D'$) is larger than the 'rear' face ($E'F'G'H'$). Subsequent orthographic projection onto the $z = 0$ plane yields the same result for $[X^*]$ as given above and illustrated in Fig. 3–28c.

Figure 3–28e, which is an orthographic projection of the distorted object of Fig. 3–28d onto the $x = 0$ plane, shows that the edges of the distorted object originally parallel to the z-axis now converge to the vanishing point at $z = -10$.

Figure 3–28b does not convey the three-dimensional character of the cube. A more satisfactory result is obtained by centering the cube. This is illustrated in the next example.

Example 3–19 Single-Point Perspective Transformation of a Centered Cube

The cube shown in Fig. 3–28a can be centered on the z-axis by translating it 1/2 unit in the x and y directions. The resulting transformation is

$$[T] = [Tr_{xy}][P_{rz}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix}$$

The translated cube is shown in Fig. 3–29a.

The transformed ordinary coordinates are

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -0.5 & 0 & 0.9 \\ 0.5 & -0.5 & 0 & 0.9 \\ 0.5 & 0.5 & 0 & 0.9 \\ -0.5 & 0.5 & 0 & 0.9 \\ -0.5 & -0.5 & 0 & 1 \\ 0.5 & -0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \\ -0.5 & 0.5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.56 & -0.56 & 0 & 1 \\ 0.56 & -0.56 & 0 & 1 \\ 0.56 & 0.56 & 0 & 1 \\ -0.56 & 0.56 & 0 & 1 \\ -0.5 & -0.5 & 0 & 1 \\ 0.5 & -0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \\ -0.5 & 0.5 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3-29b. Notice that the lines originally parallel to the z-axis connecting the corners of the front and rear faces now converge to intersect the z-axis ($x = 0$, $y = 0$) in Fig. 3-29b.

Unfortunately, the resulting display still does not provide an adequate perception of the three-dimensional shape of the object. Consequently, we turn our attention to more complex perspective transformations.

If two terms in the first three rows of the fourth column of the 4 × 4 transformation matrix are nonzero, the result is called a two-point perspective transformation. The two-point perspective transformation

$$[x \quad y \quad z \quad 1]\begin{bmatrix}1 & 0 & 0 & p\\0 & 1 & 0 & q\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} = [x \quad y \quad z \quad (px+qy+1)] \qquad (3-55)$$

with ordinary coordinates,

$$[x^* \quad y^* \quad z^* \quad 1] = \left[\frac{x}{px+qy+1} \quad \frac{y}{px+qy+1} \quad \frac{z}{px+qy+1} \quad 1\right] \qquad (3-56)$$

has two centers of projection: one on the x-axis at $[-1/p \quad 0 \quad 0 \quad 1]$ and one on the y-axis at $[0 \quad -1/q \quad 0 \quad 1]$, and two vanishing points: one on the x-axis at $[1/p \quad 0 \quad 0 \quad 1]$ and one on the y-axis at $[0 \quad 1/q \quad 0 \quad 1]$. Note that the two-point perspective transformation given by Eq. (3-55) can be obtained by concatenation of two single-point perspective transformations. Specifically,

$$[P_{pq}] = [P_p][P_q]$$
$$= [P_q][P_p]$$

where $[P_{pq}]$ is given by Eq. (3-55), $[P_p]$ by Eq. (3-53) and $[P_q]$ by Eq. (3-51). The next example shows the details of a two-point perspective projection.

Example 3-20 Two-Point Perspective Projections

Again consider the cube described in Ex. 3-18 transformed by a two-point perspective transformation with centers of projection at $x = -10$ and $y = -10$ projected onto the $z = 0$ plane. The transformation is obtained by concatenating Eqs. (3-55) and (3-27). Specifically,

$$[T] = [P_{pq}][P_z] = \begin{bmatrix}1 & 0 & 0 & p\\0 & 1 & 0 & q\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0 & p\\0 & 1 & 0 & q\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

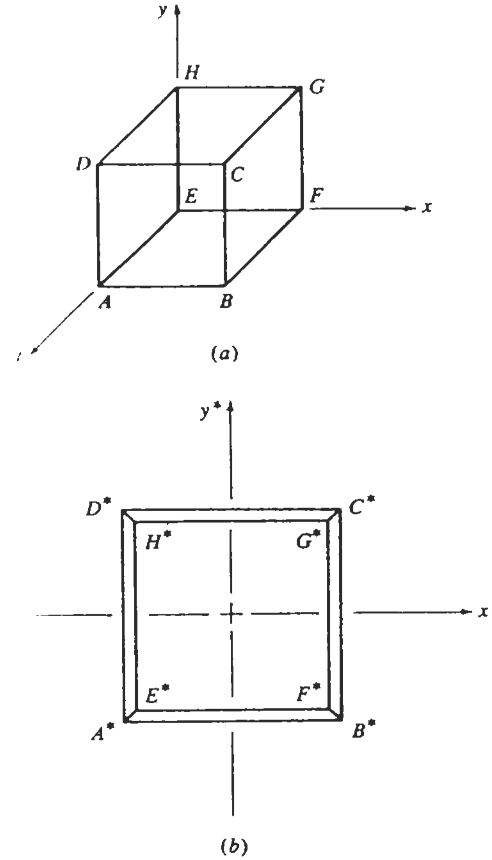


Figure 3-29 Single-point perspective projection of a centered unit cube.

Here p and q are

$$p = -1/(-10) = 0.1 \qquad q = -1/(-10) = 0.1$$

The transformed coordinates of the cube are

$$[X^*] = [X][T] = \begin{bmatrix}0 & 0 & 1 & 1\\1 & 0 & 1 & 1\\1 & 1 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 0 & 1\\1 & 0 & 0 & 1\\1 & 1 & 0 & 1\\0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0.1\\0 & 1 & 0 & 0.1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1.1 \\ 1 & 1 & 0 & 1.2 \\ 0 & 1 & 0 & 1.1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1.1 \\ 1 & 1 & 0 & 1.2 \\ 0 & 1 & 0 & 1.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.909 & 0 & 0 & 1 \\ 0.833 & 0.833 & 0 & 1 \\ 0 & 0.909 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.909 & 0 & 0 & 1 \\ 0.833 & 0.833 & 0 & 1 \\ 0 & 0.909 & 0 & 1 \end{bmatrix}.$$

The results are shown in Fig. 3-30a. The two vanishing points are at $x = 10$ and $y = 10$.

Centering the cube on the z-axis by translating -0.5 in x and y as was done in Ex. 3-18 yields the concatenated transformation matrix

$$[T] = [T_{xy}][P_{pq}][P_z]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.9 \end{bmatrix}$$

where projection onto the $z = 0$ plane has been assumed. Notice that here the overall scaling factor (see Eq. 3-4) is no longer unity, i.e., there is an apparent scaling of the cube caused by translation. The transformed coordinates are

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -0.5 & 0 & 0.9 \\ 0.5 & -0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1.1 \\ -0.5 & 0.5 & 0 & 1 \\ -0.5 & -0.5 & 0 & 0.9 \\ 0.5 & -0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1.1 \\ -0.5 & 0.5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.56 & -0.56 & 0 & 1 \\ 0.5 & -0.5 & 0 & 1 \\ 0.46 & 0.46 & 0 & 1 \\ -0.5 & 0.5 & 0 & 1 \\ -0.56 & -0.56 & 0 & 1 \\ 0.5 & -0.5 & 0 & 1 \\ 0.46 & 0.46 & 0 & 1 \\ -0.5 & 0.5 & 0 & 1 \end{bmatrix}$$

The results are shown in Fig. 3-30b.

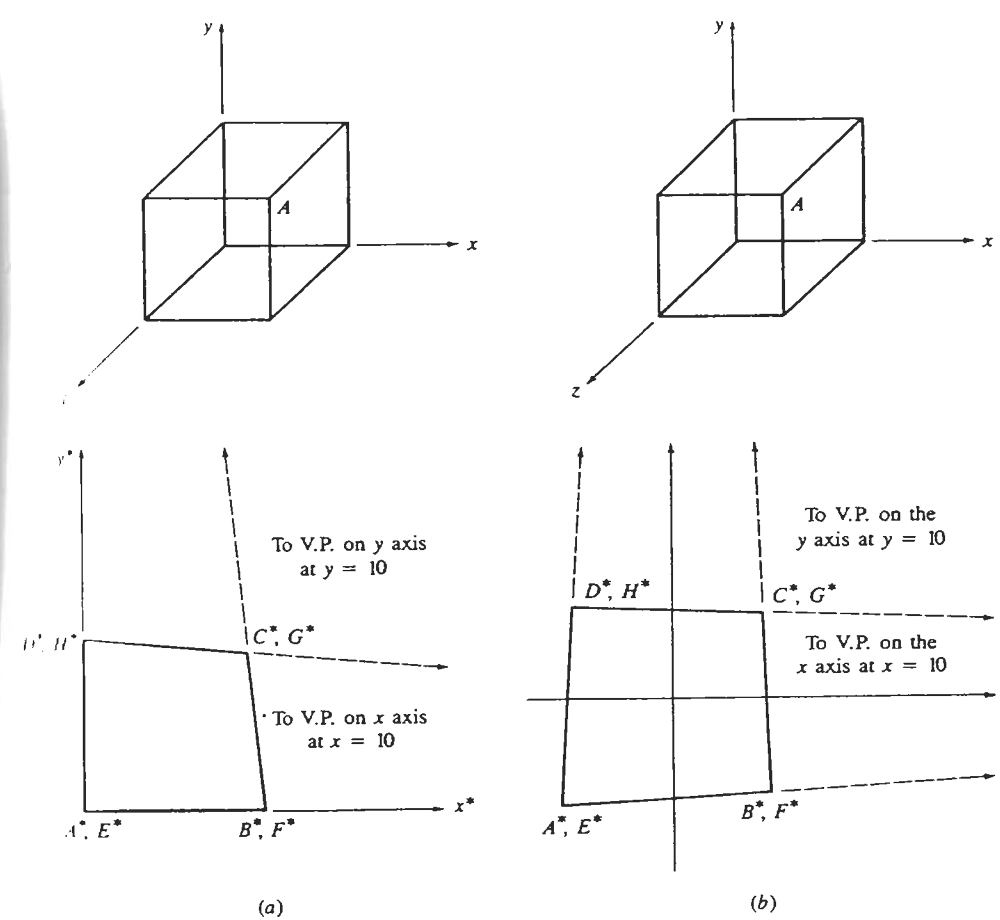


Figure 3–30 Two-point perspective projections. (a) Non-centered; (b) centered.

Again the resulting display does not provide an adequate perception of the three-dimensional shape of the object. Hence we turn our attention to three-point perspective transformations.

If three terms in the first three rows of the fourth column of the 4 × 4 transformation matrix are nonzero, then a three-point perspective is obtained. The three-point perspective transformation

$$[x \quad y \quad z \quad 1] \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad z \quad (px + qy + rz + 1)] \qquad (3-57)$$

with ordinary coordinates

$$[x^* \ y^* \ z^* \ 1] = \left[\frac{x}{px+qy+rz+1} \quad \frac{y}{px+qy+rz+1} \quad \frac{z}{px+qy+rz+1} \quad 1 \right] \tag{3-58}$$

has three centers of projection: one on the x-axis at $[-1/p \ \ 0 \ \ 0 \ \ 1]$, one on the y-axis at $[0 \ \ -1/q \ \ 0 \ \ 1]$, and one on the z-axis at $[0 \ \ 0 \ \ -1/r \ \ 1]$, and three vanishing points: one on the x-axis at $[1/p \ \ 0 \ \ 0 \ \ 1]$, one on the y-axis at $[0 \ \ 1/q \ \ 0 \ \ 1]$, and one on the z-axis at $[0 \ \ 0 \ \ 1/r \ \ 1]$.

Again, note that the three-point perspective transformation given by Eq. (3-57) can be obtained by concatenation of three single-point perspective transformations, one for each of the coordinate axes. An example illustrates the generation of a three-point perspective.

Example 3-21 Three-Point Perspective Transformation

Consider the cube described in Ex. 3-18 transformed by a three-point perspective transformation with centers of projection at $x = -10$, $y = -10$ and $z = 10$ projected onto the $z = 0$ plane. Vanishing points are at $x = 10$, $y = 10$ and $z = -10$. The transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed coordinates of the cube are

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0.9 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1.1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1.1 \\ 1 & 1 & 0 & 1.2 \\ 0 & 1 & 0 & 1.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0.909 & 0.909 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.909 & 0 & 0 & 1 \\ 0.833 & 0.833 & 0 & 1 \\ 0 & 0.909 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3-31b. The distorted object, after perspective transformation, is shown in Fig. 3-31c. Note the convergence of the edges.

Again, although mathematically correct the resulting view is not informative. Appropriate techniques for generating perspective views are discussed in Sec. 3-16.

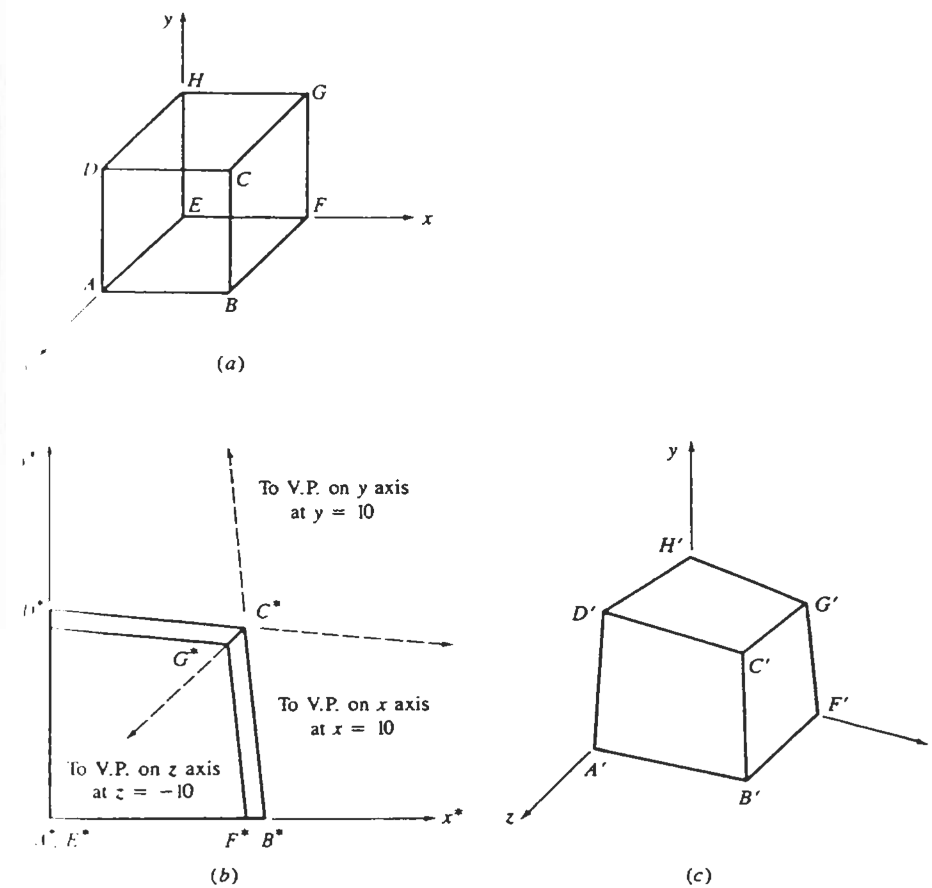


Figure 3-31 Three-point perspective. (a) The original cube; (b) perspective projection onto the $z = 0$ plane; (c) the distorted cube.

3-16 TECHNIQUES FOR GENERATING PERSPECTIVE VIEWS

The perspective projection views shown in the previous section were uninformative because in each case only one face of the cube was visible from each center of projection. For an observer to perceive the three-dimensional shape of an object from a single view, it is necessary that multiple faces of the object be visible. For simple cuboidal objects, a minimum of three faces must be visible. For a fixed center of projection with the projection plane perpendicular to the viewing direction, a single-point perspective projection, preceded by translation and/or rotation of the object, provides the required multiple face view. Then, provided the center of projection is not too close to the object, a realistic view is obtained.

First, consider simple translation of the object followed by a single-point perspective projection from a center of projection at $z = z_c$ onto the $z = 0$ plane. The required transformation is

$$[T] = [Tr_{xyz}][P_{rz}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ l & m & 0 & 1+rn \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/z_c \\ l & m & 0 & 1-n/z_c \end{bmatrix} \qquad (3-59)$$

where $r = -1/z_c$.

Equation (3–59), along with Fig. 3–32, shows that translation in the x and y directions reveals additional faces of the object. Translation in both x and y is required to reveal three faces of a simple cuboidal object. Figure 3–32 shows the results of translating an origin-centered unit cube along the line $y = x$, followed by a single-point perspective projection onto the $z = 0$ plane. Notice that the front face is shown true size and shape.

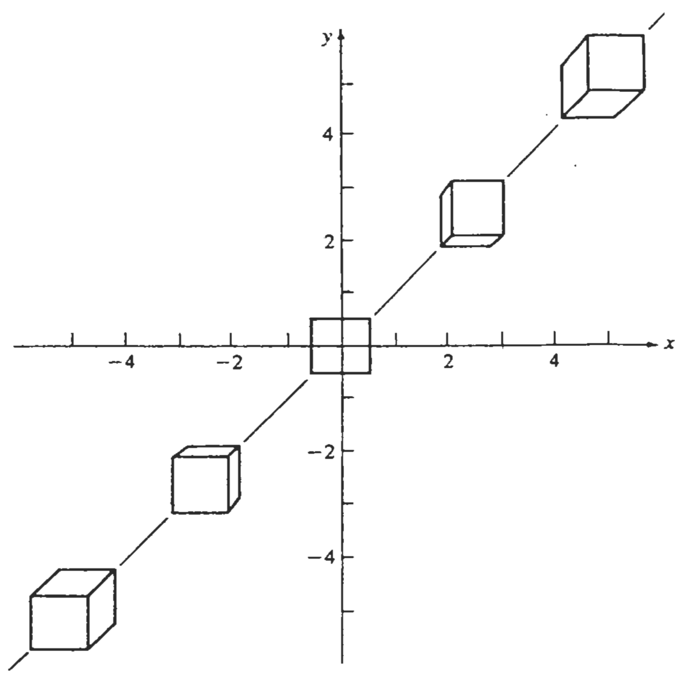


Figure 3–32 Single-point perspective projection with x and y translations.

Equation (3–59) also shows that translation in the z direction, i.e., toward or away from the center of projection, results in an apparent scale change (as shown by the term $1 - n/z_c$). This effect corresponds to physical reality, since objects that are farther away from an observer appear smaller. Notice that as the center of projection approaches infinity the scale effect disappears. Figure 3–33 schematically illustrates the effect. As shown in Fig. 3–33, the object can be on either side of the center of projection. If the object and the plane of projection are on the same side of the center of projection, as shown in Fig. 3–33, then an upright image results. However, if the object and the plane of projection are on opposite sides of the center of projection an inverted image results.

Figure 3–34 illustrates the effects of translation in all three directions. Here, a cube is translated along the three-dimensional line from $-x = -y = -z$ to $x = y = z$. Notice the apparent size increase. Also notice that the true shape but *not* the true size of the front face is shown in all views.

An example more fully illustrates these concepts.

Example 3–22 Single-Point Perspective Projection with Translation

Consider an origin-centered unit cube with position vectors given by

$$[X] = \begin{bmatrix} -0.5 & -0.5 & 0.5 & 1 \\ 0.5 & -0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 0.5 & 1 \\ -0.5 & 0.5 & 0.5 & 1 \\ -0.5 & -0.5 & -0.5 & 1 \\ 0.5 & -0.5 & -0.5 & 1 \\ 0.5 & 0.5 & -0.5 & 1 \\ -0.5 & 0.5 & -0.5 & 1 \end{bmatrix}$$

Translate the cube 5 units in the x and y directions and perform a single-point perspective projection onto the $z = 0$ plane from a center of projection at $z = z_c = 10$.

From Eq. (3–59) the concatenated transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 5 & 5 & 0 & 1 \end{bmatrix}$$

The resulting transformed position vectors are

$$[X^*] = [X][T] = \begin{bmatrix} 4.5 & 4.5 & 0 & 0.95 \\ 5.5 & 4.5 & 0 & 0.95 \\ 5.5 & 5.5 & 0 & 0.95 \\ 4.5 & 5.5 & 0 & 0.95 \\ 4.5 & 4.5 & 0 & 1.05 \\ 5.5 & 4.5 & 0 & 1.05 \\ 5.5 & 5.5 & 0 & 1.05 \\ 4.5 & 5.5 & 0 & 1.05 \end{bmatrix} = \begin{bmatrix} 4.737 & 4.737 & 0 & 1 \\ 5.789 & 4.737 & 0 & 1 \\ 5.789 & 5.789 & 0 & 1 \\ 4.737 & 5.789 & 0 & 1 \\ 4.286 & 4.286 & 0 & 1 \\ 5.238 & 4.286 & 0 & 1 \\ 5.238 & 5.238 & 0 & 1 \\ 4.286 & 5.238 & 0 & 1 \end{bmatrix}$$

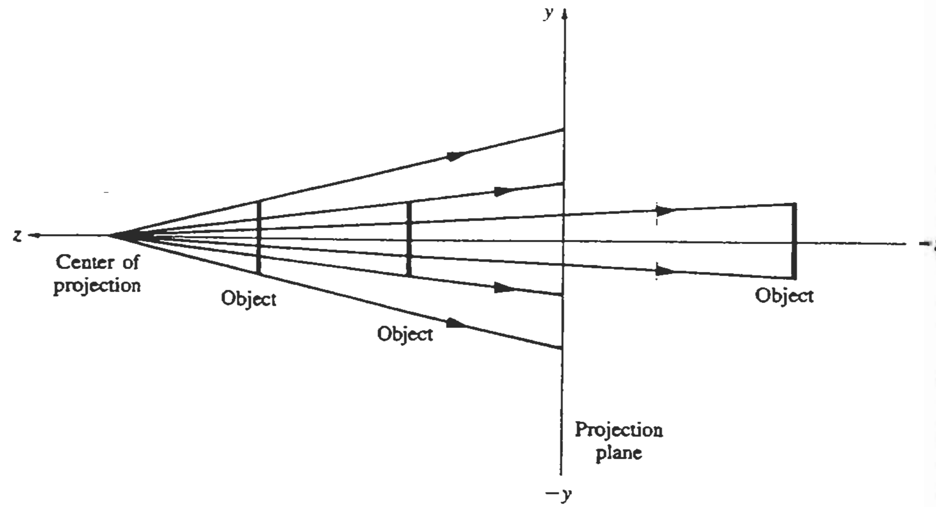


Figure 3–33 Scale effect of z translation for a single-point perspective projection.

The result is shown as the upper right hand object in Fig. 3–32.

If the original object is translated by 5 units in the x, y and z directions and a single-point perspective projection onto the $z = 0$ plane from a center of projection at $z = z_c = 20$ is performed, then from Eq. (3–59) the concatenated transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.05 \\ 5 & 5 & 0 & 0.75 \end{bmatrix}$$

Notice the overall scaling indicated by the value of 0.75 in the lower right hand element of the transformation matrix.

The resulting transformed position vectors are

$$[X^*] = [X][T] = \begin{bmatrix} 4.5 & 4.5 & 0 & 0.725 \\ 5.5 & 4.5 & 0 & 0.725 \\ 5.5 & 5.5 & 0 & 0.725 \\ 4.5 & 5.5 & 0 & 0.725 \\ 4.5 & 4.5 & 0 & 0.775 \\ 5.5 & 4.5 & 0 & 0.775 \\ 5.5 & 5.5 & 0 & 0.775 \\ 4.5 & 5.5 & 0 & 0.775 \end{bmatrix} = \begin{bmatrix} 6.207 & 6.207 & 0 & 1 \\ 7.586 & 6.207 & 0 & 1 \\ 7.586 & 7.586 & 0 & 1 \\ 6.207 & 7.586 & 0 & 1 \\ 5.806 & 5.806 & 0 & 1 \\ 7.097 & 5.806 & 0 & 1 \\ 7.097 & 7.097 & 0 & 1 \\ 5.806 & 7.097 & 0 & 1 \end{bmatrix}$$

The result is shown as the upper right hand object in Fig. 3–34.

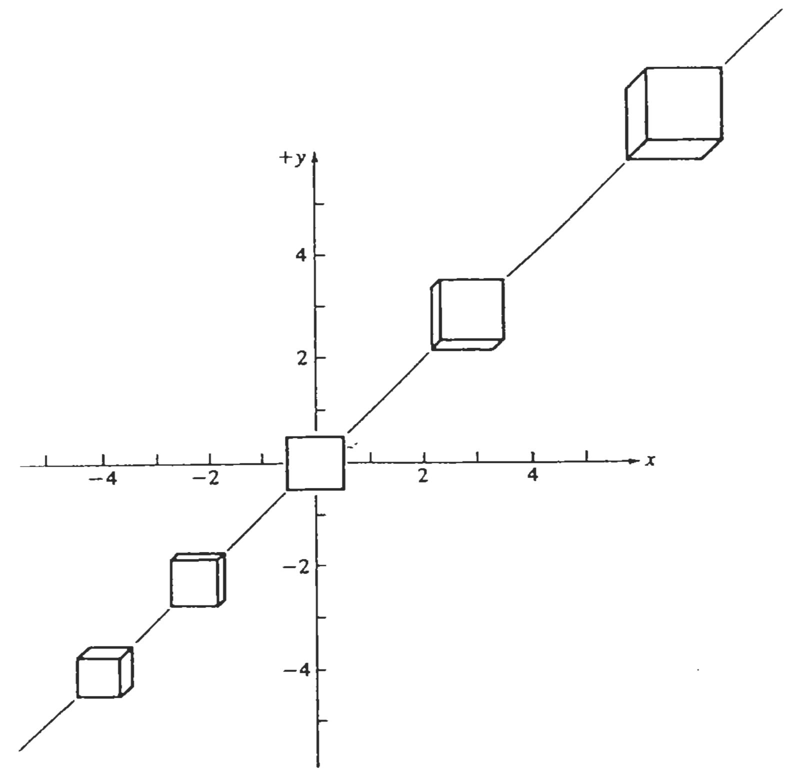


Figure 3–34 Single-point perspective projection with x, y and z translations.

Multiple faces of an object are also revealed by rotation of the object. A single rotation reveals at least two faces of an object, while two or more rotations about separate axes reveal a minimum of three faces.

The transformation matrix for rotation about the y-axis by an angle ϕ, followed by a single-point perspective projection onto the $z = 0$ plane from a center of projection at $z = z_c$, is given by

$$[T] = [R_y][P_{rz}] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & 0 & 0 & \dfrac{\sin\phi}{z_c} \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & 0 & -\dfrac{\cos\phi}{z_c} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-60)$$

Similarly, the transformation matrix for rotation about the x-axis by an angle θ, followed by a single-point perspective projection onto the $z = 0$ plane from a center of projection at $z = z_c$, is

$$[T] = [R_x][P_{rz}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\dfrac{\sin\theta}{z_c} \\ 0 & -\sin\theta & 0 & -\dfrac{\cos\theta}{z_c} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-61)$$

In both Eqs. (3–60) and (3–61) two of the perspective terms in the fourth column of the transformation matrix are nonzero. Thus, a single rotation about a principal axis perpendicular to that on which the center of projection lies is equivalent to a two-point perspective transformation. Rotation about the axis on which the center of projection lies does not have this effect. Notice that for a single rotation the perspective term for the axis of rotation is unchanged, e.g., in Eqs. (3–60) and (3–61) q and p, respectively, remain zero.

Rotation about a single principal axis does not in general reveal the minimum three faces for an adequate three-dimensional representation. In general, rotation about a single principal axis must be combined with translation along the axis to obtain an adequate three-dimensional representation. The next example illustrates this.

Example 3–23 Two-Point Perspective Projection Using Rotation About a Single Principal Axis

Consider the cube shown in Fig. 3–35a rotated about the y-axis by $\phi = 60°$ to reveal the left-hand face and translated -2 units in y to reveal the top face projected onto the $z = 0$ plane from a center of projection at $z = z_c = 2.5$.

Using Eq. (3–38) with $\phi = 60°$, Eq. (3–47) with $z_c = 2.5$ and Eq. (3–14) with $n = l = 0$, $m = -2$ yields

$$[T] = [R_y][Tr][P_{rz}]$$

$$= \begin{bmatrix} 0.5 & 0 & -0.866 & 0 \\ 0 & 1 & 0 & 0 \\ 0.866 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0 & 0.346 \\ 0 & 1 & 0 & 0 \\ 0.866 & 0 & 0 & -0.2 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

The transformed position vectors are

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 & 0.346 \\ 0 & 1 & 0 & 0 \\ 0.866 & 0 & 0 & -0.2 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & -2 & 0 & 0.8 \\ 1.366 & -2 & 0 & 1.146 \\ 1.366 & -1 & 0 & 1.146 \\ 0.866 & -1 & 0 & 0.8 \\ 0 & -2 & 0 & 1 \\ 0.5 & -2 & 0 & 1.346 \\ 0.5 & -1 & 0 & 1.346 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.083 & -2.5 & 0 & 1 \\ 1.192 & -1.745 & 0 & 1 \\ 1.192 & -0.872 & 0 & 1 \\ 1.083 & -1.25 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0.371 & -1.485 & 0 & 1 \\ 0.371 & -0.743 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–35b. The distortion is the result of the center of projection being very close to the cube. Notice the convergence of lines originally parallel to the x and z axes to vanishing points that lie on the x-axis. These vanishing points are determined in Ex. 3–25 in Sec. 3–17.

Similarly, a three-point perspective transformation is obtained by rotating about two or more of the principal axes and then performing a single-point perspective transformation. For example, rotation about the y-axis followed by rotation about the x-axis and a perspective projection onto the $z = 0$ plane from a center of projection at $z = z_c$ yields the concatenated transformation matrix

$$[T] = [R_y][R_x][P_{rz}]$$

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & \dfrac{\sin\phi\cos\theta}{z_c} \\ 0 & \cos\theta & 0 & -\dfrac{\sin\theta}{z_c} \\ \sin\phi & -\cos\phi\sin\theta & 0 & -\dfrac{\cos\phi\cos\theta}{z_c} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-62)$$

Notice the three nonzero perspective terms. The object may also be translated. If translation occurs after rotation, then the resulting concatenated transformation

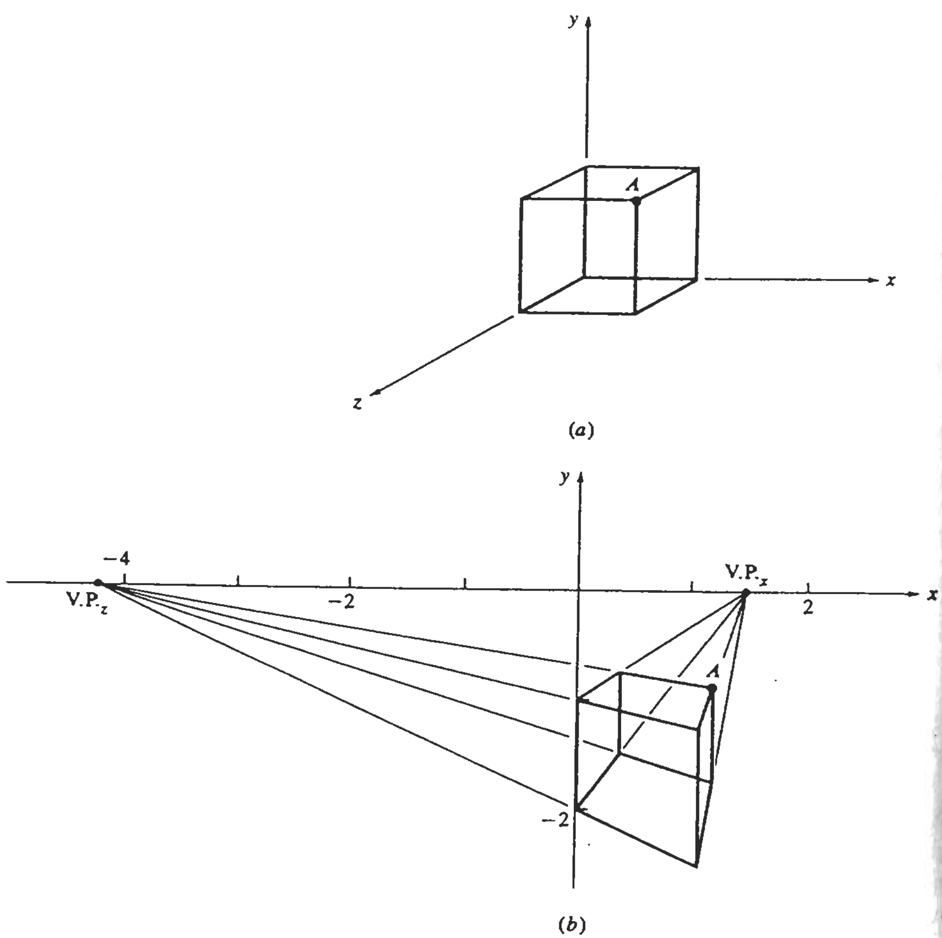


Figure 3–35 Two-point perspective projection with rotation about a single axis.

matrix is

$$[T] = [R_y][R_x][Tr][P_{rz}]$$

$$= \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & \dfrac{\sin\phi\cos\theta}{z_c} \\ 0 & \cos\theta & 0 & -\dfrac{\sin\theta}{z_c} \\ \sin\phi & -\cos\phi\sin\theta & 0 & -\dfrac{\cos\phi\cos\theta}{z_c} \\ l & m & 0 & 1-\dfrac{n}{z_c} \end{bmatrix} \qquad (3-63)$$

Here, note the apparent scaling effect of translation in z. If the order of the rotations is reversed or if translation occurs before rotation, the results are different.

Example 3–24 Three-Point Perspective Projection with Rotation About Two Axes

Consider the cube shown in Fig. 3-35a rotated about the y-axis by $\phi = -30°$, about the x-axis by $\theta = 45°$ and projected onto the $z = 0$ plane from a center of projection at $z = z_c = 2.5$.

Using Eq. (3–62) yields

$$[T] = \begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors are

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -0.612 & 0 & 0.755 \\ 0.366 & -0.966 & 0 & 0.614 \\ 0.366 & -0.259 & 0 & 0.331 \\ -0.5 & 0.095 & 0 & 0.472 \\ 0 & 0 & 0 & 1 \\ 0.866 & -0.354 & 0 & 0.859 \\ 0.866 & 0.354 & 0 & 0.576 \\ 0 & 0.707 & 0 & 0.717 \end{bmatrix} = \begin{bmatrix} -0.662 & -0.811 & 0 & 1 \\ 0.596 & -1.574 & 0 & 1 \\ 1.107 & -0.782 & 0 & 1 \\ -1.059 & 0.201 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1.009 & -0.412 & 0 & 1 \\ 1.504 & 0.614 & 0 & 1 \\ 0 & 0.986 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3-36.

From these results it is clear that one-, two- or three-point perspective transformations can be constructed using rotations and translations about and along the principal axes, followed by a single-point perspective transformation from a center of projection on one of the principal axes. These results also follow for rotation about a general axis in space. Consequently, in implementing a graphics system using a fixed center of projection object manipulation paradigm, it is only necessary to provide for a single-point perspective projection onto the $z = 0$ plane from a center of projection on the z-axis.

3–17 VANISHING POINTS

When a perspective view of an object is created a horizontal reference line, normally at eye level, as shown in Fig. 3-37a is used. Principal vanishing points

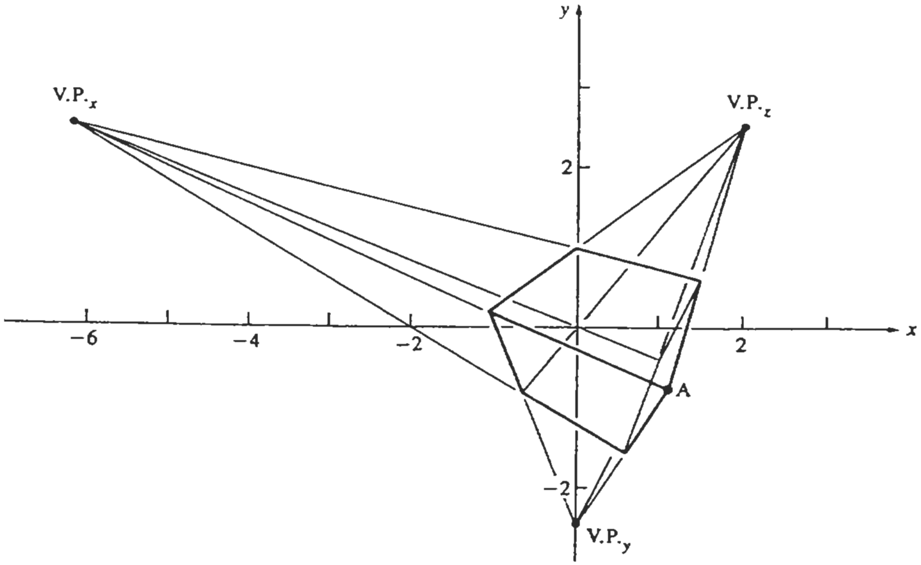


Figure 3–36 Three-point perspective projection with rotation about two axes.

are points on the horizontal reference line at which lines originally parallel to the untransformed principal axes converge. In general, different sets of parallel lines have different principal vanishing points. This is illustrated in Fig. 3–37b. For planes of an object which are tilted relative to the untransformed principal axes, the vanishing points fall above or below the horizontal reference line. These are often called trace points, as shown in Fig. 3–37c.

Two methods for determining vanishing points are of general interest. The first simply calculates the intersection point of a pair of transformed projected parallel lines. The second is more complex but numerically more accurate. Here, an object with sides originally parallel to the principal axes is transformed to the desired position and orientation. A single-point perspective projection is applied. The final concatenated transformation matrix (see Eq. 3–63) is then used to transform the points at infinity on the principal axes. The resulting ordinary coordinates are the principal vanishing points for that object. For trace points resulting from inclined planes, the points at infinity in the directions of the edges of the inclined plane are first found and then transformed.

Several examples illustrate these techniques. The first uses intersection of the transformed lines to find the vanishing points.

Example 3–25 Principal Vanishing Points by Line Intersection

Recalling Ex. 3–23, the transformed position vectors for the pair of line segments with one line through the point A (see Fig. 3-35a) originally parallel to

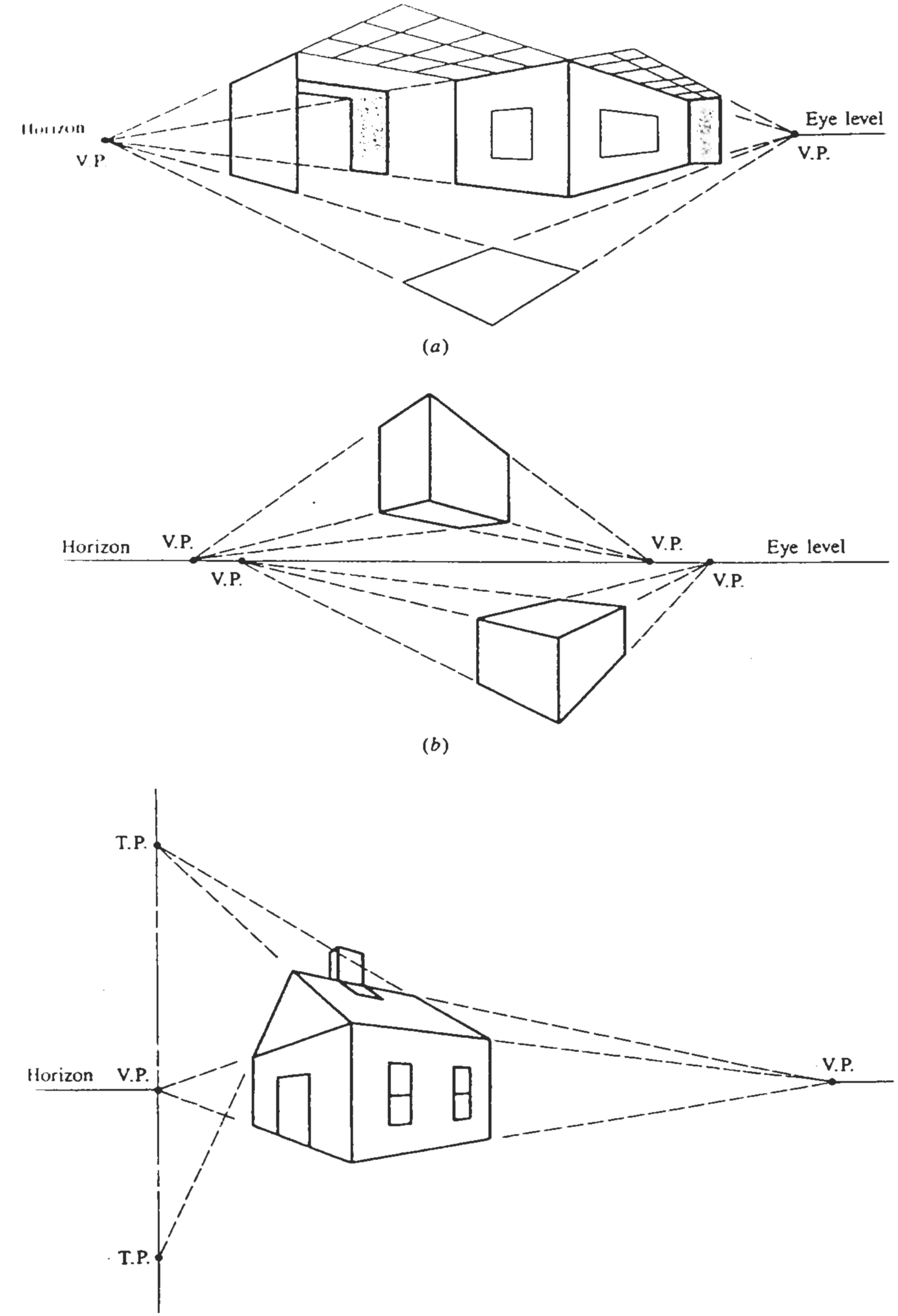


Figure 3–37 Trace points and vanishing points.

the x- and z-axes, respectively, are

$$\begin{matrix} ④ \\ ③ \\ ⑧ \\ ⑦ \end{matrix}\begin{bmatrix} 1.083 & -1.25 & 0 & 1 \\ 1.192 & -0.872 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0.371 & 0.743 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1.192 & -0.872 & 0 & 1 \\ 0.371 & -0.743 & 0 & 1 \\ 1.083 & -1.25 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{matrix} ③ \\ ⑦ \\ ④ \\ ⑧ \end{matrix}$$

Here, the numbers in circles refer to the rows in the original and transformed data matrices given in Ex. 3-23. The equations of the pair of lines originally parallel to the x-axis are

$$y = 3.468x - 5.006$$
$$y = 0.693x - 1$$

Solution yields $[\,VP_x\,] = [\,1.444 \quad 0\,]$.

The equations of the pair of lines originally parallel to the z-axis are

$$y = -0.157x - 0.685$$
$$y = -0.231x - 1$$

Solution yields $[\,VP_z\,] = [\,-4.333 \quad 0\,]$.

These vanishing points are shown in Fig. 3-35b.

The second example uses transformation of the points at infinity on the principal axes to find the vanishing point.

Example 3-26 Principal Vanishing Points by Transformation

Recalling Ex. 3-24 the concatenated complete transformation was

$$[\,T\,] = \begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transforming the points at infinity on the x-, y- and z-axes yields

$$[\,VP\,][\,T\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6.142 & 2.5 & 0 & 1 \\ 0 & -2.5 & 0 & 1 \\ 2.04 & 2.5 & 0 & 1 \end{bmatrix}$$

These vanishing points are shown in Fig. 3-36.

This third example uses transformation of the points at infinity for skew planes to find trace points.

Example 3-27 Trace Points by Transformation

Consider the simple triangular prism shown in Fig. 3-38a. The position vectors for the prism are

$$[\,X\,] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0.5 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \end{bmatrix}$$

Applying the concatenated transformation of Ex. 3-24 yields transformed position vectors

$$[\,X^*\,] = [\,X\,][\,T\,]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0.5 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -0.612 & 0 & 0.755 \\ 0.366 & -0.966 & 0 & 0.614 \\ -0.067 & -0.436 & 0 & 0.543 \\ 0 & 0 & 0 & 1 \\ 0.866 & -0.354 & 0 & 0.859 \\ 0.433 & 0.177 & 0 & 0.788 \end{bmatrix}$$

$$= \begin{bmatrix} -0.662 & -0.811 & 0 & 1 \\ 0.596 & -1.574 & 0 & 1 \\ -0.123 & -0.802 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1.009 & -0.412 & 0 & 1 \\ 0.55 & 0.224 & 0 & 1 \end{bmatrix}$$

The transformed prism is shown in Fig. 3-38b.

The direction cosines for the inclined edges of the left-hand plane forming the top of the untransformed prism are $[\,0.5 \quad 0.5 \quad 0\,]$. Thus, the point at infinity in this direction is $[\,1 \quad 1 \quad 0 \quad 0\,]$.

Similarly, $[\,-0.5 \quad 0.5 \quad 0\,]$ are the direction cosines for the inclined edges of the right-hand plane forming the top of the untransformed prism. Thus, the point at infinity in this direction is $[\,-1 \quad 1 \quad 0 \quad 0\,]$.

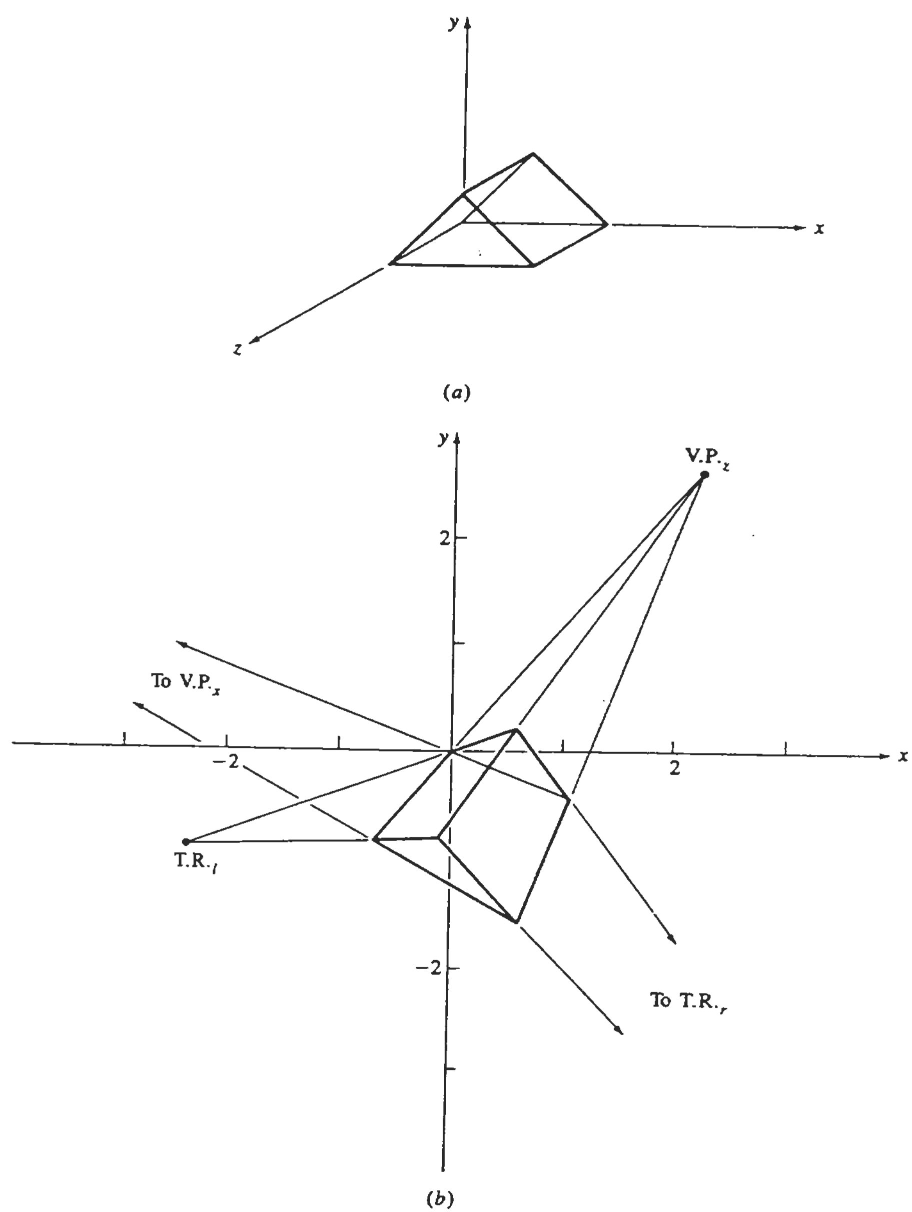


Figure 3–38 Trace points.

Transforming these infinite points along with those for the principal axes yields

$$[VP][T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0.866 & 0.354 & 0 & -0.424 \\ -0.866 & 1.061 & 0 & -0.141 \end{bmatrix}$$

$$= \begin{bmatrix} -6.142 & 2.5 & 0 & 1 \\ 0 & -2.5 & 0 & 1 \\ 2.041 & 2.5 & 0 & 1 \\ -2.041 & -0.833 & 0 & 1 \\ 6.142 & -7.5 & 0 & 1 \end{bmatrix} \begin{matrix} VP_x \\ VP_y \\ VP_z \\ TP_l \\ TP_r \end{matrix}$$

The vanishing and trace points are also shown in Fig. 3–38b. Notice that as expected VP_x, VP_y and VP_z are the same as found in Ex. 3–26.

3–18 PHOTOGRAPHY AND THE PERSPECTIVE TRANSFORMATION

A photograph is a perspective projection. The general case for a pinhole camera is illustrated in Fig. 3–39. The center of projection is the focal point of the camera lens. It is convenient to consider the creation of the original photographic negative and of a print from that negative as two separate cases.

Figure 3–40a illustrates the geometry for creation of the original negative. Here it is convenient to place the negative at the $z = 0$ plane with the center of projection and the scene located in the negative half space $z < 0$. Perspective

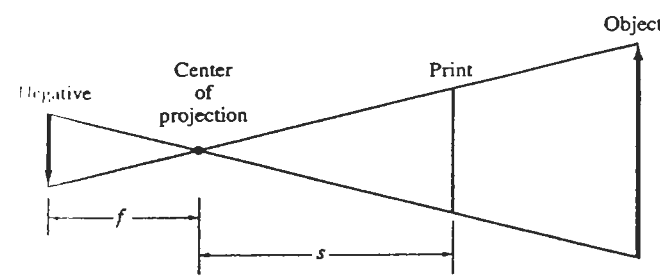


Figure 3–39 A photograph as a perspective projection.

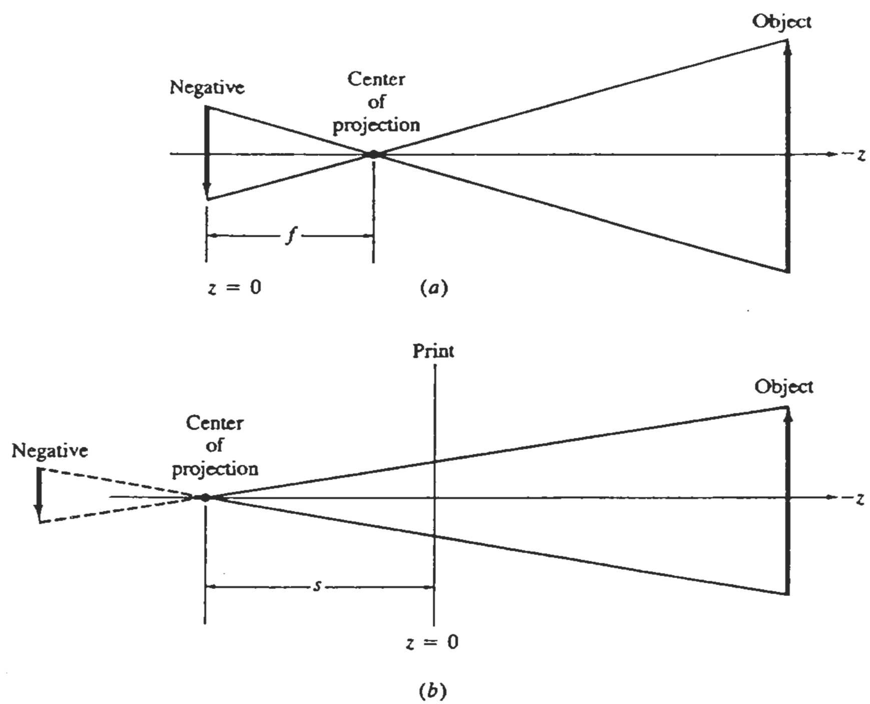


Figure 3–40 Photographic perspective geometry. (a) Creation of the original negative; (b) creation of a print.

projection onto the $z = 0$ plane (the negative) yields the transformation

$$[T_n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/f \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-64)$$

where f is the focal length of the lens. Note that an inverted image of the object is formed on the negative. Specifically,

$$[x \quad y \quad z \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/f \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad 0 \quad 1+z/f]$$

and

$$x^* = \frac{xf}{f+z} \qquad y^* = \frac{yf}{f+z}$$

Here, for $f + z < 0$, x^* and y^* are of opposite sign to x and y, and an inverted image is formed on the negative.

Figure 3-40b illustrates the geometry for creation of a print from a photographic negative. Here s is the distance from the focal point of the enlarger lens to the paper. The paper is assumed to be located at $z = 0$. The perspective projection transformation is then

$$[T_p] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/s \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3-65)$$

Note that an upright image of the object is formed on the print. Specifically,

$$[x \quad y \quad z \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/s \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[x \quad y \quad z \quad 1-\frac{z}{s}\right]$$

and

$$x^* = \frac{xs}{s-z} \qquad y^* = \frac{ys}{s-z}$$

For the object $z < 0$, $s - z > 0$ and x^*y^* are of the same sign as x, y, i.e., an upright image is formed.

3-19 STEREOGRAPHIC PROJECTION

Increasing the perception of three-dimensional depth in a scene is important in many applications. There are two basic types of depth perception cues used by the eye-brain system: monocular and binocular, depending on whether they are apparent when one or two eyes are used. The principal monocular cues are:

Perspective — convergence of parallel lines.

Movement parallax — when the head is moved laterally, near objects appear to move more against a projection plane than far objects.

Relative size of known objects.

Overlap — a closer object overlaps and appears in front of a more distant object.

Highlights and shadows.

Atmospheric attenuation of, and the inability of the eye to resolve, fine detail in distant objects.

Focusing accommodation — objects at different distances require different tension in the focusing muscles of the eye.

The principal binocular cues are:

The convergence angles of the optical axes of the eyes.

Retinal disparity — the different location of objects projected on the eye's retina is interpreted as differences in distance from the eye.

The monocular cues produce only weak perceptions of three-dimensional depth. However, because the eye-brain system fuses the two separate and distinct images produced by each eye into a single image, the binocular cues produce very strong three-dimensional depth perceptions. Stereography attempts to produce an image with characteristics analogous to those for true binocular vision. There are several techniques for generating stereo images (see Refs. 3–4 and 3–5). All depend upon supplying the left and right eyes with separate images.

Two methods, called chromatic anaglyphic and polarized anaglyphic, use filters to insure reception of correct and separate images by the left and right eyes. Briefly, the chromatic anaglyphic technique creates two images in two different colors, one for the left eye and one for the right eye. When viewed through corresponding filters the left eye sees only the left image and the right eye only the right image. The eye-brain system combines both two-dimensional images into a single three-dimensional image with the correct colors. The polarized anaglyphic method uses polarizing filters instead of color filters.

A third technique uses a flicker system to alternately project a left and a right eye view. An associated viewing device is synchronized to block the light to the opposite eye.

A fourth method, autostereoscopy, does not require any special viewing equipment. The method depends on the use of line or lenticular screens. The images are called parallax stereograms, parallax panoramagrams and panoramic parallax stereograms. The details are given in Ref. 3–5.

All of these techniques require projection of an object onto a plane from two different centers of projection, one for the right eye and one for the left eye. Figure 3–41 shows a projection of the point P onto the $z = 0$ plane from centers of projection at $E_L(-e\ ,\ 0\ ,\ d_e)$ and $E_R(e\ ,\ 0\ ,\ d_e)$ corresponding to the left and right eye, respectively.

For convenience, the center of projection for the left eye is translated so that it lies on the z-axis as shown in Fig. 3–41b. Using similar triangles then yields

$$\frac{x_L^{*\prime}}{d_e} = \frac{x'}{d_e - z}$$

and

$$x_L^{*\prime} = \frac{x'}{1 - z/d_e} = \frac{x'}{1 + rz}$$

where

$$r = -1/d_e$$

Similarly, translating the center of projection for the right eye so that it lies on the z-axis as shown in Fig. 3–41c, and again using similar triangles, yields

$$\frac{x_R^{*\prime\prime}}{d_e} = \frac{x''}{d_e - z}$$

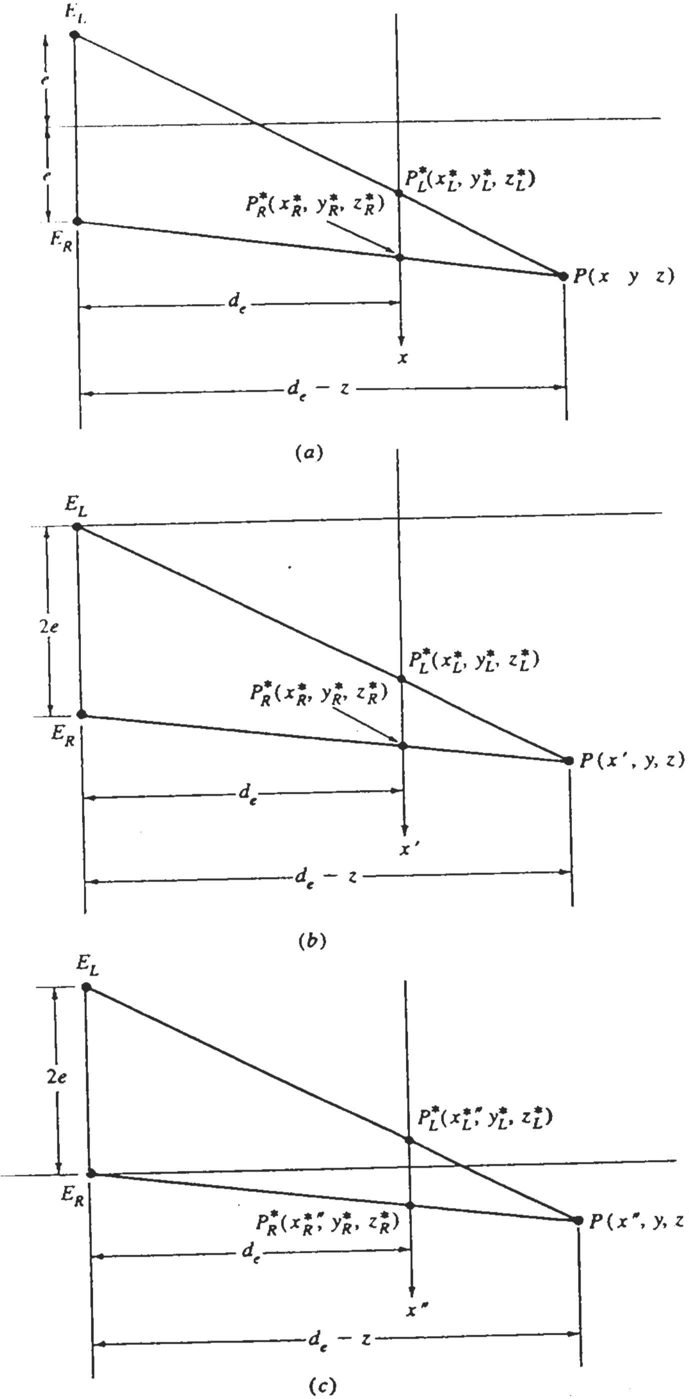


Figure 3–41 Stereographic projection onto $z = 0$.

and

$$x_R^{*''} = \frac{x''}{1 - z/d_e} = \frac{x''}{1 + rz}$$

Since each eye is at $y = 0$, the projected values of y are both

$$y^* = \frac{y}{1 - z/d_e} = \frac{y}{1 + rz}$$

The equivalent 4×4 transformation matrices for the left and right eye views are

$$[S_L] = [Tr_z][P_{rz}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/d_e \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/d_e \\ e & 0 & 0 & 1 \end{bmatrix} \qquad (3-66)$$

and

$$[S_R] = [Tr_z][P_{rz}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/d_e \\ -e & 0 & 0 & 1 \end{bmatrix} \qquad (3-67)$$

Consequently, a stereographic projection is obtained by transforming the scene using Eqs. (3-66) and (3-67) and displaying both images.

Stereographic projections are displayed in a number of ways. For many individuals a stereo image can be created without any viewing aid. One technique, which takes a bit of practice, is to first focus the eyes at infinity; then, without changing the focus, gradually move the stereo pairs, held at about arm's length, into view.

Binocular fusion of the stereo pairs is improved by using a small opaque mask, e.g., a strip of black cardboard about an inch wide. As illustrated in Fig. 3-42, the mask is placed between the eyes and the stereo pair and moved back and forth until it is in the position shown. When the mask is in the position shown, the left eye sees only the left image and the right eye only the right image of the stereo pair.

Two more formal devices for viewing stereo pairs are shown in Figs. 3-43a and 3-43b. Figure 3-43a shows a Brewster stereopticon popular in the early part of this century. Figure 3-43b shows a typical modern laboratory stereoscope. Both devices are examples of simple focal plane lens stereoscopes.

Although Eqs. (3-66) and (3-67) provide the basic transformation for generating stereo pairs, it is necessary to modify this basic transformation to accommodate the geometry of various stereo viewing devices. The geometry for a simple focal plane stereoscope is shown in Fig. 3-44. Here the stereo pairs are

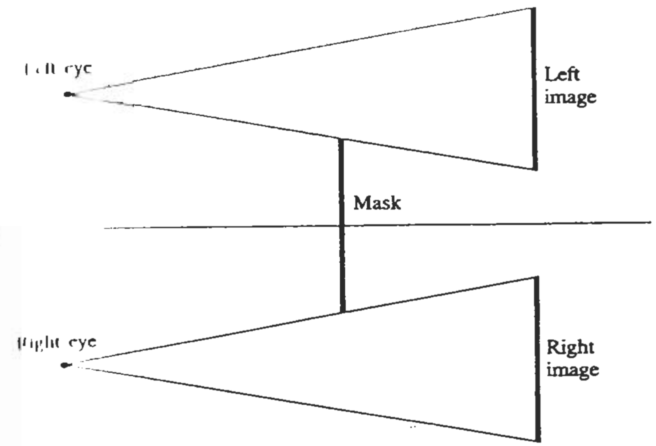


Figure 3-42 Simple stereo pair viewing method.

located at the focal distance f of the lenses. The stereo image is reconstructed at a distance I from the lenses. As mentioned previously, the stereo convergence angle γ, which is one of the strongest binocular fusion cues, is associated with the convergence of the optical axes of the eyes. For a normal human the inter-pupil distance of the eyes is about 60 millimeters. Experiments (Ref. 3-4) have shown that the strongest stereo effect occurs at a normal viewing distance of about 600 millimeters. Consequently, the strongest stereo effect occurs for $\tan(\gamma/2) = 1/5$. If, as shown in Fig. 3-44, the stereo pairs are located at the focal distance f, they must be separated by an amount $2w$ to achieve binocular fusion.

From Fig. 3-44 with $2D$ as the distance between the lenses of the stereoscope, similar triangles show that

$$\frac{D - w}{f} = \frac{D}{I} = \frac{1}{5}$$

(a) (b)

Figure 3-43 Stereoscopes. (a) Brewster stereopticon; (b) typical laboratory instrument.

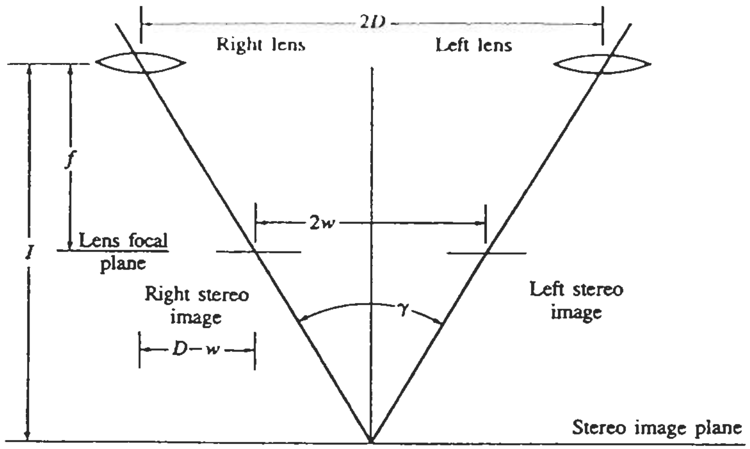


Figure 3–44 Geometry for a focal plane stereoscope.

Thus, a separation of

$$w = D - \frac{f}{5} \qquad (3-68)$$

yields the strongest stereo effect. For a stereo pair to be viewed with a simple lens stereoscope, Eqs. (3–66) and (3–67) become

$$[S_L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/f \\ w & 0 & 0 & 1 \end{bmatrix} \qquad (3-69a)$$

and

$$[S_R] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/f \\ -w & 0 & 0 & 1 \end{bmatrix} \qquad (3-69b)$$

A typical stereo pair for a simple lens stereoscope is shown in Fig. 3–45.

The separations obtained from Eq. (3–68) for typical values of D and f are quite small. Thus, only small stereo pair images are viewable. To allow viewing larger images, mirrors or prisms are used to increase the separation distance. A typical mirrored stereoscope is shown in Fig. 3–43b. The associated geometry is shown in Fig. 3–46. Again using similar triangles, the stereo pair separation is

$$\frac{w_0 - w}{f} = \frac{D}{I} = \frac{1}{5}$$

and

$$w = \frac{w_0 - f}{5} \qquad (3-70)$$

An example illustrates the generation of a typical stereo pair.

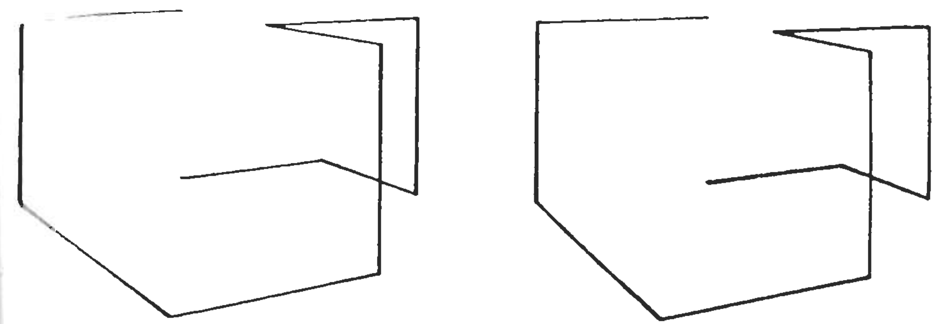


Figure 3–45 Stereo pair.

Example 3–28 Stereo Pair Generation

Consider a simple three-dimensional wire frame image with position vectors

$$[X] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 2 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

as shown in Fig. 3–45. The figure is first rotated about the y-axis by 20° and then translated −1.5 units in the z direction for viewing purposes. The resulting transformation is

$$[V] = [R_y][T_z] = \begin{bmatrix} 0.94 & 0 & -0.342 & 0 \\ 0 & 1 & 0 & 0 \\ 0.342 & 0 & 0.94 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.94 & 0 & -0.342 & 0 \\ 0 & 1 & 0 & 0 \\ 0.342 & 0 & 0.94 & 0 \\ 0 & 0 & -1.5 & 1 \end{bmatrix}$$

A stereo pair is constructed for viewing through a simple lens focal plane stereoscope. The lens separation $2D = 4$ inches and the focal length is also 4 inches. From Eq. (3–68) the stereo pair separation is

$$w = 2 - 4/5 = 1.2''$$

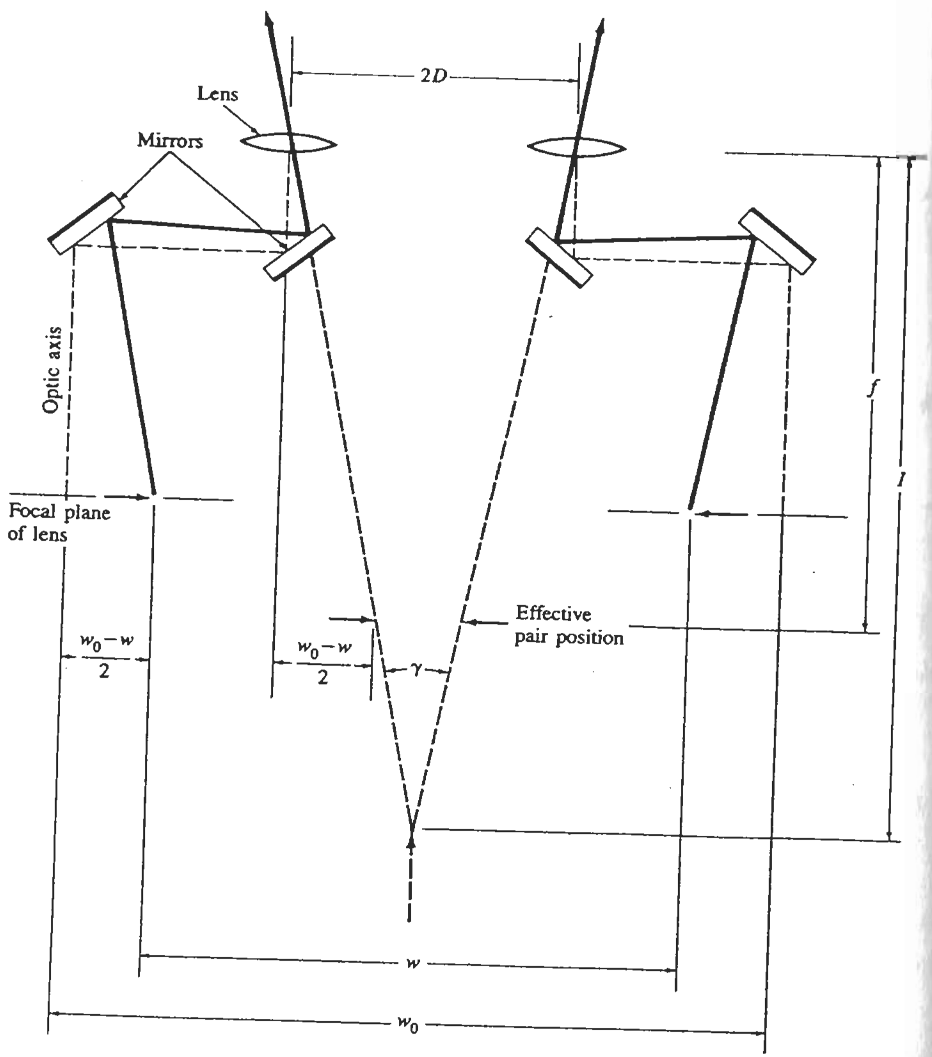


Figure 3–46 Geometry for a mirrored stereoscope.

Using Eq. (3–69), the right and left stereo image transformations are

$$[S_L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.25 \\ 1.2 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [S_R] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.25 \\ -1.2 & 0 & 0 & 1 \end{bmatrix}$$

The combined viewing and left and right image transformations are

$$[C_L] = [V][S_L] \quad \text{and} \quad [C_R] = [V][S_R]$$

$$[C_L] = \begin{bmatrix} 0.94 & 0 & 0 & 0.086 \\ 0 & 1 & 0 & 0 \\ 0.342 & 0 & 0 & -0.235 \\ 1.2 & 0 & 0 & 1.38 \end{bmatrix}$$

and

$$[C_R] = \begin{bmatrix} 0.94 & 0 & 0 & 0.086 \\ 0 & 1 & 0 & 0 \\ 0.342 & 0 & 0 & -0.235 \\ -1.2 & 0 & 0 & 1.375 \end{bmatrix}$$

The transformed position vectors for the left and right images are

$$[X_L^*] = [X][C_L] \quad \text{and} \quad [X_R^*] = [X][C_R]$$

$$[X_L^*] = \begin{bmatrix} 0.873 & 0 & 0 & 1 \\ 1.465 & 0 & 0 & 1 \\ 2.025 & 0 & 0 & 1 \\ 2.025 & 0.816 & 0 & 1 \\ 1.353 & 0.877 & 0 & 1 \\ 2.081 & 1.105 & 0 & 1 \\ 2.081 & 0 & 0 & 1 \\ 1.152 & 0 & 0 & 1 \\ 0.202 & 0 & 0 & 1 \\ 0.202 & 0 & 0 & 1 \\ 0.202 & 0.775 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [X_R^*] = \begin{bmatrix} -0.873 & 0 & 0 & 1 \\ -0.178 & 0 & 0 & 1 \\ 0.067 & 0 & 0 & 1 \\ 0.067 & 0.816 & 0 & 1 \\ -0.753 & 0.877 & 0 & 1 \\ -0.57 & 1.105 & 0 & 1 \\ -0.57 & 0 & 0 & 1 \\ -1.776 & 0 & 0 & 1 \\ -1.659 & 0 & 0 & 1 \\ -1.659 & 0 & 0 & 1 \\ -1.659 & 0.775 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–45. Note that in generating the final image the results must be scaled for each particular output device to yield the correct physical dimensions for w.

3–20 COMPARISON OF OBJECT FIXED AND CENTER OF PROJECTION FIXED PROJECTIONS

An object fixed movable center of projection technique is easily converted to the movable object fixed center of projection technique previously discussed. There are two cases of interest. The first and simpler assumes that the projection plane is perpendicular to the sight vector from the center of projection into the scene. The second eliminates the perpendicular projection plane assumption. Only the first is discussed here.

When the projection plane is perpendicular to the sight vector, the following procedure yields the equivalent movable object fixed center of projection transformation:

Determine the intersection of the sight vector and the projection plane.

Translate the intersection point to the origin.

Rotate the sight vector so that it is coincident with the $+z$-axis and pointed towards the origin (see Sec. 3-9).

Apply the concatenated transformations to the scene.

Perform a single-point perspective projection onto the $z = 0$ plane from the transformed center of projection on the z-axis.

A relatively simple example serves to illustrate the technique.

Example 3–29 Object Fixed Perspective Projection onto a Perpendicular Projection Plane

Consider the cube with one corner removed, previously discussed in Ex. 3-10. Project the cube from a center of projection at $[\,10 \quad 10 \quad 10\,]$ onto the plane passing through the point $[\,-1 \quad -1 \quad -1\,]$ and perpendicular to the sight vector as shown in Fig. 3–47.

The equation for the projection plane can be obtained from its normal (see Ref. 3-1 for alternate techniques). Here the normal is in the opposite direction to the sight vector.

The direction of the sight vector is given by

$$[\,s\,] = [\,-1 \quad -1 \quad -1\,]$$

The normal to the projection plane through $[\,-1 \quad -1 \quad -1\,]$ perpendicular to the sight vector is then

$$[\,n\,] = [\,1 \quad 1 \quad 1\,]$$

The general form of a plane equation is

$$ax + by + cz + d = 0$$

The normal to the general plane is given by

$$[\,n\,] = [\,a \quad b \quad c\,]$$

The value of d in the plane equation is obtained from any point in the plane. The equation for the projection plane through $[\,-1 \quad -1 \quad -1\,]$ is then

$$x + y + z + d = 0$$

and

$$d = -x - y - z = 1 + 1 + 1 = 3$$

Hence,

$$x + y + z + 3 = 0$$

is the equation for the projection plane.

The intersection of the sight vector and the projection plane is obtained by writing the parametric equation of the sight vector, substituting into the plane equation, and solving for the parameter value.

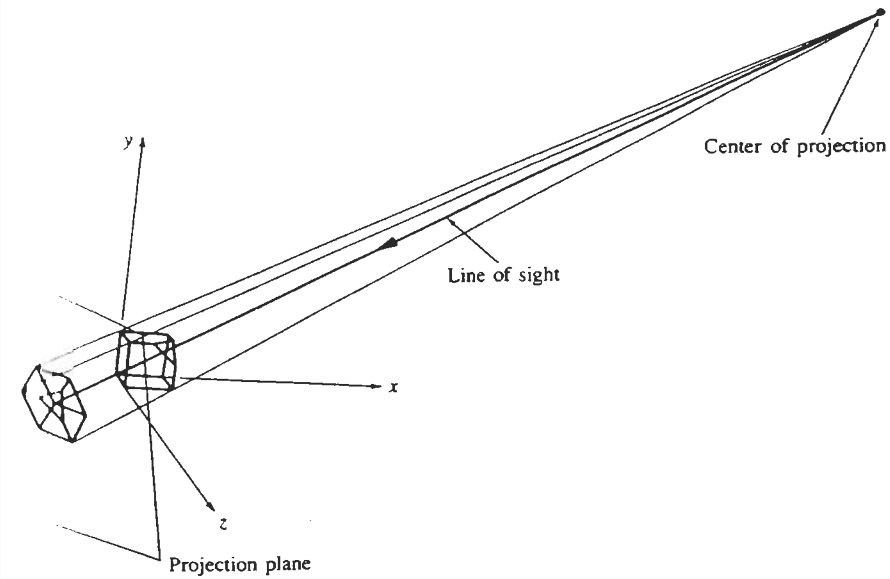


Figure 3–47 Perspective projection from movable center of projection.

The parametric equation of the sight vector is

$$[\,S(t)\,] = [\,x(t) \quad y(t) \quad z(t)\,]$$
$$= [\,10 \quad 10 \quad 10\,] + [\,-11 \quad -11 \quad -11\,]\,t \qquad 0 \le t \le 1$$

Substituting into the plane equation yields

$$x(t) + y(t) + z(t) + 3 = (10 - 11t) + (10 - 11t) + (10 - 11t) + 3 = 0$$

Solving for t yields the parameter value for the intersection point, i.e.,

$$-33t + 33 = 0 \quad \rightarrow \quad t = 1.0$$

The intersection is obtained by substituting t into $[\,S(t)\,]$. Specifically,

$$[\,I\,] = [\,S(1)\,] = [\,10 \quad 10 \quad 10\,] + [\,-11 \quad -11 \quad -11\,](1.0)$$
$$= [\,-1 \quad -1 \quad -1\,]$$

The intersection point is at $x = y = z = -1$ as expected from simple geometric considerations.

The required translation matrix is

$$[\,Tr\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

After translation the center of projection is at $[\,11 \quad 11 \quad 11\,]$ and the sight vector passes through the origin.

Using the results of Sec. 3–9, a rotation about the x-axis of $\alpha = 45°$ followed by a rotation about the y-axis by $\beta = 35.26°$ makes the sight vector coincident with the z-axis. The rotation matrices are

$$[\,R_x\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [\,R_y\,] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the concatenated transformation matrix is

$$[\,M\,] = [\,T_T\,][\,R_x\,][\,R_y\,] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ 0 & 0 & 3/\sqrt{3} & 1 \end{bmatrix}$$

Transforming the center of projection yields

$$[\,C_p\,][\,M\,] = [\,10 \quad 10 \quad 10 \quad 1\,] \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ 0 & 0 & 3/\sqrt{3} & 1 \end{bmatrix}$$

$$= [\,0 \quad 0 \quad 33/\sqrt{3} \quad 1\,]$$

The transformation for a single-point perspective projection from a center of projection at $z = 33/\sqrt{3}$ onto the $z = 0$ plane is

$$[\,P_{rz}\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3}/33 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Concatenation with $[\,M\,]$ yields

$$[\,T\,] = [\,M\,][\,P_{rz}\,] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ 0 & 0 & 3/\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3}/33 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{6} & 0 & 0 & -1/33 \\ -1/\sqrt{6} & 1/\sqrt{2} & 0 & -1/33 \\ -1/\sqrt{6} & -1/\sqrt{2} & 0 & -1/33 \\ 0 & 0 & 0 & 30/33 \end{bmatrix}$$

The transformed ordinary coordinates of the projected object are

$$[\,X^*\,] = [\,X\,][\,T\,] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.816 & 0 & 0 & -0.030 \\ -0.408 & 0.707 & 0 & -0.030 \\ -0.408 & -0.707 & 0 & -0.030 \\ 0 & 0 & 0 & 0.909 \end{bmatrix}$$

$$= \begin{bmatrix} -0.408 & -0.707 & 0 & 0.879 \\ 0.408 & -0.707 & 0 & 0.848 \\ 0.204 & -0.354 & 0 & 0.833 \\ -0.408 & 0 & 0 & 0.833 \\ -0.816 & 0 & 0 & 0.848 \\ 0 & 0 & 0 & 0.909 \\ 0.816 & 0 & 0 & 0.879 \\ 0.408 & 0.707 & 0 & 0.848 \\ -0.408 & 0.707 & 0 & 0.879 \\ 0.204 & 0.354 & 0 & 0.833 \end{bmatrix}$$

$$= \begin{bmatrix} -0.465 & -0.805 & 0 & 1 \\ 0.481 & -0.833 & 0 & 1 \\ 0.245 & -0.424 & 0 & 1 \\ -0.490 & 0 & 0 & 1 \\ -0.962 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.929 & 0 & 0 & 1 \\ 0.481 & 0.833 & 0 & 1 \\ -0.465 & 0.805 & 0 & 1 \\ 0.245 & 0.424 & 0 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–48.

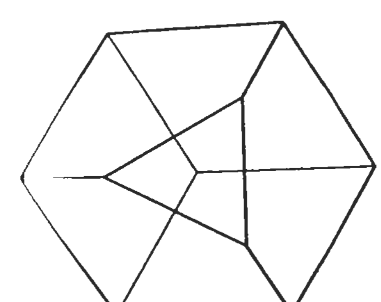


Figure 3–48 Result for Ex. 3–29.

3–21 RECONSTRUCTION OF THREE-DIMENSIONAL IMAGES

The reconstruction of a three-dimensional object or position in space is a common problem. For example, it occurs continuously in utilizing mechanical drawings which are orthographic projections. The method of reconstructing a three-dimensional object or position from two or more views (orthographic projections) given on a mechanical drawing is well known. However, the technique of reconstructing a three-dimensional position vector from two perspective projections, for example, two photographs, is not as well known. Of course, if the method is valid for perspective projections, then it is also valid for the simpler orthographic projections, and in fact for all the projections mentioned in previous sections. Further, as is shown below, if certain other information is available, then no direct knowledge about the transformation is required.

Before considering the more general problem we consider the special case of reconstruction of the three-dimensional coordinates of a point from two or more orthographic projections. Front, right-side and top orthographic views (projections) of an object are shown in Fig. 3–49. In determining the three-dimensional coordinates of point A, the front view yields values for x and y, the right-side view for y and z, and the top view for x and z, i.e.,

$$\begin{array}{rccc} \text{front}: & x_f & y_f & \\ \text{right side}: & & y_r & z_r \\ \text{top}: & x_t & & z_t \end{array}$$

Notice that two values are obtained for each coordinate. In any measurement system, in general $x_f \neq x_t$, $y_f \neq y_r$, $z_r \neq z_t$.† Since neither value is necessarily correct, the most reasonable solution is to average the values. Mathematically, the problem is said to be overspecified. Here only three independent values need be determined, but six conditions (equations) determining those values are available.

Turning now to reconstruction of three-dimensional coordinates from perspective projections, recall that the general perspective transformation is represented as a 4 × 4 matrix. Thus,

$$[x \quad y \quad z \quad 1][T'] = [x' \quad y' \quad z' \quad h]$$

where

$$[T'] = \begin{bmatrix} T'_{11} & T'_{12} & T'_{13} & T'_{14} \\ T'_{21} & T'_{22} & T'_{23} & T'_{24} \\ T'_{31} & T'_{32} & T'_{33} & T'_{34} \\ T'_{41} & T'_{42} & T'_{43} & T'_{44} \end{bmatrix}$$

†For this reason mechanical drawings are explicitly dimensioned.

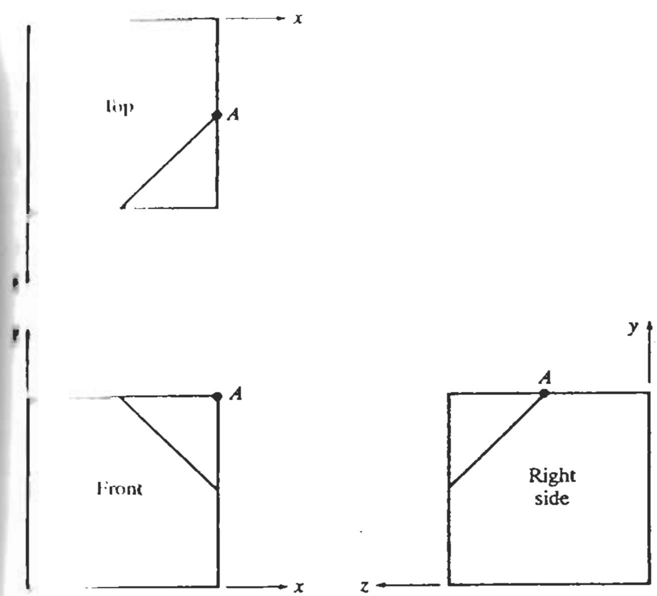


Figure 3–49 Three-dimensional reconstruction from orthographic projections.

The results can be projected onto a two-dimensional plane, say $z = 0$, using

$$[T''] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Concatenation of the two matrices yields

$$[T] = [T''][T'] = \begin{bmatrix} T_{11} & T_{12} & 0 & T_{14} \\ T_{21} & T_{22} & 0 & T_{24} \\ T_{31} & T_{32} & 0 & T_{34} \\ T_{41} & T_{42} & 0 & T_{44} \end{bmatrix}$$

It is useful to write the transformation as

$$[x \quad y \quad z \quad 1] \begin{bmatrix} T_{11} & T_{12} & 0 & T_{14} \\ T_{21} & T_{22} & 0 & T_{24} \\ T_{31} & T_{32} & 0 & T_{34} \\ T_{41} & T_{42} & 0 & T_{44} \end{bmatrix} = [x' \quad y' \quad 0 \quad h]$$

$$= h[x^* \quad y^* \quad 0 \quad 1] \qquad (3-71)$$

Note that x^* and y^* are the coordinates of the perspective projection onto the $z = 0$ plane. Projections onto the $x = 0$ or $y = 0$ planes could also be used.

Writing out Eq. (3–71) yields

$$T_{11}x + T_{21}y + T_{31}z + T_{41} = hx^* \qquad (3-72a)$$
$$T_{12}x + T_{22}y + T_{32}z + T_{42} = hy^* \qquad (3-72b)$$
$$T_{14}x + T_{24}y + T_{34}z + T_{44} = h \qquad (3-72c)$$

Using h from Eq. (3–72c) and substituting into Eqs. (3–72a) and (3–72b) yields

$$(T_{11} - T_{14}x^*)x + (T_{21} - T_{24}x^*)y + (T_{31} - T_{34}x^*)z + (T_{41} - T_{44}x^*) = 0 \qquad (3-73a)$$
$$(T_{12} - T_{14}y^*)x + (T_{22} - T_{24}y^*)y + (T_{32} - T_{34}y^*)z + (T_{42} - T_{44}y^*) = 0 \qquad (3-73b)$$

As suggested by Sutherland (Ref. 3–4), this pair of equations can be considered in three different ways. First assume T and x, y, z are known. Then there are two equations in the two unknowns x^* and y^*. Thus, they may be used to solve directly for the coordinates of the perspective projection. This is the approach taken in all the previous discussions in this chapter.

Alternately T, x^*, y^* can be assumed known. In this case two equations in the three unknown space coordinates x, y, z result. The system of equations cannot be solved. However, if two perspective projections, say two photographs, are available, then Eq. (3–73) can be written for both projections. This yields

$$(T^1_{11} - T^1_{14}x^{*1})x + (T^1_{21} - T^1_{24}x^{*1})y + (T^1_{31} - T^1_{34}x^{*1})z + (T^1_{41} - T^1_{44}x^{*1}) = 0$$
$$(T^1_{12} - T^1_{14}y^{*1})x + (T^1_{22} - T^1_{24}y^{*1})y + (T^1_{32} - T^1_{34}y^{*1})z + (T^1_{42} - T^1_{44}y^{*1}) = 0$$
$$(T^2_{11} - T^2_{14}x^{*2})x + (T^2_{21} - T^2_{24}x^{*2})y + (T^2_{31} - T^2_{34}x^{*2})z + (T^2_{41} - T^2_{44}x^{*2}) = 0$$
$$(T^2_{12} - T^2_{14}y^{*2})x + (T^2_{22} - T^2_{24}y^{*2})y + (T^2_{32} - T^2_{34}y^{*2})z + (T^2_{42} - T^2_{44}y^{*2}) = 0$$

where the superscripts 1 and 2 indicate the first and second perspective projections. Note that the transformations $[T^1]$ and $[T^2]$ need not be the same. These equations can be rewritten in matrix form as

$$[A][X] = [B] \qquad (3-74)$$

where

$$[A] = \begin{bmatrix} T^1_{11} - T^1_{14}x^{*1} & T^1_{21} - T^1_{24}x^{*1} & T^1_{31} - T^1_{34}x^{*1} \\ T^1_{12} - T^1_{14}y^{*1} & T^1_{22} - T^1_{24}y^{*1} & T^1_{32} - T^1_{34}y^{*1} \\ T^2_{11} - T^2_{14}x^{*2} & T^2_{21} - T^2_{24}x^{*2} & T^2_{31} - T^2_{34}x^{*2} \\ T^2_{12} - T^2_{14}y^{*2} & T^2_{22} - T^2_{24}y^{*2} & T^2_{32} - T^2_{34}y^{*2} \end{bmatrix}$$

$$[X]^T = [x \quad y \quad z]$$

$$[B]^T = \begin{bmatrix} T^1_{44}x^{*1} - T^1_{41} & T^1_{44}y^{*1} - T^1_{42} & T^2_{44}x^{*2} - T^2_{41} & T^2_{44}y^{*2} - T^2_{42} \end{bmatrix}$$

Equation (3–74) represents four equations in the three unknown space coordinates x, y, z. $[A]$ is not a square matrix and consequently cannot be inverted to obtain the solution for $[X]$. Again, as in the case of reconstructing three-dimensional coordinates from orthographic projections, the problem is overspecified and thus can be solved only in some mean or best-fit sense.

A mean solution is computed by recalling that a matrix times its transpose is always square. Thus, multiplying both sides of Eq. (3–74) by $[A]^T$ yields

$$[A]^T[A][X] = [A]^T[B]$$

Taking the inverse of $[[A]^T[A]]$ yields a mean solution for $[X]$, i.e.,

$$[X] = [[A]^T[A]]^{-1}[A]^T[B] \qquad (3-75)$$

If no solution for $[X]$ results, then the imposed conditions are redundant and no unique solution which yields a least error condition exists. An example illustrates this technique.

Example 3–30 Three-Dimensional Reconstruction

Assume that the measured position of a point in one perspective projection is $[0.836 \quad -1.836 \quad 0 \quad 1]$ and is $[0.6548 \quad 0 \quad 0.2886 \quad 1]$ in a second perspective projection. The first perspective projection transformation is known to be the result of a 60° rotation about the y-axis, followed by a translation of 2 units in the negative y direction. The point of projection is at $z = -1$, and the result is projected onto the $z = 0$ plane. This is effectively a two-point perspective projection. The second perspective projection is the result of a 30° rotation about each of the x- and y-axes. The point of projection is at $y = -1$ and the result is projected onto the $y = 0$ plane, i.e., effectively a three-point perspective projection. $[T^1]$ and $[T^2]$ are thus

$$[T^1] = \begin{bmatrix} 0.5 & 0 & 0 & -0.87 \\ 0 & 1 & 0 & 0 \\ 0.87 & 0 & 0 & 0.5 \\ 0 & -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [T^2] = \begin{bmatrix} 0.87 & 0 & -0.5 & 0 \\ 0.25 & 0 & 0.43 & 0.87 \\ 0.43 & 0 & 0.75 & -0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

First noting that the last two rows of $[A]$ and $[B]$ must be rewritten to account for $[T^2]$ being a projection onto the $y = 0$ plane, the A-matrix is

$$[A] = \begin{bmatrix} 1.22 & 0 & 0.45 \\ -1.59 & 1 & 0.92 \\ 0.87 & -0.32 & 0.76 \\ -0.5 & 0.18 & 0.89 \end{bmatrix}$$

and

$$[B]^T = [0.84 \quad 0.16 \quad 0.65 \quad 0.29]$$

Solution yields $[X] = [0.5 \quad 0.5 \quad 0.5]$, i.e., the center of the unit cube.

As a third way of considering Eq. (3–73) note that if the location of several points which appear in the perspective projection are known in object space and in the perspective projection, then it is possible to determine the transformation elements, i.e., the T_{ij}'s. These transformation elements can subsequently be used to determine the location of unknown points using the second technique described above. To see this, rewrite Eq. (3–73) as

$$T_{11}x + T_{21}y + T_{31}z + T_{41} - T_{14}xx^* - T_{24}yx^* - T_{34}zx^* - T_{44}x^* = 0 \qquad (3-76a)$$

$$T_{12}x + T_{22}y + T_{32}z + T_{42} - T_{14}xy^* - T_{24}yy^* - T_{34}zy^* - T_{44}y^* = 0 \qquad (3-76b)$$

Assuming that x^* and y^* as well as x, y, z are known, Eqs. (3–76*a*) and (3–76*b*) represent two equations in the 12 unknown transformation elements T_{ij}. Applying these equations to 6 noncoplanar known locations in object space and in the perspective projection yields a system of 12 equations in 12 unknowns. These equations can be solved exactly for the T_{ij}'s. Thus, the transformation that produced the perspective projection, for example, a photograph, is determined. Notice that in this case no prior knowledge of the transformation is required. If, for example, the perspective projections are photographs, neither the location nor the orientation of the camera is required. In matrix form the system of 12 equations is written as

$$\begin{bmatrix}
x_1 & 0 & -x_1x_1^* & y_1 & 0 & -y_1x_1^* & z_1 & 0 & -z_1x_1^* & 1 & 0 & -x_1^* \\
0 & x_1 & -x_1y_1^* & 0 & y_1 & -y_1y_1^* & 0 & z_1 & -z_1y_1^* & 0 & 1 & -y_1^* \\
x_2 & 0 & -x_2x_2^* & y_2 & 0 & -y_2x_2^* & z_2 & 0 & -z_2x_2^* & 1 & 0 & -x_2^* \\
0 & x_2 & -x_2y_2^* & 0 & y_2 & -y_2y_2^* & 0 & z_2 & -z_2y_2^* & 0 & 1 & -y_2^* \\
x_3 & 0 & -x_3x_3^* & y_3 & 0 & -y_3x_3^* & z_3 & 0 & -z_3x_3^* & 1 & 0 & -x_3^* \\
0 & x_3 & -x_3y_3^* & 0 & y_3 & -y_3y_3^* & 0 & z_3 & -z_3y_3^* & 0 & 1 & -y_3^* \\
x_4 & 0 & -x_4x_4^* & y_4 & 0 & -y_4x_4^* & z_4 & 0 & -z_4x_4^* & 1 & 0 & -x_4^* \\
0 & x_4 & -x_4y_4^* & 0 & y_4 & -y_4y_4^* & 0 & z_4 & -z_4y_4^* & 0 & 1 & -y_4^* \\
x_5 & 0 & -x_5x_5^* & y_5 & 0 & -y_5x_5^* & z_5 & 0 & -z_5x_5^* & 1 & 0 & -x_5^* \\
0 & x_5 & -x_5y_5^* & 0 & y_5 & -y_5y_5^* & 0 & z_5 & -z_5y_5^* & 0 & 1 & -y_5^* \\
x_6 & 0 & -x_6x_6^* & y_6 & 0 & -y_6x_6^* & z_6 & 0 & -z_6x_6^* & 1 & 0 & -x_6^* \\
0 & x_6 & -x_6y_6^* & 0 & y_6 & -y_6y_6^* & 0 & z_6 & -z_6y_6^* & 0 & 1 & -y_6^*
\end{bmatrix}
\begin{bmatrix} T_{11} \\ T_{12} \\ T_{14} \\ T_{21} \\ T_{22} \\ T_{24} \\ T_{31} \\ T_{32} \\ T_{34} \\ T_{41} \\ T_{42} \\ T_{44} \end{bmatrix} = 0 \qquad (3-77)$$

where the subscripts correspond to points with known locations. Equations (3–77) are written in more compact form as

$$[\,A'\,][\,T\,] = 0$$

Since Eqs. (3–77) are homogeneous, they contain an arbitrary scale factor. Hence T_{44} may, for example, be defined as unity and the resulting transformation normalized. This reduces the requirement to 11 equations or 5 1/2 points. If the transformation is normalized, then the last column in $[\,A'\,]$ is moved to the right-hand side and the nonhomogeneous matrix equation is solved. An example is given below.

Example 3–31 Elements for Reconstruction

As a specific example, consider the unit cube with the six known corner points in the physical plane given by

$$[\,P\,] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The corresponding points in the transformed view are marked by a dot as shown in Fig. 3–50. The corresponding transformed coordinates of these points are

$$\begin{bmatrix} 0 & -1 \\ 0.34 & -0.8 \\ 0.34 & -0.4 \\ 0 & -0.5 \\ 0.44 & -1.75 \\ 0.83 & -1.22 \end{bmatrix}$$

Equation (3–77) then becomes

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.34 & 1 & 0 & -0.34 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0 & 1 & 0.8 \\
0 & 0 & 0 & 1 & 0 & -0.34 & 1 & 0 & -0.34 & 1 & 0 & -0.34 \\
0 & 0 & 0 & 0 & 1 & 0.4 & 0 & 1 & 0.4 & 0 & 1 & 0.4 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0.5 & 0 & 0 & 0 & 0 & 1 & 0.5 \\
1 & 0 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.44 \\
0 & 1 & 1.75 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1.75 \\
1 & 0 & -0.83 & 0 & 0 & 0 & 1 & 0 & -0.83 & 1 & 0 & -0.83 \\
0 & 1 & 1.22 & 0 & 0 & 0 & 0 & 1 & 1.22 & 0 & 1 & 1.22
\end{bmatrix}
\begin{bmatrix} T_{11} \\ T_{12} \\ T_{14} \\ T_{21} \\ T_{22} \\ T_{24} \\ T_{31} \\ T_{32} \\ T_{34} \\ T_{41} \\ T_{42} \\ T_{44} \end{bmatrix} = 0$$

Solution for the 12 unknown T_{ij}'s yields

$$\begin{bmatrix} T_{11} \\ T_{12} \\ T_{14} \\ T_{21} \\ T_{22} \\ T_{24} \\ T_{31} \\ T_{32} \\ T_{34} \\ T_{41} \\ T_{42} \\ T_{44} \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \\ -0.43 \\ 0 \\ 0.5 \\ 0 \\ 0.43 \\ 0 \\ 0.25 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

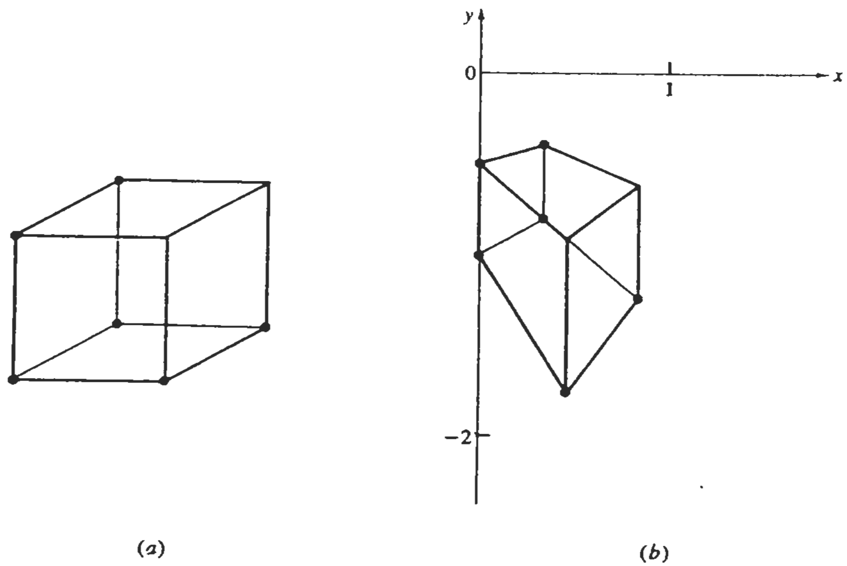


Figure 3–50 Determining the transformation from a perspective projection.

Substituting these results into the 4 × 4 [T] matrix yields

$$[T] = \begin{bmatrix} 0.25 & 0 & 0 & -0.43 \\ 0 & 0.5 & 0 & 0 \\ 0.43 & 0 & 0 & 0.25 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

3–22 REFERENCES

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CHAPTER

FOUR

PLANE CURVES

4–1 INTRODUCTION

A multitude of techniques are available for drawing and designing curves manually. A wide variety of pencils, pens, brushes, knives, etc., along with straightedges, French curves, compasses, splines, templates, etc., are used to aid the designer. Each tool has its function and use. No single tool is sufficient for all tasks. Similarly, a variety of techniques and tools are useful for curve design and generation in computer graphics. Two-dimensional curve generation techniques are discussed in this chapter. A curve is two-dimensional if it lies in its entirety in a single plane. Here, the discussion is limited to the conic sections.

4–2 CURVE REPRESENTATION

The previous two chapters treated the transformation of points. A curve may be represented as a collection of points. Provided the points are properly spaced, connection of the points by short straight line segments yields an adequate visual representation of the curve. Figure 4–1 shows two alternate point representations of the same plane curve. Points along the curve in Fig. 4–1a are equally spaced along the curve length. Notice that connection of the points by short straight

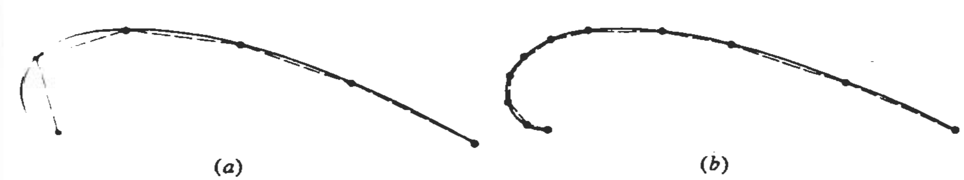

Figure 4–1 Point representations of curves. (a) Equal point density along the curve length; (b) point density increases with decreasing radius of curvature.