

II.2 Линејна алгебра - Иокс. Табачков  
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 Линејна алгебра - розово П. Сидеров  
 Записи по алгебра - син/зелен П. Сидеров

Метод на Гаус за решавање на системи  
 линејни уравнения

$$\begin{cases}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\
 \dots \\
 a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n = b_k
 \end{cases}$$

$(\alpha_1, \alpha_2, \dots, \alpha_n) = (x_1, x_2, \dots, x_n)$  с-тата е свместита

2) системата е несвм  $\Leftrightarrow$  няма решение

1.1) Уточно 1 реш.  $\Leftrightarrow$  определена е-ма

1.2) Системата е неодредена  $\Leftrightarrow$  Уповега от 1 реш.

Еквивалентни (елементарни) преобразованија

1) умнож. на  $y$ -ние с число  $\neq 0$

2) умнож. на  $y$ -ние с число  $\neq 0$  и прибавяне към друго  $y$ -ние

3) смяна на местата на 2  $y$ -ния



0.17 a) 
$$\begin{cases} x_1 + 2x_2 + 5x_3 = -9 \\ x_1 - x_2 + 3x_3 = 2 \\ 3x_1 - 6x_2 - x_3 = 25 \end{cases}$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 5 & -9 \\ 1 & -1 & 3 & 2 \\ 3 & -6 & -1 & 25 \end{array} \right) \sim$$

$\times (-3)$

$$A = \left( \begin{array}{ccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} & b_k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array} \right)$$

$k > n$

$$L_2 := L_2 - L_1$$

$$L_3 := L_3 - 3L_1$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & 5 & -9 \\ 0 & -3 & -2 & 11 \\ 0 & -12 & -16 & 52 \end{array} \right) \quad | : (-4)$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & 5 & -9 \\ 0 & -3 & -2 & 11 \\ 0 & 3 & 4 & -13 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & 5 & -9 \\ 0 & -3 & -2 & 11 \\ 0 & 0 & 2 & -2 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & 5 & -9 \\ 0 & -3 & -2 & 11 \\ 0 & 0 & 2 & -4 \end{array} \right)$$

$$\begin{cases} x_1 + 2x_2 + 5x_3 = -9 \\ -3x_2 - 2x_3 = 11 \\ x_3 = -1 \end{cases}$$

$$\begin{cases} x_1 = 2 \\ x_2 = -3 \\ x_3 = -1 \end{cases}$$

с-мата е съвместима и определ.  
 $(x_1, x_2, x_3) = (2, -3, -1)$

$$\begin{cases} x_1 + x_2 + 2x_3 = 4 \\ x_1 + x_2 + x_3 = 2 \\ x_1 + 2x_2 + 3x_3 = 7 \end{cases}$$

$$\left( \begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 3 & 7 \end{array} \right) \sim$$

$$\left( \begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -1 & -1 & -2 \\ 0 & 1 & 1 & 3 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$$\begin{cases} x_1 + x_2 + 2x_3 = 4 \\ x_2 + x_3 = 2 \\ 0 = 1 \end{cases}$$

с-мата е  
несъвместима



$$\begin{cases} x_1 - x_3 = 2 \\ 2x_1 + 5x_2 = 7 \\ 3x_1 + 5x_2 - x_3 = 9 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 5 & 0 & 7 \\ 3 & 5 & -1 & 9 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 5 & 2 & 3 \\ 0 & 5 & 2 & 3 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 5 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{cases} x_1 - x_3 = 2 \\ 5x_2 + 2x_3 = 3 \end{cases}$$

$$\begin{aligned} x_3 &= p \\ x_2 &= \frac{3-2p}{5} \end{aligned}$$

с-матрица с совм., неопр.  $(x_1, x_2, x_3) = (2+p, \frac{3-2p}{5}, p)$   $\forall p$

0.185

$$\begin{cases} 2x_1 + 3x_2 - 7x_3 + 5x_4 + 2x_5 = 0 \\ 6x_1 + 3x_2 - 19x_3 + 7x_4 + 8x_5 = 0 \\ 10x_1 + 22x_2 + 36x_3 + 33x_4 + 23x_5 = 0 \\ 6x_1 + 8x_2 - 18x_3 + 11x_4 + 37x_5 = 0 \\ 2x_1 + 13x_2 - 9x_3 + 17x_4 + 14x_5 = 0 \end{cases}$$

$$\left( \begin{array}{ccccc} 2 & 3 & -7 & 5 & 2 \\ 6 & 3 & -19 & 7 & 8 \\ 10 & 22 & -36 & 33 & 23 \\ 6 & 8 & -18 & 11 & 37 \\ 2 & 13 & -9 & 17 & 14 \end{array} \right) \sim \left( \begin{array}{ccccc} 2 & 3 & -7 & 5 & 2 \\ 0 & -6 & 2 & -8 & 2 \\ 0 & 7 & -1 & 8 & 13 \\ 0 & -1 & 3 & -4 & 31 \\ 0 & 10 & -2 & 12 & 12 \end{array} \right) \sim \left( \begin{array}{ccccc} 2 & 3 & -7 & 5 & 2 \\ 0 & -1 & 3 & -4 & 31 \\ 0 & 0 & -16 & -18 & -13 \\ 0 & 0 & 20 & -10 & 230 \\ 0 & 28 & -28 & 32 & 129 \end{array} \right)$$

$$L_2 := L_1 + L_2$$

$$L_4 \Rightarrow L_2$$

$$L_3 := 2 - 5L_1 + L_3$$

$$L_4 := L_4 - 7L_2 +$$

$$L_4 := 23L_1 + L_4$$

$$L_5 := L_5 - L_1$$

$$\sim \left( \begin{array}{ccccc} 2 & 3 & -7 & 5 & 2 \\ 0 & -1 & 3 & -4 & 31 \\ 0 & 0 & -2 & 2 & -16 \\ 0 & 0 & 2 & -2 & 23 \\ 0 & 0 & 2 & -2 & 23 \end{array} \right) \sim \left( \begin{array}{ccccc} 2 & 3 & -7 & 5 & 2 \\ 0 & -1 & 3 & -4 & 31 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

11.2.3



$$\begin{cases} 2x_1 + 3x_2 - 7x_3 + 5x_4 + 2x_5 = 0 \\ -x_2 + 3x_3 - 4x_4 + 31x_5 = 0 \\ -2x_3 + 2x_4 - 23x_5 = 0 \end{cases}$$

$$\begin{aligned} x_4 &= p & x_5 &= q \\ x_3 &= 2p - 23q \end{aligned}$$

$$x_2 = \frac{6p + 24q + 4p - 161q - 20p + 8q}{4} = -37q$$

$$(x_1, x_2, x_3, x_4, x_5) = (-37q, -\frac{2p+7q}{2}, \frac{2p-23q}{2}, p, q) \quad \forall p, q$$

$$\begin{cases} x_1 + x_2 + ax_3 = 1 \\ x_1 + ax_2 + x_3 = a \\ ax_1 + x_2 + x_3 = 1 \end{cases} \quad \begin{pmatrix} 1 & 1 & a & | & 1 \\ 1 & a & 1 & | & a \\ a & 1 & 1 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1+a & 2+a & 2+a & | & 2+a \\ 1 & a & 1 & | & a \\ a & 1 & 1 & | & 1 \end{pmatrix}$$

$$2+a \neq 0 \quad a \neq -2$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & a & 1 & | & a \\ a & 1 & 1 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & a-1 & 0 & | & a-1 \\ 0 & 1-a & 1-a & | & 1-a \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & a-1 & 0 & | & a-1 \\ 0 & 0 & 1-a & | & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ (a-1)x_2 &= a-1 \\ (1-a)x_3 &= 0 \end{aligned}$$

Сл.  $a = 1$  нека  $x_2 = p$   $x_3 = q$   $x_1 = 1-p-q$  неопределена  
 $(x_1, x_2, x_3) = (1-p-q, p, q) \quad \forall p, q$

Сл.  $a \neq 1, -2$   $x_3 = 0$   $x_2 = 1$   $x_1 = 0$   $(x_1, x_2, x_3) = (0, 1, 0)$  опрег.

Сл.  $a = -2$

$$\begin{pmatrix} 1 & -2 & 1 & | & -2 \\ -2 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & | & -2 \\ 0 & -3 & 3 & | & -3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \quad \begin{cases} x_1 - 2x_2 + x_3 = -2 \\ x_2 - x_3 = 1 \end{cases}$$

$x_3 = s$   $x_2 = 1+s$   $x_1 = s$  неопределена  
 $(x_1, x_2, x_3) = (s, 1+s, s) \quad \forall s$

2.4



$$\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = -1 \\ x_1 + x_2 + 3x_3 + x_4 = 2 \\ x_1 + x_2 + 3x_3 + (4+a)x_4 = 6+2 \\ x_1 + \dots + 2x_3 + 2x_4 = 2 \end{cases}$$

$$\left( \begin{array}{cccc|c} 1 & 2 & 3 & -2 & -1 \\ 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 3 & 4+a & 6+2 \\ 1 & 0 & 2 & 2 & 2 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{cccc|c} 1 & 2 & 3 & -2 & -1 \\ 0 & 1 & 0 & -3 & -3 \\ 0 & 1 & 0 & -6-a & -5b \\ 0 & 2 & 1 & -4 & -3 \end{array} \right) \sim \left( \begin{array}{cccc|c} 1 & 2 & 3 & -2 & -1 \\ 0 & 1 & 0 & -3 & -3 \\ 0 & 0 & 0 & a+3 & 6 \\ 0 & 0 & 1 & -2 & -3 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{cccc|c} 1 & 2 & 3 & -2 & -1 \\ 0 & 1 & 0 & -3 & -3 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & a+3 & 6 \end{array} \right) \quad \begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = -1 \\ x_2 = -3 \\ x_3 - 2x_4 = -3 \\ (a+3)x_4 = 6 \end{cases}$$

$$\text{I cr. } a \neq -3 \quad x_4 = \frac{6}{a+3} \quad a \neq -3 \quad x_3 = -3 + 2 \frac{6}{a+3} \quad x_2 = -3 + 3 \frac{6}{a+3}$$

$$x_1 = -1 + 6 \left( 1 - \frac{6}{a+3} \right) + 3 \left( 3 - 2 \frac{6}{a+3} \right) + 2 \frac{6}{a+3} \quad \text{определена}$$

$$x_3 = \frac{-3a-9+2b}{a+3} = \frac{-3a-9+3b}{a+3} \quad x_2 = \frac{-3a-9+3b}{a+3}$$

$$x_1 = 1 - \frac{6a-18+6b}{a+3} - \frac{-9a-27+6b}{a+3} + \frac{2b}{a+3} = \frac{a+3+6a+18-6b+9a+27-6b+2b}{a+3}$$

$$= \frac{-6b+2b}{a+3} = \frac{14a-10b+48}{a+3}$$

$$\text{II cr. } a = -3 \quad x_4 = 6 \quad x_3 = 6 \quad x_2 = 6 \quad x_1 = 6$$

$$\text{2.1.) } b=0 \quad x_4 = 6 \quad x_3 = 6 \quad x_2 = 6 \quad x_1 = 6$$

$$x_3 = -3 - z \quad x_2 = -3 + 3z \quad x_1 = -1 + 6 - 6z + 9 + 3z + 2z = -z + 14 = 14 - z$$

неопределена ~~за~~



0.19 g) и e)

0.20 - га капоумним

$$g) \begin{cases} (3-2\lambda)x_1 + (2-\lambda)x_2 + x_3 = \lambda \\ (2-\lambda)x_1 + (2-\lambda)x_2 + 2x_3 = 1 \\ x_1 + x_2 + (2-\lambda)x_3 = 1 \end{cases}$$

$$e) \begin{cases} \lambda x_1 + 2x_2 + 3x_3 + 2x_4 = 3 \\ 2x_1 + \lambda x_2 + 3x_3 + 2x_4 = 3 \\ 2x_1 + 3x_2 + \lambda x_3 + 2x_4 = 3 \\ 2x_1 + 3x_2 + 2x_3 + \lambda x_4 = 3 \\ 2x_1 + 3x_2 + 2x_3 + 3x_4 = \lambda \end{cases}$$

$$\text{II. 1.1} \quad \begin{cases} 3-\lambda=0 & \lambda=3 \\ -1x_3 + 1x_4 = 0 \\ x_3 = x_4 \end{cases}$$

$$\text{Реша } x_1 = p \Rightarrow x_3 = \frac{3-5p}{5} \\ -x_1 + x_2 = 0 \Rightarrow x_2 = p \Rightarrow x_1 = x_2 \\ 5x_1 + 5x_3 = 3 \\ x_3 = \frac{3-5x_1}{5}$$

$$(x_1, x_2, x_3, x_4) = (p, p, \frac{3-5p}{5}, \frac{3-5p}{5}) \quad \text{неопределена}$$

$$\text{II 1.1)} \quad \begin{cases} (2-\lambda)x_3 = \lambda-2 \\ \lambda \neq 2 \end{cases} \quad \text{II 2.1)} \quad \begin{cases} \lambda \neq 2, 3, -10 \end{cases}$$

$$x_3 = -1$$

$$x_2 = -1 \quad x_1 = -1 \quad \lambda = 10$$

$$(x_1, x_2, x_3, x_4) = (-1, -1, -1, -1)$$

$$\text{II 2.2)} \quad \begin{cases} \lambda = 2 & x_4 = -1 \\ x_2 = x_3 = 0 \\ x_2 = x_3 \end{cases}$$

$$2x_1 + 5x_2 = 5$$

$$x_1 = p \quad x_2 = \frac{5-2p}{5} = x_3$$

$$(x_1, x_2, x_3, x_4) = (p, \frac{5-2p}{5}, \frac{5-2p}{5}, -1)$$

II.2.6



$$0.20 \delta) \begin{cases} ax_1 + ax_2 + \dots + ax_{n-1} + (a+1)x_n = b_1 \\ ax_1 + ax_2 + \dots + (a+1)x_{n-1} + ax_n = b_2 \\ \vdots \\ (a+1)x_1 + ax_2 + \dots + ax_{n-1} + ax_n = b_n \end{cases}$$

$$\begin{pmatrix} a & a & a & \dots & a & a & a+1 & | & b_1 \\ a & a & a & \dots & a & a+1 & a & | & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & | & \vdots \\ a & a+1 & a & \dots & a & a & a & | & b_{n-1} \\ a+1 & a & a & \dots & a & a & a & | & b_n \end{pmatrix} \sim \begin{pmatrix} a & a & a & \dots & a & a & a+1 & | & b_1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 & | & b_2 - b_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & | & \vdots \\ 0 & 1 & 0 & \dots & -1 & 0 & 0 & | & b_{n-1} - b_2 \\ 1 & -1 & 0 & \dots & 0 & 0 & 0 & | & b_n - b_{n-1} \end{pmatrix}$$

$$\sim \begin{pmatrix} a+1 & a+1 & a+1 & \dots & a+1 & a+1 & | & \sum_{i=1}^n b_i \\ a & a & \dots & a & a+1 & a & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & | & \vdots \\ a+1 & a & a & \dots & a & a & a & | & b_n \end{pmatrix} \quad 1: (a+1) \neq 0 \quad a+1 = \frac{1}{n}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & | & \frac{1}{a+1} \sum_{i=1}^n b_i \\ a & a & a & \dots & a & a+1 & a & | & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & | & \vdots \\ a+1 & a & a & \dots & a & a & a & | & b_n \end{pmatrix} \xrightarrow{A} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & | & \frac{1}{a+1} \sum_{i=1}^n b_i \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & | & b_2 - \frac{1}{a} A \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & | & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & | & b_n - \frac{1}{a} A \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + \dots + x_{n-1} + x_n = \frac{1}{a+1} \sum_{i=1}^n b_i = A \\ x_{n-1} = b_2 - aA \\ x_{n-2} = b_3 - aA \\ \vdots \\ x_1 = b_n - aA \end{cases}$$



$$x_n = \frac{1}{na+1} \sum_{i=1}^n b_i - \sum_{j=2}^n b_j + (n-1)a \cdot \frac{1}{na+1} \sum_{i=1}^n b_i$$

$$x_n = \sum_{j=2}^n b_j \left( \frac{1}{na+1} \cdot b_1 (1 + (n-1)a) - 1 \right)$$

$$x_k = b_{n-k+1} - \frac{a}{na+1} \sum_{i=1}^n b_i, \quad k=1, \dots, n-1$$

II. cn.  $a = -\frac{1}{n}$

$$\left( \begin{array}{cccc|c} 0 & 0 & 0 & \dots & 0 & 0 & 0 & \sum_{i=1}^n b_i \\ -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} & 1 & -\frac{1}{n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & -\frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & b_n \end{array} \right)$$

II.1)  $\sum_{i=1}^n b_i \neq 0$  c-маца е несъвместима

II.2)  $\sum_{i=1}^n b_i = 0$

$$\left( \begin{array}{cccc|c} -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} & 1 & -\frac{1}{n} & b_2 \\ 0 & 0 & 0 & \dots & 1 & -1 & 0 & b_3 - b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & b_n - b_2 \end{array} \right)$$

$$-\frac{1}{n}(x_1 + x_2 + \dots + x_{n-2} + x_{n-1} + x_n) + x_{n-1} = b_2$$

$$x_{n-2} - x_{n-1} = b_3$$

$$x_1 - x_{n-1} = b_n - b_2$$

$$x_{n-2} = p \quad x_{n-1} = \frac{b_n - b_2}{2} + p \quad \dots \quad x_{n-2} = \frac{b_3 - b_2}{2} + p \quad x_n = \dots$$



## Комплексни числа

Алгебричен вид на комплексните числа

$$z = (a, b), \quad a, b - \text{реални } \in \mathbb{R}$$

$$a \rightarrow \operatorname{Re} z \text{ - реална част}$$

$$b \rightarrow \operatorname{Im} z \text{ - имагинерна част}$$

$$z = a + ib$$

единица  $i^2 = -1$

$$z_1 = a_1 + ib_1$$

$$z_2 = a_2 + ib_2$$

$$z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$$

$$\begin{matrix} + \\ - \end{matrix} z_1 = \begin{pmatrix} a_1, b_1 \end{pmatrix} \Rightarrow (a_1 \pm a_2, b_1 \pm b_2)$$

$$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$* \begin{matrix} z_1 = (a_1, b_1) \\ z_2 = (a_2, b_2) \end{matrix} \Rightarrow (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$$

$$i = (0, 1)$$

$$i^2 = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1; 0 \cdot 1 - 1 \cdot 0) = (-1, 0)$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}$$

$$z = (a, b) = (a, 0) + (0, b)$$

$ib$  - чисто имагинерно число

$\bar{z}$  - комплексно спряганото на  $z$

$$\bar{z} = a - ib$$

$$z = \bar{\bar{z}}$$

$$0 \neq \frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \cdot \frac{a_2 - ib_2}{a_2 - ib_2} =$$

$$z \bar{z} = a^2 + b^2 \in \mathbb{R}$$

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$$



$x, y \in \mathbb{R}$

$$(1+i)x - (3-2i)y = 1+6i$$

$$x+iy - 3y+2iy = 1+6i$$

$$(x-3y) + i(x+2y) = 1+6i$$

$$\begin{cases} x-3y = 1 \\ x+2y = 6 \end{cases}$$

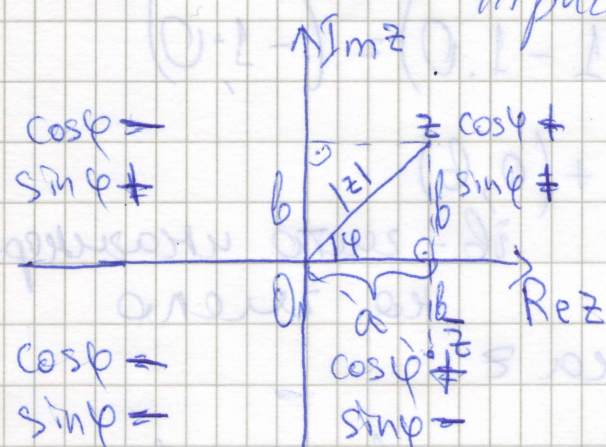
$$\begin{aligned} x &= 1+3y = 4 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} (2+3i)(1-2i) + \frac{3-4i}{1+i} &= 2+3i-4i-6i^2 + \frac{3-4i}{1+i} = \\ &= \frac{(-6i^2-i+2)(1-i)}{1-i} = (2+6) + i(3-4) + \frac{(3-4i)(1-i)}{2} = \\ &= \frac{16-2i-7i-1}{2} = \frac{15-9i}{2} \end{aligned}$$

$$i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

$$\frac{(1+i)^3 - (1-i)^4}{(4-2i)(3-2i)} = ? \quad \text{за гонимко}$$

Тригонометричен вид



$$z = a + ib$$

$$\cos \varphi = \frac{a}{|z|} \Rightarrow a = |z| \cos \varphi$$

$$\sin \varphi = \frac{b}{|z|} \Rightarrow b = |z| \sin \varphi$$

$$z = a + ib = |z|(\cos \varphi + i \sin \varphi)$$

$$\text{Arg } z = \varphi \in [0, 2\pi]$$

$$z_1 = |z_1|(\cos \varphi_1 + i \sin \varphi_1)$$

$$z_2 = |z_2|(\cos \varphi_2 + i \sin \varphi_2)$$

$$z_1 z_2 = |z_1| |z_2| (\cos \varphi_1 + i \sin \varphi_1) (\cos \varphi_2 + i \sin \varphi_2) =$$

$$= |z_1| |z_2| ((\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2) + i(\sin \varphi_1 \cos \varphi_2 + \cos \varphi_1 \sin \varphi_2)) =$$

$$= |z_1| |z_2| (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))$$

11.2.10



$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} (\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)) \quad \text{за гомашино}$$

формули на Моавър

$$z^2 = |z|^2 (\cos 2\varphi + i \sin 2\varphi) \quad \text{уопшениу} \Rightarrow z^n = |z|^n (\cos n\varphi + i \sin n\varphi)$$

$$\sqrt[n]{z} = \sqrt[n]{|z|} \left( \cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n} \right) \quad k = 0, 1, \dots, n-1$$

Да се намерат модула и аргумента на  $z$

$$z = 1 \quad |z| = \sqrt{1^2 + 0^2} = 1 \quad \text{и аргумента на } z$$

$$\cos \varphi = \frac{a}{|z|} = a = 1 > \varphi = 0 \quad (\text{Arg } z = 0)$$

$$\sin \varphi = \frac{b}{|z|} = b = 0$$

$$z = -\frac{1}{2} \quad |z| = \frac{1}{2} \quad \text{Arg } z = \pi$$

$$\cos \varphi = -1 > \varphi = \pi$$

$$\sin \varphi = 0$$

$$z = i \quad |z| = 1 \quad \text{Arg } z = \frac{\pi}{2}$$

$$z = 1 - i\sqrt{3} \quad |z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\text{Arg } z = \frac{2\pi}{3}$$

$$\cos \varphi = \frac{1}{2} > \varphi = \frac{\pi}{3}$$

$$\sin \varphi = -\frac{\sqrt{3}}{2}$$



Търсим множител от този вид в равнината  
когато съответства на комплексен числа

$$|z| = 1 \quad |z| \leq 1 \quad 1 < |z| \leq 3$$

$$\operatorname{Arg} z = 0 \quad \operatorname{Arg} z = \frac{\pi}{2} \quad \operatorname{Im} z \geq -2$$

Тригонометричен вид

$$\frac{(\sqrt{3} - i)^{15}}{(1+i)^8}$$

$$z_1 = \sqrt{3} - i \quad z_2 = 1 + i$$

$$|z_1| = 2 \left( \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right)$$

$$\cos \varphi_1 = \frac{\sqrt{3}}{2} \quad \sin \varphi_1 = -\frac{1}{2} \quad \varphi = \frac{11\pi}{6} = -\frac{\pi}{6}$$

$$z_1 = \sqrt{3} - i = 2 \left( \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right)$$

$$|z_2| = \sqrt{2}$$

$$\cos \varphi_2 = \frac{\sqrt{2}}{2} \quad \sin \varphi_2 = \frac{\sqrt{2}}{2} \quad \varphi = \frac{\pi}{4}$$

$$z_2 = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$z_1^{15} = 2^{15} \left( \cos \frac{15 \cdot 11\pi}{6} + i \sin \frac{15 \cdot 11\pi}{6} \right) = 2^{15} \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) = -2^{15}i$$

$$z_2^8 = 2^4 \left( \cos 2\pi + i \sin 2\pi \right) = 2^4$$

$$z = \frac{z_1^{15}}{z_2^8} = \frac{2^{15}}{2^4} \left( \cos \left( \frac{3\pi}{2} - 2\pi \right) + i \sin \left( \frac{3\pi}{2} - 2\pi \right) \right) =$$

$$= 2^{11} \left( \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right) = -2^{11}i$$



$$\begin{pmatrix} z_1 = -1 + i \\ z_2 = 2\sqrt{3} - 2i \end{pmatrix}^{60}$$

$$\begin{aligned} z_1 &= -1 + i = \sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \\ |z_1| &= \sqrt{(-1)^2 + 1^2} = \sqrt{2} \\ \cos \varphi_1 &= -\frac{\sqrt{2}}{\sqrt{2}} \\ \sin \varphi_1 &= \frac{\sqrt{2}}{\sqrt{2}} \end{aligned} \Rightarrow \varphi_1 = \frac{3\pi}{4}$$

$$z_2 = 2\sqrt{3} - 2i$$

$$|z_2| = 4$$

$$\cos \varphi_2 = \frac{\sqrt{3}}{2} \quad \sin \varphi_2 = -\frac{1}{2}$$

$$z = \frac{z_1}{z_2} = \frac{\sqrt{2}}{4} \left( \cos \left( \frac{3\pi}{4} + \frac{\pi}{6} \right) + i \sin \frac{11\pi}{12} \right)$$

$$z_1 = -1 + i$$

$$z_2 = 2\sqrt{3} - 2i = 4 \left( \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right)$$

$$\begin{aligned} z^{60} &= z^{-\frac{3}{2}} \cdot z^{30} = \left( \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right)^{-3} \cdot \left( \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right)^{30} \\ &= z^{-90} \left( \cos \pi + i \sin \pi \right) = -z^{90} \end{aligned}$$

$$(1 + \cos \varphi + i \sin \varphi)^n =$$

$$\begin{aligned} z &= 1 + \cos \varphi + i \sin \varphi = \cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2} + \cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2} + \\ &+ 2i \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} = 2 \cos^2 \frac{\varphi}{2} + i \sin \varphi = 2 \cos \frac{\varphi}{2} \left( \cos \frac{\varphi}{2} + i \sin \frac{\varphi}{2} \right) \end{aligned}$$

$$z^n = 2^n \cos^n \frac{\varphi}{2} \left( \cos n \frac{\varphi}{2} + i \sin n \frac{\varphi}{2} \right)$$

$$(1+i)^n \quad n \in \mathbb{N}$$

$$i^n = \begin{cases} i & n=4k+1 \\ -1 & n=4k+2 \\ -i & n=4k+3 \\ 1 & n=4k \end{cases}$$



$$(1+i)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}i + \binom{n}{2}1^{n-2}i^2 + \binom{n}{3}1^{n-3}i^3 + \dots + \binom{n}{n-1}1 \cdot i^{n-1} + \binom{n}{n}i^n = \left( \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \dots \right) + i \left( \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \dots \right)$$

$$(1-i)^n + (1+i)^n = \text{за домашню}$$

$$\left( \frac{2-2i}{-3+i} \right)^{108} = \text{за домашню}$$

$$\sqrt[n]{z} = \sqrt[n]{|z|} \left( \cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n} \right); k=0,1,\dots,n-1$$

$$\sqrt[3]{\frac{1+\sqrt{3}i}{1-i}} \quad z_1 = 1 + \sqrt{3}i \quad \cos \varphi_1 = \frac{1}{2} \Rightarrow \varphi_1 = \frac{\pi}{3}$$

$$|z_1| = 2 \quad \sin \varphi_1 = \frac{\sqrt{3}}{2}$$

$$z_2 = 1-i \quad |z_2| = \sqrt{2} \quad \cos \varphi_2 = \frac{\sqrt{2}}{2} \quad \sin \varphi_2 = -\frac{\sqrt{2}}{2} \Rightarrow \varphi_2 = -\frac{\pi}{4} = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$$z = \frac{z_1}{z_2} = \sqrt{2} \left( \cos \left( \frac{\frac{\pi}{3}}{2} + \frac{\frac{7\pi}{4}}{2} \right) + i \sin \left( \frac{\frac{\pi}{3}}{2} + \frac{\frac{7\pi}{4}}{2} \right) \right)$$

$$\sqrt[3]{z} = \sqrt[6]{2} \left( \cos \frac{\frac{7\pi}{12} + 2k\pi}{3} + i \sin \frac{\frac{7\pi}{12} + 2k\pi}{3} \right) \quad k=0,1,2$$

$$\sqrt[6]{\frac{2+2i}{3-i}} \quad z_1 = 2+2i \quad |z_1| = \sqrt{4+4} = 2\sqrt{2}$$

$$\cos \varphi_1 = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \varphi_1 = \frac{\pi}{4}$$

$$\sin \varphi_1 = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$z_2 = \sqrt{3}-i \quad |z_2| = \sqrt{3+1} = 2$$

$$\cos \varphi_2 = \frac{\sqrt{3}}{2} \Rightarrow \varphi_2 = \frac{\pi}{6}$$

$$\sin \varphi_2 = -\frac{1}{2}$$

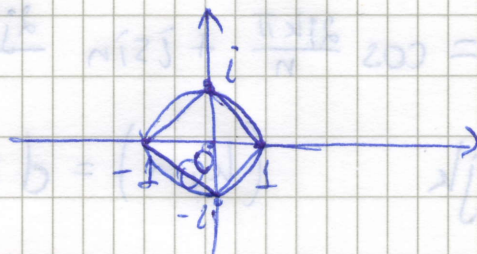
$$z = \frac{z_1}{z_2} = \sqrt{2} \left( \cos \left( \frac{\frac{\pi}{4}}{2} + \frac{\frac{\pi}{6}}{2} \right) + i \sin \left( \frac{\frac{\pi}{4}}{2} + \frac{\frac{\pi}{6}}{2} \right) \right)$$



$$\sqrt[n]{z} = \sqrt[n]{2} \left( \cos \frac{5\pi + 2k\pi}{6} + i \sin \frac{5\pi + 2k\pi}{6} \right) \quad k=0, 1, \dots, 5$$

$$\begin{aligned} x^4 + i + 1 &= 0 && \text{за домашно} \\ x^2 &= i && \text{за домашно} \\ x^n &= 1 \end{aligned}$$

$$\begin{aligned} x^n &= 1 && z = 1 = \cos 0 + i \sin 0 \\ |z| &= 1 && \\ \cos \varphi &= 1 && \\ \sin \varphi &= 0 && \varphi = 0 \end{aligned}$$



$$w_k = \sqrt[n]{x} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \quad k=0, 1, \dots, n-1$$

$n$ -th roots of 1

$$\begin{aligned} n=4 & \quad w_0 = \cos 0 + i \sin 0 = 1 \\ & \quad w_1 = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i \\ & \quad w_2 = \cos \pi + i \sin \pi = -1 \\ & \quad w_3 = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i \end{aligned}$$

0.8 a)  $w_0, w_1, \dots, w_{n-1}$  — корени на  $x^n = 1$

$$w_k^n = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} = 1$$

$$\begin{aligned} \beta^n &= \alpha \in \mathbb{C} \\ \beta &= \sqrt[n]{\alpha} \left( \cos \frac{\varphi}{n} + i \sin \frac{\varphi}{n} \right); \quad \sqrt[n]{\alpha} \in \mathbb{R} \end{aligned}$$

b)  $\alpha, \beta \in \mathbb{C}$  и  $\beta^n = \alpha \Rightarrow \beta w_0, \beta w_1, \dots, \beta w_{n-1}$  — корени на  $x^n = \alpha$

$$w_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \quad k=0, 1, \dots, n-1$$

$$\beta w_k = \sqrt[n]{\alpha} \left( \cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n} \right) \quad k=0, 1, \dots, n-1$$



$$2) \omega_1^k = \omega_k \quad \omega_1^k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \quad k=0, 1, \dots, n-1$$

$$g) \omega_k^j, \quad j=0, 1, \dots, n-1 \quad \& \quad n^{\text{th}} \text{ корень из } 1, \text{ тогда}$$

$$\omega_k^j = \cos \frac{2jk\pi}{n} + i \sin \frac{2jk\pi}{n}, \quad k=0, 1, \dots, n-1$$

$$n | jk \quad (k, n) = d > 1 \quad d | k \quad j, k_1, n_1 \in \mathbb{Z}$$

$$k = dk_1 \quad (k_1, n_1) = 1$$

$$\omega_k^j = \cos \frac{2j \cdot dk_1 \cdot \pi}{dn_1} + i \sin \frac{2j \cdot dk_1 \cdot \pi}{dn_1}$$

$$e) m \in \mathbb{Z} \quad \omega_0^m + \omega_1^m + \dots + \omega_{n-1}^m = 0 \quad \begin{matrix} n/m \\ n \nmid m \end{matrix}$$

$$1) \omega_0^m + \omega_1^m + \omega_1^{2m} + \dots + \omega_1^{(n-1)m} = n$$

$$2) n \nmid m \quad \omega_1^m + \omega_1^{2m} + \dots + \omega_1^{(n-1)m} = \frac{\omega_1^{mn} - 1}{\omega_1^m - 1} = 0$$

~~XXXXXXXXXX~~



# Детерминанты

$\{1, 2, \dots, n\}$   $1432$  - нечетная пермутация (3 инверсии)

$\{n, n-1, n-2, \dots, 3, 2, 1\}$   $(n-1) + (n-2) + \dots + 2 + 1 = \frac{n(n-1)}{2}$

$$f: M_n(F) \rightarrow F$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$B = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = b_{11}b_{22}b_{33} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{22}b_{31} - b_{13}b_{21}b_{32}$$

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} -$$

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} +$$

$$\det(A) = \sum (-1)^{[i_1 i_2 \dots i_n]}$$

$$a_{1i_1} a_{2i_2} \dots a_{ni_n} \frac{(n-1)!}{2}$$

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$(-1)^{[n-1 \dots 2, 1]} = (-1)^{\frac{(n-1)n}{2}}$$



$$A = \begin{vmatrix} 1 & 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 \end{vmatrix} = 0 + (-1) + (-1) - \{0 - (-1)\} = 0 = -1$$

1)  $\det\{A\} = \det A^T$

2) умнож. рег (стълб) с число  $\neq 0$  и да прибавим към друг рег (стълб) не променя  $\det$

3) промяна на местата на 2 реда (стълба) - сменя знака

4) рег (стълб) с нули (0)  $\Rightarrow \det = 0$

5) два пропорционални реда  $\Rightarrow \det = 0$

$$\det A = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 3 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix} = -1 \cdot 2 \cdot 3 \cdot 5 = -30$$

$$\begin{vmatrix} n & n-1 & n-2 & \dots & 3 & 2 & 1 \\ n & n-1 & n-2 & \dots & 3 & 2 & 0 \\ n & n-1 & n-2 & \dots & 3 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n & 0 & 0 & \dots & 0 & 0 & 0 \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} n!$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} ; \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} ; f(x) = \begin{vmatrix} 1-x & 2 & 2 \\ 2 & 1-x & 2 \\ 2 & 2 & 1-x \end{vmatrix}$$

за домашно



$$f(x) = \begin{vmatrix} 1-x & 2 & 2 \\ 2 & 1-x & 2 \\ 2 & 2 & 1-x \end{vmatrix} = \begin{vmatrix} 5-x & 5-x & 5-x \\ 2 & 1-x & 2 \\ 2 & 2 & 1-x \end{vmatrix} =$$

$$= (5-x) \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1-x & 2 \\ 2 & 2 & 1-x \end{vmatrix}^{x(-2)} = (5-x) \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1-x & 0 \\ 0 & 0 & -1-x \end{vmatrix} = (5-x)(1+x)^2$$

Детерминанта - тип "Пазы края"

$$\Delta_n = \begin{vmatrix} 1 & 2 & 2 & 2 & \dots & 2 \\ 4 & 2 & 0 & 0 & \dots & 0 \\ 6 & 0 & 3 & 0 & \dots & 0 \\ 8 & 0 & 0 & 4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2n & 0 & 0 & 0 & \dots & n \end{vmatrix} = \begin{vmatrix} 5 & 2 & 2 & 2 & \dots & 2 \\ 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 3 & 0 & \dots & 0 \\ 0 & 0 & 0 & 4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & n \end{vmatrix} = S \cdot n! =$$

$\begin{matrix} + & + & + & + \\ x(-2) & x(-2) & x(-2) & x(-2) \end{matrix}$

$$S = 1 - 2 + 2 - 2 + \dots + (-1)^{n-1} = (5-4n)n!$$

$$S = 1 - (n-1) \cdot 4 = 5 - 4n$$

$$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 1 & 0 & \frac{1}{3} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 & \frac{1}{n} \end{vmatrix} = \begin{vmatrix} 5 & 1 & 1 & \dots & 1 & 1 \\ 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{n} \end{vmatrix} = S \cdot \frac{1}{n!} =$$

$\begin{matrix} + & + & + \\ x(-2) & x(-3) & x(-n) \end{matrix}$

$$S = - \frac{(n+2)(n-1)}{2} = - \frac{(n+2)(n-1)}{2 \cdot n!}$$



$$A_2 = \begin{vmatrix} 1 & 1 & 1 & 1 & x-2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & x-1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & x-1 & 0 & 3^2 & 0 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 3^{n-2} & 0 & 0 & x-1 & 0 & 0 & 0 & 2 \\ 3^{n-1} & 0 & 0 & 0 & x-1 & 0 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & x-2 & 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 & x-1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 3^2 & 0 & x-1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & 0 & 0 & x-1 & 0 & 0 & 3^{n-2} & 0 \\ 2 & 0 & 0 & 0 & x-1 & 0 & 0 & 0 & 3^{n-1} \end{vmatrix} \cdot (-1)^{\frac{n(n-1)}{2}} =$$

$$= \begin{vmatrix} 5 & 1 & 1 & 1 & x-2 & 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 & x-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3^2 & 0 & x-1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & x-1 & 0 & 0 & 3^{n-1} & 0 \end{vmatrix} \cdot (-1)^{\frac{n(n-1)}{2}} = (-1)^{\frac{n(n-1)}{2}} \cdot 3^{\frac{n(n-1)}{2} - (n-1)} =$$

$$S = 1 - 2 \left( \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}} \right)$$

$$a_1 = \frac{1}{3} \quad q = \frac{1}{3} \quad S_{n-1} = a_1 \frac{q^{n-1} - 1}{q - 1} = \frac{1}{3} \frac{\frac{1}{3^{n-1}} - 1}{(-\frac{2}{3})} =$$

$$= \frac{3^{n-1} - 1}{2 \cdot 3^{n-1}} \quad S = 1 - \frac{3^{n-1} - 1}{3^{n-1}} = \frac{1}{3^{n-1}}$$



$$\Delta_n = \begin{vmatrix} 2-n & 1 & 1 & \dots & 1 & 1 \\ 1 & 2-n & 1 & \dots & 1 & 1 \\ 1 & 1 & 2-n & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 2-n \end{vmatrix} = \begin{vmatrix} 2-n & 1 & 1 & \dots & 1 & 1 \\ n-1 & 1-n & 0 & \dots & 0 & 0 \\ n-1 & 0 & 1-n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & 0 & 0 & \dots & 0 & 1-n \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1-n & 0 & \dots & 0 & 0 \\ 0 & 0 & 1-n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1-n \end{vmatrix} \cdot (1-n)^{n-1} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2-n & 1 & \dots & 1 & 1 \\ 1 & 1 & 2-n & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 2-n \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1-n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1-n \end{vmatrix}$$

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = \sum (-1)^{i_1 i_2 \dots i_n} a_{1i_1} a_{2i_2} \dots a_{ni_n}$$

$n! \quad (n-1)!$   
 $a_{ij} \rightarrow A_{ij} = (-1)^{i+j} \Delta_{ij}$

$$\Delta_n = a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in} = a_{1j} A_{1j} + a_{2j} A_{2j} + \dots + a_{nj} A_{nj}$$

$$\Delta_3 = \begin{vmatrix} 1 & 0 & 3 \\ 5 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} \xrightarrow{+ \times (-2)} \begin{vmatrix} 1 & 0 & 3 \\ -1 & 0 & -7 \\ 3 & 1 & 4 \end{vmatrix} = 1 \cdot (-1)^{(3+2)} \begin{vmatrix} 1 & 3 \\ -1 & -7 \end{vmatrix} = 7 - 3 = 4$$



$$\begin{vmatrix} 1 & 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 4 & 1 \\ -1 & 0 & 1 & 5 & -2 \\ 3 & 4 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 \end{vmatrix} = 1(-1) \begin{vmatrix} 1 & 1 & 0 & 2 \\ 3 & 0 & 4 & 3 \\ -1 & 0 & 1 & -2 \\ -1 & 0 & -3 & -5 \\ 1 & 0 & 1 & 3 \end{vmatrix} =$$

$$= (-1) \begin{vmatrix} 0 & 1 & -2 & -6 \\ 0 & 2 & 7 & 1 \\ 0 & -2 & 4 & -2 \\ 1 & 1 & 2 & 3 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -2 & -6 \\ 2 & 7 & 1 \\ -2 & 4 & -2 \end{vmatrix} =$$

$$= 1 \begin{vmatrix} 1 & -2 & -6 \\ 0 & 11 & -11 \\ 0 & 0 & -14 \end{vmatrix} = 1 \cdot (-1)^2 \begin{vmatrix} 11 & -1 \\ 0 & -14 \end{vmatrix} = -154$$

$$\Delta_n = \begin{vmatrix} x & y & 0 & \dots & 0 & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 & 0 \\ 0 & 0 & x & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & x & y \\ y & 0 & 0 & \dots & 0 & 0 & x \end{vmatrix} = x(-1)^{n+1} \begin{vmatrix} x & y & 0 & \dots & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & x & y \\ 0 & 0 & 0 & \dots & 0 & 0 & x \end{vmatrix} +$$

$$+ y(-1)^{n+2} \begin{vmatrix} 0 & y & 0 & \dots & 0 & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & x & y \\ y & 0 & 0 & \dots & 0 & 0 & x \end{vmatrix} = x^n - y \cdot y(-1)^{n+1} \begin{vmatrix} y & 0 & \dots & 0 \\ xy & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & xy \end{vmatrix}$$

$$= x^n + (-1)^{n+1} y^n$$

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$$\Delta_{n+1} = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & \dots & 1 & x_1 & 1 \\ 1 & 1 & \dots & x_2 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_n & 1 & \dots & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & 0 & \dots & 0 & x_1-1 & 0 \\ 0 & 0 & \dots & x_2-1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_n-1 & 0 & \dots & 0 & 0 & 0 \end{vmatrix} =$$

$$= (-1)^{\frac{n(n+1)}{2}} \prod_{i=1}^n (x_i - 1)$$

$$\Delta_{n+1} = \begin{vmatrix} x_1 y_1 & x_2 y_2 & x_3 y_3 & \dots & x_n y_n \\ x_1 y_2 & x_2 y_2 & x_2 y_3 & \dots & x_2 y_n \\ x_1 y_3 & x_2 y_3 & x_3 y_3 & \dots & x_3 y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 y_n & x_2 y_n & x_3 y_n & \dots & x_n y_n \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ x_1 y_1 & x_2 y_2 & x_2 y_3 & \dots & x_2 y_n \\ x_1 y_3 & x_2 y_3 & x_3 y_3 & \dots & x_3 y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 y_n & x_2 y_n & x_3 y_n & \dots & x_n y_n \end{vmatrix} - x_n \begin{vmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ x_1 y_1 & x_2 y_2 & x_2 y_3 & \dots & x_2 y_n \\ x_1 y_3 & x_2 y_3 & x_3 y_3 & \dots & x_3 y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 y_n & x_2 y_n & x_3 y_n & \dots & x_n y_n \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ x_1 y_2 - y_1 x_2 & 0 & \dots & 0 & 0 \\ x_1 y_3 - y_1 x_3 & x_2 y_2 - y_2 x_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 y_n - y_1 x_n & x_2 y_n - y_2 x_n & \dots & x_n y_n - y_n x_n & 0 \end{vmatrix} = x_1 y_n (-1)^{n+1} \begin{vmatrix} x_1 y_2 - y_1 x_2 & 0 & \dots & 0 \\ x_1 y_3 - y_1 x_3 & x_2 y_2 - y_2 x_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1 y_n - y_1 x_n & x_2 y_n - y_2 x_n & \dots & x_n y_n - y_n x_n \end{vmatrix}$$

$$= x_1 y_n (-1)^{n+1} \prod_{i=1}^{n-1} (x_i y_{i+1} - y_i x_{i+1})$$



$$\Delta_n^2 = \begin{vmatrix} x_1 & a_{12} & a_{13} & \dots & a_{1n-1} & a_{1n} \\ x_1 & x_2 & a_{23} & \dots & a_{2n-1} & a_{2n} \\ x_1 & x_2 & x_3 & \dots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & a_{n-1n} \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \end{vmatrix} = \begin{vmatrix} x_1 & a_{12} & a_{13} & \dots & a_{1n-1} & a_{1n} \\ 0 & x_2 - a_{12} & \dots & \dots & \dots & \dots \\ 0 & 0 & x_3 - a_{23} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{n-1} - a_{n-1n} & \dots \\ 0 & 0 & 0 & \dots & 0 & x_n - a_{n-1n} \end{vmatrix}$$

$$= x_1 \prod_{i=2}^n (x_i - a_{i-1i})$$

$$\Delta_{n+1}^2 = \begin{vmatrix} x & a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_1 & x & a_2 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & x & \dots & a_{n-1} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \dots & x & 1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & 1 \end{vmatrix} = \begin{vmatrix} x - a_1 & \dots & \dots & \dots & \dots & 0 \\ 0 & x - a_2 & \dots & \dots & \dots & 0 \\ 0 & 0 & x - a_3 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x - a_n & 0 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & 1 \end{vmatrix}$$

$\xrightarrow{(-1)}$

$$= (-1)^{2n+2} \prod_{i=1}^n (x - a_i)$$

$$\Delta_{n+1} = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & -x_1 & x_2 & \dots & x_{n-1} & x_n \\ 1 & +x_1 & -x_2 & \dots & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_1 & x_2 & \dots & -x_{n-1} & x_n \\ 1 & x_1 & x_2 & \dots & x_{n-1} & -x_n \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & -2x_1 & 0 & \dots & 0 & 0 \\ 1 & 0 & -2x_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & -2x_{n-1} & 0 \\ 1 & 0 & 0 & \dots & 0 & -2x_n \end{vmatrix}$$

$\xrightarrow{(-\frac{1}{2x_1})}$

$$\xrightarrow{(-\frac{1}{2x_2})}$$

$$\xrightarrow{(-\frac{1}{2x_n})}$$



$$\begin{aligned}
 &= \begin{vmatrix} S & 1 & 1 & \dots & 1 & 1 \\ 0 & -2x_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -2x_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & -2x_n \end{vmatrix} = S \prod_{i=1}^n (-2x_i) = \\
 &= (-1)^n \cdot 2^{n-1} \cdot \prod_{i=1}^n x_i = \sum_{i=1}^n \frac{1}{x_i}
 \end{aligned}$$

$$\begin{aligned}
 \Delta_n &= \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & S \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{vmatrix} \\
 &= S (-1)^{n-1} (-1)^{n-1} = S
 \end{aligned}$$

Diagram showing the expansion of the determinant \$\Delta\_n\$ using Laplace expansion along the last row. The last row is \$[0, 0, 0, \dots, 0, -1, 2]\$. The expansion is shown as a sum of terms with arrows pointing from the elements to the corresponding minors:

- \$x\_2\$ (the element 2) is multiplied by \$x\_3\$ (the element -1 in the minor) with a '+' sign.
- \$x\_3\$ (the element -1) is multiplied by \$x\_4\$ (the element 2 in the minor) with a '+' sign.
- \$x\_4\$ (the element 2) is multiplied by \$x\_{n-1}\$ (the element -1 in the minor) with a '+' sign.
- \$x\_{n-1}\$ (the element -1) is multiplied by \$x\_n\$ (the element 2 in the minor) with a '+' sign.

$$S = a_n + 2a_{n-1} + 3a_{n-2} + \dots + (n-1)a_2 + na_1$$







$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(n-1) & -(n-2) & \dots & \dots & -1 \end{vmatrix} = 1 \cdot (-1)^{n+1} \begin{vmatrix} -1 & 0 & 0 & \dots & 0 \\ -2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(n-1) & -(n-2) & \dots & \dots & -1 \end{vmatrix}$$

Let A

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \bar{a}_{11} & \bar{a}_{12} & \dots & \bar{a}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$\Delta_n = \begin{vmatrix} a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ a_2+b_1 & a_2+b_2 & \dots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix} = \begin{vmatrix} a_1 & a_1 & \dots & a_1 \\ a_2+b_1 & a_2+b_2 & \dots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix} +$$

$$+ \begin{vmatrix} b_1 & b_2 & \dots & b_n \\ a_2+b_1 & a_2+b_2 & \dots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix} = \begin{vmatrix} (-b_1) & 1 & \dots & 1 \\ a_2+b_1 & a_2+b_2 & \dots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ (-b_n) & a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix} +$$



$$\begin{vmatrix} b_1 & b_2 & \dots & b_n \\ a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix} = a_1 \begin{vmatrix} 1 & 1 & \dots & 1 \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \dots & b_n \end{vmatrix} + \dots$$

$$\begin{vmatrix} b_1 & b_2 & b_3 & \dots & b_n \\ a_1 & a_1 & a_1 & \dots & a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \dots & a_n \end{vmatrix} = 0$$

$$\Delta_1 = a_1 + b_1$$

$$\Delta_2 = \begin{vmatrix} a_1+b_1 & a_2+b_1 \\ a_1+b_1 & a_2+b_2 \end{vmatrix}$$

$$n \geq 3 \Rightarrow \Delta_n = 0$$

$$\Delta_2 = (a_1+b_1)(a_1+b_2) - (a_1+b_2)(a_1+b_1)$$

$$\Delta_n = \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{vmatrix}$$

$$+ \begin{vmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_n \\ x_2y_1 & x_2y_2 & \dots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \dots & x_ny_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1y_1 & x_1y_2 & \dots & x_1y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \dots & x_ny_n \end{vmatrix} + \begin{vmatrix} (-x_1) & y_1 & y_2 & \dots & y_n \\ x_1 & 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-x_n) & 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_2y_1 & x_2y_2 & \dots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \dots & x_ny_n \end{vmatrix} + x_1 \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix} = 0$$

$$= x_2 \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ y_1 & y_2 & y_3 & \dots & y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1 & y_2 & y_3 & \dots & y_n \end{vmatrix} + \dots$$

$$\Delta_n = 0 \quad n \geq 3$$



$$n_1 \mid 1 + xy \quad \Delta_1 = 1 + x_1 y_1 \quad \Delta_2 = \begin{vmatrix} 1 + x_1 y_1 & 1 + x_1 y_2 \\ 1 + x_2 y_1 & 1 + x_2 y_2 \end{vmatrix}$$

$$\Delta_2 = (1 + x_1 y_1)(1 + x_2 y_2) - (1 + x_2 y_1)(1 + x_1 y_2)$$

$$\Delta_n = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & -x_1 & x_2 & \dots & x_{n-1} & x_n \\ 1 & +x_1 & -x_2 & \dots & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_1 & x_2 & \dots & -x_{n-1} & -x_n \\ 1 & x_1 & x_2 & \dots & x_{n-1} & -x_n \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & -2x_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -2x_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & -2x_n \end{vmatrix}$$

Annotations:   
 - In the first matrix, a curved arrow from the first row to the last row is labeled  $x(-x_1)$ .   
 - In the second matrix, a curved arrow from the first row to the last row is labeled  $x(\frac{1}{2x_1})$ .

$$= \begin{vmatrix} 0 & 1 & \dots & 1 & 1 \\ 0 & -2x_1 & \dots & 0 & 0 \\ 0 & 0 & -2x_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & -2x_n \end{vmatrix}$$

$$S = \frac{1}{2x_1} + \dots + \frac{1}{2x_n}$$

$$S = \sum_{i=1}^{n+1} \frac{1}{2x_i}$$

$$\Delta_{n+1} = S (-2)^n (x_1 - x_2 - \dots - x_n)$$

$$\Delta_{n+1} = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ x_1 & a_1 & 0 & \dots & 0 & 0 \\ x_2 & x_2 & a_2 & \dots & 0 & 0 \\ x_3 & x_3 & x_3 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-1} & \dots & x_{n-1} & 0 & 0 \\ x_n & x_n & \dots & x_n & a_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & a_1 & 0 & \dots & 0 & 0 \\ x_2 & x_2 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n & x_n & x_n & \dots & x_n & a_n \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_1 & 0 & \dots & 0 & 0 \\ 0 & x_2 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_n & x_n & \dots & x_n & a_n \end{vmatrix}$$

Annotation: A curved arrow from the first row to the last row in the first matrix is labeled  $x(-x_n)$ .



$$= \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & a_1 - x_1 & -x_1 & \dots & -x_1 & x_1 \\ 0 & 0 & a_2 - x_2 & \dots & -x_2 & -x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_n - x_n \end{vmatrix} - a_1 a_2 \dots a_n = \prod_{i=1}^n (a_i - x_i) - a_1 a_2 \dots a_n$$

$$\Delta_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{vmatrix} = p \cdot \Delta_{n-1} + q \cdot \Delta_{n-2} =$$

$$= 2 \cdot (-1)^2 \Delta_{n-1} + 1 \cdot (-1)^3 \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{vmatrix} = 2 \cdot \Delta_{n-1} - 1 \cdot \Delta_{n-2}$$

$$= 2 \Delta_{n-1} - 1 \Delta_{n-2}$$

$$x^2 - px - q = 0$$

$d_1, d_2$  - корни

$$+p = d_1 + d_2$$

$$-q = d_1 \cdot d_2$$

$$\Delta_n = (d_1 + d_2) \Delta_{n-1} + d_1 d_2 \Delta_{n-2} \quad (d_1 + d_2 = 1)$$

$d_1 \neq d_2$

$$\Delta_n = C_1 d_1^n + C_2 d_2^n$$

$C_1, C_2$  - const.

$$\Delta_1 = C_1 d_1 + C_2 d_2$$

$$+p = d_1 + d_2 = +3 \Rightarrow$$

$$\Delta_2 = C_1 d_1^2 + C_2 d_2^2$$

$$-q = d_1 d_2 = -1$$

$$d_1 = d_2 \Rightarrow \Delta_n = (C_1 + n C_2) d_1^n = (n+1) d_1^n \Rightarrow d_1 = d_2 = 1$$

$$\Delta_n = C_1 + n C_2$$

$$\Delta_1 = C_1 + C_2 = 2$$

$$\Rightarrow C_1 = C_2 = 1$$

$$\Delta_2 = C_1 + 2 C_2 = 3$$

$$\Delta_n = \{n+1\}$$

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$$\Delta_n = \begin{vmatrix} 3 & 2 & 0 & 0 & \dots & 0 & 0 \\ 1 & 3 & 2 & 0 & \dots & 0 & 0 \\ 0 & 1 & 3 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 3 \end{vmatrix} = 3\Delta_{n-1} + 2(-1)^3$$

$$\begin{vmatrix} 1 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 3 & 2 & 0 & \dots & 0 & 0 \\ 0 & 1 & 3 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 3 \end{vmatrix} =$$

$$= 3\Delta_{n-1} - 2\Delta_{n-2}$$

$$p=3 \quad q=-2$$

$$x^2 - px + q = 0$$

$$x_1 + x_2 = p = 3$$

$$x_1 x_2 = q = -2$$

$$x_1 = 2 \quad x_2 = 1$$

$$\Delta_n = C_1 2^n + C_2$$

$$\Delta_1 = 3 = 2(C_1 + C_2)$$

$$\Delta_2 = 7 = 4C_1 + C_2$$

$$\Delta_n = 2^{n+1} - 1$$

$$\Rightarrow C_1 = 2 \quad C_2 = -1$$

$$C_1 + C_2 = 1$$

$$\begin{vmatrix} p & q \\ q & -1-p & q \\ & q & -1-p & q \\ & & \ddots & \ddots & q \\ & & & q & -1-p & q \end{vmatrix}$$

$$\begin{vmatrix} x+p & d & 0 & \dots & 0 \\ 1 & x+p & d & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & x+p \end{vmatrix}$$

Детерминанта на Вандермонд

$$W_n = W(x_1, x_2, \dots, x_n) = \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & \dots & x_{n-1} & x_n \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & \dots & x_{n-1}^2 & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & x_4^{n-1} & \dots & x_{n-1}^{n-1} & x_n^{n-1} \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$W_{n-1} = \prod (x_i - x_j)$$



$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \dots & x_n - x_1 \\ 0 & (x_2 - x_1)x_2 & (x_3 - x_1)x_2 & \dots & x_n(x_2 - x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_2 - x_1) & \dots & x_n^{n-2}(x_2 - x_1) \end{vmatrix} = 1(-1)^2$$

$$= 1(-1)^2 \prod_{i=2}^n (x_i - x_1)$$

$$W_{n-1} = \prod_{1 \leq j < i \leq n-1} (x_i - x_j)$$

$$\begin{vmatrix} 1 & 1 \\ x_{n-1} & x_n \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_2 & x_3 & \dots & x_n \\ x_2^2 & x_3^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2} & x_3^{n-2} & \dots & x_n^{n-2} \end{vmatrix}$$

$$W_{n-1} = \prod_{1 \leq j < i \leq n} (x_i - x_j)$$

$$W_{n+1} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & n+1 \\ 1 & 2^2 & 3^2 & \dots & (n+1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-1} & 3^{n-1} & \dots & (n+1)^{n-1} \\ 1 & 2^n & 3^n & \dots & (n+1)^n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & n \\ 0 & 2 \cdot 1 & 3 \cdot 2 & \dots & n(n+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 2^{n-1} \cdot 1 & 3^{n-1} \cdot 2 & \dots & (n+1)^{n-1} \cdot n \\ 0 & 2^{n-1} \cdot 1 & 3^{n-1} \cdot 2 & \dots & (n+1)^{n-1} \cdot n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 2 & 3 & 4 & \dots & (n+1) \\ 2^2 & 3^2 & \dots & (n+1)^2 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{n-2} & 3^{n-2} & \dots & (n+1)^{n-2} \\ 2^{n-1} & 3^{n-1} & \dots & (n+1)^{n-1} \end{vmatrix}$$

$$= n! \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & n-1 \\ 3 \cdot 1 & 4 \cdot 2 & \dots & (n+1)(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ 3^{n-3} \cdot 1 & 4^{n-3} \cdot 2 & \dots & (n+1)^{n-3}(n-1) \\ 3^{n-2} \cdot 1 & 4^{n-2} \cdot 2 & \dots & (n+1)^{n-2}(n-1) \end{vmatrix} = n!(n-1)! \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 3 & 4 & 5 & \dots & n+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3^{n-2} & 4^{n-2} & 5^{n-2} & \dots & (n+1)^{n-2} \end{vmatrix}$$



$$\Rightarrow W_{n+1} = n!$$

$$W_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_2+1 & x_2+1 & \dots & x_n+1 \\ x_2^2+x_2 & x_2^2+x_2 & \dots & x_n^2+x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-1}+x_2 & \dots & x_n^{n-1}+x_n \end{vmatrix} \times \begin{vmatrix} 1 & \dots & 1 \\ x_2+1 & \dots & x_n+1 \\ x_2(x_2+1) & \dots & x_n(x_n+1) \\ \vdots & \ddots & \vdots \\ x_2^{n-2}(x_2+1) & \dots & x_n^{n-2}(x_n+1) \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & x_2-x_1 & x_3-x_1 & \dots & x_n-x_1 \\ 0 & (x_2-x_1)(x_2+1) & (x_3-x_1)(x_3+1) & \dots & (x_n-x_1)(x_n+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & x_n^{n-3}(x_n-x_1)(x_n+1) \end{vmatrix} = \prod_{1 \leq i \leq n} (x_i - x_1) \begin{vmatrix} 1 & \dots & 1 \\ (x_2+1) & \dots & (x_n+1) \\ x_2(x_2+1) & \dots & x_n(x_n+1) \\ \vdots & \ddots & \vdots \\ x_2^{n-3}(x_2+1) & \dots & x_n^{n-3}(x_n+1) \end{vmatrix}$$

$$x_2^{n-2}(x_2+1) - x_1 x_2^{n-3}(x_2+1) = x_2^{n-3}(x_2+1)(x_2 - x_1)$$

$$W_2 = \begin{vmatrix} 1 & 1 \\ x_{n+1}+1 & x_n+1 \end{vmatrix} = x_n - x_{n+1}$$

$$\prod_{1 \leq j < i \leq n} (x_i - x_j)$$

$$\Delta_n = \begin{vmatrix} 1 & f_1(x_1) & f_2(x_1) & \dots & f_{n-1}(x_1) \\ 1 & f_1(x_2) & f_2(x_2) & \dots & f_{n-1}(x_2) \\ 1 & f_1(x_3) & f_2(x_3) & \dots & f_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(x_n) & f_2(x_n) & \dots & f_{n-1}(x_n) \end{vmatrix} =$$

$$f_i(x) = x^i + a_{i1}x^{i-1} + a_{i2}x^{i-2} + \dots + a_{i,i-1}x + a_{ii}$$



$$f_1(x) = x + a_{11}$$

$$f_2(x) = x^2 + a_{21}x + a_{22}$$

$$f_n(x) = x^n + a_{n1}x + a_{n2}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & x_2 - x_1 & (x_2 - x_1)(x_2 + x_1 + a_{21}) & \dots & (x_2 - x_1)A_1 \\ 1 & x_3 - x_1 & (x_3 - x_1)(x_3 + x_1 + a_{21}) & \dots & (x_3 - x_1)A_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & (x_n - x_1)(x_n + x_1 + a_{21}) & \dots & (x_n - x_1)A_{n-1} \end{vmatrix} = \prod_{i=2}^n (x_i - x_1)$$

$$i=2, \dots, n$$

$$f_1(x_i) - f_1(x_1) = x_i + a_{11} - x_1 - a_{11} = x_i - x_1$$

$$f_2(x_i) - f_2(x_1) = x_i^2 + a_{21}x_i + a_{22} - (x_1^2 + a_{21}x_1 + a_{22}) =$$

$$= (x_i - x_1)(x_i + x_1) + a_{21}(x_i - x_1) = (x_i - x_1)(x_i + x_1 + a_{21})$$

$$f_{n-1}(x_i) - f_{n-1}(x_1) = x_i^{n-1} + a_{n-1,1}x_i^{n-2} + \dots + a_{n-1,n-2}x_i +$$

$$+ a_{n-1,n-1} - (x_1^{n-1} + a_{n-1,1}x_1^{n-2} + \dots + a_{n-1,n-2}x_1 + a_{n-1,n-1}) =$$

$$= (x_i - x_1) \cdot A = (x_i - x_1)(x_i^{n-2} + x_i^{n-3} + \dots + x_1^{n-2} + a_{n-1,1})$$

$$\begin{vmatrix} 1 & x_1 + x_2 + a_{21} + \dots & (x_2 - x_1)A_1 \\ 1 & x_1 + x_3 + a_{21} & (x_3 - x_1)A_2 \\ \vdots & \vdots & \vdots \\ 1 & x_1 + x_n + a_{21} & A_{n-1} \end{vmatrix} = \prod_{j=2}^n (x_i + x_j)$$

$$= \prod_{j=2}^n (x_i + x_j) \begin{vmatrix} 1 & x_1 + x_2 + a_{21} & A_1 \\ 0 & x_3 - x_2 & (x_3 - x_2)B_1 \\ \vdots & \vdots & \vdots \\ 0 & x_n - x_2 & (x_n - x_2)B_{n-2} \end{vmatrix}$$

$$= W_n = \prod_{1 \leq j \leq i \leq n} (x_i - x_j)$$

11.2.35



$$\begin{vmatrix} \cos(d_1 - \beta_1) & \cos(d_1 - \beta_2) & \dots & \cos(d_1 - \beta_n) \\ \cos(d_2 - \beta_1) & \cos(d_2 - \beta_2) & \dots & \cos(d_2 - \beta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(d_n - \beta_1) & \cos(d_n - \beta_2) & \dots & \cos(d_n - \beta_n) \end{vmatrix} =$$

$$\cos(d_i - \beta_j) = \cos d_i \cos \beta_j + \sin d_i \sin \beta_j$$

$$= \begin{vmatrix} \cos d_1 \cos \beta_1 + \sin d_1 \sin \beta_1 & \dots & \cos d_1 \cos \beta_n + \sin d_1 \sin \beta_n \\ \cos d_2 \cos \beta_1 + \sin d_2 \sin \beta_1 & \dots & \cos d_2 \cos \beta_n + \sin d_2 \sin \beta_n \\ \vdots & \ddots & \vdots \\ \cos d_n \cos \beta_1 + \sin d_n \sin \beta_1 & \dots & \cos d_n \cos \beta_n + \sin d_n \sin \beta_n \end{vmatrix}$$

$$= \begin{vmatrix} \cos \beta_1 & \cos \beta_2 & \dots & \cos \beta_n \\ \sin \beta_1 & \sin \beta_2 & \dots & \sin \beta_n \end{vmatrix} \begin{vmatrix} \cos d_1 & \dots & \cos d_n \\ \sin d_1 & \dots & \sin d_n \end{vmatrix}$$

$$n \geq 3 \Rightarrow \Delta_n \neq 0$$

$$n \geq 1 \rightarrow$$

$$n \geq 2 \rightarrow$$

с багете на регоче

$$A_n = \begin{vmatrix} a & 0 & 0 & \dots & 0 & 0 & b \\ 0 & a & 0 & \dots & 0 & b & 0 \\ 0 & 0 & a & \dots & b & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & b & 0 & \dots & 0 & a & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & a \end{vmatrix} = (-1)^{1+n} a \begin{vmatrix} a & 0 & 0 & \dots & 0 & b & 0 \\ 0 & a & 0 & \dots & b & 0 & 0 \\ 0 & 0 & a & \dots & 0 & a & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & a \end{vmatrix} +$$



$$+ b(-1)^{2n+1} \begin{vmatrix} 0 & 0 & \dots & 0 & a \\ a & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a \end{vmatrix} = a(-1)^{4n-2} \cdot a \cdot \Delta_{2n-2}$$

$$+ b(-1)^{4n} \Delta_{2n-2} = a^2 \cdot \Delta_{2n-2} - b^2 \Delta_{2n-2} = \Delta_{2n-2} (a^2 - b^2)$$

$$\Delta_2 = \begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} = a^2 - b^2$$

$$\Delta_{2n} = (a^2 - b^2)^n$$

$$\Delta_n = \begin{vmatrix} hx - 1 & 0 & \dots & 0 \\ hxc & h & \dots & 0 \\ hxc^2 & hxc & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ hxc^n & hxc^{n-1} & \dots & h \end{vmatrix} = (-1)^2 h \begin{vmatrix} h-1 & \dots & 0 \\ hxc & h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ hxc^{n-1} & hxc^{n-2} & \dots & h \end{vmatrix}$$

$$+ (-1)^n \begin{vmatrix} hxc & h & \dots & 0 \\ hxc^2 & hxc & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ hxc^n & hxc^{n-1} & \dots & h \end{vmatrix} = h \Delta_n + xc \Delta_n = (h+xc) \Delta_n =$$

$$= (h+xc)^n \Delta_1 = (h+xc)^n \cdot h$$



$$\Delta_n = \begin{vmatrix} x & y & y & - & - & - & - & y & y \\ z & x & y & - & - & - & - & y & y \\ z & x & x & - & - & - & - & y & y \\ - & - & - & - & - & - & - & - & - \\ z & z & z & - & - & - & - & x & y \\ z & z & z & - & - & - & - & z & x \end{vmatrix} \xrightarrow{(-1)} \begin{vmatrix} x+y-y & y+0 & y+0 & - & - & - & - & y+0 & y+0 \\ z-x & x-y & 0 & - & - & - & - & 0 & 0 \\ z-x & z-y & x-y & - & - & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ z-x & z-y & - & - & - & - & - & x-y & 0 \\ z-x & z-y & - & - & - & - & - & z-y & x-y \end{vmatrix}$$

$$\Rightarrow y \begin{vmatrix} 1 & 1 & 1 & - & - & - & - & 1 & 1 \\ z & x & y & - & - & - & - & y & y \\ z & z & x & - & - & - & - & y & y \\ - & - & - & - & - & - & - & - & - \\ z & z & z & - & - & - & - & x & y \\ z & z & z & - & - & - & - & z & x \end{vmatrix} \xrightarrow{(-z)} \begin{vmatrix} x & y & 0 & 0 & - & - & - & 0 & 0 \\ z & x & y & - & - & - & - & y & y \\ z & z & x & - & - & - & - & y & y \\ - & - & - & - & - & - & - & - & - \\ z & z & z & - & - & - & - & x & y \\ z & z & z & - & - & - & - & z & x \end{vmatrix}$$

$$\Rightarrow y \begin{vmatrix} 1 & 1 & 1 & - & - & - & - & 1 & 1 \\ 0 & x-z & y-z & - & - & - & - & y-z & y-z \\ 0 & 0 & 0 & x-z & - & - & - & y-z & y-z \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & - & - & x-z & y-z \\ 0 & 0 & 0 & - & - & - & - & 0 & x-z \end{vmatrix} + (x-y) \Delta_{n-1} = y(x-z)^{n-1} + (x-y) \Delta_{n-1}$$

$$y(x-z)^{n-1}$$

$$= y(x-z)^{n-1} + (x-y)(y(x-z)^{n-2} + (x-y) \Delta_{n-2}) =$$

$$= y[(x-z)^{n-1} + (x-y)(x-z)^{n-2}] + (x-y)^2 \Delta_{n-2} =$$

$$= \frac{(x-z)^n - (x-y)^n}{y-z} y + (x-y)^{n-1} x$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$$



$$\Delta_2 = \begin{vmatrix} x & y \\ z & xc \end{vmatrix} = x^2 - yz$$

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 subject: РМ I group  
 Text: имена, бр. No

$$\text{Сроч.} \times 27 = 107$$

$$2 \text{ контр.} \times 107 = 214$$

$$307$$

Писмен. црнит  $\Rightarrow 607$

$$\det A \det B = \det (AB)$$

Детерминант матрицы

$A, B \in M_{n \times n}(F), \lambda \in F$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

$$1) A \pm B = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1n} \pm b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} \pm b_{n1} & a_{n2} \pm b_{n2} & \dots & a_{nn} \pm b_{nn} \end{pmatrix}$$

$$2) \lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \dots & \lambda a_{nn} \end{pmatrix}$$

3)  $(A+B)+C = A+(B+C) = A+B+C$  - ассоциативен закон

4)  $A+B = B+A$  - коммутативен закон



5)  $\lambda(A+B) = \lambda A + \lambda B$  - дистрибутивен закон

$$(\lambda + \mu)A = \lambda A + \mu A$$

$$\begin{matrix} A \in M_{n \times k} \\ B \in M_{k \times s} \end{matrix} \Rightarrow A \cdot B \in M_{n \times s}$$

1)  $A(B \cdot C) = (A \cdot B)C = ABC$  - асоциативен закон

2)  $AB \neq BA$

$$\textcircled{1} \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 5 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 3 \cdot 0 + 1 \cdot (-1) & 3 \cdot 5 + 1 \cdot 6 \\ -1 \cdot 0 + 2 \cdot (-1) & -1 \cdot 5 + 2 \cdot 6 \end{pmatrix} = \begin{pmatrix} -1 & 21 \\ -2 & 7 \end{pmatrix}$$

$$\textcircled{2} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} (2, -6, 7) = \begin{pmatrix} 6 & -18 & 21 \\ -2 & 6 & -7 \\ 4 & -12 & 14 \end{pmatrix}$$

$$\textcircled{3} (1 \ -4 \ 5) \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = (-18)$$

$$\begin{aligned} \textcircled{4} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} &= \begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} -11 & 1 \\ 7 & -2 \end{pmatrix} \end{aligned}$$

$$A = A \cdot E = E \cdot A$$



$$\# \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$$

$$1) \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$$

$$2) \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$$

$$3) \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (n+1)a \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$$

$$\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix}$$

$$A^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{cases} n=2k & E=A \\ n=2k+1 & A \end{cases}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E - \text{единичная матрица}$$

$$\boxed{A \cdot E = E \cdot A = A}$$



$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$$

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}$$

$$\begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} = \begin{pmatrix} \cos 3\alpha & -\sin 3\alpha \\ \sin 3\alpha & \cos 3\alpha \end{pmatrix}$$

$$\begin{matrix} n \geq 2 \\ \left| \begin{array}{cccc} \cos(\alpha_1 - \beta_1) & \cos(\alpha_1 - \beta_2) & \dots & \cos(\alpha_1 - \beta_n) \\ \cos(\alpha_2 - \beta_1) & \cos(\alpha_2 - \beta_2) & \dots & \cos(\alpha_2 - \beta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\alpha_n - \beta_1) & \cos(\alpha_n - \beta_2) & \dots & \cos(\alpha_n - \beta_n) \end{array} \right| = \end{matrix}$$

$$= \begin{vmatrix} \cos \alpha_1 & \sin \alpha_1 & 0 & \dots & 0 \\ \cos \alpha_2 & \sin \alpha_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos \alpha_n & \sin \alpha_n & 0 & \dots & 0 \end{vmatrix} \begin{vmatrix} \cos \beta_1 & \cos \beta_2 & \dots & \cos \beta_n \\ \sin \beta_1 & \sin \beta_2 & \dots & \sin \beta_n \\ 0 & 0 & \dots & 0 \end{vmatrix}$$

$$\cos(\alpha_i - \beta_j) = \cos \alpha_i \cos \beta_j + \sin \alpha_i \sin \beta_j$$

$$AB - BA, f(B)$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$f(x) = 2x^2 - 3x + 4$$



$$B^2 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & -1 \\ 3 & 1 & 3 \end{pmatrix}$$

$$J(B) = \begin{pmatrix} 2 & 0 & -6 \\ 0 & 4 & -2 \\ 6 & 2 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 3 & -3 \\ 3 & -3 & 0 \\ 3 & 0 & 6 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} =$$

$$A \cdot B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 5 & 0 \\ 2 & -1 & -3 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -6 & 2 \\ -1 & 2 & -1 \\ 3 & 3 & -2 \end{pmatrix}$$

$$AB - BA = \begin{pmatrix} -2 & 8 & -3 \\ 1 & 3 & 1 \\ -1 & -4 & -1 \end{pmatrix}$$

$$J(B) = \begin{pmatrix} 3 & -3 & -3 \\ -3 & 11 & -2 \\ 3 & 2 & 4 \end{pmatrix}$$

$\text{tr} A$  — следа на матрица  
сума на елем. от главния  
диагонал

$$[A, B] = AB - BA$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A \in M_2(F)$$

$$f(x) = x^2 - (\text{tr} A)x + \det A$$

$$f(A) = 0$$

~~$f(A)$~~



$$\begin{aligned}
 f(A) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - (a_{11} + a_{22}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \\
 &+ (a_{11}a_{22} - a_{12}a_{21}) = \begin{pmatrix} \cancel{a_{11}^2 + a_{11}a_{22}} & \cancel{a_{12}(a_{11} + a_{22})} \\ \cancel{a_{21}(a_{11} + a_{22})} & \cancel{a_{22}^2 + a_{12}a_{21}} \end{pmatrix} - \\
 &- \begin{pmatrix} \cancel{a_{11}^2 + a_{11}a_{22}} & \cancel{a_{12}(a_{11} + a_{22})} \\ \cancel{a_{21}(a_{11} + a_{22})} & \cancel{a_{22}^2 + a_{12}a_{21}} \end{pmatrix} + \begin{pmatrix} \cancel{a_{11}a_{22} - a_{12}a_{21}} & 0 \\ 0 & \cancel{a_{11}a_{22} - a_{12}a_{21}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
 \end{aligned}$$

$$f(A) = 0$$

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix}$$



$$(a_1 \dots a_n) \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} = a_1 (b_{11} \ b_{12} \ \dots \ b_{1n}) +$$

$$+ a_2 (b_{21} \ b_{22} \ \dots \ b_{2n}) + \dots + a_n (b_{n1} \ b_{n2} \ \dots \ b_{nn}).$$

E-единична матрица  $E = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$E_{ij} \in M_n(F)$  матрична ~~матр~~ <sup>матр</sup> единича

$$\tilde{i} \rightarrow \begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & \dots & 0 \end{pmatrix}$$

Да се докаже  $E_{pq} E_{rs} = \delta_{qz} E_{ps} \begin{cases} E_{ps} & q=z \\ 0 & q \neq z \end{cases}$

$$E_{pq} \cdot E_{rs} =$$

Да се докаже  $C \in M_n(F) \quad \forall A \in M_n(F) : (CA = AC) \Rightarrow$

$\Rightarrow C$  е скаларна (единичната умножена с число)

$$C = \lambda E \quad \lambda \in F \quad AC = A \lambda E = \lambda A E = \lambda E A = CA$$

$$C = (C_{ij})_{n \times n} \quad 1 \leq p \neq q \leq n$$

$$E_{pq} C E_{qs} = C \cdot E_{pq} E_{qs} = C \cdot E_{ps} =$$

$$= \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix} \xrightarrow{p \rightarrow q} \begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

$$E_{pp} C E_{qq} = C E_{pp} E_{qq} = 0$$



$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix} = \sum_{i,j=1}^n c_{ij} E_{ij}$$

$$E_{pp} C E_{qq} = E_{pp} \left( \sum_{i,j=1}^n c_{ij} E_{ij} \right) E_{qq} = \sum_{i,j=1}^n c_{ij} E_{pp} E_{ij} E_{qq}$$

$\Rightarrow C$  е диагонална матрица

$$E_{pq} \cdot C \cdot E_{qp} = C \cdot E_{pp} = \left( \sum_{i=1}^n c_{ii} E_{ii} \right) \cdot E_{pp} = c_{pp} E_{pp}$$

$$E_{pq} \cdot \left( \sum_{i=1}^n c_{ii} E_{ii} \right) E_{qp} = E_{pq} \cdot c_{qq} E_{qq} E_{qp} = c_{qq} \cdot E_{pq} \cdot E_{qp} = c_{qq} E_{pp}$$

$$A \in M_2(F)$$

$$a) A^2 = 0$$

$$b) A^2 = E$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} a^2+bc=0 \\ a(a+d)=0 \\ c(a+d)=0 \\ bc+d^2=0 \end{cases}$$

$$1) a=b=c=d=0$$

$$2) a^2=d^2 \quad |a|=|d| \quad a=\pm d$$

$$a=d \Rightarrow b=c=0$$

$$a=-d \Rightarrow bc=-a^2$$

$$\begin{vmatrix} a & b \\ c & -a \end{vmatrix} = -a^2 - bc$$

$$\det A = 0$$



$$d) A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\det A)^2 = \det A^2 = 1$$

$$\det A = \pm 1$$

$$ad - bc = \pm 1$$

$$\begin{cases} a^2 + bc = 1 \\ bc + d^2 = 1 \\ ab + bd = 0 \\ ac + dc = 0 \end{cases} \quad \begin{cases} a^2 + bc = 1 \\ bc + d^2 = 1 \\ b(a+d) = 0 \\ c(a+d) = 0 \end{cases} \quad \begin{aligned} &1) a \neq -d \Rightarrow b = c = 0 \\ &\quad a^2 = d^2 = 1 \\ &\quad a = d = \pm 1 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$2) a + d = 0 \quad a = -d \Rightarrow b \neq 0 \quad c \neq 0$$

$$d^2 = 1 - bc \geq 0 \quad bc \leq 1$$

$$-a^2 - bc = \pm 1$$

$$-a^2 - bc = -1$$

$$a^2 + bc = -1 = 1 \quad \text{нельзя}$$

$$-a^2 - bc = -1$$

$$a^2 + bc = 1$$

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad \det A = -1$$



$$A \neq E, A - E \text{ — обратная } (\det(A+E) = \det(A-E) = 0) \\ \Rightarrow A^2 = E$$

$$\cancel{A-E} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix} \quad \cancel{\det(A+E) =}$$

$$\cancel{A-E} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}$$

$$\det(A+E) = (a+1)(d+1) - bc = 0$$

$$\det(A-E) = (a-1)(d-1) - bc = 0$$

$$\det(A+E) = ad + a + d + 1 - bc = 0$$

$$\det(A-E) = ad - d - a + 1 - bc = 0$$

$$\begin{cases} ad + a + d + 1 - bc = 0 \\ ad - d - a + 1 - bc = 0 \end{cases}$$

$$\begin{cases} ad + a + d + 1 - bc = 0 \\ 2ad + 2 = 0 \end{cases}$$

$$\begin{cases} ad + a + d + 1 - bc = 0 \\ ad = -1 \end{cases}$$

$$\begin{cases} a + d - bc = 0 \\ a = -\frac{1}{d} \end{cases}$$

$$2a + 2d = 0$$

$$a = -d$$

$$\det(A) = ad - bc \stackrel{?}{=} -1$$



## Елементарни преобразувания

- 1) умнож. на ред с число  $\neq 0$
- 2) умнож. на ред с число  $\neq 0$  и прибавяне към друг ред
- 3) смяна на местата на два реда

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$A \in M_n(F)$$

$$\begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

## Матрици на елементарни преобразувания

$$\begin{pmatrix} 1 & & & 0 \\ & \lambda & & \\ 0 & & 1 & \\ & & & \ddots \end{pmatrix}$$

$$1) X = A_i(\lambda) = E + (\lambda - 1)E_{ii}$$

$$2) X = B_{ij}(\lambda) = E + \lambda E_{ji}$$

$$3) X = C_{ij}(\lambda) = E - E_{ii} - E_{jj} + E_{ij} + E_{ji}$$

$$X \cdot \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{j1} & \dots & a_{jn} \\ a_{i1} & \dots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{j1} + \lambda a_{i1} & \dots & a_{jn} + \lambda a_{in} \\ a_{i1} & \dots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

$$\begin{pmatrix} X_1 & X_2 & \dots & X_k \end{pmatrix} A = B$$

$$XA = E \\ X = A^{-1}$$

$$\begin{pmatrix} 1 & & & 0 \\ & \lambda & & \\ 0 & & 1 & \\ & & & \ddots \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & 0 \\ & \lambda & & \\ 0 & & 1 & \\ & & & \ddots \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{j1} & \dots & a_{jn} \\ a_{i1} & \dots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{j1} & \dots & a_{jn} \\ a_{i1} & \dots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$



# Матричные уравнения. Обратная матрица

$$A, B \in M_n(F) \quad X \in M_n(F) = ?$$

$$1) A \cdot X = B \mid A^{-1} \text{ обратная } A \text{ неособенна} \Leftrightarrow \det A \neq 0 \\ \Rightarrow \exists A^{-1} = A^{-1} \cdot A = A^{-1} \cdot A = E$$

$$\underbrace{A^{-1} \cdot A}_E X = A^{-1} B \Rightarrow X = A^{-1} B$$

~~$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$~~

$$(A|B) \sim \dots \sim \left( E \mid \begin{matrix} X \\ \parallel \\ A^{-1}B \end{matrix} \right)$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 4 & 9 \\ 2 & 2 & 6 \\ -1 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & | & 2 & 2 & 6 \\ 2 & 1 & 3 & | & 2 & 4 & 9 \\ 1 & -1 & 1 & | & -1 & 3 & 2 \end{pmatrix} \xrightarrow{+2} \begin{pmatrix} 1 & 1 & 2 & | & 2 & 2 & 6 \\ 0 & -1 & -1 & | & 0 & 2 & 3 \\ 0 & -2 & -1 & | & -3 & 1 & -4 \end{pmatrix} \xrightarrow{+2} \begin{pmatrix} 1 & 1 & 2 & | & 2 & 2 & 6 \\ 0 & 1 & 1 & | & 2 & 0 & 3 \\ 0 & 0 & 1 & | & 1 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & | & -1 & 1 & 1 \\ 0 & 1 & 0 & | & 1 & -1 & 1 \\ 0 & 0 & 1 & | & 1 & 1 & 2 \end{pmatrix} \quad X = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 4 & 2 \end{pmatrix} X = \begin{pmatrix} 4 & 2 & 2 \\ 1 & -1 & 5 \\ 5 & 1 & 7 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & -1 & | & 4 & 2 & 2 \\ 0 & -1 & 3 & | & 1 & -1 & 5 \\ 1 & 1 & 2 & | & 5 & 1 & 7 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 2 & -1 & | & 4 & 2 & 2 \\ 0 & -1 & 3 & | & 1 & -1 & 5 \\ 0 & -1 & 3 & | & 1 & -1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & | & 4 & 2 & 2 \\ 0 & -1 & 3 & | & 1 & -1 & 5 \\ 0 & 0 & 0 & | & 0 & 0 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 - x_3 = 4 \\ -x_2 + 3x_3 = 1 \\ y_1 + 4y_2 - y_3 = 2 \\ -y_2 + 3y_3 = -1 \\ z_1 + 2z_2 - z_3 = 2 \\ -z_2 + 3z_3 = 5 \end{cases}$$

$$\begin{aligned} x_3 &= p & x_2 &= 3p - 1 & x_1 &= 6 - 5p \\ y_3 &= q & y_2 &= 3q + 1 & y_1 &= 5q \\ z_3 &= r & z_2 &= 3r - 5 & z_1 &= 12 - 5r \end{aligned} \quad X = \begin{pmatrix} 6-5p & -5q & 12-5r \\ 3p-1 & 3q+1 & 3r-5 \\ p & q & r \end{pmatrix}$$

$$2) XA = B \quad A^{-1} = \text{отсюда}$$

$$X A A^{-1} = B A^{-1}$$

$$X = B A^{-1}$$

$$(AB)^T = B^T \cdot A^T$$

$$(XA)^T = B^T$$

$$A^T X^T = B^T$$

$$(A^T | B^T) \sim \dots \sim (E | X^T)$$



$$X \begin{pmatrix} 1 & 2 & 1 \\ 2 & -2 & 3 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 13 \\ 5 & -2 & 7 \\ -3 & 8 & 7 \end{pmatrix} \quad A \cdot X = E$$

$$\begin{pmatrix} 1 & 2 & -3 & | & 5 & 5 & -3 \\ 2 & -2 & 1 & | & -1 & -2 & 8 \\ -3 & 1 & 2 & | & 13 & 7 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & | & 5 & 5 & -3 \\ 0 & 0 & -7 & | & 11 & 12 & -8 \\ 0 & -1 & -5 & | & 8 & 2 & 10 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & | & 5 & 5 & -3 \\ 0 & -1 & -5 & | & -8 & -2 & -10 \\ 0 & 0 & -37 & | & -37 & 0 & -74 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & | & 5 & 5 & -3 \\ 0 & 1 & 5 & | & 8 & 2 & 10 \\ 0 & 0 & 1 & | & 1 & 0 & 2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & | & 5 & 5 & 3 \\ 0 & 1 & 0 & | & 3 & 2 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & 2 & 1 & 3 \\ 0 & 1 & 0 & | & 3 & 2 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 2 \end{pmatrix} \Rightarrow$$

$$X = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & - & 1 & 1 \\ 0 & 1 & 1 & - & 1 & 1 \\ 0 & 0 & 1 & - & 1 & 1 \\ \hline 0 & 0 & 0 & - & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 2 & 1 & 0 & - & - & 0 & 0 \\ 1 & 2 & 1 & - & - & 0 & 0 \\ 0 & 1 & 2 & - & - & 0 & 0 \\ \hline 0 & 0 & 0 & - & - & 1 & 2 \end{pmatrix}$$



$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left( \begin{array}{ccc} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{array} \right) \begin{matrix} X \\ Y \end{matrix} = \left( \begin{array}{ccc} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{array} \right) \begin{matrix} Z \\ Z \\ Z \end{matrix} = \left( \begin{array}{ccc} 2 & 0 & -2 \\ 18 & 12 & 9 \\ 23 & 15 & 11 \end{array} \right)$$

$$\left( \begin{array}{ccc} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{array} \right) \begin{matrix} X \\ Y \end{matrix} = \left( \begin{array}{ccc} 2 & 0 & -2 \\ 18 & 12 & 9 \\ 23 & 15 & 11 \end{array} \right) \begin{matrix} Z \\ Z \\ Z \end{matrix}$$

$$\left( \begin{array}{ccc|ccc} 2 & -3 & 1 & 2 & 0 & -2 \\ 4 & -5 & 2 & 18 & 12 & 9 \\ 5 & -7 & 3 & 23 & 15 & 11 \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 2 & -3 & 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 16 & 12 & 13 \\ 0 & 1 & 1 & 36 & 30 & 32 \end{array} \right) \sim$$

$$\left( \begin{array}{ccc|ccc} 2 & -3 & 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 16 & 12 & 13 \\ 0 & 0 & 1 & 22 & 18 & 19 \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 2 & 0 & 0 & 22 & 18 & 19 \\ 0 & 1 & 0 & 16 & 12 & 13 \\ 0 & 0 & 1 & 22 & 18 & 19 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & 9 & 9 \\ 0 & 1 & 0 & 16 & 12 & 13 \\ 0 & 0 & 1 & 22 & 18 & 19 \end{array} \right) = \begin{matrix} X \\ Y \\ Z \end{matrix}$$

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$$X \begin{pmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 9 & 9 \\ 14 & 12 & 13 \\ 22 & 18 & 19 \end{pmatrix}$$

$$\left( \begin{array}{ccc|ccc} 9 & 1 & 1 & 11 & 14 & 22 \\ 7 & 1 & 1 & 9 & 12 & 18 \\ 6 & 2 & 1 & 9 & 13 & 19 \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 3 & -1 & 0 & 2 & 1 & 3 \\ 1 & -1 & 0 & 0 & -1 & -1 \\ 6 & 2 & 1 & 9 & 13 & 19 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 2 & 4 \\ 1 & -1 & 0 & 0 & -1 & -1 \\ 6 & 2 & 1 & 9 & 13 & 19 \end{array} \right) \xrightarrow{\times(6)} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & -1 & -1 \\ 6 & 2 & 1 & 9 & 13 & 19 \end{array} \right) \sim$$

$$\xrightarrow{\times(6)} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 & 7 & 7 \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array} \right)$$

$$X = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}$$

Обратная матрица

$A \in M_n(F)$  - обратная  $\Leftrightarrow A$  е неособенная матрица  
 $(\det A \neq 0) \quad \exists A^{-1}: A \cdot A^{-1} = A^{-1} \cdot A = E$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \quad A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ - & - & - & - \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

$$A \cdot X = E$$



$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ad - bc \neq 0$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

~~$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$~~  $A \cdot X = E$

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 2 & 3 & 4 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 2 & 3 & 4 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & 1 & -2 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & 1 & -2 & 0 \\ 0 & 1 & 0 & | & +2 & 2 & -1 \\ 0 & 0 & 1 & | & -2 & -1 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 2 & -1 \\ -2 & -1 & 1 \end{pmatrix}$$



$$A_2 = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \quad \left( \begin{array}{ccc|ccc} 3 & -1 & -1 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array} \right) \sim$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{array} \right) \begin{array}{l} \\ \cdot \frac{1}{4} \\ \cdot \frac{1}{4} \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array} \right)$$

$$A^{-2} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$$

$$A^{-1} = \left( \begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{array} \right) \sim \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{array} \right) \sim$$

$$\left( \begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & -2 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \cdot \frac{1}{3} \\ \\ \\ \end{array}$$



$$\sim \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 4 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \text{null} & \text{null} & \text{null} & \text{null} \\ 0 & 1 & 0 & 0 & \text{null} & \text{null} & \text{null} & \text{null} \\ 0 & 0 & 1 & 0 & \text{null} & \text{null} & \text{null} & \text{null} \\ 0 & 0 & 0 & 1 & \text{null} & \text{null} & \text{null} & \text{null} \end{array} \right)$$

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & -1 & -1 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & -1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

$$A^{-1} \left( \begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ & & & & & & & \end{array} \right)$$

$$\sim \left( \begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$\sim \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array} \right)$$



$$\sim \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array} \right)$$

$$\sim \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array} \right)$$

$$\sim \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

$$A_2 = \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & & & & 0 & 0 \\ 0 & 1 & 1 & 0 & & & & 0 & 0 \\ 0 & 0 & 1 & 1 & & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & & 1 & 1 & & \\ 0 & 0 & 0 & 0 & & 0 & 1 & & \end{array} \right)$$

$$A^{-1}_2 = \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & & & & 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 1 & 0 & & & & 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 1 & 1 & & & & 0 & 0 & 1 & 0 & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & & 1 & 1 & & 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 & & 0 & 0 & 0 & 0 & & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{c|cccc|cccc} E & 1 & -1 & -1 & -1 & & (-1)^{n_1} & (-1)^{n_2} \\ & 0 & 1 & -1 & 1 & & (-1)^{n_1} & (-1)^{n_2} \\ & & & & & & & \\ & & & & & & & \\ & 0 & & & & & 1 & -1 \\ & 0 & 0 & 0 & 0 & & 1 & -1 \\ & 0 & 0 & 0 & 0 & & 0 & 1 \end{array} \right)$$



$$\sim \left( \begin{array}{cccc|cc} 1 & a & a^2 & a^3 & \dots & a^{n-2} & a^{n-1} & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & a & a^2 & \dots & a^{n-2} & a^{n-1} & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & a & \dots & a^{n-3} & a^{n-2} & \dots & \dots & \dots & \dots & \dots \\ \hline 0 & 0 & 0 & 0 & \dots & 1 & a & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 1 \end{array} \right) \sim$$

$$\sim \left( E \left| \begin{array}{cccc|cc} 1-a & 0 & \dots & 0 & 0 & 0 \\ 0 & 1-a & \dots & 0 & 0 & 0 \\ 0 & \dots & \dots & 1-a & 0 & 0 \\ 0 & \dots & \dots & 0 & 1-a & 0 \\ 0 & \dots & \dots & 0 & 0 & 1 \end{array} \right. \right)$$

$$\sim \left( \begin{array}{cccc|cc} 1 & 2 & 3 & 4 & \dots & n-1 & n & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 3 & \dots & n-2 & n-1 & 0 & 1 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \dots & 1 & 2 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 1 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{cccc|cc} 1 & 1 & \dots & 1 & 1 & 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 1 & 1 & 0 & 1 & -1 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 1 & 1 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 1 \end{array} \right) \sim \left( E \left| \begin{array}{cccc|cc} 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 1 & -2 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 1 \end{array} \right. \right)$$



$$\begin{pmatrix} 1+a & 1 & 1 & \dots & 1 & 1 \\ 1 & 1+a & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ n & 1 & 2 & 3 & \dots & n-2 & n-1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 3 & 4 & 5 & 6 & \dots & 1 & 2 \\ 2 & 3 & 4 & 5 & \dots & n & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}, A^t =$$

$$A = -A^t \text{ (антисиметрична)}$$

$$A^t = \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2+c^2+d^2 & 0 & 0 \\ 0 & 0 & a^2+b^2+c^2+d^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

$$A^t = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$$

$$(a^2+b^2+c^2+d^2)E = A \cdot A^t$$

$$A^{-1} = ?$$

$$A \cdot A^{-1} = E$$

$$A^{-1} = \frac{E}{A^2+b^2+c^2+d^2} \cdot A^t$$

$$\det A = \det A^t \Rightarrow \det A \cdot \det A^t = \det(AA^t) = (a^2+b^2+c^2+d^2)^4$$

$$\det A = (a^2+b^2+c^2+d^2)^2$$

Ранг на с-ма вектори,  
ранг на матрица

МИНОР  
 $A \in M_{k \times n}(F)$   
 $k \leq n$   
 $1 \leq m \leq k \leq n$

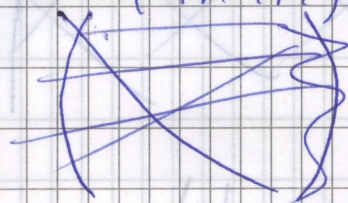
$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$



$$\begin{aligned} a_1 &= (2, 1, -3) \\ a_2 &= (3, 1, -5) \\ a_3 &= (1, 0, -7) \\ a_4 &= (4, 2, -1) \\ a_5 &= (1, 0, -2) \end{aligned}$$

$$\begin{pmatrix} 2 & 1 & -3 \\ 3 & 1 & -5 \\ 1 & 0 & -7 \\ 4 & 2 & -1 \\ 1 & 0 & -2 \end{pmatrix}$$

Максимальная  
линейно-независимая  
подсистема  
(МЛНП)



$$\begin{aligned} & \xrightarrow{\times(-2)} \\ & \begin{pmatrix} a_3 & 1 & 0 & -7 \\ a_1 & 2 & 1 & -3 \\ a_2 & 3 & 1 & -5 \\ a_4 & 4 & 2 & -1 \\ a_5 & 1 & 0 & -2 \end{pmatrix} \xrightarrow{\begin{matrix} \times(-2) \\ \times(-3) \\ \times(-4) \end{matrix}} \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 11 \\ 0 & 1 & 16 \\ 0 & 2 & 27 \\ 0 & 0 & 5 \end{pmatrix} \sim \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 11 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 11 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank}(a_1, a_2, a_3, a_4, a_5) = 3$$

например МЛНП образуют  
 $\{a_1, a_2, a_3\}$

$$\begin{aligned} a_1 &= (3, 5, -13, 11) \\ a_2 &= (3, -1, 3, -3) \\ a_3 &= (3, 2, -5, 4) \\ a_4 &= (3, 8, -21, 18) \end{aligned}$$

$$\begin{pmatrix} 3 & 5 & -13 & 11 \\ 3 & -1 & 3 & -3 \\ 3 & 2 & -5 & 4 \\ 3 & 8 & -21 & 18 \end{pmatrix} \xrightarrow{\begin{matrix} \times(-1) \\ \times(-2) \\ \times(-3) \end{matrix}} \begin{pmatrix} 3 & 5 & -13 & 11 \\ 0 & -6 & 16 & -14 \\ 0 & -3 & 8 & -7 \\ 0 & 3 & -8 & 7 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 3 & 5 & -13 & 11 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix}$$

$$\text{rank}(a_1, a_2, a_3, a_4) = 2$$

МЛНП - например  $\{a_1, a_2\}$



$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad z(A) = 2$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -i & -1 & i & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i & 1 \\ i^2 & i^2 & (-1)^2 & (-i)^2 & 1 \\ 1^3 & i^3 & (-1)^3 & (-i)^3 & 1 \end{pmatrix} \Rightarrow \begin{matrix} i & -i \\ i^2 & -1 & (-i)^2 & -1 \\ i^3 & -i & (-i)^3 & i \end{matrix}$$

$$\Rightarrow w(1, i, -1, -i) \neq 0 \Rightarrow z(A) = 4$$

$$w(x_1, x_2, \dots, x_n) = \prod_{1 \leq j \leq i \leq n} (x_i - x_j)$$

$$A = \begin{pmatrix} 14 & -27 & 49 & 113 \\ 43 & -82 & -145 & 340 \\ -29 & 55 & 96 & -224 \\ 128 & -245 & -438 & 1014 \end{pmatrix} \xrightarrow{\substack{x(-3) \\ + \\ x(-9) \\ + \\ x(-3) \\ + \\ x(-9)}} \sim \begin{pmatrix} 14 & -27 & 49 & 113 \\ 1 & -1 & 2 & 1 \\ -1 & 1 & -2 & -1 \\ 0 & 2 & 3 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 14 & -27 & -49 & 113 \\ 1 & -1 & 2 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad z(A) = 3$$

$$A = \begin{pmatrix} 2 & -1 & 3 & 1 \\ -1 & 2 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 3 & -3 & 2 & \lambda \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 3 & 5 & 3 \\ 0 & -3 & -5 & -3 \\ 3 & -3 & 2 & \lambda \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 3 & 1 \\ 3 & -3 & 2 & \lambda \\ 0 & 3 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$z(A) = 3$$



$$A = \begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & -1 & -1 \\ 1 & -1 & 1-\lambda & -1 \\ 1 & -1 & -1 & 1-\lambda \end{pmatrix} \sim \begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 2-\lambda & 2-\lambda & 0 & 0 \\ 2-\lambda & 0 & 2-\lambda & 0 \\ 2-\lambda & 0 & 0 & 2-\lambda \end{pmatrix} \sim \begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 2-\lambda & 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & \lambda & 0 \\ 0 & 2-\lambda & 0 & 2-\lambda \end{pmatrix}$$

$\xrightarrow{\substack{R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2}}}$

$$\sim \begin{pmatrix} -\lambda-2 & 0 & 1 & 1 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix}$$

1)  $\lambda \neq \pm 2 \quad z(A) = 4$

2)  $\lambda = -2 \quad z(A) = 3$

$\lambda = 2 \quad z(A) = 1$

$$A = \begin{pmatrix} \lambda & 1 & 2 & 3 & \dots & n-2 & 1 \\ 1 & \lambda & 2 & 3 & \dots & n-2 & 1 \\ 1 & 2 & \lambda & 3 & \dots & n-2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & 4 & \dots & n-1 & \lambda \end{pmatrix} \sim \begin{pmatrix} \lambda-1 & -1 & -1 & \dots & -1 & 1 \\ 0 & \lambda-2 & -1 & \dots & -1 & 1 \\ 0 & 0 & \lambda-3 & \dots & -1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda-n+1 & 1 \end{pmatrix}$$

$\xrightarrow{\substack{R_1 \leftarrow R_1 + R_2 \\ R_2 \leftarrow R_2 + R_3 \\ \vdots \\ R_{n-1} \leftarrow R_{n-1} + R_n}}}$

$$A = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & \lambda & \dots & 0 & 0 \\ 1 & \lambda & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \lambda & \lambda & \dots & \lambda & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix} \sim \begin{pmatrix} \frac{n-1}{\lambda} & 1 & 1 & \dots & 1 & 1 \\ 0 & -\lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -\lambda \end{pmatrix}$$

$\xrightarrow{\substack{R_1 \leftarrow R_1 + R_2 \\ R_2 \leftarrow R_2 + R_3 \\ \vdots \\ R_{n-1} \leftarrow R_{n-1} + R_n}}}$

$\text{npn } \lambda = 0 \quad z_g(A) = 2$   
 $\text{npn } \lambda \neq 0 \quad z_g(A) = n$

④



$$\text{rg}(a_1, a_2, \dots, a_n) = ?$$

$$a_1(1, \lambda, \dots, \lambda, \lambda)$$

$$a_2(\lambda-1, 2, \dots, \lambda, \lambda)$$

$$\vdots$$

$$a_n(\lambda-n+1, \lambda, \dots, \lambda, n)$$

$$\begin{pmatrix} 1 & \lambda & \lambda & \dots & \lambda & \lambda \\ \lambda-1 & 2 & \lambda & \dots & \lambda & \lambda \\ \lambda-2 & \lambda & 3 & \dots & \lambda & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda-n+2 & \lambda & \lambda & \dots & n-1 & \lambda \\ \lambda-n+1 & \lambda & \lambda & \dots & \lambda & n \end{pmatrix} \sim \begin{pmatrix} 1 & \lambda & \lambda & \dots & \lambda & \lambda \\ -\lambda & \lambda-2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda+n-1 & 0 & 0 & \dots & \lambda-n+1 & 0 \\ -\lambda+n & 0 & 0 & \dots & 0 & \lambda-n \end{pmatrix}$$

$$\sim \begin{pmatrix} \lambda + \lambda(n-1) & 1 & \dots & \lambda & \lambda \\ 0 & \lambda-2 & \dots & 0 & 0 \\ 0 & \dots & \lambda-3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \lambda-n & \lambda \end{pmatrix}$$

det  $\lambda \neq 2, 3, \dots, n-1, n, -\frac{1}{n-1}$   $\text{rg}(a_1, a_n) = n$

$\lambda = 2, 3, \dots, n-1, n, -\frac{1}{n-1}$

$\text{rg}(a_1, \dots, a_n) = n-1$



# Лин. пространства

Численое поле

$$F \neq \{\emptyset\}$$

$+, -, \cdot, \div$  на число  $\neq 0$

$$\mathbb{Q}, \mathbb{R}, \mathbb{C}$$

$$V \neq \{\emptyset\}$$

$$\forall a, b \in V, \lambda \in F$$

$$1) a + b \in V$$

елем. на  $V$  - вектори

$$2) \lambda a \in V$$

елем. на  $F$  - скалари

$$\textcircled{1} \text{ Нека } V = \mathbb{R}^+ \oplus, \otimes, F = \mathbb{R}, a, b \in V, \lambda \in F$$

$$1) a \oplus b = ab \in V$$

$$2) \lambda \otimes a = a^\lambda \in V$$

$$1) (a \oplus b) \oplus c \stackrel{?}{=} a \oplus (b \oplus c)$$

$$(ab) \oplus c = (ab)c = a(bc) = a \oplus (b \oplus c) \quad \checkmark$$

$$2) a \oplus b \stackrel{?}{=} b \oplus a$$

$$ab = ba \quad \checkmark$$

$$3) \exists! \vec{0} \in V: a \oplus \vec{0} = \vec{0} \oplus a = a$$

$$a \vec{0} = a \Rightarrow \vec{0} = 1 \quad \checkmark$$

$$4) a \oplus x = \vec{0}, x \in V$$

$$ax = 1$$

$$x = a^{-1} \quad \checkmark$$

$$5) 1 \otimes a = a$$

$$a^1 = a \quad \checkmark$$

$$6) \lambda \otimes (a \oplus b) \stackrel{?}{=} \lambda \otimes a \oplus \lambda \otimes b$$

$$\lambda \otimes (ab) = (ab)^\lambda = a^\lambda \cdot b^\lambda = (\lambda \otimes a) \oplus (\lambda \otimes b) \quad \checkmark$$

$$7) (\lambda + \mu) \otimes a \stackrel{?}{=} \lambda \otimes a \oplus \mu \otimes a = a^\lambda \cdot a^\mu = a^{\lambda + \mu} \quad \checkmark$$

$$8) \lambda \otimes (\mu \otimes a) \stackrel{?}{=} (\lambda \mu) \otimes a$$

$$(a^\mu)^\lambda = a^{\mu \cdot \lambda} \quad \checkmark$$



$F^n$  - нареденa n-орка

$$\alpha, \beta \in F^n, \lambda \in F$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$\beta = (\beta_1, \beta_2, \dots, \beta_n)$$

$$1) \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n) \in F^n$$

$$2) \lambda \alpha = (\lambda \alpha_1, \lambda \alpha_2, \dots, \lambda \alpha_n)$$

$V = M_{n \times n}(F)$ ,  $F$  - числово поле

$$P_{n+1} = \{ f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n, a_i \in F, i=1, \dots, n \}$$

$$\deg f \leq n$$

Линейно подпространство

$V$  - лине. пространство над  $F$

$$\{0\} \neq W \subseteq V$$

$$w_1, w_2 \in W, \lambda \in F$$

$$1) w_1 + w_2 \in W$$

$$2) \lambda w_1 \in W$$

$M_{n \times n}(F)$  - лине. пр-ство

$$W \subseteq M_{n \times n}(F): \forall w \in W:$$

$$\lambda \in F^n$$

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$w = \gamma, \beta \in W, \lambda \in F.$$

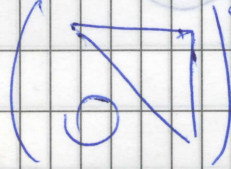
$$\vec{0} = (0, 0, \dots, 0) \in W$$

$$\beta = (0, \beta_2, \dots, \beta_n)$$

$$\gamma = (0, \gamma_2, \dots, \gamma_n)$$

$$\Rightarrow \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ 0 & 0 & \dots & 0 \\ w_{n1} & w_{n2} & \dots & w_{nn} \end{pmatrix}$$

$W$  - лине. подпр-ство на  $M_{n \times n}(F)$



$$1) \beta + \gamma = (0, \beta_2 + \gamma_2, \dots, \beta_n + \gamma_n) \in W$$

$$2) \lambda \beta = (0, \lambda \beta_2, \dots, \lambda \beta_n)$$



$U$ -тип пр-ства  
 произволка  
 е подпр.?  
 сечение

доказание от подпр-ства на  $V$



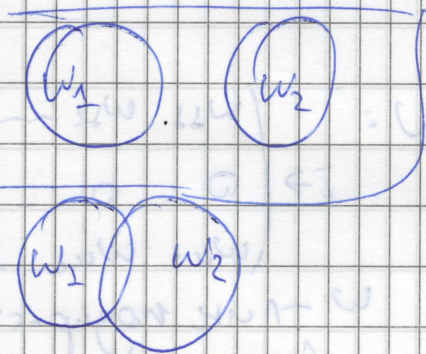
$$\begin{aligned} v_1, v_2 \in W_1 \cap W_2 &\Rightarrow \\ &\Rightarrow v_1 \in W_1 \text{ и } v_2 \in W_2 \\ &\quad v_2 \in W_1 \text{ и } v_1 \in W_1 \end{aligned}$$

$$1) \text{ и } \begin{aligned} v_1 + v_2 \in W_1 \\ v_1 + v_2 \in W_2 \end{aligned} \Rightarrow v_1 + v_2 \in W_1 \cap W_2$$

$$2) \lambda v_1 \in W_1 \text{ и } \lambda v_1 \in W_2 \Rightarrow \lambda v_1 \in W_1 \cap W_2$$



$U$  на 2 подпростр- на  $V$  е подпростр. тогава  
 обединение когато  $W_1 \subseteq W_2$  или  $W_2 \subseteq W_1$



$$v_1 \in W_1 / W_2$$

$$v_2 \in W_2 / W_1$$

$$v_1 + v_2 \in W_1 \Rightarrow v_2 \in W_1$$

$$\in W_2 \Rightarrow v_1 \in W_2$$



линейная комбинация

$$\begin{matrix} v_1, & \dots, & v_n \in V \\ \lambda_1, & \dots, & \lambda_n \in F \end{matrix}$$

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n \in F$$

линейная оболочка

$$A = \{a_1, \dots, a_k\} \in V$$

$$l(A) = \{ \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_k a_k \mid \lambda_i \in F \}$$

$l(A)$  — подпространство на  $V$ ?

$$v_1, v_2 \in l(A)$$

$$v_1 = \lambda_1 a_1 + \dots + \lambda_k a_k$$

$$\lambda_i, \mu_i \in F$$

$$v_2 = \mu_1 a_1 + \dots + \mu_k a_k$$

$$1) v_1 + v_2 = (\lambda_1 + \mu_1) a_1 + \dots + (\lambda_k + \mu_k) a_k \in l(A)$$

$$2) \lambda v_1 = \lambda_1 \lambda a_1 + \dots + \lambda_k \lambda a_k \in l(A)$$

$\forall \{v_1, v_2, \dots, v_m\}$  — подпространство, содержащее  $A$

$$\cap v_i = l(A)$$

$$\uparrow l(A) \subseteq \cap v_i$$

$$\cap v_i \subseteq l(A)$$

линейная зависимость

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0$$

$$\text{Ако } \forall \lambda_i = 0, i \geq 1, \dots, n$$

$$\{a_1, a_2, \dots, a_n\} \text{ — ЛЗ}$$

Когда некое  $i$  от  $\lambda_i \neq 0$

$$\{a_1, a_2, \dots, a_n\} \text{ — ЛЗ}$$

линейная независимость



$$\{a, b, c\} = \{a, 2a, c\}$$

$$b = 2a$$

$$2a - b + 0c = \vec{0}$$

$$\lambda_1 \neq 0$$

$$a_1 = -\frac{\lambda_2}{\lambda_1} a_2 - \frac{\lambda_3}{\lambda_1} a_3 - \dots - \frac{\lambda_n}{\lambda_1} a_n$$

Базис на линейно пространство  $(V)$   
с-на от вектори  $A = \{a_1, \dots, a_n\} \in V$

1) ЛНБ

2)  $\forall v \in V$  може да се изрази като линейна комбинация на векторите от  $A$ .

$\dim V = n$  (за всеки базис има  $n$  др. вектори)

$$v = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_k e_k$$

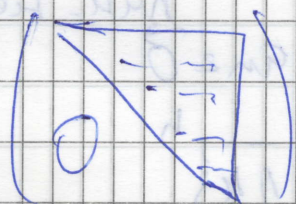
$v \in \{\lambda_1, \lambda_2, \dots, \lambda_k\}$

$$\begin{array}{l} \mathbb{F}^n \\ \left| \begin{array}{l} e_1 = (1, 0, \dots, 0) \\ e_2 = (0, 1, \dots, 0) \\ \vdots \\ e_n = (0, 0, \dots, 1) \end{array} \right. \end{array} \quad \lambda_1 e_1 + \dots + \lambda_n e_n = \vec{0} = 0 \cdot e_1 + 0 \cdot e_2 + \dots + 0 \cdot e_n$$

$$M_{n \times n}(F) \quad E_{ij}, \quad i=1, \dots, k, \quad j=1, \dots, n$$

$$\dim M_n(F) = n^2$$

$$T_n(F)$$



$$\frac{n(n+1)}{2} = \dim T_n(F)$$



$$\begin{aligned} e_1 &= x e^0 \\ e_2 &= x e^1 \\ e_3 &= x e^2 \\ &\vdots \\ e_{n+1} &= x e^n \end{aligned}$$

$\{e_1, e_2, e_3\}$  - базис на  $V$

$$g_1 = e_1 - 2e_2 + 4e_3$$

$$g_2 = e_1 + e_2 - 2e_3 \quad - \text{базис на } V$$

$$g_3 = 3e_1 + 4e_2$$

$$\lambda_1, \lambda_2, \lambda_3 = 0:$$

$$\lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 = \vec{0}$$

$$\lambda_1(e_1 - 2e_2 + 4e_3) + \lambda_2(e_1 + e_2 - 2e_3) + \lambda_3(3e_1 + 4e_2) = \vec{0}$$

$$(\lambda_1 + \lambda_2 + 3\lambda_3)e_1 + (-2\lambda_1 + \lambda_2 + 4\lambda_3)e_2 + (4\lambda_1 - 2\lambda_2)e_3 =$$

$$= 0e_1 + 0e_2 + 0e_3$$

$$\lambda_1 + \lambda_2 + 3\lambda_3 = 0$$

$$-2\lambda_1 + \lambda_2 + 4\lambda_3 = 0$$

$$4\lambda_1 - 2\lambda_2 = 0$$

матрица на преход

$$\begin{pmatrix} g_1 & g_2 & g_3 \\ 1 & 1 & 3 \\ -2 & 1 & 4 \\ 4 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 10 \\ 0 & -6 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & 4 \end{pmatrix} \quad \begin{matrix} \text{+} \\ \text{+} \\ \text{+} \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 10 \\ 0 & -3 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$\Rightarrow \{g_1, g_2, g_3\}$  - лнк. независими, 3 на брой, 3-мерно простр.  
 $\Rightarrow$  базис

$\{e_1, e_2, e_3\}$  - базис на  $V$

$$a_1 = e_1 + e_2$$

$$a_2 = e_1 + e_2 + 3e_3$$

$$a_3 = e_1 + e_2$$

$$b_1 = a_1 + 2a_2 - a_3$$

$$b_2 = a_1 - a_2$$

$$b_3 = -a_1 + 2a_2 + 2a_3$$

$$Te \rightarrow a = ?$$

$$Ua \rightarrow b = ?$$

$$Qe \rightarrow b = ?$$

$$xe = 2e_1 - 3e_3$$

$$xa = ?$$

$$xe = ?$$



$$\begin{aligned} e_1 &= x^0 \\ e_2 &= x^1 \\ e_3 &= x^2 \end{aligned}$$

$$T = \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$U_{a \rightarrow b} = \begin{pmatrix} b_1 & b_2 & b_3 \\ 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 0 & 2 \end{pmatrix}$$

$$Q_{a \rightarrow b} T_{e \rightarrow a} \cdot U_{a \rightarrow b} =$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 & -1 \\ 3 & 0 & 1 \\ 5 & -3 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} e$$

$$\begin{pmatrix} 1 & -1 & 1 & | & 2 \\ 1 & 1 & 0 & | & 0 \\ 0 & 3 & 1 & | & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & | & 2 \\ 0 & 2 & -1 & | & -2 \\ 0 & 3 & 1 & | & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & | & 2 \\ 0 & 5 & 0 & | & -3 \\ 0 & 3 & 1 & | & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 3 & 1 & | & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

$$x_a = a_1 - a_2$$

Петри 152. - 172. контрольно в 023ага

$$\{e_1, e_2, e_3\}, \{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$$

$$a_1 = e_1 - e_3$$

$$a_2 = 2e_1 + e_2 + 3e_3$$

$$a_3 = -e_1 + 2e_2$$

$$b_1 = e_1 - e_2 + 5e_3$$

$$b_2 = 4e_1 + 7e_2 + 9e_3$$

$$b_3 = 2e_1 + 3e_2 + 2e_3$$

$$x = e_1 + 5e_2 + 2e_3$$

$$e \rightarrow a$$

$$\parallel 270$$



$e_1, e_2, e_3$  — базис на  $V$

$$\begin{cases} a_1 = e_1 - e_3 \\ a_2 = 2e_1 + e_2 + 3e_3 \\ a_3 = -e_1 + 2e_2 \end{cases}$$

$$\begin{cases} b_1 = e_1 - e_2 + 5e_3 \\ b_2 = 4e_1 + 7e_2 + 9e_3 \\ b_3 = 2e_1 + 3e_2 + 2e_3 \end{cases}$$

$$x_0 = e_1 + 5e_2 + 2e_3$$

$T_{e \rightarrow a} \quad Q_{e \rightarrow b}, \quad 4a \rightarrow b = ?$   
 $x_{a_1}, x_{b_1} = ?$

$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = \vec{0}$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\lambda_1(e_1 - e_3) + \lambda_2(2e_1 + e_2 + 3e_3) + \lambda_3(-e_1 + 2e_2) = 0e_1 + 0e_2 + 0e_3$$

$$\begin{cases} \lambda_1 + 2\lambda_2 - \lambda_3 = 0 \\ \lambda_2 + 2\lambda_3 = 0 \\ -\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0 \end{cases} \quad T_{e \rightarrow a} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 3 & 0 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 5 & -1 \end{pmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow \{a_1, a_2, a_3\} \text{ — нхз базис}$$

$$T_{e \rightarrow a} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 3 & 0 \end{pmatrix}$$

$$Q_{e \rightarrow b} = \begin{pmatrix} 1 & 4 & 2 \\ -1 & 7 & 3 \\ 5 & 9 & 2 \end{pmatrix}$$



$$T_{e \rightarrow a} \cdot U_{a \rightarrow b} = Q_{e \rightarrow b}$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 7 & 3 \\ -1 & 3 & 0 & 5 & 9 & 2 \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 7 & 3 \\ 0 & 5 & -1 & 6 & 13 & 4 \end{array} \right) \sim \begin{matrix} /(-5) \\ /(-2) \end{matrix}$$



$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & -5 & 3 & -10 & -4 \\ 0 & 1 & 2 & -1 & 7 & 3 \\ 0 & 0 & -11 & 11 & -22 & -11 \end{array} \right) \sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -10 & -4 \\ 0 & 1 & 2 & -1 & 7 & 3 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array} \right) \begin{array}{l} \uparrow \\ \downarrow \end{array}$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array} \right) \quad U_{A \rightarrow B} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix} \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix}$$

$$x e_1 = e_1 + 5e_2 + 2e_3 = x a_1 = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$$

$$\lambda_1 (e_1 - e_3) + \lambda_2 (2e_1 + e_2 + 3e_3) + \lambda_3 (-e_1 + 2e_2) = 8e_1 + 5e_2 + 2e_3$$

$$\begin{cases} \lambda_1 + 2\lambda_2 - \lambda_3 = 8 \\ \lambda_2 + 2\lambda_3 = 5 \\ -\lambda_1 + 3\lambda_2 = 2 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & 1 & 2 & 5 \\ -1 & 3 & 0 & 2 \end{array} \right) \sim \begin{array}{l} \uparrow \\ \downarrow \end{array}$$

$$T_{e \rightarrow a} \cdot \begin{pmatrix} x_a \end{pmatrix} = \begin{pmatrix} x_e \end{pmatrix}$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & 1 & 2 & 5 \\ 0 & 5 & -1 & 10 \end{array} \right) \begin{array}{l} \uparrow \\ \downarrow \end{array} \sim \left( \begin{array}{ccc|c} 1 & 0 & -5 & -9 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -11 & -22 \end{array} \right) \begin{array}{l} \uparrow \\ \downarrow \end{array}$$

$$\sim \left( \begin{array}{ccc|c} 1 & 0 & -5 & -9 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right) \begin{array}{l} \uparrow \\ \downarrow \end{array} \sim \left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

$$x a_1 = a_1 + a_2 + 2a_3$$

1.2.72



$$Qe \rightarrow b = \begin{pmatrix} 1 & 4 & 2 \\ -1 & 7 & 3 \\ 5 & 9 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} \xrightarrow{+1(-5)} \sim \begin{pmatrix} 1 & 4 & 2 \\ 0 & 11 & 5 \\ 0 & -11 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix} \xrightarrow{+}$$

$$\sim \begin{pmatrix} 1 & 4 & 2 \\ 0 & 11 & 5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix} \xrightarrow{+1(-5)} \xrightarrow{+1(-4)} \sim \begin{pmatrix} 1 & 4 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \xrightarrow{+1(-4)} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

За управление:

$e_1, e_2, e_3$  - базис на  $V$   
 $a_1, a_2, a_3$  - базис на  $W$

$$\begin{aligned} a_1 &= 2e_1 + 2e_2 - e_3 \\ a_2 &= 2e_1 - e_2 + 2e_3 \\ a_3 &= -e_1 + 2e_2 + 2e_3 \end{aligned}$$

$$V_e = 2e_1 + 2e_2 + 2e_3$$

$V_a = ?$   $T_e \rightarrow a$

$$\begin{cases} \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0 \\ \lambda_1(2e_1 + 2e_2 - e_3) + \lambda_2(2e_1 - e_2 + 2e_3) \\ + \lambda_3(-e_1 + 2e_2 + 2e_3) \end{cases}$$

$$T_e \rightarrow a \cdot V \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$T_e \rightarrow a \cdot V \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \xrightarrow{+1(-2)} \xrightarrow{+1(-2)} \xrightarrow{+1(-2)}$$

$$\sim \begin{pmatrix} 2 & 2 & -1 & 2 \\ 0 & -3 & 3 & 0 \\ 0 & 6 & 0 & 2 \end{pmatrix} \xrightarrow{+1(-2)} \sim \begin{pmatrix} 2 & 2 & -1 & 2 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 6 & 2 \end{pmatrix} \xrightarrow{+1(-1)} \sim$$



$$\sim \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 3 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 3 & | & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 3 & | & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -\frac{1}{2} & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & \frac{1}{3} \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & 1 & -1 & | & 0 \\ 1 & 0 & -1 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 2 & 2 & -1 & | & 2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & | & \frac{2}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & \frac{2}{3} \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & \frac{2}{3} \end{pmatrix}$$

$$v_a = \frac{2}{3} a_1 + \frac{2}{3} a_2 + \frac{2}{3} a_3$$

$$a_1 = (1, -1, 2, 3)$$

$$a_2 = (2, -2, 1, 1)$$

$$\begin{pmatrix} a_1 & a_2 \\ 1 & 2 \\ -1 & -2 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & -3 \\ 0 & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & -3 \\ 0 & 0 \end{pmatrix} \begin{matrix} e_1 & e_2 \\ 0 & * \\ 1 & * \\ * & * \\ * & 1 \end{matrix}$$

$\{a_1, a_2, e_2, e_4\}$ -Basis  $\mathbb{R}^4$



$$b = (3, 6, 9)$$

$$a_1 = (1, 2, 3)$$

$$a_2 = (-1, 0, 2)$$

$$a_3 = (1, 4, 8)$$

$$b \in \langle a_1, a_2, a_3 \rangle$$

$$b = d_1 a_1 + d_2 a_2 + d_3 a_3$$

$$e_1, e_2, e_3 - \text{Bazue ka } V$$

$$d_1(e_1 + 2e_2 + 3e_3) + d_2(-e_1 + 2e_3) + d_3(e_1 + 4e_2 + 8e_3) = 3e_1 + 6e_2 + 9e_3$$

$$\begin{cases} d_1 - d_2 + d_3 = 3 \\ 2d_1 + 4d_3 = 6 \\ 3d_1 + 2d_2 + 8d_3 = 9 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 1 & | & 3 \\ 2 & 0 & 4 & | & 6 \\ 3 & 2 & 8 & | & 9 \end{pmatrix} \xrightarrow{R_2 - 2R_1, R_3 - 3R_1} \begin{pmatrix} 1 & -1 & 1 & | & 3 \\ 0 & 2 & 2 & | & 0 \\ 0 & 5 & 5 & | & 0 \end{pmatrix} \xrightarrow{R_3 - 2.5R_2} \begin{pmatrix} 1 & -1 & 1 & | & 3 \\ 0 & 2 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & | & 3 \\ 0 & 2 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 1 & 3 & | & 3 \\ 0 & 2 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & -1 & 1 & | & 3 \\ 0 & 2 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & | & 3 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{cases} d_1 - d_2 + d_3 = 3 \\ d_2 + d_3 = 0 \end{cases}$$

$$\begin{aligned} d_3 &= p & d_2 &= -p \\ d_1 &= 3 - 2p \end{aligned}$$

$$b = (3 - 2p)a_1 - pa_2 + pa_3$$

$$a_1 = (1, 1, -1) \quad a_2 = (0, -2, 1) \quad a_3 = (1, 1, 1) \quad b = (0, -2, 1)$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 1 & | & -2 \\ 1 & 1 & 1 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 1 & | & -2 \\ 0 & 0 & 2 & | & 1 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & -1 & 0 & | & -2 \\ 0 & -2 & 1 & | & -2 \\ 0 & 0 & 2 & | & 1 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 1 & | & -2 \\ 0 & 0 & 2 & | & 1 \end{pmatrix}$$



2

$$x_3 = -2 + \frac{2(\lambda+1)}{\lambda+3} = \frac{-2\lambda-6+2\lambda+8}{\lambda+3}$$

$$= \frac{-2(\lambda+1+1)}{\lambda+3}$$

$$x_2 = \frac{-\lambda - 4 - 2(M + \lambda + 1)}{M + 3} = \frac{-3\lambda - 2M - 6}{M + 3} = \frac{-3(\lambda + 2)}{M + 3}$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -2 & \mu & -2 \\ -1 & 1 & 1 & \lambda \end{array} \right) \xrightarrow{R_2 - R_1, R_3 + R_1} \left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & -2 & \mu & -2 \\ 0 & -1 & \mu+1 & \lambda-2 \end{array} \right) \xrightarrow{R_2 \leftrightarrow R_3} \left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & \mu+1 & \lambda-2 \\ -1 & -2 & \mu & -2 \end{array} \right) \xrightarrow{R_3 + R_2} \left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & \mu+1 & \lambda-2 \\ 0 & -3 & 2\mu+1 & \lambda-2 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -\mu-2 & 2-2\lambda+2 & 0 & 0 \\ 0 & -1 & \mu+1 & \lambda-2 & 0 & 0 \end{array} \right) \xrightarrow{R_1 - R_2} \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\mu-2 & 2-2\lambda+2 & 0 & 0 \\ 0 & -1 & \mu+1 & \lambda-2 & 0 & 0 \end{array} \right) \xrightarrow{R_2 \leftrightarrow R_3} \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & \mu+1 & \lambda-2 & 0 & 0 \\ 0 & 0 & -\mu-2 & 2-2\lambda+2 & 0 & 0 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & \mu+1 & \lambda-2 \\ 0 & 0 & -\mu-3 & -2\lambda+2 \end{array} \right) \quad \begin{array}{l} \lambda_1 + \lambda_3 = 0 \\ -\lambda_2 + (\mu+1)\lambda_3 = \lambda-2 \\ (-\mu-3)\lambda_3 = -2\lambda+2 \end{array}$$

1.1 (a)  $M = 3$   ~~$M = 3$~~   $-\lambda_2 - 2\lambda_3 = -1$   
 $0 \quad \lambda_3 = 0$   
 ~~$\lambda_3 = 2\lambda_1 - 1$~~   $\lambda_3 = -\lambda_1$



$$\lambda_3 = 2 \quad \lambda_1 = -2 \quad \lambda_2 = 1 - 2\lambda \Rightarrow b = -2a_1 + (1-2\lambda)a_2 + 2a_3$$

$$12cn) \quad \mu = -3 \quad \lambda \neq 1 \quad 0 \quad \lambda_3 = -2\lambda + 2 \neq 0$$

$$b = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$$

$$2cn) \quad \mu = -3 \Rightarrow \lambda_3 = \frac{-2\lambda + 2}{-11 - 3}$$

$$\lambda_1 = -\lambda_3 = \frac{+2\lambda - 2}{-11 - 3}$$

$$\lambda_2 = (\mu + 1)\lambda_3 - \lambda + 2 = -\lambda + 2 \mp \frac{(\mu + 1)(2 - 2\lambda)}{-11 - 3}$$

За упражнение:

$$a_1 = (1, 2, 3, 4)$$

$$a_2 = (7, 14, 20, 27)$$

$$a_3 = (5, 10, 16, 19)$$

$$v = (2, 1, 5, 5)$$

$e_1, e_2, \dots, e_n$  - базис на  $V$ ,  $\dim V = n$   
 $k \in \mathbb{N}$ ,  $k \leq n$

$$\Rightarrow \text{Ако } V_1 = \langle e_1, e_2, \dots, e_k \rangle$$

$$V_2 = \langle e_{k+1}, e_{k+2}, \dots, e_n \rangle$$

$$\text{Да се покаже, че } V = V_1 \oplus V_2$$

$$\Leftarrow \text{Ако } V = V_1 \oplus V_2$$

$e_1, e_2, \dots, e_k$  - базис на  $V_1$   
 $e_{k+1}, e_{k+2}, \dots, e_n$  - базис на  $V_2$

Да покаже, че  $\Rightarrow e_1, e_2, \dots, e_k, e_{k+1}, \dots, e_n$  - базис на  $V \Rightarrow \dim V = \dim V_1 + \dim V_2$



$$V = V_1 + V_2 + \dots + V_k$$

$$\forall v \in V: v = v_1 + v_2 + \dots + v_k, \quad v_i \in V_i$$

Ако представянето е египербо

$$\Rightarrow V = V_1 \oplus V_2 \oplus \dots \oplus V_k$$

$$V_i \cap V_j = \{\vec{0}\}$$

$$\Rightarrow 1) V = V_1 + V_2$$

$$\forall v \in V: v = \lambda_1 e_1 + \lambda_2 + \dots + \lambda_k e_k + \lambda_{k+1} e_{k+1} + \dots + \lambda_n e_n$$

$$\in V_1$$

$$\in V_2$$

$$2) u \in V_1 \cap V_2$$

$$u \in V_1$$

$$u = \lambda_1 e_1 + \dots + \lambda_k e_k$$

$$u \in V_2$$

$$u = \lambda_{k+1} e_{k+1} + \dots + \lambda_n e_n$$

$$\Rightarrow \lambda_1 e_1 + \dots + \lambda_k e_k - \lambda_{k+1} e_{k+1} - \dots - \lambda_n e_n = \vec{0}$$

$$e_1, \dots, e_n - \text{база на } V \Rightarrow \lambda_i = 0$$

$$u = \vec{0}$$

$$\Rightarrow V = V_1 \oplus V_2$$

$$\Leftarrow V = V_1 \oplus V_2$$

$$v_1 \in V_1 \Rightarrow v_1 = \mu_1 e_1 + \dots + \mu_k e_k \neq \vec{0}$$

$$v_2 \in V_2 \Rightarrow v_2 = \mu_{k+1} e_{k+1} + \dots + \mu_n e_n \neq \vec{0}$$

$$v_1 + \lambda v_2 = \mu_1 e_1 + \dots + \mu_k e_k + \lambda \mu_{k+1} e_{k+1} + \dots + \lambda \mu_n e_n = \vec{0}$$

$$v_1 = -\lambda v_2 \Rightarrow v_1 \in V_1 \cap V_2, \text{ но}$$

$$V = V_1 \oplus V_2 \Rightarrow v_1 = \vec{0} = v_2$$

$$\Rightarrow \text{ЛНБ} \Rightarrow e_1, e_2, \dots, e_n - \text{база на } V$$



$\forall W$  подпр-на  $V$ ,  $\exists U$ -подпр., такава че  $V = U \oplus W$

$W = \{e_1, \dots, e_k\} \rightarrow$  базис

$V = \{e_1, \dots, e_k, e_{k+1}, \dots, e_n\} \rightarrow$  базис

$U = \{e_{k+1}, \dots, e_n\}$

$$V = \{e_1\} \oplus \{e_2\} \oplus \dots \oplus \{e_n\}$$

Нека  $V = M_n(F)$ ,  $S$ -множеството на всички симетрични матрици  $A = A^t$

$T$ -множ. на всички антисиметрични  $A = -A^t$

a)  $S$  и  $T$  са подпространства

b)  $\dim S = \frac{n(n+1)}{2}$ ,  $\dim T = \frac{n(n-1)}{2}$

b)  $V = S \oplus T$

a)  $\{0\} \neq S \subset V$

1)  $A, B \in S \Rightarrow A+B \in S$

2)  $\lambda A \in S$

$A = (a_{ij}) \Rightarrow A$  е симетрична

$\Leftrightarrow a_{ij} = a_{ji}$

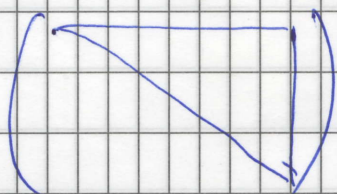
~~За~~  $T$  е аналогично  $B \in T, B = (b_{ij}) \Rightarrow b_{ij} = -b_{ji}$

$\Rightarrow S$  и  $T$  са подпр-ства

b)  $\{E_{ij} \mid i, j = 1, \dots, n\}$  - базис на  $V$

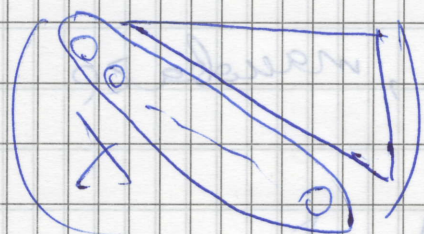
$S \rightarrow E_{ij} = E_{ji}$

$T \rightarrow E_{ij} = -E_{ji}$



$\dim S = n(n-1) + \dots + 2 + 1 = \frac{n(n+1)}{2}$





$$\dim T = \frac{n(n-1)}{2}$$

$$g) A \in S: A = a_{11}E_{11} + a_{12}E_{12} + \dots + a_{nn}E_{nn}$$

$$A = \sum_{i,j} a_{ij} E_{ij} = E_{ij} + E_{ji}$$

$$B \in T: B = E_{ij} - E_{ji}$$

$$C \in V: C = 2 E_{ij}$$

Q8.1 2013: фундаментална с-ма от решения (ФСР)

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n = 0$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) > \text{решения.}$$

$$\beta = (\beta_1, \beta_2, \dots, \beta_n)$$

$$1) \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n)$$

$$2) \lambda \alpha = (\lambda \alpha_1, \lambda \alpha_2, \dots, \lambda \alpha_n)$$

множ-от решения е лнн. пр-ство

ФСР ~~е~~ представлява базис на с-мата от решенията.

II.2.80



$$\textcircled{1} \begin{cases} 2x_1 + 3x_2 + 5x_3 - x_4 = 0 \\ x_1 + 2x_2 + 2x_3 - 3x_4 = 0 \\ -3x_1 - 8x_2 - 4x_3 + 13x_4 = 0 \\ 2x_1 + x_2 + 7x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 2 & 3 & 5 & -1 \\ 1 & 2 & 2 & -3 \\ -3 & -8 & -4 & 13 \\ 2 & 1 & 7 & 0 \end{pmatrix} \begin{matrix} \xrightarrow{+2} \\ \xrightarrow{+3} \\ \xrightarrow{+3} \\ \xrightarrow{-2} \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 2 & -3 \\ 0 & -1 & 1 & 5 \\ 0 & -2 & 2 & 4 \\ 0 & -3 & 3 & 6 \end{pmatrix} \xrightarrow{\times(-2)} \sim \begin{pmatrix} 1 & 2 & 2 & -3 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W: \begin{cases} x_1 + 2x_2 + 2x_3 - 3x_4 = 0 \\ -x_2 + x_3 + 5x_4 = 0 \\ x_4 = 0 \end{cases}$$

бр. неизв. -  $\text{rg}(A) =$   
= бр. базисни вектори

$$v_1 = \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ \boxed{-4} & \boxed{1} & \boxed{1} & \boxed{0} \end{matrix}$$

$$v_1 = (-4, 1, 1, 0) - \text{ФСР}$$

$W$  е подпростр. на  $F^4$

$W$  е подпр. свъз. на  $F^6$   $W \subseteq F^6$

$$W: x_3 = 0$$

$$\begin{matrix} \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} \end{matrix}$$

$$\dim W = 5$$

$$\begin{aligned} e_1 &= (1, 0, 0, 0, 0, 0) \\ e_2 &= (0, 1, 0, 0, 0, 0) \\ e_3 &= (0, 0, 0, 0, 0, 0) \\ e_4 &= (0, 0, 1, 0, 0, 0) \\ e_5 &= (0, 0, 0, 1, 0, 0) \\ e_6 &= (0, 0, 0, 0, 1, 0) \end{aligned}$$



$$\begin{cases} 2x_1 + x_2 + 0x_3 - 2x_4 = 0 \\ -2x_1 + 5x_2 - 10x_3 + 5x_4 = 0 \\ 6x_1 + 7x_2 - 5x_3 - x_4 = 0 \\ 4x_1 + 4x_2 + 3x_3 - 9x_4 = 0 \end{cases}$$

$$\begin{pmatrix} 2 & 1 & 1 & -2 \\ -2 & 5 & 10 & 5 \\ 6 & 7 & -5 & -1 \\ 4 & 4 & 3 & -9 \end{pmatrix} \xrightarrow{\begin{matrix} +(-3) \\ +(-2) \end{matrix}} \sim$$

$$\sim \begin{pmatrix} 2 & 1 & 1 & -2 \\ 0 & 6 & 8 & 3 \\ 0 & 4 & -8 & 5 \\ 0 & 2 & 1 & -5 \end{pmatrix} \xrightarrow{1/3} \sim$$

$$\begin{pmatrix} 2 & 1 & 1 & -2 \\ 0 & 2 & -3 & 1 \\ 0 & 4 & -8 & 5 \\ 0 & 2 & 1 & -5 \end{pmatrix} \xrightarrow{\begin{matrix} \times(-2) \\ - \end{matrix}} \sim$$

$$\begin{pmatrix} 2 & 1 & 1 & -2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & -6 \end{pmatrix} \xrightarrow{1/2} \sim$$

$$\sim \begin{pmatrix} 2 & 1 & 1 & -2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 + x_2 + x_3 - 2x_4 = 0 \\ 2x_2 - 3x_3 + x_4 = 0 \\ -2x_3 + 3x_4 = 0 \end{cases}$$

$$v_1 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -5 & 14 & 12 & 8 \end{pmatrix}$$

$$\begin{aligned} a_1 &= (2, -3, 0, 1, -2) \\ a_2 &= (3, 0, 1, -1, 2) \\ a_3 &= (7, -6, 1, 1, -2) \end{aligned}$$

$l(a_1, a_2, a_3) = W =$  пространство  
от решений  
на однородной  
с-ма

$$\begin{pmatrix} 2 & -3 & 0 & 1 & -2 \\ 3 & 0 & 1 & -1 & 2 \\ 7 & -6 & 1 & 1 & -2 \end{pmatrix} \xrightarrow{\begin{matrix} - \\ - \end{matrix}} \sim \begin{pmatrix} 2 & -3 & 0 & 1 & -2 \\ -4 & 6 & 0 & -2 & 4 \\ 7 & -6 & 1 & 1 & -2 \end{pmatrix} \xrightarrow{\begin{matrix} \times(2) \\ - \end{matrix}} \sim$$

$$\sim \begin{pmatrix} 2 & -3 & 0 & 1 & -2 \\ 7 & -6 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$l(a_1, a_2, a_3) = l(a_1, a_3)$$



$$U: \begin{cases} 2x_1 - 3x_2 + x_4 - 2x_5 = 0 \\ 7x_1 - 6x_2 + x_3 + x_4 - 2x_5 = 0 \end{cases}$$

$$W \oplus U = F^5$$

|       | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|-------|-------|-------|-------|-------|-------|
| $v_1$ | 1     | 0     | -5    | -2    | 0     |
| $v_2$ | 0     | 1     | 3     | 3     | 0     |
| $v_3$ | 0     | 0     | 0     | 2     | 1     |

$$\cancel{y_1 - 5y_3 - 2y_4 = 0}$$

$$W: \begin{cases} y_1 - 5y_3 - 2y_4 = 0 \\ y_2 + 3y_3 + 3y_4 = 0 \\ 2y_4 + y_5 = 0 \end{cases}$$

$$\begin{cases} a_1 = (3, 5, -1, 2) \\ a_2 = (2, 3, 3, 1) \\ a_3 = (1, 1, 7, 0) \\ a_4 = (1, 3, -15, 2) \end{cases}$$

$$W = \langle a_1, a_2, a_3, a_4 \rangle$$

$$\begin{pmatrix} 3 & 5 & -1 & 2 \\ 2 & 3 & 3 & 1 \\ 1 & 1 & 7 & 0 \\ 1 & 3 & -15 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -1 & 2 \\ 1 & 2 & -4 & 1 \\ 1 & 1 & 7 & 0 \\ 1 & 3 & -15 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 3 & 3 & 1 \\ -1 & -1 & -7 & 0 \\ 1 & 1 & 7 & 0 \\ -3 & -3 & -21 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 3 & 1 \\ +1 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{cases} 2x_1 + 3x_2 + 3x_3 + x_4 = 0 \\ x_1 + x_2 + 7x_3 = 0 \end{cases}$$

|       | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|-------|-------|-------|-------|-------|
| $v_1$ | -1    | 1     | 0     | -1    |
| $v_2$ | 7     | 0     | 1     | 4     |

$$\begin{cases} -y_1 + y_2 = 0 \\ -y_1 + y_3 - y_4 = 0 \end{cases} \quad \begin{cases} -2y_1 + 2y_2 - y_3 = 0 \\ -2y_1 + 2y_2 - y_3 = 0 \end{cases}$$

$$\begin{cases} -y_1 + y_2 = 0 \\ -7y_1 + y_3 + 4y_4 = 0 \end{cases}$$

11.2.83



④  $U, W$  — подпространства  $F^n$

$$1) \begin{cases} U = \ell(a_1, a_2, \dots) \\ W = \ell(b_1, b_2, \dots) \end{cases} \quad \text{даны на } U+W$$

$$U \cap W$$

$$2) U: \begin{vmatrix} X & C \end{vmatrix}$$

$$W: \begin{vmatrix} X & C \end{vmatrix}$$

$$3) U = \ell(a_1, a_2, \dots)$$

$$W: \begin{vmatrix} X & C \end{vmatrix}$$

$$1) U+W = \ell(a_1, a_2, \dots) + \ell(b_1, b_2, \dots) = \ell(a_1, a_2, \dots, b_1, b_2, \dots)$$

$$U \cap W =$$

↓  
хомогенный  
С-ММ

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

$$U = \ell(a_1, a_2, a_3)$$

$$a_1 = (1, -1, 0, 2, 0)$$

$$a_2 = (0, 1, -2, 0, 0)$$

$$a_3 = (2, 1, 0, 0, 1)$$

$$W = \ell(b_1, b_2, b_3)$$

$$b_1 = (1, 0, -2, +2, 0)$$

$$b_2 = (2, -1, -2, 4, 0)$$

$$b_3 = (1, 2, 1, 0, 1)$$



$$\begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 2 & 0 \\ 2 & -1 & -2 & 4 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{\begin{matrix} \times (-2) \\ + \\ - \end{matrix}} \begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{-}$$

$$\begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \left\{ \begin{array}{l} \text{База на } U+W \text{ образуют} \\ c_1 = (1, -1, 0, 2, 0) = a_1 \\ c_2 = (0, 1, -2, 0, 0) = a_2 \\ c_3 = (2, 1, 0, 0, 1) = a_3 \\ c_4 = (-1, 1, 1, 0, 0) \end{array} \right. \quad \text{NHЗ}$$

$$\begin{cases} x_1 - x_2 + 2x_4 = 0 \\ x_2 - 2x_3 = 0 \\ 2x_1 + x_2 + x_5 = 0 \end{cases} \quad \begin{matrix} v_1 \\ v_2 \end{matrix} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 2 & 2 & 1 & 0 & -6 \\ -2 & 0 & 0 & 1 & 4 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -2 & 2 & 0 \\ 2 & -1 & -2 & 4 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{\begin{matrix} -2y_1 \\ -2y_1 \\ -2y_1 \end{matrix}} \begin{cases} 2y_1 + 2y_2 + y_3 - 6y_5 = 0 \\ -2y_1 + y_4 + 4y_5 = 0 \\ 2y_1 + 2y_2 + y_3 + y_4 + 7y_5 = 0 \\ -2y_1 + y_4 + 2y_5 = 0 \end{cases} \quad b_1, b_2, b_3 - \text{NHЗ}$$

$$\begin{cases} x_1 - 2x_3 + 2x_4 = 0 \\ 2x_1 - x_2 - 2x_3 + 4x_4 = 0 \\ x_1 + 2x_2 + x_3 + x_5 = 0 \end{cases} \quad \begin{matrix} u_2 \\ u_1 \end{matrix} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline -2 & 0 & 0 & 1 & +2 \\ 2 & 2 & 1 & 0 & -7 \end{array}$$



$$\begin{pmatrix} 2 & 2 & 1 & 0 & -6 \\ -2 & 0 & 0 & 1 & 4 \\ 2 & 2 & 1 & 0 & -7 \\ -2 & 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} 2 & 2 & 1 & 0 & -6 \\ -2 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\sim}$$

$$\begin{pmatrix} 2 & 2 & 1 & 0 & -6 \\ -2 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} 2y_1 + 2y_2 + y_3 - 6y_5 = 0 \\ -2y_1 + y_4 + 4y_5 = 0 \\ y_5 = 0 \end{cases}$$

|       | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|-------|-------|-------|-------|-------|
| $d_1$ | 1     | 0     | -2    | 2     |
| $d_2$ | 0     | 1     | -2    | 0     |

Задание на  $U \cap W$  определить

$$\begin{cases} d_1 = (1, 0, -2, 2, 0) \\ d_2 = (0, 1, -2, 0, 0) \end{cases}$$

2)  $U: \begin{cases} x_1 - 3x_2 + 2x_3 - x_4 = 0 \\ 2x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 + 2x_2 + x_3 - 2x_4 = 0 \end{cases} \quad W: \begin{cases} 3x_1 - x_2 - 3x_3 + x_4 = 0 \end{cases}$

D.P.

Задание на  $U+W, U \cap W$

3)  $U: \ell(a_1, a_2)$   
 $a_1 = (1, 1, 2, 2)$   
 $a_2 = (1, -1, 1, -2)$

$W: \begin{cases} x_1 + x_2 + 2x_3 + 2x_4 = 0 \\ x_1 + x_2 - 2x_3 - 2x_4 = 0 \end{cases}$

Задание на  $U+W$   
 ОПР на  $W$ !

$$\begin{cases} x_1 + x_2 + 2x_3 + 2x_4 = 0 \\ x_1 + x_2 - 2x_3 - 2x_4 = 0 \end{cases}$$

$W = \ell(v_1, v_2)$

|       | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|-------|-------|-------|-------|-------|
| $v_1$ | -1    | 1     | 0     | 0     |
| $v_2$ | 0     | 0     | -1    | 1     |



$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & -2 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{1/2} \begin{pmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Stąd na  $U+W$  odpowiadają

$$b_1 = (0, 1, 0, 2)$$

$$b_2 = (-1, 1, 0, 0)$$

$$b_3 = (0, 0, -1, 1)$$

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

Stąd na  $U \cap W$

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + 2x_3 + 2x_4 = 0 \\ x_1 + 2x_3 + 2x_4 = 0 \end{cases}$$

|       | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|-------|-------|-------|-------|-------|
| $u_1$ | -2    | 0     | 1     | 0     |
| $u_2$ | 0     | -2    | 0     | 1     |

$$U: \begin{cases} -2x_1 + x_3 = 0 \\ -2x_2 + x_4 = 0 \end{cases}$$

$$\begin{cases} x_1 + x_2 + 2x_3 + 2x_4 = 0 \\ x_1 + x_2 - 2x_3 + 2x_4 = 0 \\ -2x_1 + x_3 = 0 \\ -2x_2 + x_4 = 0 \end{cases} \sim \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & -2 & -2 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \xrightarrow{\times (-2)} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 1 & 0 \\ -2 & -2 & 1 & 1 \end{pmatrix}$$



$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} \underline{x_1 + x_2} = 0 \\ \underline{x_3 + x_4} = 0 \\ \underline{-2x_1 + x_3} = 0 \end{cases}$$

$$C_1 \begin{bmatrix} 1 & -1 & 2 & -2 \end{bmatrix}$$

Базис на  $U \cap W$   $C_1 = a_1 = (1, -1, 2, -2)$

2. P.  $U = \langle a_1, a_2, a_3 \rangle$   
 $a_1 = (1, 2, 1, 2)$   
 $a_2 = (2, 1, 2, 1)$   
 $a_3 = (1, 2, 3, 4)$

$$W = \begin{cases} x_1 + x_2 + 2x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - 5x_4 = 0 \end{cases}$$

Базис на  $U+W$  и  $U \cap W$

15. 2013г.

$$a_1 = (2, 8, -3, 14)$$

$$a_2 = (-1, 2, 3, 5)$$

$$a_3 = (-1, 14, 6, 29)$$

$$a_4 = (0, 12, 3, 24)$$

$$U = \langle a_1, a_2, a_3, a_4 \rangle$$

$$W = \begin{cases} x_2 + x_3 = 0 \\ 10x_1 + 7x_2 + -8x_4 = 0 \end{cases}$$

? Базис на  $U+W$  и  $U \cap W$

$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline b_1 & 0 & -1 & 1 & -\frac{7}{8} \\ b_2 & 1 & 0 & 0 & \frac{5}{4} \\ \hline & 2 & 8 & -3 & 14 \\ & -1 & 2 & 3 & 5 \\ & -1 & 14 & 6 & 29 \\ & 0 & 12 & 3 & 24 \\ & 0 & -8 & 8 & -7 \\ & 4 & 0 & 0 & 5 \end{array}$$

$$1.8$$

$$1.4$$

$$1.(-2)$$

$$1.(-4)$$

$$1.2$$

$$W = \langle b_1, b_2 \rangle \begin{bmatrix} 0 & -8 & 8 & -7 \\ 4 & 0 & 0 & 5 \end{bmatrix}$$

11.2.88



$$\sim \begin{pmatrix} 0 & 12 & 3 & 24 \\ -1 & 2 & 3 & 5 \\ 0 & 12 & 3 & 24 \\ 0 & 12 & 3 & 24 \\ 0 & -8 & 8 & -7 \\ 0 & 8 & 12 & 25 \end{pmatrix} \xrightarrow{1:3} \sim \begin{pmatrix} 0 & 4 & 1 & 8 \\ -1 & 2 & 3 & 8 \\ 0 & 0 & 20 & 18 \\ 0 & 8 & 12 & 25 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{1:2} \sim$$

$$\sim \begin{pmatrix} 0 & 4 & 1 & 8 \\ -1 & 2 & 3 & 8 \\ 0 & 0 & 10 & 9 \\ 0 & 0 & 10 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rg}(A) = 3$   
 $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$   
 $3 = 2 + 2 - 1$

$a_1, a_2, b_1$  — вектора базиса на  $U+W$

базис на  $U \cap W$

|       | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|-------|-------|-------|-------|-------|
| $z_1$ | -19   | 0     | -8    | 1     |
| $z_2$ | -10   | 1     | -4    | 0     |

$$U: \begin{cases} 4x_2 + x_3 + 8x_4 = 0 \\ -x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \end{cases}$$

$$W: \begin{cases} x_2 + x_3 = 0 \\ 10x_1 + 7x_2 - 8x_4 = 0 \end{cases}$$

$$U: \begin{cases} -19x_1 - 8x_3 + x_4 = 0 \\ -10x_1 + x_2 - 4x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 10 & 7 & 0 & -8 \\ -19 & 0 & -8 & 1 \\ -10 & 1 & -4 & 0 \end{pmatrix} \xrightarrow{1: (-7)} \sim \begin{pmatrix} 0 & 1 & 1 & 0 \\ -14 & 7 & -64 & 0 \\ -19 & 0 & -8 & 1 \\ -10 & 1 & -4 & 0 \end{pmatrix} \sim$$



$$\sim \left( \begin{array}{cccc|c} 10 & 0 & 5 & 0 & 1:5 \\ -72 & 0 & -36 & 0 & 1: (-36) \\ -19 & 0 & -8 & 1 & \\ -10 & 1 & -4 & 0 & \end{array} \right) \sim \left( \begin{array}{cccc|c} 2 & 0 & 1 & 0 & \\ -19 & 0 & 1 & 0 & \\ -19 & 0 & -8 & 1 & \\ -10 & 1 & -4 & 0 & \end{array} \right) \sim$$

$$\sim \left( \begin{array}{cccc|c} 2 & 0 & 2 & 0 & \\ -19 & 0 & -8 & 1 & \\ -10 & 1 & -4 & 0 & \\ 0 & 0 & 0 & 0 & \end{array} \right) \quad \begin{cases} 2x_1 + x_3 = 0 \\ -19x_1 - 2x_3 + x_4 = 0 \\ 10x_1 + x_2 - 4x_3 = 0 \end{cases}$$

$$\begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & -22 & 2 & -3 & 1: (-1) \end{array}$$

1 22 -2 3 → базис на  $U \cap W$

Линейни оператори

$M_1, M_2$  — множества

$\varphi: M_1 \rightarrow M_2$

$\varphi(a) = a'$

$V_1, V_2$  — линеар оператор над  $F$

$a, b \in V, \lambda \in F$

1)  $\varphi(a+b) = \varphi(a) + \varphi(b)$

2)  $\varphi(\lambda a) = \lambda \varphi(a)$

$\varphi: V_1 \rightarrow V_2$

→ линейно изображение

$\varphi: V \rightarrow V \Rightarrow \varphi$  е линеар оператор

$\varphi$  е линеар изображение

$\{e_1, e_2, \dots, e_n\}$  — базис на  $V$

$\varphi(e_1) = \dots$

$\varphi(e_n) = \dots$

11.2.90



$$\ker \varphi = \left( \begin{array}{c} \varphi(e_1) \\ \vdots \\ \varphi(e_n) \end{array} \right)$$

sgpo  $\ker \varphi = \{a \in V, \varphi(a) = \vec{0}\}$   
 od paz  $\operatorname{Im} \varphi = \{b \in V, \exists a \in V, \varphi(a) = b\}$

①  $P_{n+1} = \{f(x) = a_0 x^n + \dots + a_n, a_i \in A\}$

$\deg f \leq n \quad D(f(x)) = f'(x), f(x) \in P_{n+1}$

$\left. \begin{array}{l} e_1 = 1 \\ e_2 = x \\ \vdots \\ e_{n+1} = x^n \end{array} \right\} \text{ bazis na } P_{n+1}$

Dann  $D$  e nirk operator?

Mapyruosa na  $D$  cupemo bazisa?

$\ker D = ? \quad \operatorname{Im} D = ?$

1)  $D(f(x) + g(x)) \stackrel{?}{=} D(f(x)) + D(g(x)) \quad f, g \in P_{n+1}$

$f'(x) + g'(x) = f'(x) + g'(x)$

2)  $D(\lambda f(x)) \stackrel{?}{=} \lambda D(f(x)) \Rightarrow D$  e nirk operator

$D(e_1) = 0 = 0e_1 + 0e_2 + \dots + 0e_{n+1}$

$D(e_2) = 1 = 1e_1 + 0e_2 + \dots + 0e_{n+1}$

$D(e_3) = 2x = 0e_1 + 2e_2 + 0e_3 + \dots + ne_n + 0e_{n+1}$



$$D(e_{n+1}) = n x e^{n-1} = 0 e_1 + \dots + n e_n + 0 e_{n+1}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\text{Ker } D = \{ f(x) \in P_{n+1}, D(f(x)) = 0 \} = \{ a_n \in F \}$$

$$\text{Im } D = \{ f'(x), \forall f(x) \in P_{n+1}, D(f(x)) = f'(x) \} = P_n$$

(2)  $M_2(F)$

$E_{11}, E_{12}, E_{21}, E_{22}$  - базис

$\varphi(A) = A + A^t$  - линейный оператор?

$\text{Ker } \varphi = ?$

$\text{Im } \varphi = ?$

$$1) \varphi: M_2(F) \rightarrow M_2(F)$$

линейность:  $A+B \in M_2(F)$

$$\varphi(A) = A + A^t \in M_2(F)$$

$\lambda \in F$

$$\forall A \in M_2(F)$$

$$1) \varphi(A+B) = \varphi(A) + \varphi(B) ?$$

$$\varphi(A+B) = (A+B) + (A+B)^t = A+B + A^t + B^t =$$

$$= (A+A^t) + (B+B^t)$$

$$2) \varphi(\lambda A) = (\lambda A) + (\lambda A)^t = \lambda A + \lambda A^t = \lambda(A+A^t) = \lambda \varphi(A)$$

11.2.92



$$\varphi(E_{11}) = E_{11} + E_{11}^t = 2E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$$

$$\varphi(E_{12}) = E_{12} + E_{12}^t = E_{12} + E_{21} + 0E_{11} + 0E_{22}$$

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi(E_{21}) = E_{21} + E_{21}^t = 0E_{11} + E_{12} + E_{21} + 0E_{22}$$

$$\varphi(E_{22}) = E_{22} + E_{22}^t = 0E_{11} + 0E_{12} + 0E_{21} + 2E_{22}$$

$$A = \begin{pmatrix} \varphi E_{11} & \varphi E_{12} & \varphi E_{21} & \varphi E_{22} \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\text{Ker } \varphi = \{ A \in M_2(F), \varphi(A) = 0 \} = \{ A \in M_2(F), A + A^t = 0 \}$$

$A = -A^t$  — антисимметричны

$$\text{Im } \varphi = \{ \varphi(A) \in M_2(F), \exists A \in M_2(F), \varphi(A) = A + A^t \}$$

— все симметричные матрицы

③  $\varphi: M_2(F) \rightarrow M_2(F)$   
линейный оператор?

$$\varphi(A) = A + E$$

$$\varphi(A+B) = (A+B) + E$$

$$\varphi(A) + \varphi(B) = A + B + 2E$$



$$(4) \varphi(X) = AXB$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$K = \mathbb{Z}_2(F)$$

$\varphi$ -линейный оператор

матрица на  $\varphi$  в базисе

$$\varphi(X) : M_2(F) \rightarrow M_2(F) \quad x, y \in M_2(F)$$

$$1) \varphi(x+y) = A(x+y) \cdot B = AxB + AYB = \varphi(x) + \varphi(y)$$

$$2) \varphi(\lambda x) = A(\lambda x)B = A\lambda xB = \lambda AxB \Rightarrow \varphi - \text{линейный оператор}$$

$$\varphi(E_{11}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} =$$

$$= 1E_{11} + 1E_{12} + 3E_{21} + 3E_{22}$$

$$\varphi(E_{12}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 6 & -3 \end{pmatrix} =$$

$$= 2E_{11} - 1E_{12} + 6E_{21} - 3E_{22}$$

$$\varphi(E_{21}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} =$$

$$= 2E_{11} + 2E_{12} + 4E_{21} + 4E_{22}$$

$$\varphi(E_{22}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} =$$

$$= 4E_{11} - 2E_{12} - 8E_{21} - 4E_{22}$$

$$\Rightarrow A_\varphi = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & -1 & 2 & -2 \\ 3 & 6 & 4 & 8 \\ 3 & -3 & 4 & -4 \end{pmatrix}$$



⑤  $e_1, e_2$  - базис на  $V$   
 $\varphi \in \text{Hom } V$

$\varphi(\{1\}e_1, \{2\}e_2) = \{1\}e_1 + (-\{1\} + 2\{2\})e_2$   
 Матрица на  $\varphi$  в базиса  $e_1' = e_1 + e_2$   
 $e_2' = -2e_1 - e_2$

$V = 2e_1' + 3e_2'$   
 $\{1\}, \{2\} \in F$   
 $\varphi(V)_{e_1'} = ?$

1) Нека  $\{1\} \geq 1, \{2\} \geq 0$   
 $\varphi(e_1) = e_1 - e_2$

$$A_e = \begin{pmatrix} \varphi(e_1) & \varphi(e_2) \\ 1 & 0 \\ -1 & 2 \end{pmatrix}$$

2)  $\{1\} \geq 0, \{2\} \geq 1$   
 $\varphi(e_2) = 2e_2$

$$A_e = \begin{pmatrix} \varphi(e_1') & \varphi(e_2') \\ 1 & -2 \\ 1 & 0 \end{pmatrix}$$

$$\varphi(e_1) = \varphi(e_1 + e_2) = \varphi(e_1) + \varphi(e_2) = e_1 + e_2$$

$$\varphi(e_2') = \varphi(-2e_1 - e_2) = -2\varphi(e_1) - \varphi(e_2) = -2(e_1 - e_2) - 2e_2 = -2e_1 + 2e_2 - 2e_2 = -2e_1$$

$$T_{e \rightarrow e'} = \begin{pmatrix} e_1' & e_2' \\ 1 & -2 \\ 1 & 0 \end{pmatrix}$$

$$T'_{e \rightarrow e'} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

$$A_{e'} = T^{-1} A_e T_{e \rightarrow e'}$$

$$\varphi(V_{e'}) = A_{e'} \cdot V_{e'} = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

$$\varphi(V) = -4e_1' + 2e_2'$$



⑥  $e_1, e_2$  - базис на  $V$   
 $\varphi \in \text{Hom } V$

$$A = \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix} \text{ в базиса } \left| \begin{array}{l} a_1 = -3e_1 + 7e_2 \\ a_2 = e_1 - 2e_2 \end{array} \right.$$

$\psi \in \text{Hom } V$

$$B = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \text{ в базиса } \left| \begin{array}{l} b_1 = 6e_1 - 7e_2 \\ b_2 = -5e_1 + 6e_2 \end{array} \right.$$

$\varphi, \psi$

← отсюда можно в стандартный базис  $e_1, e_2$

$$Te \rightarrow a = \begin{pmatrix} a_1 & a_2 \\ -3 & 1 \\ 7 & -2 \end{pmatrix} \quad Ue \rightarrow b = \begin{pmatrix} b_1 & b_2 \\ 6 & -5 \\ -7 & 6 \end{pmatrix}$$

$$\varphi \rightarrow Ae = Te \rightarrow a \cdot A \cdot T^{-1} \quad (\text{умножаемое отсюда с } T, a \text{ отсюда с } T^{-1})$$

$$\psi \rightarrow Be = Ue \rightarrow b \cdot B \cdot U^{-1}$$

$\varphi, \psi$

$$\varphi(\psi(v)) = \varphi(\psi(v))$$

$$Ae = Be = \dots ?$$



21.1.2013. а)  $\pi^2 = \pi$   $\pi \in \text{Hom } V$   
 $\exists V_1$  и  $V_2$  - подпространства на  $V$ ,  $V = V_1 \oplus V_2$  и  
 $\pi(v) = v$ ,  $\forall v \in V_1$   
 $\pi(v) = 0$ ,  $\forall v \in V_2$   $\pi$  - проектор

$v_1, v_2, \dots, v_k$  - базис на  $V_1$   
 $v_{k+1}, \dots, v_n$  - базис на  $V_2$

Матрицата на  $\pi$  в базиса на  $V$   
 $\pi^2(v) = \pi(v)$   $\mathcal{E}(v) = v$  за  $\forall v \in V$   
 $\pi(\pi - \mathcal{E})(v) = \vec{0}$   $\mathcal{E}$  - идентитет

- 1)  $(\pi - \mathcal{E})(v) = \vec{0} \rightarrow v_1$ ,  $\forall v \in V \Leftrightarrow \pi = \mathcal{E} \Rightarrow \pi(v) = v$
- 2)  $(\pi - \mathcal{E})(v) = w \in V \rightarrow V_2$ ,  $w \neq 0$   
 $\pi(w) = \vec{0} \Rightarrow w \in \text{Ker } \pi = \{v \in V \mid \pi(v) = \vec{0}\}$

$1 \Rightarrow v \in \text{Im } \pi = \{\pi(v) \in V \mid \exists v \in V : \pi(v) = v\}$   
 $\forall v \in V : v = \pi(v) + (v - \pi(v)) \Rightarrow v = v_1 + v_2$   
 $\in V_1$   $\in V_2$

$$V_1 \cap V_2 = \{0\} \Rightarrow V_1 \oplus V_2 = V$$

$$v = \underbrace{\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k}_{V_1} + \underbrace{\lambda_{k+1} v_{k+1} + \dots + \lambda_n v_n}_{V_2}$$

$$\varphi(v) = \lambda_1 \varphi(v_1) + \lambda_2 \varphi(v_2) + \dots + \lambda_k \varphi(v_k) + \dots + \lambda_{k+1} \varphi(v_{k+1}) + \dots + \lambda_n \varphi(v_n) = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k$$

$\vec{0}$



$$A = \left( \begin{array}{c|c} V_1 & V_2 \\ \hline E_k & 0 \\ \hline 0 & 0_{n-k} \end{array} \right)$$

8)  $\eta \in \text{Hom } V$

$\eta$  - отражение, инволюция

$$\eta^2 = E$$

$$V = V_1 \oplus V_2$$

$$\eta(v) = v, \quad \forall v \in V_1$$

$$\eta(v) = -v, \quad \forall v \in V_2$$

$$(\eta - E)(\eta + E)(v) = 0$$

9)  $e_1, e_2, \dots, e_n$  - базис на  $V$

$a_1, a_2, \dots, a_n$  - базис на  $V$

$b_1, b_2, \dots, b_n$  - произвольные векторы от  $V$

$$a_1 = a_{11}e_1 + \dots + a_{1n}e_n$$

$$b_1 = b_{11}e_1 + \dots + b_{1n}e_n$$

$$a_i = a_{i1}e_1 + \dots + a_{in}e_n$$

$$b_i = b_{i1}e_1 + \dots + b_{in}e_n$$

$$A = (a_{ij})_{n \times n}$$

$$B = (b_{ij})_{n \times n}$$

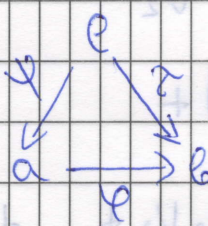
$\varphi \in \text{Hom } V$

$$\varphi(a_i) = b_i$$

$$C = B A^{-1}$$

$$i = 1, \dots, n$$

- матрица на  $\varphi$  в базисах  $\{e_1, \dots, e_n\}$



$$\varphi(e_i) = a_i \Leftrightarrow A$$

$$\tau(e_i) = b_i \Leftrightarrow B$$

$$\varphi \psi = \tau$$



$$CA = B \quad (\text{отгъщо } A^{-1})$$

$$C = BA^{-1}$$

$e_1, e_2, e_3$  - базис на  $V$      $\varphi \in \text{Hom } V$      $\varphi(a_i) = b_i, i=1, 2, 3$

$A = ?$  - матрицата на  $\varphi$  в базис  $\{e_1, e_2, e_3\}$

$$a_1 = e_1 + 2e_2 + e_3$$

$$b_1 = 4e_1 + 2e_2 + 5e_3$$

$$a_2 = 2e_1 + e_3$$

$$b_2 = e_1 + e_2$$

$$a_3 = e_1 + e_3$$

$$b_3 = e_3$$

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow a_1, a_2, a_3 \text{ - базис на } V$$

$$\Rightarrow \exists A^{-1}$$

$$B = \begin{pmatrix} b_1 & b_2 & b_3 \\ 4 & 1 & 0 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix} \quad A' \rightarrow \left( \begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\substack{R_1 \leftarrow R_1 - 2R_3 \\ R_2 \leftarrow R_2 - 2R_3}} \sim$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & 0 & 0 \end{array} \right) \xrightarrow{\substack{R_1 \leftarrow R_1 - \frac{1}{2}R_2 \\ R_3 \leftarrow R_3 - R_2}} \sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} & 2 \end{array} \right) \sim$$

$$\sim \left( \begin{array}{ccc|ccc} & 0 & \frac{1}{2} & 1 \\ E & 1 & 0 & -1 \\ & -1 & -\frac{1}{2} & 2 \end{array} \right)$$



$$\frac{1}{2} \begin{pmatrix} 4 & 1 & 0 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -2 \\ -2 & 1 & 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 4 & -2 \\ 2 & 2 & -2 \\ -2 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ -1 & 2 & 2 \end{pmatrix}$$

$e_1, e_2, e_3, e_4 \rightarrow$  базис на  $V$   
 $\varphi \in \text{Hom } V$

$$A = \begin{pmatrix} -1 & 2 & -3 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & -2 & -2 & -1 \end{pmatrix} \quad \begin{array}{l} \text{базис на } \text{Ker } \varphi, \text{Im } \varphi \\ \text{Ker } \varphi \cap \text{Im } \varphi \\ \text{Ker } \varphi + \text{Im } \varphi \end{array}$$

1)  $\text{Ker } \varphi \quad v \in V \quad Av = \vec{0}$   
 $v = (x_1, x_2, x_3, x_4)$

$$\begin{cases} -x_1 - 2x_2 - 3x_3 - 2x_4 = 0 \\ x_3 + x_4 = 0 \\ x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ -x_1 - 2x_2 - 2x_3 - x_4 = 0 \end{cases} \quad A = \begin{pmatrix} -1 & -2 & -3 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & -2 & -2 & -1 \end{pmatrix} \rightsquigarrow$$

$$\sim \begin{pmatrix} -1 & -2 & -3 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & -2 & -3 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow$$

$$\sim \begin{pmatrix} -1 & -2 & -3 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline a_1 & -2 & 1 & 0 & 0 \\ \hline a_2 & 1 & 0 & -1 & 1 \\ \hline \end{array}$$



2)  $\text{Im } \varphi$

$$v_1 = \varphi(e_1), \quad v_2 = \varphi(e_2), \quad v_3 = \varphi(e_3), \quad v_4 = \varphi(e_4)$$

$$l(v_1, v_2, v_3, v_4)$$

$$\begin{pmatrix} -1 & 0 & 1 & -1 \\ -2 & 0 & 2 & -2 \\ -3 & 1 & 2 & -2 \\ -2 & 1 & 1 & -1 \end{pmatrix} \xrightarrow{\substack{1. (-2) \\ 2. (+) \\ 3. (+)}} \sim \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{4. (-)}$$

$$\sim \begin{pmatrix} -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Jazue ka } \text{Im } \varphi$$

$$v_1 = (-1, 0, 1, -1)$$

$$v_3 = (-1, 1, 0, 0)$$

$$\varphi \rightarrow A = \begin{pmatrix} 1 & -5 & 2 & 1 \\ 2 & -3 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & -9 & 3 & 0 \end{pmatrix}$$

$$\varphi \rightarrow B = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 3 & 5 \\ 2 & 0 & -1 & 11 \end{pmatrix}$$

$$\text{Ker } \varphi \cap \text{Im } \varphi, \quad \text{Ker } \varphi + \text{Im } \varphi$$



22.1.2013г.

# Собствени вектори. Собствени стойности на линейен оператор

$f_A(\lambda) = \det(A - \lambda E)$  - характеристичен полином на  $A$   
(характеристични корени)

$A$  - матрица на оператора в базиса на  $V$   
 $A = (a_{ij})_{n \times n}$

$$v \in V \quad \varphi \in \text{Hom } V \quad \varphi(v) = \lambda v, \quad \lambda \in F$$

$$(\varphi - \lambda E)(v) = 0$$

$v$  е собствен вектор на  $\varphi$ ,  
 $\lambda$  е съответната собствена стойност

① характеристичните корени на  $A$

$$A = \begin{pmatrix} 1 & -1 & 5 \\ 1 & 1 & 8 \\ 0 & 0 & 3 \end{pmatrix}$$

$$f_A(\lambda) = \det(A - \lambda E) = \begin{vmatrix} 1-\lambda & -1 & 5 \\ 1 & 1-\lambda & 8 \\ 0 & 0 & 3-\lambda \end{vmatrix} =$$

$$= (3-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda) [(1-\lambda)^2 + 1] = -\lambda^3 + 5\lambda^2 - 8\lambda + 60$$

$$\lambda_1 = 3, \quad \lambda_{2,3} = 1 \pm i$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = (-1)^n \lambda + (-1)^{n-1} \text{tr } A \cdot \lambda^{n-1} + \dots + \det A$$

|    |    |    |    |
|----|----|----|----|
| -1 | 5  | -8 | 6  |
| 3  | -1 | 2  | -2 |
| 0  | 0  | 3  | -1 |

11.2.102



$$\textcircled{2} A = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \quad \chi_A(\lambda) = \begin{vmatrix} \cos \alpha - \lambda & \sin \alpha \\ \sin \alpha & \cos \alpha - \lambda \end{vmatrix} =$$

$$= \lambda^2 - 2\lambda \cos \alpha + 1 = 0$$

$$D = \cos^2 \alpha - 1 = \sin^2 \alpha$$

$$\lambda_{1,2} = \cos \alpha \pm \sin \alpha$$

$\textcircled{3} \mathbb{R}^3$  - none,  $V = \mathbb{R}^3$  - лине. пространство  
 $e_1, e_2, e_3$  - базис  
 $\varphi \in \text{Hom } V$   $d_1, d_2, d_3 \in \mathbb{R}$

$$\varphi(d_1 e_1 + d_2 e_2 + d_3 e_3) = (-6d_1 - 8d_2 + 16d_3)e_1 +$$

$$+ (-4d_1 - 2d_2 + 8d_3)e_2 + (-4d_1 - 4d_2 + 10d_3)e_3$$

а)  $A$  - матрица к  $\varphi$ , характеристич. корни к  $A$

б) базис к  $V$  от собственных векторов к  $\varphi$  и матрица к  $T^{-1}AT = D$  - диагональ

в)  $A^{99} = ?$

$$\text{а) } d_1 = 1, d_2 = d_3 = 0$$

$$\varphi(e_1) = -6e_1 + 4e_2 - 3e_3$$

$$A = \begin{pmatrix} -6 & -8 & 16 \\ -4 & -2 & 8 \\ -3 & -4 & 10 \end{pmatrix}$$

$$d_1 = d_3 = 0, d_2 = 1$$

$$\varphi(e_2) = -8e_1 - 2e_2 - 4e_3$$

$$d_1 = d_2 = 0, d_3 = 1$$

$$\varphi(e_3) = 16e_1 + 8e_2 + 10e_3$$



$$d) \det A(\lambda) = \begin{vmatrix} -6-\lambda & -8 & 16 \\ -4 & -2-\lambda & 8 \\ -4 & -4 & 10-\lambda \end{vmatrix} = (6+\lambda)(2+\lambda)(10-\lambda) + 512 -$$

$$-64(2+\lambda) - 32(6+\lambda) - 32(10-\lambda) =$$

$$= -\lambda^3 + 2\lambda^2 + 68\lambda + 120 + 512 - 128 - 64\lambda - 192 - 32\lambda - 320 + 32\lambda = -\lambda^3 + 2\lambda^2 + 68\lambda + 8 = 0$$

$$-\lambda^2(\lambda - 2) + 4(\lambda - 2) = (\lambda - 2)^2(2 - \lambda)(\lambda + 2) = 0$$

$$\lambda_{1,2} = 2 \quad \lambda_3 = -2$$

$$\varphi(v) = \lambda v \quad (\varphi - \lambda E)(v) = \vec{0}$$

$$1) \lambda_{1,2} = 2$$

$$\begin{pmatrix} -6-2 & -8 & 16 \\ -4 & -2-2 & 8 \\ -4 & -4 & 10-2 \end{pmatrix} \sim \begin{pmatrix} -8 & -8 & 16 \\ -4 & -4 & 8 \\ -4 & -4 & 8 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & -4 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_1 + x_2 - 2x_3 = 0$$

|       | $x_1$ | $x_2$ | $x_3$ |
|-------|-------|-------|-------|
| $v_1$ | -1    | 1     | 0     |
| $v_2$ | 2     | 0     | 1     |

$$2) \lambda_3 = -2$$

$$\begin{pmatrix} -6+2 & -8 & 16 \\ -4 & -2+2 & 8 \\ -4 & -4 & 10+2 \end{pmatrix} \sim \begin{pmatrix} -4 & -8 & 16 \\ -4 & 0 & 8 \\ -4 & -4 & 12 \end{pmatrix} \begin{matrix} /:-4 \\ /:-4 \\ /:-4 \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & -4 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \sim \begin{pmatrix} 1 & 2 & -4 \\ 0 & 8 & -8 \\ 0 & -4 & 7 \end{pmatrix} \begin{matrix} /:-8 \\ /:-8 \end{matrix} \sim \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\begin{cases} x_1 + 2x_2 - 4x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$

$$[v_3] \begin{matrix} x_1 & x_2 & x_3 \\ 2 & 1 & 1 \end{matrix}$$

$$T = \begin{pmatrix} v_1 & v_2 & v_3 \\ -1 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad T^{-1}AT = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$b) \quad T^{-1}AT = D$$

$$A = TDT^{-1}$$

$$A^{99} = \underbrace{TDT^{-1}TDT^{-1} \dots TDT^{-1}}_{99 \text{ mal}} = TDT^{-1} = 2^{98} \underbrace{T \cdot D \cdot T^{-1}}_A$$

$$D^{99} = 2^{98} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 2^{98} D$$

$$A^{99} = 2^{98} A$$

$$④ \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 1 & 0 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = (1-\lambda)^2 \lambda^2$$

$$\lambda_{1,2} = 1$$

$$\lambda_{3,4} = 0$$

$$1) \lambda_{1,2} = 1$$

$$(A - E) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \quad v = (0, 0, 0, 1)$$

~~2)  $\lambda_{3,4}$~~ 

$$2) \lambda_{3,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} x_1 = 0 & v_3 = (0, 1, 0, 0) \\ x_4 = 0 & v_4 = (0, 0, 1, 0) \end{matrix}$$

$$\textcircled{5} \quad \varphi(\beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3) = (2\beta_1 - 2\beta_2 - 4\beta_3)e_1 + (-2\beta_1 + 5\beta_2 - 2\beta_3)e_2 + (-4\beta_1 - 2\beta_2 + 2\beta_3)e_3$$

$e_1, e_2, e_3$  - базис на  $V$

$\varphi \in \text{Hom } V$

базис на  $V$ , в който матрицата на  $\varphi$  е диагонална.

$$A = \begin{pmatrix} e_1 & e_2 & e_3 \\ 2 & -2 & -4 \\ -2 & 5 & -2 \\ -4 & -2 & 2 \end{pmatrix} \quad \det A(\lambda) = \begin{vmatrix} 2-\lambda & -2 & -4 \\ -2 & 5-\lambda & -2 \\ -4 & -2 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda)^2(5-\lambda) - 16 - 16 - 16(5-\lambda) - 4(2-\lambda) - 4(2-\lambda) =$$

$$= (2-\lambda)^2(5-\lambda) - 32 - 16(5-\lambda) - 8(2-\lambda) =$$

$$= (2-\lambda)((2-\lambda)(5-\lambda) - 8) - 16(2+5-\lambda) =$$

$$= (2-\lambda)(\lambda^2 - 7\lambda + 2) - 16(-\lambda + 7) =$$

$$= -\lambda^3 + 7\lambda^2 - 2\lambda + 2\lambda^2 + 14\lambda + 4 + 16\lambda - 112 = -\lambda^3 + 9\lambda^2 + 22\lambda - 108$$

11.2.106



$$\lambda_{1,2} = 6 \quad \lambda_3 = -3$$

$$1) \lambda_3 = -3$$

$$\begin{pmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -2 & -4 \\ 18 & 0 & -18 \\ -9 & 0 & 9 \end{pmatrix}$$

$$\begin{cases} 5x_1 - 2x_2 - 4x_3 = 0 \\ x_1 - x_3 = 0 \end{cases}$$

$$\begin{array}{c|c|c|c} & x_1 & x_2 & x_3 \\ \hline v_3 & 2 & 1 & 2 \end{array}$$

$$2) \lambda_{1,2} = 6$$

$$\begin{pmatrix} -4 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -4 \end{pmatrix} =$$

$$2x_1 + x_2 + 2x_3 = 0$$

$$\begin{array}{c|c|c|c} & x_1 & x_2 & x_3 \\ \hline v_1 & 0 & -2 & 1 \\ v_2 & 1 & -2 & 0 \end{array}$$

$$T = \begin{pmatrix} v_3 & v_1 & v_2 \\ 2 & 0 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$(6) \quad A = \begin{pmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \dots & -a_1 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$



$$f_A(\lambda) = \begin{vmatrix} -a_{n-1}-\lambda & -a_{n-2} & -a_{n-3} & \dots & -a_1 & -a_0 \\ 1 & -\lambda & 0 & \dots & 0 & 0 \\ 0 & 1 & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\lambda \\ 0 & 0 & 0 & \dots & 0 & 1 & -\lambda \end{vmatrix}$$

Diagram showing the expansion of the determinant along the first row, with arrows indicating the signs and the resulting terms:  $+ \lambda^n$ ,  $+ \lambda^{n-1}$ ,  $+ \lambda^{n-2}$ , ...,  $+ \lambda$ ,  $+ a_0$ .

$$= \begin{vmatrix} -a_{n-1}-\lambda & -a_{n-2} & -a_{n-3} & \dots & a_1 & a_0 \\ 1 & -\lambda & 0 & \dots & 0 & 0 \\ 0 & 1 & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

$$= (-1)^{n+1} (-1)^n (\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0)$$

$$\cancel{S} \quad S = -(a_0 + a_1\lambda + a_2\lambda^2 + \dots + (a_{n-1})\lambda^{n-1})$$



$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -y_1 & x_1 & 0 & \dots & 0 & 0 \\ 0 & -y_2 & x_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & -y_n & x_n \end{pmatrix} =$$

$\frac{x_1}{y_1} \dots \frac{x_n}{y_n}$   
 $\frac{x_n x_{n-1}}{y_{n-1} y_n}$   
 $\frac{x_1}{y_1} \dots \frac{x_n}{y_n}$

$$= \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} & S \\ -y_1 & x_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -y_2 & x_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & -y_n & 0 \end{vmatrix} = (-1)^{n+1} S (-1)^n y_1 \dots y_n =$$

$$= (-1) \left( a_n + a_{n-1} \frac{x_n}{y_n} + \dots + a_1 \frac{x_2 x_3 \dots x_n}{y_2 y_3 \dots y_n} + a_0 \frac{x_1 \dots x_n}{y_1 \dots y_n} \right) \times (y_1 y_2 \dots y_n) = (-1) (a_0 x_1 x_2 \dots x_n + a_1 y_1 x_2 \dots x_n + \dots + a_{n-1} y_1 y_2 \dots y_{n-1} x_n + a_n y_1 \dots y_n)$$