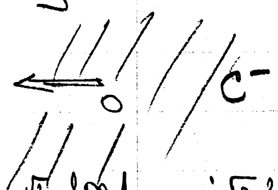
31.03.2014г. КА - лек.

Функцията z^α

$$z^\alpha = e^{\alpha\cdot\log z} = e^{\alpha(\log_0 z + i\cdot 2k\pi)} = e^{\alpha\cdot\log_0 z}\cdot e^{i2k\alpha\pi}$$

зависи от това.

$z \in \mathbb{C}^- = \mathbb{C}\setminus\{t\le 0\}$



Примери: 1) $1^{\sqrt{2}} = e^{\sqrt{2}\cdot\log 1} = e^{i\sqrt{2}2k\pi}$, $k\in\mathbb{Z}$.

2) $i^i = e^{i\log i} = e^{i(i\pi/2 + i2k\pi)} = e^{-(\frac{\pi}{2}+2k\pi)} \in \mathbb{R}$, $k\in\mathbb{Z}$.

3) $i^{-i} = e^{-i\log(-i)} = e^{-i(-i\frac{\pi}{2}+i2l\pi)} = e^{-\pi/2+2l\pi}$, $l\in\mathbb{Z}$.

! $i^i\cdot i^{-i} \neq i^0 = 1$

! $i^i\cdot i^{-i} = e^{2m\pi}$, $m\in\mathbb{Z}$.

Свойства:

1) $z_1^\alpha\cdot z_2^\alpha = (z_1\cdot z_2)^\alpha$

2) $z^\alpha\cdot z^\beta \supsetneq z^{\alpha+\beta}$

$z^\alpha = e^{\alpha\cdot\log z} = e^{\alpha(\log_0 z + i2k\pi)}$

$z^\beta = e^{\beta\cdot\log z} = e^{\beta(\log_0 z + i2l\pi)}$, $k,l\in\mathbb{Z}$.

$z^\alpha\cdot z^\beta = e^{(\alpha+\beta)\cdot\log_0 z}\cdot e^{i(2k\alpha+2l\beta)\pi}$

$z^{\alpha+\beta} = e^{(\alpha+\beta)\log z} = e^{(\alpha+\beta)\cdot(\log_0 z + i\cdot 2m\pi)}$, $m\in\mathbb{Z}$.

$= e^{i(\alpha+\beta)\log_0 z}\cdot e^{i2(\alpha+\beta)m\pi}$.

Ако $k=l=m$ имаме „=" $\Rightarrow z^{\alpha+\beta}\subset z^\alpha\cdot z^\beta$

Включването е строго.

Пр.: $z=1$, $\alpha=\beta=1/2$ $\quad 1^{\frac{1}{2}} = \{-1,1\}$

$1^{\frac{1}{2}+\frac{1}{2}} = 1^1 = \{1\}$ $\quad 1^{\frac{1}{2}}\cdot 1^{\frac{1}{2}} = \{-1,1\}$.

3) $(z^\alpha)^\beta \overset{?}{\neq} z^{\alpha\beta}$

Пр.: $z=1$, $\alpha=2$, $\beta=\frac{1}{2}$.

$$(1^2)^{\frac{1}{2}} = 1^{\frac{1}{2}} = \{-1,1\}$$

$$1^{2\cdot\frac{1}{2}} = 1^1 = \{1\}.$$

4) Всеки еднозначен клон на z^α в $\mathbb{C}^- = \mathbb{C}\setminus\{z\le 0\}$ е холоморфна ф-ция.

$$(z^\alpha)_0 := e^{\alpha\cdot\log_0 z}, \quad z\in\mathbb{C}^-.$$

$$(z^\alpha)'_0 = \left(e^{\alpha\log_0 z}\right)' = \frac{\alpha\cdot e^{\alpha\log_0 z}}{z} = \frac{\alpha\cdot e^{\alpha\cdot\log_0 z}}{e^{\log_0 z}} = \alpha\cdot e^{(\alpha-1)\cdot\log_0 z}.$$

$$(z^\alpha)' = \alpha\cdot z^{\alpha-1}$$

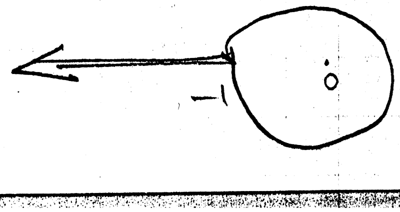
$$(z^\alpha)'_0 = \frac{\alpha\cdot(z^\alpha)_0}{z}$$

Бином на Нютон.

(Развитие в ред около 0 на функцията $(1+z)^\alpha_0$ (главният клон))

$$\underset{f(z)}{\overset{(1+z)^\alpha_0}{\shortparallel}} = e^{\alpha\log_0(1+z)} \quad z\in\mathbb{C}\setminus(-\infty;-1]$$

(защото имаме $z+1\le 0$)



Нека $f(z) = c_0 + c_1 . z + \dots + c_n . z^n + \dots$

$c_0 = f(0) = 1$

Имаме, че $f'(z) = \dfrac{\alpha . e^{\alpha . \log_0 (1+z)}}{1+z}$

$$\Rightarrow (1+z) f'(z) = \alpha . f(z)$$

Тогава

$$(1+z) . (c_1 + 2 . c_2 z + \dots + (n-1) c_{n-1} z^{n-2} + n . c_n z^{n-1} + \dots) =$$

$$= \alpha . (1 + c_1 . z + c_2 z^2 + \dots + c_{n-1} z^{n-1} + c_n . z^n + \dots)$$

приравняваме коефициентите:

z^0: $c_1 = \alpha$

z^1: $2c_2 + c_1 = \alpha c_1 \Rightarrow c_2 = \dfrac{(\alpha-1) c_1}{2} = \dfrac{\alpha . (\alpha-1)}{2}.$

z^2: $3c_3 + 2c_2 = \alpha . c_2 \Rightarrow c_3 = \dfrac{(\alpha-2) . c_2}{3} = \dfrac{\alpha . (\alpha-1)(\alpha-2)}{3!}.$

. . . .

z^{n-1}: $n . c_n + (n-1) c_{n-1} = \alpha c_n \Rightarrow c_n = \dfrac{(\alpha-n+1) c_{n-1}}{n} =$

$$= \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{n!}$$

$\Rightarrow$ Def. $\binom{n}{0} = 1$, $\binom{n}{\alpha} = \dfrac{\alpha . (\alpha-1) \dots (\alpha-n+1)}{n!}.$

Нека $\varphi(z) = \sum_{n=1}^{\infty} \binom{n}{\alpha} . z^n$

$R_{cx} = ?$

ще използваме крит. на Даламбер.

$$\left| \frac{c_{n+1}}{c_n . z^n} \right| = \frac{\left| \binom{n+1}{\alpha} \right| . |z|^{n+1}}{\left| \binom{n}{\alpha} \right| . |z|^n} =$$

$$= |z| . \frac{|\alpha(\alpha-1) \dots (\alpha-n+1)(\alpha-n)| . n!}{(n+1)! . |\alpha . (\alpha-1) \dots (\alpha-n+1)|} =$$

$= |z| \cdot \frac{|\alpha - n|}{n+1} \xrightarrow[n\to\infty]{} |z|.$

$\Rightarrow R_{cx} = 1 \Rightarrow \varphi(z)$ е холоморфна в единичния кръг, т.е. в $|z|<1$.

Освен това φ удовлетворява у-нието от по-горе:

$$(1+z)\cdot f'(z) = \alpha \cdot f(z)$$

$$\Rightarrow \begin{matrix} (1+z)\cdot\varphi'(z) = \alpha\cdot\varphi(z) \\ (1+z)\cdot f'(z) = \alpha\cdot f(z) \end{matrix} \Longrightarrow \begin{matrix} f(z)\cdot\varphi'(z) = f'(z)\cdot\varphi(z) \\ \Rightarrow \left(\frac{f(z)}{\varphi(z)}\right)' = 0 \end{matrix}$$

Т.к. $\frac{f}{\varphi}$ е холом. в $|z|<1$,

$$\Rightarrow \frac{f(z)}{\varphi(z)} = \mathrm{const} = \frac{f(0)}{\varphi(0)} = 1, \quad |z|<1.$$

$$\Rightarrow (1+z)^{\alpha}_{0} = \sum_{n=0}^{\infty} \binom{n}{\alpha} \cdot z^n, \quad |z|<1.$$

$$\binom{n}{\alpha} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!},$$ ако $\alpha \in \mathbb{Z}$ това е крайна сума.

Функцията $\operatorname{arctg} z$

Df) Всяко р-ние на у-то $\operatorname{tg} w = z$ се нар. $\operatorname{arctg} z$.

$$\operatorname{tg} w = z \iff \frac{\sin w}{\cos w} = z \iff$$

$$\iff \frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} = z$$

$$\iff \frac{e^{i2w} - 1}{i(e^{i2w} + 1)} = z$$

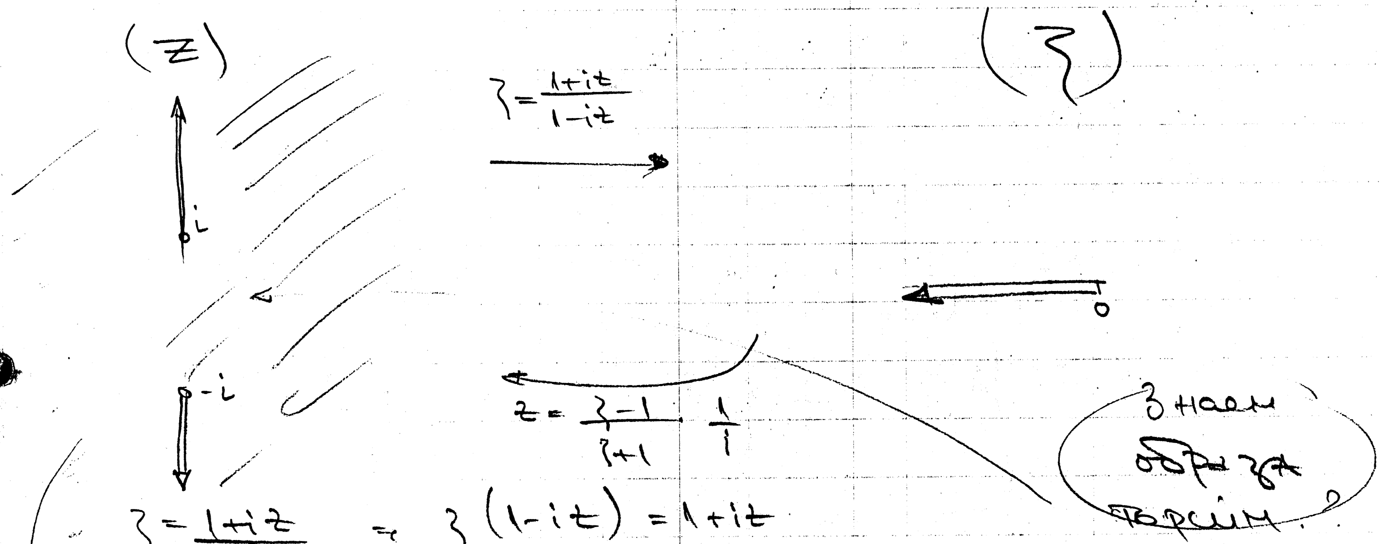
$$\Leftrightarrow e^{i2w}-1 = iz\left(e^{i2w}+1\right) \Leftrightarrow (1-iz)\cdot e^{i2w} = 1+iz$$

$$\overset{z\neq\pm i}{\Leftrightarrow} e^{i2w} = \frac{1+iz}{1-iz}$$

$$\Rightarrow i2w = \log\frac{1+iz}{1-iz}, \text{ т.е. } w = \operatorname{arctg} z = \frac{1}{2i}\cdot\log\frac{1+iz}{1-iz}, \quad z\neq\pm i$$

ζ

Област, в която можем да отделим еднозначен клон на $\operatorname{arctg} z$:



$$\zeta = \frac{1+iz}{1-iz} \Rightarrow \zeta(1-iz) = 1+iz$$

$$\zeta - i\zeta z = 1+iz$$

$$\zeta - 1 = zi(1+\zeta)$$

$$z = \frac{\zeta-1}{i(1+\zeta)}$$

През безкр. точка

$$z(0) = -\frac{1}{i} = i$$

$$z(\infty) = \frac{1}{i} = -i$$

Областта е: $\mathbb{C} \setminus \{(-i\infty, -i] \cup [i; +i\infty)\}$.

$$\operatorname{arctg}_0 z = \frac{1}{2i}\cdot\log_0\frac{1+iz}{1-iz}$$

$$\operatorname{arctg} z = \operatorname{arctg}_0 z + k\pi, \quad k\in\mathbb{Z}.$$

Тв: Всеки еднозначен клон на $\operatorname{arctg} z$ е холоморфна ф. в областта

$$\mathbb{C} \setminus \{ -i.\infty, -i] \cup [i; +i.\infty) \}$$

и $\quad (\operatorname{arctg} z)' = \dfrac{1}{1+z^2}.$

Д-во: $\quad \operatorname{arctg}_0 z = \dfrac{1}{2i} \cdot \log_0 \dfrac{1+iz}{1-iz}.$

① $\log_0 \dfrac{(1+iz)}{1-iz} = \log_0 (1+iz) - \log_0 (1-iz) + i2k\pi, k \in \mathbb{Z}$

$z := 0 \Rightarrow 0 = 0 - 0 + i2k\pi \Rightarrow k = 0$

$\Rightarrow \log_0 \dfrac{1+iz}{1-iz} = \log_0 (1+iz) - \log_0 (1-iz)$

② $(\operatorname{arctg} z)' = \dfrac{1}{2i} \left[\log_0 (1+iz) - \log_0 (1-iz) \right]' =$

$= \dfrac{1}{2\cancel{i}} \left[\dfrac{\cancel{i}}{1+iz} - \dfrac{-\cancel{i}}{1-iz} \right] =$

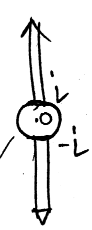
$= \dfrac{1}{(1+iz)(1-iz)} = \dfrac{1}{1+z^2}.$

Развитие на $\operatorname{arctg}_0 z$ в ред около 0

$f(z) = \operatorname{arctg}_0 z$

$f'(z) = \dfrac{1}{1+z^2} = 1 - z^2 + z^4 - \dots, \quad |z| < 1.$

$\varphi(z) = z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \dots, \quad |z| < 1,$



Rсх. $= 1$

(Коши-Адамар)

$\Rightarrow \varphi(z)$ е холоморфна в $|z| < 1$

и $\varphi'(z) = 1 - z^2 + z^4 - \dots = \dfrac{1}{1+z^2} = f'(z),$

в $|z| < 1$

максимален кръг около 0 ще разбиваме !?

$\Rightarrow \quad f(z)-\varphi(z)=const=f(0)-\varphi(0)=0-0=0$

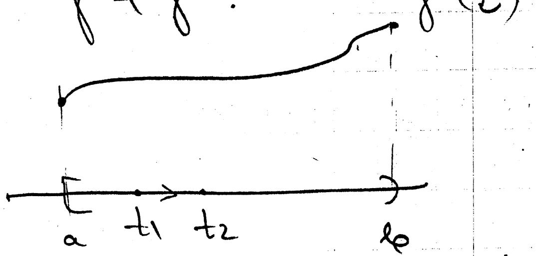
$\Rightarrow \quad \operatorname{arctg} z = z-\frac{z^3}{3}+\frac{z^5}{5}-\dots, \quad |z|<1.$

Интеграл върху крива

Д.: $\gamma: z=\gamma(t): [a;b] \to \mathbb{C}$ — крива

$\gamma^* = \{z\in\mathbb{C} : z=\gamma(t), t\in[a;b]\}$ — носител → образ на точките

$\gamma \neq \gamma^*$



$\gamma(a)$ - начало $\qquad \gamma(b)$ - край

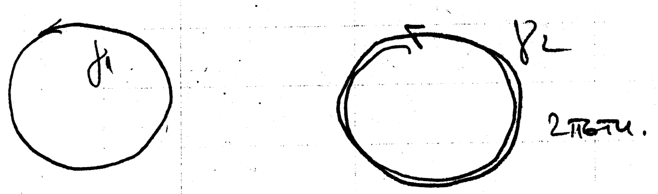
$z_1 \prec z_2 \Leftrightarrow t_1 < t_2$

Кривата има и посока, докато носителят е само мн. от точки.

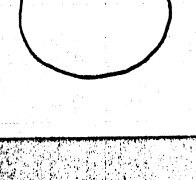
Пр.: $\gamma_1: z=\gamma_1(t)=e^{it}, \; t\in[0;2\pi]$

$\gamma_2: z=\gamma_2(t)=e^{i2t}, t\in[0;2\pi]$

$\gamma_1^* = \gamma_2^*$



Ако $\gamma_2(t)=e^{-it}, \; t\in[0;2\pi]$



Ако $\gamma(a)=\gamma(b)$ $\Rightarrow$ кривата е <u>затворена</u>

Ако $t_1 \neq t_2 \Rightarrow \gamma(t_1) \neq \gamma(t_2)$

$\Rightarrow$ γ е <u>Жорданова</u> крива (проста крива)

т.е. няма самопресичане

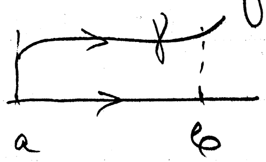
01.04.2013г. <u>Интеграл</u> <u>в/у крива</u>

① Крива в $\mathbb{C}$.

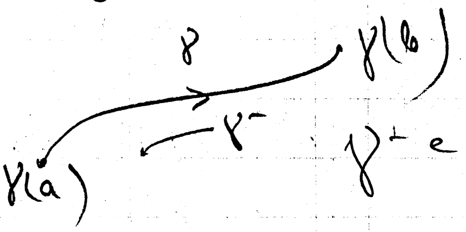
$\gamma: z=\gamma(t): [a;b] \longrightarrow \mathbb{C}$

$\gamma^* = \{ z \in \mathbb{C} : z=\gamma(t), t\in[a;b] \}$

$\gamma \neq \gamma^*$



$\gamma^- = \gamma(a+b-t)$, $t\in[a;b]$



γ^- е в другата посока

$(\gamma^-)^* = \gamma^*$.

γ е <u>гладка</u>, ако $\exists$ непр. $\gamma'(t)$, $\forall t\in[a;b]$

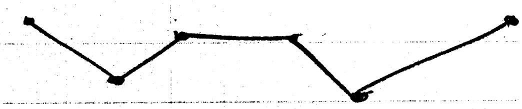
и $\gamma'(t) \neq 0$, $t\in[a;b]$.

γ е <u>частично гладка</u>, ако

$\exists\ a=a_0<a_1<a_2<\ldots<a_{n-1}<a_n=b$, така че

$\gamma|_{[a_{k-1},a_k]}$, $k=1,\ldots n$ е гладка крива.

пр. за части. гладка:



(начупена линия)

Ако γ е гладка, то тя има крайна дължина (<u>ректифицируема</u> крива)

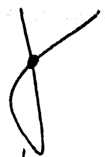
$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

② ф-ция в/у крива

$\gamma: z = \gamma(t) : [a;b] \longrightarrow \mathbb{C}$

$f: \gamma^* \longrightarrow \mathbb{C}$.

$z \in \gamma^*$, $f(z) := f(\gamma(t))$, $t \in [a;b]$.



③ <u>D:)</u> Ако $f: [a;b] \longrightarrow \mathbb{C}$

$$\int_a^b f := \int_a^b \operatorname{Re} f + i \cdot \int_a^b \operatorname{Im} f$$

<u>D:)</u> Нека $\gamma: [a;b] \longrightarrow \mathbb{C}$ е гладка крива и $f: \gamma^* \longrightarrow \mathbb{C}$ е непрекъсната

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) . \gamma'(t)\,dt$$

Ако γ е частично гладка и $\gamma = \bigcup_{k=1}^n \gamma_k$, то

$$\int_\gamma f := \sum_{k=1}^n \int_{\gamma_k} f.$$

Св-ва на $\int_{\gamma} f$.

Д:) Нека $\gamma_1: z=\gamma_1(t):[a;b]\to\mathbb{C}$

$\gamma_2: z=\gamma_2(\tau):[c,d]\to\mathbb{C}$

са гладки криви.

К., че две криви са еквивалентни

$\gamma_1 \sim \gamma_2$, ако $\exists$ ф-ция $\tau=\lambda(t)$, $\lambda: [a;b]\to[c;d]$ такава, че $\lambda(a)=c$, $\lambda(b)=d$

$\gamma_2(\lambda(t))=\gamma_1(t)$, $\forall t\in[a;b]$

и $\exists$ непр. $\lambda'(t)>0$, $t\in[a;b]$.

1) Ако $\gamma_1 \sim \gamma_2$ са гладки криви, то

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

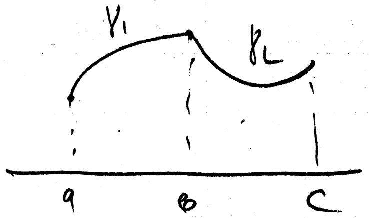
2) Линейност на интеграла:

$$\int_{\gamma}(\alpha f+\beta g)=\alpha\int_{\gamma} f+\beta\int_{\gamma} g, \quad \forall \alpha,\beta\in\mathbb{C}.$$

3) Адитивност отн. γ.

Ако $\gamma_1: z=\gamma_1(t):[a;b]\to\mathbb{C}$

$\gamma_2: z=\gamma_2(t):[b;c]\to\mathbb{C}$ и $\gamma_1(b)=\gamma_2(b)$



$\gamma_1\cup\gamma_2$

$$\int_{\gamma_1\cup\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

По-общо по деф.:

$$\int_{\gamma_1\cup\gamma_2\cup\ldots\cup\gamma_n} f := \sum_{k=1}^{n} \int f.$$

4) $$\int_{\gamma^-} f = -\int_{\gamma} f.$$

5) Оценка на модула на интеграла: $\left|\int_\gamma f\right|$

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|,$$

$$\int_\gamma |f(z)|\,|dz| = \int_a^b |f(\gamma(t))|.|\gamma'(t)|\,dt$$

Д-во: Нека $\lambda = \int_\gamma f(z)\,dz$. Имаме, че $\lambda = |\lambda|.e^{i\varphi}$, а $|\lambda| = \lambda e^{-i\varphi}$.

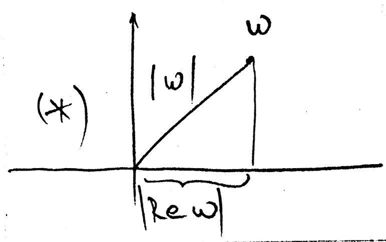
Тогава $$\left|\int_\gamma f(z)\,dz\right| = Re\left\{\left|\int_\gamma f(z)\,dz\right|\right\} =$$

$$= Re\left\{e^{-i\varphi}\int_\gamma f(z)\,dz\right\} = Re\left\{\int_\gamma e^{-i\varphi} f(z)\,dz\right\} =$$

$$= Re\left\{\int_a^b e^{-i\varphi} f(\gamma(t))\gamma'(t)\,dt\right\} =$$

$$= \int_a^b Re\left\{e^{-i\varphi} f(\gamma(t))\gamma'(t)\right\}dt \overset{(*)}{\le}$$

$$\overset{(*)}{\le} \int_a^b \left|e^{-i\varphi} f(\gamma(t))\gamma'(t)\right|dt =$$

(*) 

$$Re\,w \le |Re\,w| \le |w|$$

$$= \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\, dt .$$

В частност: ако $|f(z)| \le M$, $z \in \gamma$,
то $\left| \int_\gamma f(z)\, dz \right| \le M\, l(\gamma)$.

Примери:

① $\int_\gamma (z-a)^n dz$, $n \in \mathbb{Z}$, $\gamma: \{ |z-a| = R \}$

$\gamma: z = a + R.e^{it}$, $t \in [0; 2\pi]$

$$\int_\gamma f(z)\, dz = \int_0^{2\pi} R^n e^{int}.R.i.e^{it} dt =$$

$$= iR^{n+1}. \int_0^{2\pi} e^{i(n+1)t} dt =$$

$$= \begin{cases} n = -1, & = 2\pi i \\ n \neq -1, & \frac{1}{n+1} . R^{n+1} \int_0^{2\pi} d e^{i(n+1)t} = \\ & = \frac{R^{n+1}}{n+1} \left(e^{i2(n+1)\pi} - e^0 \right) = 0 \end{cases}$$

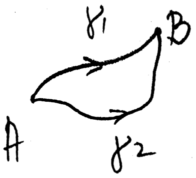
! $\int_\gamma (z-a)^n dz$, $n \in \mathbb{Z}$ $= \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1 \end{cases}$;

(2) $\int\limits_{\gamma} z^n dz$, $n \geq 0,\ n \in \mathbb{Z}$ $\gamma : [a;b] \to \mathbb{C}$ – произволна гладка кр.

Нека $\gamma(a) = A$, $\gamma(b) = B$.

$$\int\limits_{\gamma} z^n dz = \int_a^b [\gamma(t)]^n \gamma'(t)\, dt =$$

$$= \frac{1}{n+1} \cdot \int_a^b d(\gamma(t))^{n+1} = \frac{B^{n+1} - A^{n+1}}{n+1}$$



$$\int\limits_{\gamma_1} z^n = \int\limits_{\gamma_2} z^n$$

В частност, ако кривата γ е затворена:

$$\int\limits_{\gamma} z^n dz = 0.$$

При какви условия интегралът от ф-цията <u>не</u> зависи от кривата

$\Updownarrow$

$\int$ от затв. крива $= 0$ (?