26.02.2014г. КА - упражнение 1

Комплексни числа

$\mathbb{C}$ - множеството на комплексните числа.

$z \in \mathbb{C} \Rightarrow z = x + iy$ - алгебричен вид на z

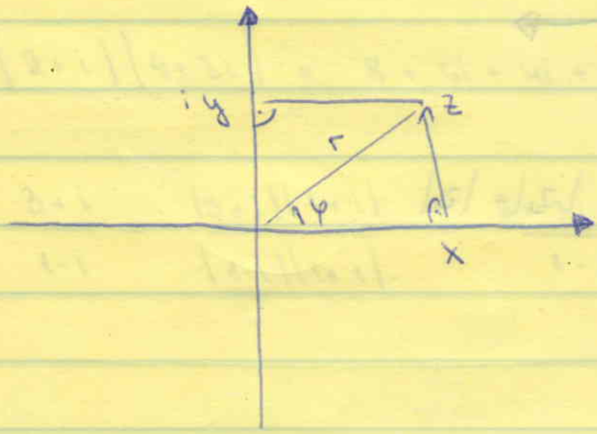
$x, y \in \mathbb{R}$, $i \in \mathbb{C}$, $i^2 = -1$

$x = \mathrm{Re}\, z$ - реална част на z

$y = \mathrm{Im}\, z$ - имагинерна част на z

Между $\mathbb{C}$ и мн. от точки в равнината $\exists$ взаимно еднозначно съответствие.

$z = x + iy \longleftrightarrow (x, y)$



$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ $\quad z = r(\cos\varphi + i\sin\varphi)$ (тригонометричен вид на z)

$r = |z|$ модул на z

$\varphi + 2k\pi = \arg z$ - аргумент на z

A.A.

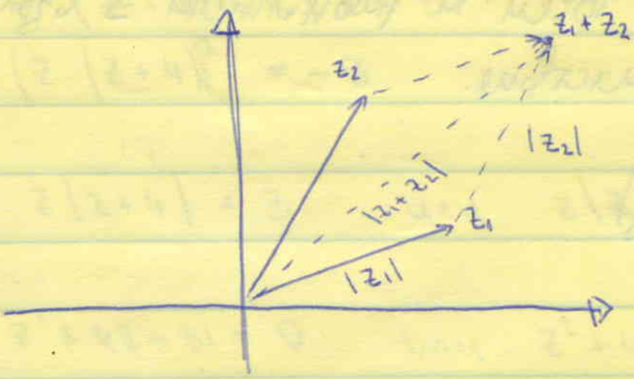
$e^{i\varphi} \overset{def}{=} \cos\varphi + i\sin\varphi$

$z = r e^{i\varphi}$

$e^{i(\varphi_1+\varphi_2)} = e^{i\varphi_1} \cdot e^{i\varphi_2}$

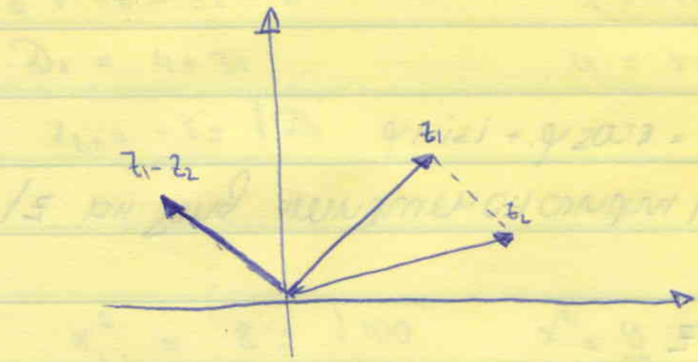
$e^{i(\varphi_1-\varphi_2)} = \dfrac{e^{i\varphi_1}}{e^{i\varphi_2}}$ за $\varphi_1, \varphi_2 \in \mathbb{R}$

$z_1 = r_1 e^{i\varphi_1}$
$z_2 = r_2 e^{i\varphi_2}$ $\Rightarrow$ $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\varphi_1+\varphi_2)}$
$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \cdot e^{i(\varphi_1-\varphi_2)}$



$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2| \le |z_1| + |z_2|$

неравенство на Δ



$z = x + iy$

$\bar{z} = x - iy$ - спрегнато на z

$z + \bar{z} = 2x$ $\qquad$ $z \cdot \bar{z} = |z|^2 = x^2 + y^2$

$$\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$$

$$\overline{z_1 . z_2} = \overline{z}_1 . \overline{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$$

$$|z| = \sqrt{x^2 + y^2}$$

задача:

$\boxed{1.1}$

Да се запише в алгебричен вид числото:

а) $(2+i)(4+3i)$

б) $\frac{(3+i)}{1-i}$

$$(2+i)(4+3i) = 8 + 6i + 4i + 3i^2 = 5 + 10i$$

$$\frac{3+i}{1-i} = \frac{(3+i)(1+i)}{(1-i)(1+i)} = \frac{3+3i+i+i^2}{1-i^2} = \frac{2+4i}{2} = 1+2i$$

задача 1.2

Д $a = 2+i$, $b = 1-2i$. Да се пресметне $a^2 + b^2$, $|b|$, $a\overline{b}$, $\frac{a}{b}$

$$a^2 + b^2 = (2+i)^2 + (1-2i)^2 = 4 + \cancel{4i} + i^2 + 1 - 4i + \cancel{4}i^2 = 0 - 4i$$

$$|b| = \sqrt{1+4} = \sqrt{5}$$

$a\bar{b} = (2+i)(1+2i) = 2+5i-2 = 5i$

$\frac{a}{b} = \frac{2+i}{1-2i} \cdot = \frac{a\bar{b}}{b\bar{b}} = \frac{5i}{5} = i$

Задача 1.3

Да се запише в тригонометричен вид числото

а) -1

б) $-i$

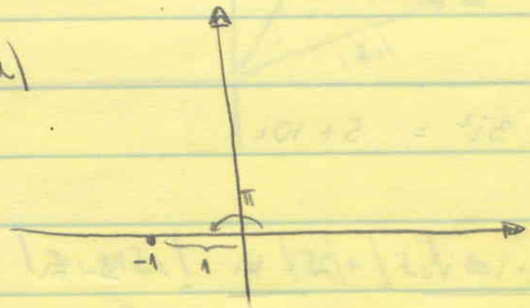
в) $1+i$

г) $\sqrt{3}-i$

д) $-\sqrt{2}-i\sqrt{2}$

ж) $z = 1-\cos\alpha + i\sin\alpha,\ \alpha \in \mathbb{R}$

а)

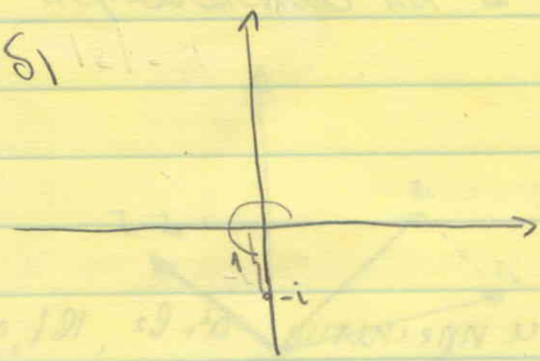


$|-1| = 1$

$\arg(-1) = \pi$

$-1 = 1 . e^{i\pi}$

б)

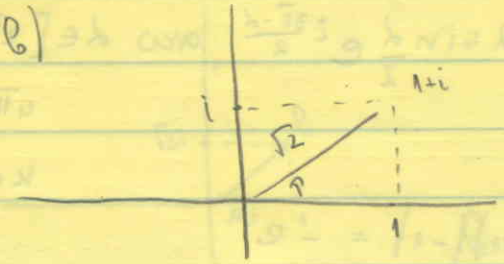


$|-i| = 1$

$\arg(-i) = -\frac{\pi}{2}$

$-i = 1 . e^{i(-\frac{\pi}{2})}$

б)

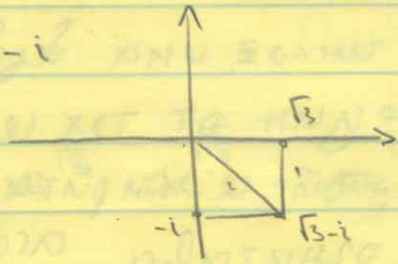


$|1+i| = \sqrt{2}$

$\arg(1+i) = \dfrac{\pi}{4}$

$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

г) $\sqrt{3} - i$

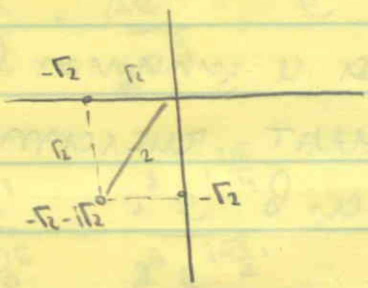


$|\sqrt{3} - i| = 2$

$\arg(1+i) = \dfrac{\pi}{6}$

$\sqrt{3} - i = 2e^{i\frac{\pi}{6}}$

g) $-\sqrt{2} - i\sqrt{2}$



$|-\sqrt{2} + i\sqrt{2}| = \sqrt{2+2} = 2$

$\arg(-\sqrt{2} - i\sqrt{2}) = -\dfrac{3\pi}{4}$

$-\sqrt{2} + i\sqrt{2} = 2e^{i\left(-\frac{3\pi}{4}\right)}$

ж) $1 - \cos\alpha + i\sin\alpha =$

$= 2\sin^2\frac{\alpha}{2} + i2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} =$

$= 2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2} + i\cos\frac{\alpha}{2}\right) =$

$= 2\sin\frac{\alpha}{2}\left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right] =$

$= 2\sin\frac{\alpha}{2}\, e^{i\frac{\pi-\alpha}{2}}$ $\qquad \sin\frac{\alpha}{2} \begin{cases} \geq 0, & \text{ако } \alpha \in [4k\pi, 2\pi + 4k\pi] \\ \leq 0, & \text{ако } \alpha \in [2\pi + 4k\pi, 4\pi + 4k\pi] \end{cases}$

$k \in \mathbb{Z}$

триг. вид на $z = \begin{cases} 2\sin\frac{\alpha}{2}\, e^{i\frac{\pi-\alpha}{2}}, & \text{ако } \alpha \in [4k\pi, 2\pi+4k\pi] \\ -2\sin\frac{\alpha}{2}\, e^{i\frac{3\pi-\alpha}{2}}, & \text{ако } \alpha \in [2\pi+4k\pi, 4\pi+4k\pi] \end{cases}$ (1)

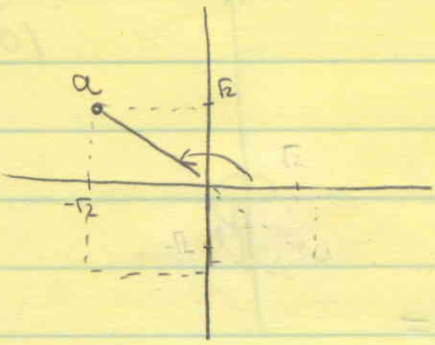
$k \in \mathbb{Z}$

(1) използвахме, че $1 = (-1)(-1) = -e^{i\pi}$

Всяко $\neq 0$ комплексно число z има безброй м-н. аргументи, но само един от тях се намира в интервала $[-\pi, \pi]$, той се нарича главен аргумент на z и се означава $\arg_0 z$ $[-\pi, \pi]$

задача 1.4

Нека $a = -\sqrt{2} + i\sqrt{2}$, $b = 1 + i\sqrt{3}$, да се запишат в тригоном. вид с главен аргумент числата $a, b, \bar{a}, \bar{b}, ab, \frac{a}{b}, \frac{a^7}{b^6}$ и $a+b$



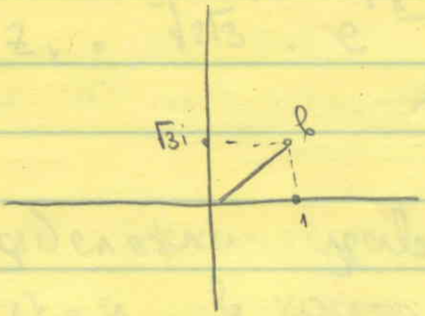
$|-\sqrt{2} + i\sqrt{2}| = 2$

$\arg_0 a = \frac{3\pi}{4}$

$(-\sqrt{2} + i\sqrt{2}) = 2e^{i\frac{3\pi}{4}}$

$(-\sqrt{2} - i\sqrt{2}) = 2e^{i(-\frac{3\pi}{4})}$

спрегнатите числа имат = модули и противоп. аргументи



$$|1+\sqrt{3}i| = 2$$

$$\arg_0 b = \frac{\pi}{3}$$

$$b = (1+\sqrt{3}i) = 2e^{i\frac{\pi}{3}}$$

$$\overline{b} = (1-\sqrt{3}i) = 2e^{-i\frac{\pi}{3}}$$

$\rightsquigarrow$

$$ab = 2e^{i\frac{3\pi}{4}} \cdot 2e^{i\frac{\pi}{3}} = 4e^{i\frac{13\pi}{12}} = 4e^{i\left(\frac{13\pi}{12}-2\pi\right)} = 4e^{-i\frac{11\pi}{12}}$$

$$\frac{13\pi}{12} = \pi + \frac{\pi}{12} \qquad e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$e^{i(\varphi+2\pi)} = e^{i\varphi}$$

$\rightsquigarrow$

$$\frac{a}{b} = \frac{\cancel{2}e^{i\frac{3\pi}{4}}}{\cancel{2}e^{i\frac{\pi}{3}}} = e^{i\frac{5\pi}{12}}$$

$\rightsquigarrow$

$$\frac{a^7}{b^6} = \frac{2^{\cancel{7}}e^{i\frac{21\pi}{4}}}{\cancel{2^6}e^{i\frac{18\pi}{3}}} = 2e^{i\frac{13\pi}{4}} = 2e^{i\left(\frac{13\pi}{4}-4\pi\right)} = 2e^{-i\frac{3\pi}{4}}$$

$$\frac{13\pi}{4} = \frac{12\pi}{4} + \frac{\pi}{4} = 3\pi + \frac{\pi}{4}$$

$\rightsquigarrow$ ако компл. числа имат $=|\ |$ те а + и - лесно

$$a+b = 2e^{i\frac{3\pi}{4}} + 2e^{i\frac{\pi}{3}} = 2e^{i\left(\frac{\frac{3\pi}{4}+\frac{\pi}{3}}{2}\right)}\left(e^{i\left(\frac{\frac{3\pi}{4}-\frac{\pi}{3}}{2}\right)} + e^{-i\left(\frac{\frac{3\pi}{4}-\frac{\pi}{3}}{2}\right)}\right) =$$

$$= 2e^{i\frac{13\pi}{24}}\left(e^{i\frac{5\pi}{24}} + e^{-\frac{5\pi}{24}}\right) \overset{(*)}{=} 2e^{i\frac{13\pi}{24}}\, 2\cos\frac{5\pi}{24} =$$

$$= \underbrace{4\cos\frac{5\pi}{24}}_{>0}\, e^{i\frac{13\pi}{24}}$$

(*) $e^{i\varphi} = \cos\varphi + i\sin\varphi \quad (1)$

$e^{-i\varphi} = \cos\varphi - i\sin\varphi \quad (2) \quad \varphi \in \mathbb{R}$

$(1) + (2) \Rightarrow \cos\varphi = \dfrac{e^{i\varphi} + e^{-i\varphi}}{2}$

$(1) - (2) \Rightarrow \sin\varphi = \dfrac{e^{i\varphi} - e^{-i\varphi}}{2i}$

Формула на Моавър:

$z^n = a, \quad n \in \mathbb{N}, \quad a \in \mathbb{C}$

$a = 0 \Rightarrow z = 0$

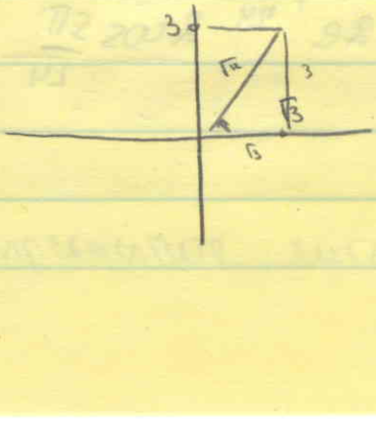
$a \neq 0 \Rightarrow a = re^{i\varphi}$ и тогава решенията на (1) са числата $z_k = \sqrt[n]{r}\, e^{i\frac{\varphi + 2k\pi}{n}}$, $k = 0, 1, 2, \dots, n-1$

С формулата на Моавър се изважда $n^{\text{ти}}$ корен на a.

задача 1.5

е) да се реши уравнението $z^3 = \sqrt{3} + 3i$

$|\sqrt{3} + 3i| = \sqrt{12} = 2\sqrt{3}$



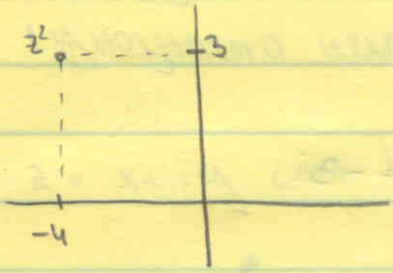
$\sin\alpha = \dfrac{3}{\sqrt{12}} = \dfrac{\cancel{3}\sqrt{12}}{\cancel{12}4} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \Rightarrow \alpha = \dfrac{\pi}{3}$

$\arg(\sqrt{3} + 3i) = \dfrac{\pi}{3}$

$$z_k = \sqrt[3]{2\sqrt{3}} \cdot e^{i\frac{\frac{\pi}{3}+2k\pi}{3}}, \quad k = 0, 1, 2;$$

забележка: уравненията от вида $z^2 = a$ или $z^3 = a$, в които a няма хубав тригоном. вид е по-добре да се решават без да се използва формулата на Моавър,

пример: $z^2 = -4+3i$



$|z^2| = 5$

$\arg(z^2) = \frac{\pi}{2} + \arccos\frac{3}{5}$

$-4+3i$ няма хубав тригон. вид $\Rightarrow$ няма да използваме Моавър, а ще запишем

$z = x+iy \qquad z^2 = -4+3i \Leftrightarrow x^2 - y^2 + i2xy = -4+3i$

$\Leftrightarrow \begin{cases} x^2 - y^2 = -4 \\ 2xy = 3 \end{cases}$ оттук намираме x и y

Корените на уравнението $az^2+bz+c=0$, $a,b,c\in\mathbb{C}$, $a\neq 0$ са $z_{1,2}=\dfrac{-b\pm\sqrt{D}}{2a}$, където $D=b^2-4ac$ и $\sqrt{D}$ е кой да е от корените на $u^2=D$

задача 1.4

д) Да се реши уравнението $z^4+8z^3+16z^2+9=0$ (1)

$z^2(z^2+8z+16)+9=0$

$z^2(z^2+8z+16)=-9$

$(z(z+4))^2=-9$

$z(z+4)=3i$ или $z(z+4)=-3i$

$\Rightarrow$

(2) $z^2+4z-3i=0$ или $z^2+4z+3i=0$ (3)

ако z_1 и z_2 са корените на (2), то корените на (3) са $\bar{z}_1$ и $\bar{z}_2$

$z^2+4z-3i=0$

$D_1=4+3i$

$z_{1,2}=-2\pm\sqrt{D_1}$

$u^2=4+3i \iff x^2-y^2+i2xy=4+3i$

$u=x+iy$

$\iff \begin{cases} x^2-y^2=4 \\ 2xy=3 \end{cases}$

$\iff \begin{cases} x^2-\dfrac{9}{4x^2}=4 \\ y=\dfrac{3}{2x} \end{cases}$

$x^2_{1,2}=\dfrac{8\pm\sqrt{100}}{4}$ $\quad x^2=\dfrac{9}{2}$ $\quad x=\pm\dfrac{3\sqrt{2}}{2}$ $\quad y=\dfrac{3}{2x}=\pm\dfrac{\sqrt{2}}{2}$

$u=\pm\left(\dfrac{3}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right)$ реш. (2): $z_{1,2}=-2\pm\left(\dfrac{3}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right)$

реш (1): $z_{1,2}$, $\bar{z}_1$, $\bar{z}_2$