07.VII.2013г. Консултация

① $\mathbb{Z}[\sqrt{2}] = \{a_0(\sqrt{2})^n + a_1(\sqrt{2})^{n-1} + \ldots + a_{n-1}\sqrt{2} + a_n \mid a_i \in \mathbb{Z}, i = 1, \ldots, n, n \in \mathbb{N}\} =$
$= \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$

② $\mathbb{Z}[\sqrt[3]{2}] = \{a_0(\sqrt[3]{2})^n + a_1(\sqrt[3]{2})^{n-1} + \ldots + a_{n-1}\sqrt[3]{2} + a_n \mid a_n \in \mathbb{Z}, n = 1, 2, \ldots, n,$
$n \in \mathbb{N}\} = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Z}\}$

$\sqrt[3]{2} \quad (\sqrt[3]{2})^2 = \sqrt[3]{4} \quad (\sqrt[3]{2})^3 = 2$

③ $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ - поле

$\mathbb{Q}(\sqrt{2}) = \left\{\frac{P(\sqrt{2})}{Q(\sqrt{2})} \mid Q(\sqrt{2}) \neq 0\right\} = \left\{\frac{a_0(\sqrt{2})^n + a_1(\sqrt{2})^{n-1} + \ldots + a_n}{b_0(\sqrt{2})^m + b_1(\sqrt{2})^{m-1} + \ldots + b_m} \;\middle|\; \begin{matrix} a_i \in \mathbb{Q}, i = 1, \ldots, n \\ b_j \in \mathbb{Q}, j = 1, \ldots, m \end{matrix} \;\middle|\; m, n \in \mathbb{N}\right\} = \left\{\frac{a + b\sqrt{2}}{c + d\sqrt{2}} \mid a, b, c, d \in \mathbb{Q}\right\} =$

пръстен

$= \{p + q\sqrt{2} \mid p, q \in \mathbb{Q}\}$

~~$m, n \in \mathbb{N}$~~

$$\frac{a + b\sqrt{2}}{c + d\sqrt{2}} \cdot \frac{c - d\sqrt{2}}{c - d\sqrt{2}} = \frac{c - 2bd - (bc - ad)\sqrt{2}}{c^2 - 2d^2} = \underbrace{\frac{ac - 2bd}{c^2 - 2d^2}}_{\in \mathbb{Q}} +$$

$$+ \underbrace{\frac{bc - ad}{c^2 - 2d^2}}_{\in \mathbb{Q}}\sqrt{2} = p + q\sqrt{2}, \quad p, q \in \mathbb{Q}$$

$\mathbb{Z}[\sqrt{2}] \overset{?}{\neq} \mathbb{Z}(\sqrt{2})$

(4) $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \left\{ \frac{P(\sqrt{2},\sqrt{3})}{Q(\sqrt{2},\sqrt{3})} \mid Q(\sqrt{2},\sqrt{3}) \neq 0 \right\} = \left\{ \frac{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}}{e+f\sqrt{2}+g\sqrt{3}+h\sqrt{6}} \mid a,\dots,h \in \mathbb{Q} \right\} =$

$= \left\{ \frac{A+B\sqrt{2}+C\sqrt{3}+D\sqrt{6}}{E} \mid A,B,C,D,E \in \mathbb{Q} \right\} =$

$P(\sqrt{2},\sqrt{3}) = a_{0,0}(\sqrt{2})^n(\sqrt{3})^m + a_{1,0}(\sqrt{2})^{n-1}(\sqrt{3})^m + \dots = a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}$

$(\sqrt{2})^4 = 2^2$

$(\sqrt{2})^5 = 2^2\sqrt{2}$

$= \{ A'+B'\sqrt{2}+C'\sqrt{2}+D'\sqrt{6} \mid A',B',C',D' \in \mathbb{Q} \}$

(5) $M_n(F)$, F-поле

$\{ n\times n$ матрици с елемент от полето $F\}$

(6) $GL_n(F) = \{ n\times n$ неособени матрици с елемент от $F\}$

(7) $SL_n(F) = \{ n\times n$ матрици с детерминанта $=1$ и елемент от $F\}$

(8) $\mathbb{Z}(\sqrt{2})$ - пръстен

$(1+\sqrt{2})$ - идеал $= \{ (a+b\sqrt{2})(1+\sqrt{2}) \mid a,b \in \mathbb{Z} \} = \{ a+2b+(a+b)\sqrt{2} \mid a,b \in \mathbb{Z} \}$

(9) G - група

$Z(G) = \{ g \in G \mid ga = ag \ \forall a \in G \}$ - комутира с всеки елемент от G

център на групата G

λE

(10) S_n - всевъзможните биекции с единица

(11) $\mathbb{Z}_n = \{\overline{k}, k = 0, 1, \ldots, n-1\}$

$\overline{k} = \{k + tn \mid t \in \mathbb{Z}\}$

$\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$

(12) $\boxed{6.1}$ сборника При каква зависимост между коефициентите p, q, r на полинома $f = x^3 + px^2 + qx + r$ между корените му x_1, x_2, x_3 съществува зависимостта:

а) $x_1 + x_2 = x_3$ б) $x_1 x_2 = x_3$

Решение: а)
$$x_1 + x_2 + x_3 = -p$$
$$x_1 x_2 + x_1 x_3 + x_2 x_3 = q$$
$$x_1 x_2 x_3 = -r$$
$$x_1 + x_2 = x_3$$

$$2x_3 = -p \Rightarrow x_3 = -\frac{p}{2}$$

$$x_1 x_2 + \underbrace{(x_1 + x_2)}_{x_3} x_3 = q$$

$$x_1 x_2 + \frac{p^2}{4} = q \Rightarrow x_1 x_2 = q - \frac{p^2}{4}$$

$$x_1 x_2 \left(-\frac{p}{2}\right) = -r$$

$$\left(q - \frac{p^2}{4}\right)\frac{p}{2} = r \quad 1.8$$

$$4qp - p^3 = 8r$$

(13) $\boxed{2.38}$ $\mathbb{F}$-числово поле

$G = \{(a,b,c) \mid a,b,c \in \mathbb{F},\ a \neq 0,\ b \neq 0\}$.

$(a_1, b_1, c_1) \circ_1 (a_2, b_2, c_2) = (a_1 a_2,\ b_1 b_2,\ a_1 c_2 + c_1 b_2)$

а) G - неабелева група

1) $[(a_1,b_1,c_1)(a_2,b_2,c_2)](a_3,b_3,c_3) = (a_1,b_1,c_1)[(a_2,b_2,c_2)(a_1,b_1,c_1)]$

2) $\exists$ неутрален елемент $(x,y,z) \in G : (a,b,c) \circ_1 (x,y,z) = (a,b,c)$

$\qquad \parallel$

$(ax, by, az+cy) = (a,b,c)$

$\cancel{a}x = \cancel{a} \Rightarrow x = 1$

$\cancel{b}y = \cancel{b} \Rightarrow y = 1$

$az + cy = c \Leftrightarrow az + c = c \Rightarrow az = 0 \Rightarrow z = 0$

$e = (1,1,0)$

$(a,b,c) \circ_1 (1,1,0) = (a, b, a.0 + c.1) = (a,b,c)$

3) $\exists$ обратен елемент $(u,v,w) \in G : (a,b,c) \circ_1 (u,v,w) = (1,1,0)$

$\qquad \parallel$

$(au, bv, aw + cv) = (1,1,0)$

$au = 1 \Rightarrow u = \frac{1}{a}$

$bv = 1 \Rightarrow v = \frac{1}{b}$

$aw + cv = 0 \Leftrightarrow aw = -\frac{c}{b} \Rightarrow w = -\frac{c}{ab}$

$(a,b,c)^{-1} = \left(\frac{1}{a}, \frac{1}{b}\ -\frac{c}{ab}\right)$

$\Rightarrow G$ е група

$(1,2,1) \circ_1 (1,1,1) = (1,2,2)$

$\neq$

$(1,1,1) \circ_1 (1,2,1) = (1,2,3)$

б) $H=\{(1,b,c) \mid b,c \in F, b\neq 0\}$ е неабелева нормална подгрупа на G,
$G/H \cong F^*$

Нека G е група и $H \subseteq G \ (H \neq \emptyset)$

Казваме, че H е подгрупа на $G \ (H \leq G)$, ако:

$$\left.\begin{array}{l} 1)\ \forall a,b \in H \Rightarrow \cancel{ab}\ \cancel{a \circ b \in H}\ a \circ b \in H \\ 2)\ \forall a \in H \Rightarrow a^{-1} \in H \end{array}\right\} \forall a,b \in H \Rightarrow a^{-1}b \in H$$

Казваме, че H е нормална подгрупа на $G \ (H \trianglelefteq G)$, ако:

$$gH = Hg \quad \forall g \in G \iff gHg^{-1}=H$$

$$gH=\{gh \mid h \in H\}$$

$$gh \neq hg$$

$$gh = h'g$$

$$\left.\begin{array}{l} 1)\ (1,b_1,c_1) \circ_1 (1,b_2,c_2) = (1, b_1b_2, c_2+c_1b_2) \in H \\ 2)\ (1,b,c)^{-1} = \left(\frac{1}{1}, \frac{1}{b}, -\frac{c}{1.b}\right) = \left(1, \frac{1}{b}, -\frac{c}{b}\right) \in H \end{array}\right\} \Rightarrow H \leq G \text{ (подгрупа)} \text{ (I вариант)}$$

$$(1,b_1,c_1)^{-1} \circ_1 (1,b_2,c_2) = \left(1, \frac{1}{b_1}, -\frac{c_1}{b_1}\right) \circ_1 (1,b_2,c_2) =$$

$$= \left(1, \frac{b_2}{b_1}, -\frac{c_1b_2}{b_1}+c_2\right) \in H \Rightarrow H \leq G \quad \text{(II вариант)}$$

$(1,2,1)$
$(1,1,1)$ $\Rightarrow$ ~~H ⊴ G~~ H е неабелева подгрупа

Th (за хомоморфизми): Нека G_1 и G_2 са групи. Нека $\varphi: G_1 \to G_2$ е хомоморфизъм и $\operatorname{Ker}\varphi = H$. Тогава $H \trianglelefteq G_1$ и $G_1/H \cong \operatorname{Im}\varphi$

Търсим $\varphi: G \to \mathbb{F}^*$, φ - хомоморфизъм и

$\operatorname{Ker}\varphi = \{(a,b,c) \in G \mid \varphi((a,b,c)) = 1\} = H$

$(a,b,c) \xrightarrow{\varphi} x \in \mathbb{F}^*$ ($\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ мултипликативна група ($\circ_2 = *$))

$(1,b,c) \xrightarrow{\varphi} 1$

$\varphi((a,b,c)) \overset{def.}{=} a$

$\varphi((a_1,b_1,c_1) \circ_1 (a_2,b_2,c_2)) = \varphi((a_1,b_1,c_1)) \circ_2 \varphi((a_2,b_2,c_2))$

$\qquad a_1 a_2 \qquad = \qquad a_1 \; * \; a_2$

~~$\Rightarrow \varphi$ е хомоморфизъм~~

Нека $(a,b,c) \in \operatorname{Ker}\varphi$

$1 = \varphi((a,b,c)) = a \Rightarrow (a,b,c) = (1,b,c) \in H \Rightarrow \operatorname{Ker}\varphi \subseteq H$

От Th. (за хомоморфизмите) $\Rightarrow H \trianglelefteq G$

$$\left.\begin{array}{l} G/H \cong \operatorname{Im}\varphi \\ \mathbb{F}^* \supseteq \operatorname{Im}\varphi \\ \text{Нека } x \in \mathbb{F}^*. \text{ Тогава } \varphi((x,1,0)) = x \Rightarrow \operatorname{Im}\varphi \supseteq \mathbb{F}^* \end{array}\right\} \Rightarrow G/H \cong \mathbb{F}^*$$

(14) Да се докаже, че ако d е цяло число, което не е точен квадрат, то $\mathbb{Z}[\sqrt{d}]/(\sqrt{d}) \cong \mathbb{Z}_{|d|}$

Решение: $\mathbb{Z}[\sqrt{d}] = \{a+b\sqrt{d} \mid a, b \in \mathbb{Z}\}$

$$(\sqrt{d}) = \{(a+b\sqrt{d}).\sqrt{d} \mid a, b \in \mathbb{Z}\} = \cancel{\{A+B\sqrt{d} \mid A, B \in \mathbb{Z}}$$

$$= \{\underbrace{a}_{B}\sqrt{d} + \underbrace{bd}_{A} \mid a, b \in \mathbb{Z}\} = \{A+B\sqrt{d} \mid A, B \in \mathbb{Z}, A \equiv 0 (\text{mod } |d|)\}$$

$$\overset{\shortparallel}{\{a+b\sqrt{d} \in \mathbb{Z}[\sqrt{d}] \mid \varphi(a+b\sqrt{d}) = \overline{0}\}}$$

Търсим $\varphi: \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}_{|d|}$, $\operatorname{Ker}\varphi = (\sqrt{d})$

$\varphi(a+b\sqrt{d}) = \overline{k}$, $k \in \{0, 1, \ldots, |d|-1\}$

$\varphi(A+B\sqrt{d}) = \overline{0}$

d/A

$\varphi(a+b\sqrt{d}) = \overline{a} \in \mathbb{Z}_{|d|}$

1) $\varphi((a+b\sqrt{d})+(c+e\sqrt{d})) = \varphi((a+c)+(b+e)\sqrt{d}) = \overline{a+c} = \overline{a} + \overline{c} = \cancel{\varphi(a+b\sqrt{d})}$
$= \varphi(a+b\sqrt{d}) + \varphi((c+e\sqrt{d}))$

2) $\varphi((a+b\sqrt{d}).(c+e\sqrt{d})) = \varphi(ac+be.d+(ae+bc)\sqrt{d}) = \overline{ac+bed} =$
$= \overline{ac} + \overline{b.e.d} = \overline{a}.\overline{c} + \overline{b}.\overline{e}.\underset{\overline{0}}{\overline{d}} = \overline{a}.\overline{c} = \varphi((a+b\sqrt{d})).\varphi((c+e\sqrt{d}))$

$\Rightarrow \varphi$ е хомоморфизъм на пръстени

Очевидно $(\sqrt{d}) \subseteq \operatorname{Ker}\varphi \Leftrightarrow$ Нека $a+b\sqrt{d} \in (\sqrt{d}) \Rightarrow a \equiv 0 (\text{mod } |d|)$

Тогава $\varphi(a+b\sqrt{d}) = \overline{a} = \overline{0} \Rightarrow a+b\sqrt{d} \in \operatorname{Ker}\varphi$

Нека $a+b\sqrt{d} \in \operatorname{Ker}\varphi$

$\overline{0} = \varphi(a+b\sqrt{d}) = \overline{a} \Rightarrow \overline{a} = 0$, т.е. $a \equiv 0 (\text{mod } |d|) \Rightarrow a+b\sqrt{d} \in (\sqrt{d})$

$\Rightarrow \operatorname{Ker}\varphi = (\sqrt{d})$

От Th. (за хомоморфизмите) $\Rightarrow \mathbb{Z}[\sqrt{d}]/(\sqrt{d}) \cong \operatorname{Im}\varphi$

$\left.\begin{array}{l} \operatorname{Im}\varphi \subseteq \mathbb{Z}_{|d|} \\ \text{Нека } \overline{k} \in \mathbb{Z}_{|d|} : \varphi((k+0\sqrt{d})) = \overline{k} \Rightarrow \mathbb{Z}_{|d|} \subseteq \operatorname{Im}\varphi \end{array}\right\} \operatorname{Im}\varphi = \mathbb{Z}_{|d|} \Rightarrow$

$\Rightarrow \mathbb{Z}[\sqrt{d}]/(\sqrt{d}) \cong \mathbb{Z}_{|d|}$

(15) $2^{19.73} \equiv 2 \pmod{19.73}$

$2^{19-1} \equiv 1 \pmod{19}$

$\overset{\shortparallel}{2^{18}}$

Th. (Euler): Нека $a, n \in \mathbb{N}$ и $n \not| a$

Тогава $a^{\varphi(n)} \equiv 1 \pmod n$.

Th. (Fermat): Нека p-просто и $p \not| a$.

Тогава $a^{p-1} \equiv 1 \pmod p$

$\varphi(n)$ - брой на естествените числа $< n$ и взаимно прости с n.

$\varphi(p) = p-1$ $\qquad \varphi(ab) = \varphi(a).\varphi(b) \quad (a,b)=1$

$n = p_1^{\alpha_1} \dots p_k^{\alpha_k} \qquad \varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$

$\varphi(n) = \varphi(p_1^{\alpha_1} \dots p_k^{\alpha_k}) = \varphi(p_1^{\alpha_1})\,\varphi(p_2^{\alpha_2}) \dots \varphi(p_k^{\alpha_k}) =$

$= (p_1^{\alpha_1} - p_2^{\alpha_2}) \dots (p_k^{\alpha_k} - p_k^{\alpha_{k-1}}) = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}(1-\frac{1}{p_1})(1-\frac{1}{p_2}) \dots (1-\frac{1}{p_k})$

$\varphi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2}) \dots (1-\frac{1}{p_k})$

От Th. (Euler) за $n = 19.73$ и $a=2$ имаме:

$2^{\varphi(19.73)} \equiv 1 \pmod{19.73}$

$\varphi(19.73) = \varphi(19).\varphi(73) = (19-1).(73-1)$

$2^{(19-1)(73-1)} \equiv 1 \pmod{19.73}$

$18.72 = (19-1)(73-1) = 19.73 - 19 - 73 + 1 = 19.73 - 91$

$2^{19.73-91} \equiv 1 \pmod{19.73} \longleftarrow$

$91 = 19 - 73 - 1$

$2^{91} = 2^{19}.2^{72} = 2^{19+72}$

$2^{72} \equiv 1 \pmod{73}$ $\Rightarrow$ $2^{19}.2^{72} \equiv 2 \pmod{19.73}$

$2^{19} \equiv 2 \pmod{19}$ $\qquad \underset{2^{91}}{\shortparallel}$

$73 \mid 2^{72}-1$

$19 \mid 2^{19}-2$

$19.73 \mid (2^{72}-1)(2^{19}-2) \Rightarrow (2^{72}-1)(2^{19}-2) \equiv 0(19.73)$

$2^{72}.2^{19} - 2^{73} - 2^{19} + 2 \equiv 0(19.73)$

$2^{19.73} \equiv 2^{91} \equiv 2(19.73)$

(16) Нека p-просто и $0 \le k \le p-1$. Тогава $\binom{p-1}{k} \equiv (-1)^k (\bmod p)$

Решение: $\binom{p-1}{k} = \frac{(p-1)(p-1-1)\dots(p-1-k+1)}{k!} = \frac{(p-1)\dots(p-k)}{k!}$

$$\left.\begin{array}{l} p-1 \equiv -1 \ (\bmod p) \\ p-2 \equiv -2 \ (\bmod p) \\ \vdots \\ p-k \equiv -k \ (\bmod p) \end{array}\right\} \Rightarrow (p-1)\dots(p-k) \equiv (-1)^k . k! \ (\bmod p) \Rightarrow$$

$(k!, p) = 1$, защото $0 \le k \le p-1$

$\Rightarrow \binom{p-1}{k} \equiv (-1)^k \ (\bmod p)$

(17) Нека p-просто, $p \nmid a$ и $a^p \equiv \pm 1 (\bmod p)$. Тогава $a^p \equiv \pm 1 (p^2)$

Решение: $a^p \equiv 1 \ (\bmod p) \Longleftrightarrow a^p - 1 \equiv 0 (p)$

$a^p - 1 = (a-1)(a^{p-1} + a^{p-2} + \dots + a + 1)$

От Th. (Ферма): $a^{p-1} \equiv 1 \ (\bmod p)$

$$\left.\begin{array}{l} a^p \equiv a \ (\bmod p) \\ a^p \equiv 1 \ (\bmod p) \end{array}\right\} a \equiv 1 \ (\bmod p) \Leftrightarrow a - 1 \equiv 0 (p) \Leftrightarrow$$

$\Leftrightarrow p \mid a-1$

От Th. (Fermat): $a^{p-1} \equiv 1 \ (\bmod p)$

$a^{p-2} \equiv 1 (p)$

$a^{p-3} \equiv 1 (p)$

$\vdots$

$a^2 \equiv 1 (p)$

$a \equiv 1 (p)$

$1 \equiv 1 (p)$

$a^{p-1} + a^{p-2} + \dots + a^2 + a + 1 \equiv 1 + 1 + \dots + 1 \equiv p \equiv 0 \ (\bmod p)$

$p \mid (a^{p-1} + \dots + a^1 + a^0)$. Тогава $p^2 \mid (a-1)(a^{p-1} + \dots a^1 + 1) = a^p - 1$

$\Rightarrow a^p \equiv 1 \ (\bmod p^2)$

(18) Нека $f(x) = x^{2n+1} - 1$

$\boxed{8.4}$ г) $g = x^{6n+4} + x^{4n+3} + x^{2n+2} - 6$.

$R(f,g) = ?$

Решение: Ако $\alpha_1, \dots, \alpha_{2n+1}$ са корени на f и $\beta_1, \dots, \beta_{6n+4}$ са корени на g, то $R(f,g) = 1^{6n+4} \cdot \prod_{i=1}^{2n+1} g(\alpha_i)$

$$\| \\ (-1)^{(2n+1)(6n+4)} R(g,f)$$

$$R(g,f) = 1^{2n+1} \prod_{j=1}^{6n+4} f(\beta_j)$$

$$R(f,g) = \prod_{i=1}^{2n+1} g(\alpha_i)$$

$$g(\alpha_i) = \alpha_i^{6n+4} + \alpha_i^{4n+3} + \alpha_i^{2n+2} - 6$$

тъй като α_i е корен на f, то $f(\alpha_i) = 0 \Leftrightarrow \alpha_i^{2n+1} - 1 = 0 \Leftrightarrow$

$\Leftrightarrow \alpha_i^{2n+1} = 1 \quad /.\alpha_i /.\alpha_i^{2n+2}$

$\alpha_i^{2n+2} = \alpha_i$

$\alpha_i^{4n+3} = \alpha_i^{2n+2} = \alpha_i$

$\alpha_i^{6n+4} = \alpha_i^{4n+3} = \alpha_i$

$g(\alpha_i) = 3\alpha_i - 6 = 3(\alpha_i - 2)$, $i = 1, \dots, 2n+1$

$$R(f,g) = 3^{2n+1} \prod_{i=1}^{2n+1} (\alpha_i - 2) = (-1)^{2n+1} \cdot 3^{2n+1} \prod_{i=1}^{2n+1} (2 - \alpha_i) =$$

$$= -3^{2n+1} \cdot g(2) = -3^{2n+1}(2^{6n+4} + 2^{4n+3} + 2^{2n+2} - 6)$$

$$R(f,g) = a_0^s b_0^n \prod_{i=1}^{n} \prod_{j=1}^{s} (\alpha_i - \beta_j)$$

(19) Да се намерят стойностите на параметъра λ, за които полиномът $f = x^3 + x^2 + \lambda - 2$ има кратен корен.

Нека $f = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$; $\alpha_1, \dots, \alpha_n$ – корени на f

$$\mathcal{D}(f) = a_0^{2n-2} \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2$$

$$\mathcal{D}(f) = a_0^{2n-2} \begin{vmatrix} S_0 & S_1 & \dots & S_{n-1} \\ S_1 & S_2 & \dots & S_n \\ \vdots & \vdots & & \vdots \\ S_{n-1} & S_n & \dots & S_{2n-2} \end{vmatrix} \qquad S_k = x_1^k + x_2^k + \dots + x_n^k$$

$\sigma_1, \sigma_2, \dots, \sigma_n$

$$S_k - \sigma_1 S_{k-1} + \sigma_2 S_{k-2} - \dots + (-1)^{k-1} \sigma_{k-1} S_1 + (-1)^k k \sigma_k = 0$$

формули на Нютон

$S_0 = n$

$S_1 - 1.\sigma_1 = 0 \Rightarrow S_1 = \sigma_1$

$S_2 - S_1 \sigma_1 + 2\sigma_2 = 0$

$S_2 = \sigma_1^2 - 2\sigma_2$

$\vdots$

Решение: $S_0 = 3$ $\qquad \sigma_1 = -3$

$S_1 = \sigma_1 = -3$ $\qquad \sigma_2 = 0$

$S_2 = 9 - 2.0 = 9$ $\qquad \sigma_3 = -\lambda + 2$

$S_3 - \sigma_1 S_2 + \sigma_2 S_1 - 3\sigma_3 = 0$

$S_3 = -27 - 3\lambda + 6 = -21 - 3\lambda$

$S_4 - \sigma_1 S_3 + \sigma_2 S_2 - \sigma_3 S_1 + 4\sigma_4 = 0$

$S_4 = 63 + 9\lambda + 3\lambda - 6 = 57 + 12\lambda$

$$\mathcal{D}(f) = 1^4 \begin{vmatrix} 3 & -3 & 9 \\ -3 & 9 & -21-3\lambda \\ 9 & -21-3\lambda & 57+12\lambda \end{vmatrix} = 27 \begin{vmatrix} 1 & -1 & 3 \\ -1 & 3 & -7-\lambda \\ 3 & -7-\lambda & 19+4\lambda \end{vmatrix} =$$

$$=27\begin{vmatrix} 1 & -1 & 3 \\ 0 & 2 & -4-\lambda \\ 0 & -4-\lambda & 10+4\lambda \end{vmatrix} = 27\cdot 1 \begin{vmatrix} 2 & -4-\lambda \\ -4-\lambda & 10+4\lambda \end{vmatrix} = 27\left(20+8\lambda-(4+\lambda)^2\right) =$$

$$= 27(20+8\lambda-16-8\lambda-\lambda^2) = 27(-\lambda^2+4) = -27(\lambda-2)(\lambda+2)$$

f има кратен корен $\iff \mathcal{D}(f)=0 \iff -27(\lambda-2)(\lambda+2)=0$

$\iff \lambda=\pm 2$

$$f = x^3+3x^2-4 = x^3+2x^2+x^2-4 = x^2(x+2)+(x+2)(x-2) =$$

$$= (x+2)(x^2+x-2) = (x+2)^2(x-1)$$

x^3+px+q

(20) Нека $\Sigma = \frac{x_1}{(1+x_1)^2} + \frac{x_2}{(1+x_2)^2} + \frac{x_3}{(1+x_3)^2}$. Да се изрази Σ като функция на коефициентите p и q на полинома $f=x^3+px+q$ (x_1, x_2, x_3 са корени на f)

Решение: $x_1+x_2+x_3=0$

$x_1x_2+x_1x_3+x_2x_3=p$

$x_1x_2x_3=-q$

$f=a_0x^n+a_1x^{n-1}+\dots+a_n$, $\alpha_1,\dots,\alpha_n$ - корени на f.

$$f(a)=a_0\prod_{i=1}^{n}(a-\alpha_i)$$

$$\sum_{i=1}^{n}\frac{1}{a-\alpha_i} = \frac{f'(a)}{f(a)}$$

$$\sum_{i=1}^{n}\frac{1}{(a-\alpha_i)^2} = \frac{(f'(a))^2-f(a)\cdot f''(a)}{(f(a))^2}$$

$$\frac{x_1+1-1}{(1+x_1)^2} = \frac{\cancel{x_1+1}}{(x_1+1)^{\cancel{2}}} - \frac{1}{(x_1+1)^2} = \frac{1}{1+x_1} - \frac{1}{(1+x_1)^2}$$

$$\Sigma = \frac{1}{1+x_1} - \frac{1}{(1+x_1)^2} + \frac{1}{1+x_2} - \frac{1}{(1+x_2)^2} + \frac{1}{1+x_3} - \frac{1}{(1+x_3)^2} =$$

$$= \sum_{i=1}^{3} \frac{1}{1+x_i} - \sum_{i=1}^{3} \frac{1}{(1+x_i)^2} = -\sum_{i=1}^{3} \frac{1}{-1-x_i} - \sum_{i=1}^{3} \frac{1}{(-1-x_i)^2} =$$

$$= -\frac{f'(-1)}{f(-1)} - \frac{(f'(-1))^2 - f(-1)f''(-1)}{(f(-1))^2} = -\frac{3+p}{-1-p+q} - \frac{(3+p)^2 + 6(-1-p+q)}{(-1-p+q)^2}$$

$f(-1) = -1-p+q$

$f'(-1) = 3+p$

$f''(-1) = -6$

(21) $f = a_0x^n + a_1x^{n-1} + \dots + a_n \in \mathbb{Z}[x]$, $\alpha_1, \dots, \alpha_n$ - корени на f

Ако съществува p - просто, такова че:

1) $p \nmid a_0$ и $p \mid a_1, p \mid a_2, \dots, p \mid a_n$

2) $p^2 \nmid a_m$

то f е неразложим над $\mathbb{Z}$

$f = x^4 + 6x^2 + 3x + 3$

(22) Да се докаже, че полиномът $f = x^5 - x^2 + 1$ е неразложим над $\mathbb{Q}$

Решение:

~~1x1x1x1x1x~~
~~1x1x1x2x~~
~~1x1x3x~~
~~1x4x~~
~~1x 2x 2~~

$1 \times 1 \times 1 \times 1 \times 1$ ×
$1 \times 1 \times 1 \times 2$ ×
$1 \times 1 \times 3$ ×
1×4 ×
$1 \times 2 \times 2$ ×
2×3

$\frac{\pm 1}{\pm 1}$ $f(1) = 1 \neq 0$
$f(-1) = -1 \neq 0$

Допускане, че $f = (x^2 + ax + b)(x^3 + cx^2 + dx + e)$

$x^5 - x^2 + 1 = x^5 + (c+a)x^4 + (d + ac + b)x^3 + (e + ad + bc)x^2 +$
$+ (bd + ae)x + be$

$$\begin{cases} c+a=0 \\ d+ac+b=0 \\ e+ad+bc=-1 \\ bd+ae=0 \\ be=1 \end{cases}$$

$be=1 \longrightarrow b=e=1$

$be=1 \longrightarrow b=e=-1$

I) $b=e=1$

$$\begin{cases} a+c=0 \longrightarrow c=-a \\ d+ac=-1 \\ 1+ad+c=-1 \\ d+a=0 \longrightarrow d=-a \end{cases}$$

$-a-a^2=-1$

$a^2+a-1=0$

$a_{1,2}=\dfrac{-1\pm\sqrt{5}}{2} \notin \mathbb{Z}$

II) $b=e=-1$

$$\begin{cases} a+c=0 \Rightarrow c=-a \\ d+ac=1 \\ -1+ad-c=-1 \\ d+a=0 \Rightarrow d=-a \end{cases}$$

$-a-a^2=1$

$a^2+a+1=0$

$a_{1,2}=\dfrac{-1\pm i\sqrt{3}}{2} \notin \mathbb{Z}$

(23) Нека $f=a_0x^n+a_1x^{n-1}+\dots+a_n \in F[x]$

$\alpha \in L$ е k-кратен корен на $f \Leftrightarrow f(\alpha)=f'(\alpha)=\dots=f^{(k-1)}(\alpha)=0$

и $f^{(k)}(\alpha)\neq 0$

L - разширение на F