20. V. 2013г. ВА - лекция

Дискриминанта и резултанта

$f(x) = a_0x^n + \dots + a_n \in F[x]$, $a_0 \neq 0$

Корени $\alpha_1, \alpha_2, \dots \alpha_n \in L \supseteq F$

Дискриминанта на f:

$$\mathcal{D}(f) = a_0^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2 \in L$$

$\mathcal{D}(f) = 0 \iff f$ има кратен корен

Тв) $\mathcal{D}(f) = a_0^{n-2} (-1)^{\frac{n(n-1)}{2}} f'(\alpha_1) f'(\alpha_2) \dots f'(\alpha_n)$

Док $f(x) = a_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$

$f'(x) = a_0(x-\alpha_2)\dots(x-\alpha_n) + a_0(x-\alpha_1)(x-\alpha_3)\dots(x-\alpha_n) + \dots$

$f'(\alpha_1) = a_0(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n)$

$f'(\alpha_2) = a_0(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\dots(\alpha_2-\alpha_n)$

$$\begin{aligned}\mathcal{D}(f) = a_0^{n-2} \cdot & a_0(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)\dots(\alpha_1-\alpha_n) \times \\ & (-1)\, a_0 (\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)\dots(\alpha_2-\alpha_n) \times \\ & (-1)^2\, a_0 (\alpha_3-\alpha_1)(\alpha_3-\alpha_2)(\alpha_3-\alpha_4)\dots(\alpha_3-\alpha_n) \times \\ & (-1)^3\, a_0 (\alpha_4-\alpha_1)(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)\dots(\alpha_4-\alpha_n) \times \dots \times \\ & (-1)^{n-1} a_0 (\alpha_n-\alpha_1)(\alpha_n-\alpha_2)\dots(\alpha_n-\alpha_{n-1}) = \\ & = a_0^{n-2}(-1)^{\frac{n(n-1)}{2}} f'(\alpha_1)\dots f'(\alpha_n)\end{aligned}$$

Сл. $\mathcal{D}(f) \in F$

$$W = W(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \ldots & \lambda_n^2 \\ & & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix} = \prod_{i>j} (\lambda_i - \lambda_j)$$

(Вандермонд)

$$a_0^{2n-2} W^2 = D(f)$$

(прилича на дискриминантата от предната страница)

$$S_k = \lambda_1^k + \lambda_2^k + \ldots + \lambda_n^k$$

$$S_0 = n$$

Тв.

$$D(f) = a_0^{2n-2} \begin{vmatrix} S_0 & S_1 & S_2 & \ldots & S_{n-1} \\ S_1 & S_2 & S_3 & \ldots & S_n \\ S_2 & S_3 & S_4 & \ldots & S_{n+1} \\ \vdots & & & & \\ S_{n-1} & S_n & S_{n+1} & \ldots & S_{2n-2} \end{vmatrix}$$

Док. Да се пресметне $W \cdot W^t$ (по правилото ред x ред)

$$W \cdot W = \begin{vmatrix} n & S_1 & S_2 & \ldots & S_{n-1} \\ S_1 & S_2 & S_3 & \ldots & S_n \\ & & & & \\ S_{n-1} & \ldots & & & S_{2n-2} \end{vmatrix}$$

пример

$f(x) = x^2 + px + q \rightarrow \lambda_1, \lambda_2$ (корени)

$S_0 = 2$

$S_1 = \lambda_1 + \lambda_2 = -p$

$S_2 = \lambda_1^2 + \lambda_2^2 = \cancel{\ldots} - pS_1 - 2q = p^2 - 2q$

$$\Leftarrow \begin{cases} \lambda_1^2 + p\lambda_1 + q = 0 \\ \lambda_1^2 = -p\lambda_1 - q \\ \lambda_2^2 = -p\lambda_2 - q \end{cases}$$

$$D = \begin{vmatrix} 2 & -p \\ -p & p^2 - 2q \end{vmatrix} = 2p^2 - 4q - p^2 = p^2 - 4q$$

Th на Фейер

f 2π периодични, непр.

ще построим $\sigma_n(x) = \dfrac{s_0(x) + s_1(x) + \dots + s_n(x)}{n+1}$

ако $a_n \to 0$, то $\dfrac{a_1 + \dots + a_n}{n} \to a$ | Коши,

$\sigma_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$

Th на Фейер: $\sigma_n(x) \rightrightarrows f(x) \quad (-\infty, +\infty)$

Лема: $\sigma_n(x) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x+t)\dfrac{\sin^2 \frac{n+1}{2}t}{2(n+1)\sin^2\frac{t}{2}}\,dt$

$\sigma_n(x)$ участва $\displaystyle\sum_{k=1}^{n} \dfrac{\sin\frac{2n+1}{2}t \sin\frac{t}{2}}{2\sin\frac{t}{2}\cdot\sin\frac{t}{2}} = \sum \dfrac{\cos - \cos}{2\sin^2\frac{t}{2}} = \dfrac{1-\cos}{\quad} =$

Док. на Th, $|\sigma_n(x) - f(x)| = ?$

В лемата полагаме $f(x) \equiv 1$

$\cancel{\sigma_n(x)} =$

$1 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \dfrac{\sin^2\frac{n+1}{2}t}{2(n+1)\sin^2\frac{t}{2}}$

$|\sigma_n(x) - f(x)| = \dfrac{1}{\pi}\left|\displaystyle\int_{0}^{\pi} \left(f(x+t) - f(x)\right)\dfrac{\sin^2\frac{n+1}{2}t}{2(n+2)\sin^2\frac{t}{2}}\,dt\right| \le$

$A_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |f(x+t) - f(x)|\dfrac{\sin^2\frac{n+1}{2t}}{2(n+1)\sin^2\frac{t}{2}}\,dt$

$f(x)$ е непр. $\Rightarrow$ е равн. непр. във $\forall$ краен затворен интервал.

По зададено ε, намираме δ: $|t| < \delta$, то

$|f(x+t) - f(x)| < \dfrac{\varepsilon}{3}$ за $\forall x$

$$A_n \le \underbrace{\frac{1}{\pi}\int_{-\delta}^{\delta} |f(x+t)-f(x)| \frac{\sin^2\frac{n+1}{2}}{2\sin^2\frac{t}{2}}\,dt}_{I_1} + \underbrace{\frac{1}{\pi}\int_{\delta}^{\pi}\dots}_{I_2} + \underbrace{\int_{-\pi}^{-\delta}\dots}_{I_3}$$

$$I_1 < \frac{\varepsilon}{3}\int_{-\delta}^{\delta}\frac{\sin^2\frac{n+1}{2}}{2\sin^2\frac{t}{2}}dt \le \frac{\varepsilon}{3}\int_{0}^{\pi}\frac{\sin^2\frac{n+1}{2}}{2\sin^2\frac{t}{2}}dt = \frac{\varepsilon}{3}\cdot 1$$

$$I_2 = \int_{\delta}^{\pi} \underbrace{|f(x+t)-f(x)|}_{\le 2M} \frac{\overbrace{\sin^2\frac{2n+1}{2}t}^{\le 1}}{2(n+1)\underbrace{\sin^2\frac{t}{2}}_{\ge \sin^2\frac{\delta}{2}}}dt \le 2M\frac{1}{2(n+1)\sin^2\frac{\delta}{2}\pi}\int_{\delta}^{\pi}dt =$$

f е непр. $\Rightarrow$ $\exists M: |f(x)| \le M$ $\quad \frac{\delta}{2} < \frac{t}{2} < \frac{\pi}{2} \Rightarrow \sin\frac{t}{2} > \sin\frac{\delta}{2}$

$$= \frac{M(\pi-\delta)}{\pi(\sin^2\frac{\delta}{2}\,2(n+1))} \le \frac{M}{(\sin^2\frac{\delta}{2})2(n+1)}$$

по ε нам. δ и обр. $I_1 < \frac{\varepsilon}{3}$ фикс. δ и $I_2 \le \frac{M}{2(n+1)\sin\frac{\delta}{2}} \xrightarrow[n\to\infty]{} 0$

$|I_3| < \frac{\varepsilon}{3} \Rightarrow A_n < \varepsilon$ $\quad \Rightarrow I_2 < \frac{\varepsilon}{3}$ за

дост. гол.

Заб. Може да се док. същото за f-и непр. с изключение на краен брой точки, Следствие (Th на Вайерщрас)

Нека $f(x)$ е непр. в $[-\pi;\pi]$, $f(-\pi)=f(\pi)$

за $\forall\, \varepsilon>0\ \exists$ тригоном. полином $T(x)$: $|f(x)-T(x)|<\varepsilon$,
за $\forall\, x\in[-\pi;\pi]$

т.е f - непр. може да бъде равномерно приближена със тригонометричен полином.

Следствие 1:
По ε нам. T $|f(x)-\sigma_n(x)|<\varepsilon$ (от Th на Фейер)
тригонометр. полином.

Следствие 2: ($(a_1,a_2)(b_1,b_2)$ - скаларно)

$\sum a_i b_i$ $\quad f(x), g(x)$ $\quad \int_{-\pi}^{\pi} f(x)g(x)\,dx$ $\quad f\perp g$, ако $\int = 0$

/ $\vec{a}\perp\vec{b}$ $(a,b)=0$ /

Нека $f(x)\perp T(x)$, $\forall$ тригоном. полином и нека f е непр $[-\pi,\pi]$
$f(-\pi)=f(\pi)$

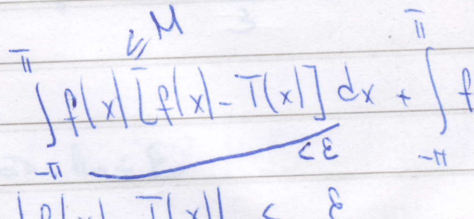
тогава $f(x)\equiv 0$

Док. $$\int_{-\pi}^{\pi} f^2(x)\,dx = \int_{-\pi}^{\pi} \underbrace{f(x)}_{\le M}\underbrace{[f(x)-T(x)]}_{<\varepsilon}\,dx + \underbrace{\int_{-\pi}^{\pi} f(x)\cdot T(x)\,dx}_{=0} \quad (1)$$

По ε нам. T: $|f(x)-T(x)| < \dfrac{\varepsilon}{M}$

където M: $|f(x)|\le M$

$\int_{-\pi}^{\pi} f^2(x)\,dx = 0$



$f^2(x_0)>0$ $\quad$ имаме околност в която f е $+$ в $(x_0-\delta, x_0+\delta)$ $f^2(x) > \dfrac{f^2(x_0)}{2}$

$\Rightarrow +$, но имаме пък $=$ от (1) $\Rightarrow \ge$