$$(x_1^2+x_1x_2+x_2^2)(x_1^2+x_1x_3+x_3^2)(x_2^2+x_2x_3+x_3^2)=$$

$$=\left(x_1^2-\frac{q}{x_2x_3}\cdot x_2+x_2^2\right)\left(x_1^2-\frac{q}{x_2}+x_3^2\right)\left(x_2^2-\frac{q}{x_1}+x_3^2\right)=$$

$$=\left(x_1^2-\frac{q}{x_3}+x_2^2\right)\times$$

$$\times\Big(x_1^2x_2^2-q x_1+x_1^2x_3-$$

$$-qx^2+\frac{q^2}{x_1x_2}-\frac{qx_3^2}{x_2}+$$

$$+x_2^2x_3^2-\frac{q}{x_1}x_3^2+x_3^4\Big)=$$

$$x_1+x_2+x_3=0$$
$$x_1x_2+x_1x_3+x_2x_3=+p$$
$$x_1x_2x_3=-q$$

$$x_1=\frac{-q}{x_2x_3}$$

$$=$$

$$(x_1^2+x_1x_2+x_2^2)\big((x_1x_2)^2+x_1x_2x_3\cdot x_1+(x_1x_2)^2+$$

$$+x_1x_2x_3\cdot x_2+x_1x_2x_3x_3+x_1x_3^3+(x_2x_3)^2+x_2x_3^3+$$

$$+x_3^4\big)=$$

$$=\left((-x_3)^2+x_1x_2\right)\left(x_1x_2x_3(x_1+x_2+x_3)+2(x_1x_2)^2+\right.$$

$$\left.+x_3^2(x_1x_3+x_2^2+x_2x_3+x_3^2)\right)=$$

$$=\left((-x_3)^2+x_1x_2\right)\Big(-q\cdot 0+2(x_1x_2)^2+$$

~~$+x_3(x_3(x_1+x_2^2+x_3+x_3)+$~~

$$(x_1+x_2)^2\big(x_1x_3+x_2(x_1+x_3)+x_2x_3+$$

$$+x_3(x_1+x_2)\big)\Big)=$$

$= 4q^2 + q x_1 x_3 (x_2 + x_3) - q + q(x_1 + x_2) = q(4q + q + x_1 x_3^2 - 1 + x_1 + x_2) =$

$= q(5q + x_1(x_3^2 + 1) - 1 + x_2) = q(5q + x_1 x_3 (x_1 + x_2) + x_1 - 1 + x_2) =$

$=$

$5x \equiv 7 \pmod 9$

~~$x \equiv 5 \pmod 9$~~

$x \equiv 5 \pmod 9$

$x \equiv 7 \pmod{10}$
$x \equiv 2 \pmod 5$
$x \equiv 5 \pmod 9$

5	5
10	1
15	6
20	2
25	7
30	
35	
40	
45	
50	

$2x \equiv 7 \pmod 9$

$x \equiv 8 \pmod 9$

2	2
4	4
6	6
8	8
10	1
12	3
14	5
16	7
18	0
20	2

$x \equiv 7 \pmod{10}$
$x \equiv 2 \pmod 5$
$x \equiv 8 \pmod 9$

$x \equiv 2 \pmod{90}$
$x \equiv 17 \pmod{90}$

$$= (x_3^2 + x_1x_2)\left(2x_1^2x_2^2 + (x_1+x_2)^2(x_1x_3 - x_1x_2 - x_2x_3 + x_2x_3 -\right.$$

$$\left.- x_1x_3 - x_2x_3 - x_2x_3\right) =$$

$$= (x_3^2 + x_1x_2)\left(2x_1^2x_2^2 \mp (x_1+x_2)^2(x_1x_2 + x_2x_3)\right) =$$

$$= 2x_1^2x_2^2x_3^2 - x_3^2(x_1+x_2)^2(x_1x_2 + x_2x_3) +$$

$$+ 2x_1^3x_2^3 - (x_1x_2)(x_1+x_2)^2(x_1x_2 + x_2x_3) =$$

$$= 2q^2 + x_3(x_1+x_2)^3(x_1x_2 + x_2x_3) +$$

$$+ 2(x_1x_2x_3)^2 - x_3^2(x_1^2 + 2x_1x_2 + x_2^2)(x_1x_2 + x_2x_3) =$$

$$= 2q^2 + x_2x_3(x_1+x_2)^3(x_1 + x_3) +$$

$$+ 2(x_1x_2x_3)^2 - x_2(x_1^2x_3^2 + 2q \cdot x_3 - x_2^2x_3^2)(x_1 + x_3) =$$

$$= 2q^2 + x_2x_3(x_1+x_2)(x_1^2 + 2x_1x_2 + x_2^2)(x_1 + x_3) +$$

$$+ 2(x_1x_2x_3)^2 - (x_1x_2x_3 \cdot x_1x_3 + 2q\,x_2x_3 - x_2^2x_3^2)(x_1 + x_3) =$$

$$= 2q^2 - q - x_2x_3(x_1+x_3)\left(x_1(x_1+x_2) + x_2(x_1+x_2)\right)(x_1+x_3) +$$

$$+ 2q^2 - (q\,x_1x_3 + 2q\,x_2x_3 - x_2x_3^2)(x_1 + x_3) =$$

$$= 4q^2 - q - q - x_2x_3^2\{(x_1+x_2)^2(x_1+x_3) - x_3(q x_1 + 2x_2 - x_2x_3)(x_1+x_3) =$$

$$= 4q^2 - 2q + x_2^2x_3^3 - x_3(q x_1^2 + 2x_1x_2 - q + q x_1x_3 + 2x_2x_3 - x_2x_3^2$$

$$= 4q^2 - 2q + x_3(x_2^2 - q x_1^2 + 2x_1x_2 - q + q x_1x_3 + 2x_2x_3 - x_2x_3^2) =$$

$$= 4q^2 - 2q + x_2^2x_3 - q x_1^2x_3 + 2q - q x_3 + q x_1x_3^2 + 2x_2x_3^2 - x_2x_3^2 =$$

$$= 4q^2 - q x_1x_3(x_1 - x_3) + x_2x_3^2 + x_2^2x_3 - q x_3 =$$

$$= 4q^2 - q x_1x_3(x_1 - x_3) + x_2x_3(x_3 + x_2) - q x_3 =$$

$$= 4q^2 - q\left(\frac{x_2}{q}\right)(x_1 - x_3) + x_2x_3(-x_1) - q\left(-\frac{q}{x_1x_2}\right) - q x_3 =$$

$$-q\,x_1x_3$$

б) фактор групата $G/H \cong F^*$

~~$\varphi(G) = Im\varphi$~~ $\varphi: G/H \to F^*$

$\varphi: G/H \to Im\varphi$ || $\varphi: (G, \circ) \longrightarrow (R, *)$

$H \subset G$ $H \trianglelefteq G$

$Ker\varphi = H$

$Ker\varphi = (1, b, c)$ $\varphi(a,b,c) = (1,b,c)$ $(1,1,0) = a$

$Im\varphi = (1, 1, 0)$

$\varphi(a_1, b_1, c_1) \circ \varphi(a_2, b_2, c_2) = (a_1a_2, b_1, b_2, c_1a_2 +$

$\varphi(x) = 1 - x$

$\varphi(x) \circ \varphi(y) =$

$= (1-x) \circ (1-y)$

$\varphi(a_1, b_1, c_1) \circ \varphi(a_2, b_2, c_2)$

$H = \{(1, b, c)\}$

Нека $x \in H$: $\varphi(x) = 1$

$(F, +)$ 1) $\varphi((1,b,c)) = 1$ / 0

$(F, *)$ 2) $\varphi(a,b,c) \neq 1$ / 0

$\varphi((a, b, c)) = 1$ $a \neq 1$

1 $\varphi((1, b, c)) = 0$ $b - c$

$\varphi((a, b, c)) \neq 0$ $a \neq 1$

$\varphi((1, b, c)) = 1$ a

$\varphi((a, b, c)) \neq 1$ a

5.1 a) $f = x^5 - 2x^3 + x^2 - 3x + 1$ $F = \mathbb{Q}$

$g = x^3 - 3x + 1$

$f = q \cdot g + r$

$$x^5 - 2x^3 + x^2 - 3x + 1 \;\lfloor\; x^3 - 3x + 1$$

$- \; x^5 - 3x^3 + x^2$ $\quad x^2 + 1$

$x^3 \quad - 3x + 1$

$- \; x^3 \quad - 3x + 1$

0

$q = x^2 + 1$

$(f, g) = g = x^3 - 3x + 1$

$uf + vg = (f, g)$

$u(x^5 - 2x^3 + x^2 - 3x + 1) + v(x^3 - 3x + 1) = x^3 - 3x + 1$

$u = 0 \quad v = 1$

б) $f = x^4 - 2x^3 + 2x - 4$ $r = -4x^2 + 10x - 4$

$g = x^3 - 2x^2 + 4x - 8$ $q = x$

$$x^4 - 2x^3 + 2x - 4 \;\lfloor\; x^3 - 2x^2 + 4x - 8$$

$- \; x^4 - 2x^3 + 4x^2 - 8x$ $\quad x + \quad q$

$-4x^2 + 10x - 4$ (r)

$\deg r \leq \deg g$

$g | r$

$$x^3 - 2x^2 + 4x - 8 \;\lfloor\; -4x^2 + 10x - 4$$

$- \; x^3 - \frac{5}{2}x^2 + x$ $\quad -\frac{1}{4}x - \frac{1}{8} \quad q_1$

$\frac{1}{2}x^2 + 3x - 8$

$- \; \frac{1}{2}x^2 - \frac{5}{4}x + \frac{1}{2}$

$r_1 \quad \frac{17}{4}x - \frac{17}{2}$

$r | r_1$

$-4x^2 + 10x - 4 \;|\; \frac{17}{4}x - \frac{17}{2}$

$-4x^2 + 8x \qquad -\frac{16}{17}x + \frac{128}{17}$

$2x - 4 \qquad q_2$

$-2x - 4$

$r_2 = 0$

~~(f, g)~~

$(r | r_1) = r_1$

$r = q_2 r_1 + r_2$

$g = q_1 r + r_1$

$f = q \cdot g + r$

$(f | g) = \frac{4}{17} r_1 = \frac{4}{17}\left(\frac{17}{4}x - \frac{17}{2}\right) = x - 2 = d$

$uf + vg = d$

$uf + vg = x - 2$

$r_2 = g - r_1 \cdot q_2$

$r_1 = g - r \cdot q_1$

$r = f - g \cdot q$

$r_1 = g - r \cdot q_1$

$r_2 = r - r_1 \cdot q_2$

$0 = r - r_1 q_2$

$r = r_1 \cdot q_2$

$\frac{17}{4}x - \frac{17}{2} = f - g(x^3 - 2x^2 + 4x - 8)$

$r_1 = g - r q_1$

$r_1 = g - r \cdot \left(-\frac{1}{4}x - \frac{1}{8}\right)$

$r = f - g \cdot q$

$f — : g$

$g : — r$

$r — r_1$

$r = \frac{g}{r_1} \cdot \frac{1}{-\frac{1}{4}x - \frac{1}{8}}$

$r = (g - r_1) \cdot \frac{1}{-\frac{1}{4}x - \frac{1}{8}}$

$r = \left[g - \left(\frac{14}{7} + \frac{17}{2}\right)\right] \frac{1}{-\frac{1}{4}x - \frac{1}{8}}$

$uf - v \cdot g = x - 2$

$\left(\frac{17}{4}x - \frac{17}{2}\right) = g - r$

$r_1 = \frac{-4x^2 + 10x - 4}{-\frac{16}{17}x + \frac{8}{17}} = \frac{r}{q_1}$

$g = q_1 r + \frac{r}{q_1}$

$r = f - gq = f - gx$

$g = q_1 (f - gx) + r_1$

$r_1 = g - q_1(f - gx) = g + \left(\frac{1}{4}x + \frac{1}{8}\right)(f - gx) =$

$= g + \frac{1}{4}xf + \frac{1}{8}f - \frac{1}{4}x^2 g - \frac{1}{8}gx =$

$= \left(\frac{1}{4}x + \frac{1}{8}\right)f - \frac{1}{8}\left(2x^2 - x + 8\right)g = d = (x-2)$

$u = \frac{1}{4}x + \frac{1}{8}$ $\quad v = -\frac{1}{8}\left(2x^2 - x + 8\right)$

$19 \equiv -5 \pmod{24}$

-29

b) $f = x^5 - 3x^3 + 2x^2 + x - 2$

$g = x^3 - 3x + 1$

$$\begin{array}{l|l} f & g \\ x^5 - 3x^3 + 2x^2 + x - 2 & x^3 - 3x + 1 \\ -\;x^5 - 3x^3 + x^2 & q: x^2 \\ \hline r: x^2 + x - 2 & \end{array}$$

$$\begin{array}{l|l} g & r \\ x^3 - 3x + 1 & x^2 + x - 2 \\ -\;x^3 + x^2 - 2x & q_1: x - 1 \\ \hline -x^2 - x + 1 & \\ -\;-x^2 - x + 2 & \\ \hline r_1 = -1 & \end{array}$$

$(g|r) = (-1)(-1)$

$$\begin{array}{l|l} r & r_1 \\ x^2 + x - 2 & -1 \\ -\;x^2 + x - 2 & q_2 = -x^2 - x + 2 \\ \hline r_2 = 0 & \end{array}$$

$r_1 = g - r \cdot q_1$

$r = f - g \cdot q$

$\frac{(-1)}{(-1)} r_1 = g - (f - g \cdot q) \cdot q_1 =$

$= g - (f - g \cdot x^2) \cdot (x-1) =$

$= g - f(x-1) + g\, x^2(x-1) =$

$= -(x-1)f + (x^2(x-1)+1)g \quad / \cdot (-1)$

$u = -x + 1 = x - 1$

$v = x^2(x-1)^{+1} = -x^2(x-1) + 1$

2) $f = \bar{3}x^5 + x^4 + \bar{3}x^3 + \bar{4}$

$g = \bar{2}x^4 + \bar{2}x^3 + \bar{2}x + \bar{3}$

$\mathbb{Z}_5$

$$\begin{array}{l} \bar{3}x^5 + x^4 + \bar{3}x^3 + \bar{4} \quad \Big| \ \bar{2}x^4 + \bar{2}x^3 + \bar{2}x + \bar{3} \\ -\ \bar{3}x^5 + \bar{2}x^4 + \bar{3}x^2 + \bar{2}x \qquad q = -\bar{1}x - \bar{1} = 4x + 4 \\ \hline \bar{3}x^4 + \bar{3}x^3 - \bar{3}x^2 - \bar{2}x + \bar{4} \\ -\ \bar{3}x^4 + \bar{3}x^3 - \bar{2}x^2 - \bar{3}x \\ \hline r = \bar{3}x + \bar{2} \end{array}$$

$q | r$

$$\bar{2}x^4 + \bar{2}x^3 + \bar{2}x + \bar{3} \ \Big| \ \bar{3}x + \bar{2}$$

x^3

-5					
-4	-1	4			
-3	-2	3			
-2	-3	2			
-1	-4	1			
0	0		0	0	0
1	1	-4			
2	2	-3			
3	3	-2			
4	4	-1			
5					

$$\begin{array}{l} \bar{3}x^5 + \bar{1}x^4 + \bar{3}x^3 + \bar{4} \quad \Big| \ \bar{2}x^4 + \bar{2}x^3 + \bar{2}x + \bar{3} \quad (g) \\ -\ \bar{3}x^5 + \bar{3}x^4 + \bar{3}x^2 + \bar{2}x \qquad q: \bar{4}x + \bar{4} \\ \hline \bar{3}x^4 + \bar{3}x^3 + \bar{2}x^2 + \bar{3}x + \bar{4} \\ -\ \bar{3}x^4 + \bar{3}x^3 + \bar{3}x + \bar{2} \\ \hline r: \bar{2}x^2 + \bar{2} \end{array}$$

$$\begin{array}{l} \bar{2}x^4 + \bar{2}x^3 + \bar{2}x + \bar{3} \quad \Big| \ \bar{2}x^2 + \bar{2} \quad (r) \\ -\ \bar{2}x^4 + \bar{2}x^2 \qquad q_1: x^2 + x + \bar{4} \\ \hline \bar{2}x^3 + \bar{3}x^2 + \bar{2}x + \bar{3} \\ -\ \bar{2}x^3 + \bar{2}x \\ \hline \bar{3}x^2 + \bar{3} \\ -\ \bar{3}x^2 + \bar{3} \\ \hline r_1: 0 \end{array}$$

$r = f - (\bar{4}x + \bar{4})g$

$uf + vg = \bar{2}x^2 + \bar{1} \ | \cdot (\bar{3})$

$\bar{3}f + \bar{3}(x + \bar{1})g = x^2 + \bar{1}$

$u = \bar{3}$

$v = \bar{3}(x + \bar{1})$

$f = q \cdot g + r$

$r = f - q \cdot g$

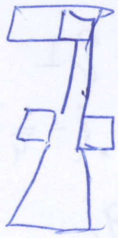
$r_1 = g - q_1 \cdot r =$

$= g - (x^2 + x + \bar{4})(f - (\bar{4}x + \bar{4})g) =$

$=$

9) $f = x^4 + 3x^3 - \bar{4}x^2 + x + \bar{1}$ $\quad F = \mathbb{Z}_7$

$g = x^4 - x^3 + x - \bar{3}$



$$\begin{array}{l} f: \quad x^4 + x^3 + \bar{3}x^2 + x + \bar{1} \\ - \; x^4 + \bar{6}x^3 + x + \bar{4} \\ \hline r: \bar{2}x^3 + \bar{3}x^2 + \bar{4} \end{array} \qquad \begin{array}{l} g \\ \underline{| x^4 + \bar{6}x^3 + x + \bar{4}} \\ q: 1 \end{array}$$

$$\begin{array}{l} g: \quad x^4 + \bar{6}x^3 + x + \bar{4} \\ - \; x^4 + \bar{5}x^3 + \bar{2}x \\ \hline \quad x^3 + \bar{6}x + \bar{4} \\ - \; x^3 + \bar{5}x^2 + \bar{2} \\ \hline r_1: \bar{2}x^2 + \bar{6}x + \bar{2} \end{array} \qquad \begin{array}{l} r \\ \underline{| \bar{2}x^3 + \bar{3}x^2 + \bar{4} |} \\ q_1: \bar{4}x + \bar{4} \end{array}$$

$$\begin{array}{l} r: \quad \bar{2}x^3 + \bar{3}x^2 + \bar{4} \\ - \; \bar{2}x^3 + \bar{6}x^2 + \bar{2}x \\ \hline \quad \bar{4}x^2 + \bar{5}x + \bar{4} \\ - \; \bar{4}x^2 + \bar{5}x + \bar{4} \\ \hline r_2: \quad 0 \end{array} \qquad \begin{array}{l} r_1 \\ \underline{| \bar{2}x^2 + \bar{6}x + \bar{2}} \\ q_2: x + \bar{2} \end{array}$$

$g + (\bar{3}x + \bar{3})f + g(\bar{1} + \bar{4}x + \bar{4})$
$\bar{4}x + \bar{5}$
$\bar{2}x + \bar{6}$

$r = f - q \cdot g$

$r_1 = g - q_1 r$

$r_1 = g + (\bar{3}x + \bar{3})(f + \bar{6}g)$

~~$r_1 = g + \bar{3}xf + \bar{3}f +$~~

$\bar{3}(x + \bar{1})f$ ~~$+ \bar{3}(x + \bar{2})g$~~

$+ (\bar{4}(x + \bar{1}) + \bar{1})g = \bar{4}\, r_1$

$\bar{5}(x + \bar{1})f + (\bar{4}(x + \bar{1}) + \bar{4})g = \bar{4}\, r_1$

$\bar{4}x + \bar{5}$

$\bar{5}(x + \bar{1})f +$ ~~$\bar{4}(x + \bar{2})g$~~ $= d$

$\bar{4}(\bar{4}(x + \bar{1}) + 1)g$

$u = \bar{5}(x + \bar{1})$

$v = \bar{2}x + \bar{2} + \bar{4} = \bar{2}x + \bar{6}$

5.2 $f, g \in F[x] \quad u, v \in F[x]$

$uf + vg = (f, g) = d$

$? = (u, v) = d_1$

~~$f | d \quad uf | d$~~ $\quad d | f \quad d | uf$

~~$g | d \quad vg | d$~~ $\quad d | g \quad d | vg$

d_1 - НОД (1)

$d_1 | uf \quad d_1 | u$

$d_1 | vg \quad d_1 | v$

6.1 $f = x^3 + px^2 + qx + r$

a) $x_1 + x_2 = x_3$

б) $x_1 x_2 = x_3$

$x_1 + x_2 + x_3 = -p$

$x_1x_2 + x_1x_3 + x_2x_3 = +q$

$x_1 x_2 x_3 = -r$

~~$x_1 + x_2 = -p - x_3$~~

~~$x_3(x_1 + x_2) + x_1 x_2 = +q$~~

~~$x_3(-p - x_3) - \frac{r}{x_3} = q$~~

$x_1 + x_2 = -p - x_3$

$x_3 = -p - x_3$

$2x_3 = -p$

$x_3 = -\frac{p}{2}$

$x_1 x_2 + x_3(x_1 + x_2) = q$

$x_1 x_2 + 2x_3 = q$

$x_1 x_2 = q + p$

$x_1^2 x_2 + x_1 x_2^2 = -r$

$x_1 x_2 - p = q$

$x_1 x_2 = -\frac{r}{x_3} = -\frac{2r}{p}$

$q + p = -\frac{2r}{p}$

$p^2 + qp + 2r = 0$