$\boxed{5.}$

Матрична игра 2×2

- първия има оптимална чиста стратегия $\Rightarrow$ втория има оптимална чиста стратегия

$$p(x,y) = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$\cancel{p(x^*, y^*)}$

$$\underset{\forall x \in X}{p(x, y^*)} \le \underset{\|}{p(x^*, y^*)} \le \underset{\forall y \in Y}{p(x^*, y)}$$

$x^* = (1, 0)$

$$a_{11}y_1^* + a_{12}y_2^* \le a_{11}y_1 + a_{12}y_2$$

$f(y) = a_{11}y_1 + a_{12}y_2$ $\quad \cancel{\min f(y) = a_{11}y_1}$ $\quad y_1 + y_2 = 1 \quad y_2 = 1 - y_1$

$$\min_{y_1 \in [0,1]} f(y) = \min a_{11}y_1 + (1-y_1)a_{12} = (a_{11} - a_{12})y_1 + a_{12}$$

I сл. $a_{11} - a_{12} > 0 \Rightarrow y_1^* = 0$

II сл. $a_{11} - a_{12} < 0 \Rightarrow y_1^* = 1$

III сл. $a_{11} - a_{12} = 0$; $\{a_{11} = a_{12}\} = V$

$$\begin{pmatrix} v & v \\ a & b \end{pmatrix} \begin{matrix} v \\ a \end{matrix}$$
$$\quad v \quad \max(v, b)$$

Нека $a \le b$

$\bar{x} = (0, 1)$

$$p(\bar{x}, y^*) = a y_1^* + b y_2^* \le v(y_1^* + y_2^*) = \boxed{v}$$

$\boxed{a} = a(y_1^* + y_2^*)$

6. игра

матрица $A = \{a_{ij}\}_{2009\times 2009}$

$a_{ij} = i - j$ за $\forall i,j$

~~∀ стратегия~~ оптимална на I е оптимална за II

$a_{ji} = j - i = -a_{ij}$

$\boxed{A^T = -A}$ антагонистични игри

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{2009} \end{pmatrix}$$

$$p(x,y) = x^T A y \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_{2009} \end{pmatrix}$$

$$(p(x,y))^T = (x^T A y)^T = y^T A^T x = -y^T A x$$

$$- \left| \begin{array}{l} p(x,x) = x^T A x \\ p(x,x) = -x^T A x \end{array} \right. \qquad \begin{array}{l} 2x^T A x = 0 \\ x^T A x = 0 = p(x,x) \end{array}$$

$$\max_{x \in X} \left(\min_{y \in Y} p(x,y) \right)$$

$$\underbrace{\min_{y \in Y} p(x,y)}_{v(x)} \le 0 = p(x,x)$$

$$\left. \begin{array}{l} \max\limits_{x \in X} v(x) \le 0 \\ v_{II} \ge 0 \end{array} \right\} \Rightarrow v = 0 \qquad \text{За оптимални: } v_I = v_{II} \Rightarrow v = 0$$

$$\underset{i = 1 \dots 2009}{p(i, y^*)} \le 0 \le \underset{j = 1 \dots 2009}{p(x^*, j)}$$

x^*

$$\boxed{p(x^*, i) \ge 0 \iff p(i, x^*) \le 0}$$

$$\sum_{j=1}^{2009} a_{ji} x_i^* \ge 0 \Leftrightarrow \sum_{j=1}^{2009} a_{ij} x_j^*$$

$$\sum_{j=1}^{2009} - a_{ij} x_j^* \ge 0$$

$$\sum_{j=1}^{2009} a_{ij} x_j^* \le 0$$

$$i = j$$

$\boxed{7}$ игра от матрица

$$\begin{array}{cc|l} & & \min(x,y) \\ A = \begin{pmatrix} 4 & -3 \\ 2 & 1 \\ -1 & 3 \\ 0 & -1 \\ -3 & 0 \end{pmatrix} & & \begin{array}{l} -3 \\ 1 \\ -1 \\ -1 \\ -3 \end{array} \\ 4 \quad 3 & & \\ \max(x) \quad \max(y) & & \end{array}$$

$\max(\min(x,y))$

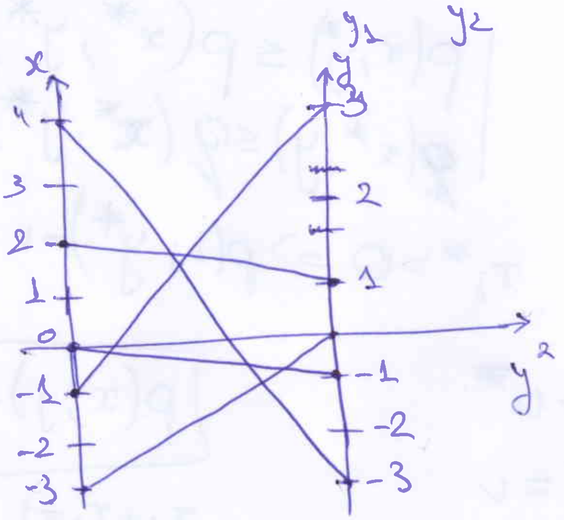
$\min(\max(x), \max(y))$

$y_1^* > 0$

$y_2^* > 0$

$x^*\left(0, \frac{4}{5}, \frac{1}{5}, 0, 0\right)$

$y^*\left(\frac{2}{5}, \frac{3}{5}\right)$

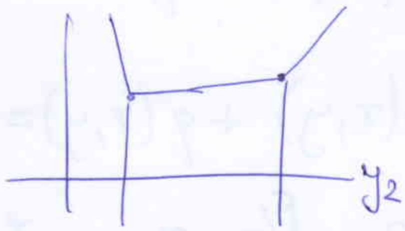


за $y \to$ max траектория в най-ниската точка

за $x \to$ min траектория в най-високата точка

I вар.

II вар.



$$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

$$\begin{cases} 2x_1 - x_2 = V \\ +x_1 + 3x_2 = V \\ x_1 + x_2 = 1 \end{cases} \qquad \begin{cases} x_1 - 4x_2 = 0 \\ x_1 + x_2 = 1 \end{cases} \qquad \boxed{x_1^* = \frac{4}{5} \quad x_2^* = \frac{1}{5} \quad V = \frac{7}{5}}$$

$$\begin{cases} 2y_1 + y_2 = V \\ -y_1 + 3y_2 = V \\ y_1 + y_2 = 1 \end{cases} \qquad 3y_1 - 2y_2 = 0 \qquad \boxed{y_1^* = \frac{2}{5} \quad y_2^* = \frac{3}{5}}$$

8

$$A = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

max печалби I играч u^* max печалби II играч v^*

	y_1	y_2
x_1	4 / 3	2 / 4
x_2	5 / 1	3 / 2

равновесие по Неш

$$u_i(x_i, \cancel{y}^*) \le u_i(x_i^*, \cancel{y}^*)$$

$\forall x_i \in X$

$u_1(x,y) = axy$

$x^* = (0,1)$

$y^* = (0,1)$

$$p(x,y^*) \le p(x^*,y^*)$$
$$q(x^*,y) \le q(x^*,y^*)$$

$x_i^* > 0 \Rightarrow p(i; y^*) = v$

$$p(1,y^*) = p(2,y^*) = u^*$$
$$q(x^*,1) = q(x^*,2) = v$$

$$\boxed{p(x;j) = \sum_{i=1}^{m} a_{ij} x_i}$$

$x_1 + x_2 = 1$

$y_1 + y_2 = 1$

9 $p(x,y) + q(x,y) = 0$ – деф. на игра с нулева сума

$$X = \left\{ (x_1, x_2, \dots, x_n) \in R^m ; \ x_i > 0; \ i = 1, \dots, m; \ \sum_{i=1}^{m} x_i = 1 \right\}$$

$$Y = \left\{ (y_1, y_2, \dots, y_n) \in R^n : \ y_j > 0; \ j = 1, \dots, n; \ \sum_{j=1}^{n} y_j = 1 \right\}$$

$$p(x,y^*) \le p(x^*,y^*)$$
$$q(x^*,y) \le q(x^*,y^*)$$
$$-p(x^*,y) \le -p(x^*,y^*)$$

$$\boxed{p(x,y^*) \le p(x^*,y^*) \le p(x^*,y)}$$

$p(x^*,y^*) = v$

$v_I = \max_{x \in X} \left(\min_{y \in Y} p(x,y) \right)$

$\|$

$v_{II} = \min_{y \in Y} \left(\max_{x \in X} p(x,y) \right)$

$\|$

v

$\boxed{10}$

биматрична игра

$$A = p(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$
$$B = q(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_i y_j$$

} очаквани печалби

$$F: X \times Y \to X \times Y$$
$$\boxed{F(x,y) \to (\bar{x}, \bar{y})}$$

$x = (x_1, x_2, \dots, x_m)$

$y = (y_1, y_2, \dots, y_n)$

$$\boxed{\begin{aligned} \bar{x}_i &= \frac{x_i + \max\{0, p(i,y) - p(x,y)\}}{1 + \sum_{k=1}^{m} \max\{0, p(k,y) - p(x,y)\}} \\ \bar{y}_j &= \frac{y_j + \max\{0, q(x,j) - q(x,y)\}}{1 + \sum_{l=1}^{n} \max\{0, q(x,l) - q(x,y)\}} \end{aligned}}$$

$$p(x, y^*) \underset{<}{\overset{>}{\leq}} p(x^*, y^*)$$
$$q(x^*, y) \leq q(x^*, y^*)$$ – неподвижната точка

$\boxed{11}$ 4 от 6-те ~~теореми~~ аксиоми на Неш

$(S, (u^*, v^*)) \to (\bar{u}, \bar{v})$

u^*, v^* – индивидуалния максимум

S – max, ако си сътрудничим

1. $(\bar{u}, \bar{v}) \in S$
2. $\bar{u} \geq u^*$
   $\bar{v} \geq v^*$
3. $\nexists$ вектор $(\tilde{u}, \tilde{v}) \in S$
   $\tilde{u} \geq \bar{u}$, $\tilde{v} \geq \bar{v}$ или $\tilde{u} > \bar{u}$, $\tilde{v} \geq \bar{v}$
4. $(S, (u^*, v^*)) \to (\bar{u}, \bar{v}) \in T$ – изпъкнало, ~~ограничено~~ ограничено, затворено
   $(T, (u^*, v^*)) \to (\bar{u}, \bar{v})$

5. $L: \begin{cases} u = \alpha_1 u + \beta_1 \\ v = \alpha_2 v + \beta_2 \end{cases}$ $\alpha_1, \alpha_2 > 0$
   лин. оператор

$(L(S), L(u^*, v^*)) \to L(\bar{u}, \bar{v})$

$$\begin{cases} \max g(u,v) \\ g(u,v) = (u - u^*)(v - v^*) \\ u \geq u^* \\ v \geq v^* \\ (u,v) \in S \end{cases}$$

6. Ако от $(u,v) \in S \Rightarrow (v,u) \in S$,
   $u^* = v^*$
   $\Rightarrow (S, (u^*, u^*)) \to (\bar{v}, \bar{v})$

$\boxed{12}$

$N = \{1, 2, \dots, n\}$

$V_1 : 2^N \to \mathbb{R}$

$|2^N| = 2^n$

1. $v(\emptyset) = 0$

2. $v(S \cup T) \geq v(S) + v(T)$ за $S \cap T = \emptyset$ > характеристични показатели (коалиция)

$x = (x_1, x_2, \dots, x_n)$ – вектор да е подялба

1. $x_1 \geq v(\{i\})$
2. $\sum_{i=1}^{n} x_i = v(N)$

$\boxed{13}$ $y = (y_1, y_2, \dots, y_n)$ $y \underset{S}{\geq} x$ 1. $y_i > x_i \quad i \in S$

2. $\sum_{j \in S} y_j \leq v(S)$

$\boxed{14}$ $a = (330, 490, 180)$

$b = (330, 500, 190)$

$v(\{1\}) = 200$

$v(\{2\}) = 300$

$v(\{3\}) = 0$

$v(\{1,2\}) = 800$

$v(\{2,3\}) = 650$

$v(\{1,3\}) = 500$

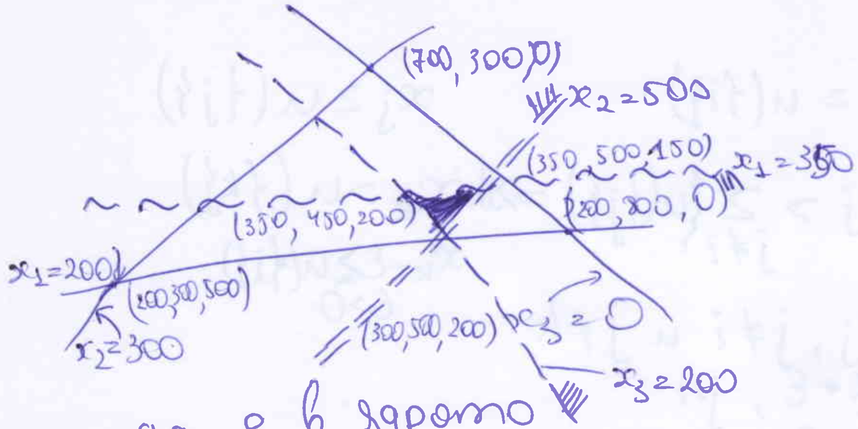
$v(\{1,2,3\}) = 1000$

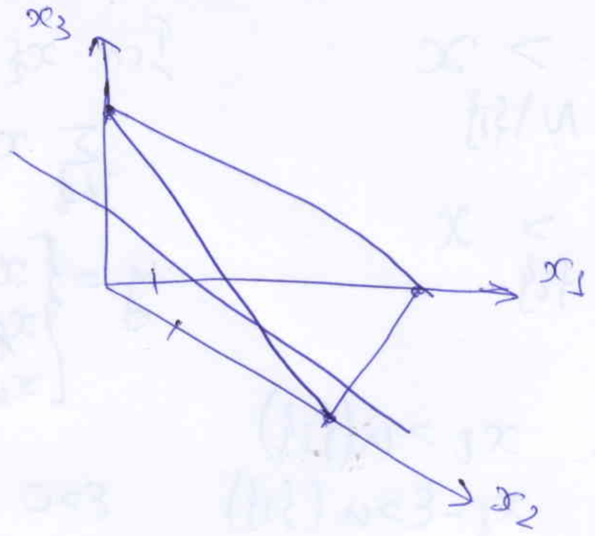
$v(\{1\}) + v(\{3\}) < v(\{1,3\})$ – характеристична игра

$x \in C(v) \Leftrightarrow \begin{cases} \sum_{i \in S} x_i \geq v(S) & \forall S \subseteq N \\ \sum_{i=1}^{n} x_i = v(N) \end{cases}$

$x \in \mathbb{R}^n$

$b \notin C(v)$ $\quad b_1 + b_2 + b_3 = 1020 > 1000 = v(\{1,2,3\})$





за да е в ядрото

$$\sum_{i \in S} x_i \geq v(S)$$

$$\left| \begin{array}{l} x_1 + x_2 \geq 800 \\ x_2 + x_3 \geq 650 \\ x_1 + x_3 \geq 500 \end{array} \right. \iff \begin{array}{l} x_3 \leq 200 \\ x_1 \leq 350 \\ x_2 \leq 500 \end{array}$$

a, b, c

$$C(v) = \left\{ x \in R^n : \begin{array}{c} x = \alpha_1 a + \alpha_2 b + \alpha_3 c \\ \alpha_1 \geq 0 \\ \alpha_2 \geq 0 \\ \alpha_3 \geq 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 1 \end{array} \right\}$$

15 а) $\boxed{\sum_{i=1}^{n} u(\{i\}) < u(N)}$ $N = \{1, \dots, n\}$

б) $u(N) = u(\{i\}) + u(N \setminus \{i\})$

$C(u) = \emptyset$

Допускане ~~...~~ противното, т.е. $\exists x \in C(u)$ $\left\{ \begin{array}{l} x_i \geq u(\{i\}) \\ \sum_{i \in S} x_i \geq u(S) \Rightarrow \\ \sum_{i=1}^{n} x_i = u(N) \end{array} \right.$

$\Rightarrow \underbrace{\sum x_i}_{u(N)} \geq \underbrace{\sum u(\{i\})}_{\wedge\, u(N)}$

$x_i \geq u(\{i\})$

$u(N) = \sum_{i=1}^{n} x_i > \sum_{i=1}^{n} u(\{i\}) < u(N)$

$u(\{i\}) + u(N \setminus \{i\}) > \sum_{j=1}^{n} u(\{j\})$

$u(N \setminus \{j\}) > \sum_{\substack{j=1 \\ j \neq i}}^{n} u(\{j\})$

$y \underset{N\setminus\{i\}}{>} x$

$y \underset{\{i\}}{>} x$

I сл. $x_i = u(\{i\})$ $\quad x_j \geq u(\{j\})$

$\sum_{j\neq i} x_j > \sum_{j\neq i} u(\{j\}) \Rightarrow \exists k: x_k > u(\{k\})$

$x_k - \varepsilon \geq u(\{i\})$, $\varepsilon > 0$

$y = \begin{cases} x_j, & j\neq i \text{ и } j\neq k \\ x_i + \varepsilon, & j = i \\ x_k - \varepsilon \end{cases}$

II сл. $x_i > u(\{i\})$

$x_i - \varepsilon > u(\{i\})$ $\quad \varepsilon > 0$

$y_j = \begin{cases} x_i - \varepsilon, & j = i \\ x_j + \frac{\varepsilon}{|N\setminus\{i\}|}, & j\neq i \end{cases}$

$y \underset{N\setminus\{i\}}{>} x$

$\sum_{j\neq i} y_i \leq u(N(\{i\}))$ $\quad \cancel{\varepsilon x_i}$ $|x_i - u(\{i\}) > \varepsilon$

$\boxed{\sum_{j\neq i} x_j + \varepsilon \overset{?}{\leq} u(N\setminus\{i\})}$

16 биматрични игри ; равновесие на Неш

$A = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 2 & 1 \end{pmatrix}$ $\quad y_1 = 0 \quad y_3 = 0$

$B = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix}$ $\quad y_1 \quad y_2 \quad y_3$

$p(x,y) = \cancel{\sum\sum a_{ij} x_i}$ $\sum\sum a_{ij} x_i y_j$

$q(x,y) = \sum\sum b_{ij} x_i y_j$

(x^*, y^*)

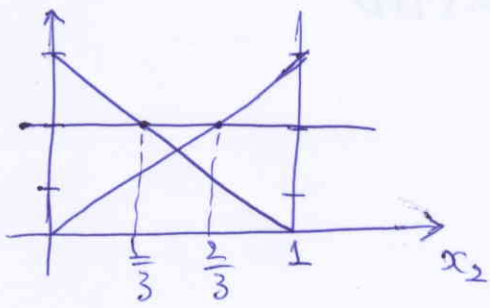
$p(x, y^*) \leq p(x^*, y^*)$

$q(x^*, y) \leq q(x^*, y^*)$

$\begin{cases} 2y_1^* + 4y_2^* + 5y_3^* = v_1 \\ 4y_1^* + 2y_2^* + y_3^* = v_1 \end{cases}$ $\quad \sum y_i = 1$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad x_1 > 0$ $\Rightarrow p(1, y^*) = p(2, y^*) = v_1$

$x_2 > 0 \Rightarrow$



I сл. $x_2 = \frac{1}{3}$ $\quad x_1 = \frac{2}{3} \Rightarrow y_3 = 0$

II сл. $x_2 = \frac{2}{3}$ $\quad x_1 = \frac{1}{3} \Rightarrow y_1 = 0$

$y_2 > 0; y_1 = 0$ $\quad v_1$