Th Ако положението на равновесие $x=a$ притежава ф-я на Лепунов, то то е устойчиво полож. на равновесие. Ако тя е строга, то то е асимптотически устойчиво полож. на равновесие.

Пример: $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$

$$v(x,y) = x^2 + y^2$$

$$L_f v = \frac{dv}{dx} \cdot \dot{x} + \frac{dv}{dy} \cdot \dot{y} =$$

$$= -2xy + 2xy = 0 = \dot{v}$$

$v(x,y) \geq 0$, $v(0,0) = 0$

$L_f v = 0$

$\Rightarrow v(x,y)$ не е строга ф-я на Лепунов $\Rightarrow$ $x=y=0$ е устойчива пол. на равновесие

Пример: $\begin{cases} \dot{x} = -y - x\ln(1+x^2+y^2) & |\,x \\ \dot{y} = x - y\ln(1+x^2+y^2) & |\,y \end{cases}$

$$\frac{d}{dt}\frac{1}{2}(x^2+y^2) = x\dot{x} + y\dot{y} = -(x^2+y^2)\ln(1+x^2+y^2)$$

$\cancel{v(x^2+}y$

$x = y = 0$

$V = x^2 + y^2$ - ф-я на Ляпунов

$V > 0$, като $V(0,0) = 0$

$$L_f V = 2x\dot{x} + 2y\dot{y} =$$
$$= -2(x^2+y^2)\ln(1+x^2+y^2) <$$
$$< 0$$

$\Rightarrow V(x,y)$ е строга ф-я на Ляпунов

$\Rightarrow$ т. $x = y = 0$ е асимптотически устойчиво полож. на равновесие

?. с + пол. коорд.

16.01.2014г.

Лекция

Зад Пресметнете периодичните решения на

(*) $\ddot{x} + \sin x = \mu \sin 2t$

$x(0) = 0 = \dot{x}(0)$

с точност $O(\mu^2)$

Решение: Нека

$$x(t) = x_0(t) + \mu x_1(t) + O(\mu^2)$$

$$\begin{cases} 0 = x_0(0) + \mu x_1(0) \Rightarrow x_0(0) = x_1(0) = 0 \\ \dot{x}(t) = \dot{x}_0(t) + \mu \dot{x}_1(t) + \dots \\ \dot{x}(0) = 0 \Rightarrow \dot{x}_0(0) = \dot{x}_1(0) = 0 \end{cases}$$

$$\sin(\underbrace{\mu x_1 + \mu^2 x_2 + \dots}_{x})$$

$$= x + O(x^3) = \mu x_1 + \dots + O(\mu x_1 + \dots)^3 =$$
$$= \mu x_1 + O(\mu^2) \approx \mu x_1$$

Заместваме в (*):

$$\ddot{x}_0 + \mu \ddot{x}_1 = -\sin(x_0 + \mu x_1) + \mu \sin 2t$$

Приравняваме пред μ^0 (полагаме $\mu = 0$):

$$\left| \begin{array}{l} \ddot{x}_0 = -\sin x_0 \\ x_0(0) = \dot{x}_0(0) = 0 \end{array} \right\} x_0 \equiv 0$$

Приравняваме пред μ:

$$\left| \begin{array}{l} \ddot{x}_1 = -x_1 + \sin 2t \\ x_1(0) = 0, \dot{x}_1(0) = 0 \end{array} \right.$$

Решаване ком. у-е:

$\ddot{x}_1 = -x_1 \Rightarrow x_1 = A\cos t + B\sin t$

Търсим частно решение на

$\ddot{x}_1 + x_1 = \sin 2t$

Такова е $x_1 = -\frac{1}{3}\sin 2t$ и

общото решение е

$x_1 = A\cos t + B\sin t - \frac{1}{3}\sin 2t$

От $x_1(0) = \dot{x}_1(0) = 0$ следва, че

$x_1 = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t$

Окончателно, приближеното решение е

$$x \approx \mu\left[\frac{2}{3}\sin t - \frac{1}{3}\sin 2t\right] + 0$$

$t = 0 \Rightarrow A = 0$

$\dot{x}_1 = -A\sin t + B\cos t$
$-\frac{2}{3}\cos 2t$

$t = 0 \quad 0 = B - \frac{2}{3}$

$B = \frac{2}{3}$

$$x = \mu\left(\frac{2}{3}\sin t \cancel{*} - \frac{1}{3}\sin 2t\right) + O(\mu^2)$$

$$\Leftrightarrow$$

$$x \approx \mu\left[\frac{2}{3}\sin t - \frac{1}{3}\sin 2t\right]$$

21.01.2014г.

Упражнение

Непрекъснатата и диференцируема зависимост по нач. данни и параметър

133зад. Намерете първите 2 члена от развитието по степените на малкия параметър μ на решенията на уравненията със съответните начални условия:

а) (1) $\ddot{x} = x + \mu(\dot{x})^2$

(2) $x(0) = 1,\ \dot{x}(0) = \mu$

Търсим решение $x(t)$ във вида

$$\cancel{*} \quad x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) +$$

$$+ \mu^3 x_3(t) \ldots$$

$$(*)\quad x(t) = x_0(t) + \mu x_1(t) + O(\mu^2)$$

зад.

Слагате $(*)$ в (1), (2) и съставяте задача на Коши пред μ^0 и μ^1

$$\dot{x}(t) = \dot{x}_0(t) + \mu \dot{x}_1(t) + O(\mu^2)$$
$$\ddot{x}(t) = \ddot{x}_0(t) + \mu \ddot{x}_1(t) + O(\mu^2)$$

$$\dot{x}^2 = \dot{x}_0^2 + 2\dot{x}_0 x_1 \mu + O(\mu^2)$$

От $x(0) = 1 \Rightarrow$

$$x_0(0) + \mu x_1(0) + O(\mu^2) = 1$$

От $\dot{x}(0) = \mu$

$$\dot{x}_0(0) + \mu \dot{x}_1(0) + O(\mu^2) = \mu$$

Разглеждате тези начални условия като равенства на редове по μ

$$\mu^0: \left| \begin{array}{l} x_0(0) = 1 \\ \dot{x}_0(0) = 0 \end{array} \right.$$

$$\mu^1: \left| \begin{array}{l} x_1(0) = 0 \\ x_1(0) = 1 \end{array} \right.$$

$$\mu^0: \left| \begin{array}{l} \ddot{x}_0 = x_0 \\ x_0(0) = 1, \quad \dot{x}_0(0) = 0 \end{array} \right.$$

$$\ddot{x}_0 - x_0 = 0$$

ХУ: $\lambda^2 - 1 = 0$

$$\lambda_{1,2} = \pm 1$$

ФСР $\quad e^{\lambda_1 t}, e^{\lambda_2 t}$

$$\{e^t, e^{-t}\}$$

$$x_0(t) = c_1 e^t + c_2 e^{-t}$$

от $x_0(0) = 1$

$$\Rightarrow c_1 + c_2 = 1$$

от

$\dot{x}_0(0) = 0$

$$\Rightarrow c_1 - c_2 = 0$$

$$c_1 = c_2 = \frac{1}{2}$$

$$\boxed{x_0 = \frac{1}{2}\left(e^t + e^{-t}\right)}$$

пред μ^1:

№1: $\begin{cases} \ddot{x}_1 = x_1 + \dot{x}_0^2 \\ x_1(0) = 0 \\ \dot{x}_1(0) = 1 \end{cases}$

$$\ddot{x}_1 - x_1 = \frac{1}{4}\left(e^{2t} + e^{-2t} - 2\right)$$

$$x_1(t) = x_h + \tilde{x}$$

x_h - реш. на ХУ

$\tilde{x}$ - едно частно решение на нехомогенното

$$x_h = c_1 e^{t} + c_2 e^{-t}$$

Търсим $\tilde{x}$ с метода на Лагранж

$$\tilde{x} = c_1(t) \cdot e^{t} + c_2(t) e^{-t}$$

$$c_1, c_2 = ?$$

Решаване системата

$$\dot{c}_1 e^{t} + \dot{c}_2 e^{-t} = 0$$

$$\dot{c}_1 (e^{t})' + \dot{c}_2 (e^{-t})' = \frac{1}{4}\left(e^{2t} + e^{-2t} - 2\right)$$

$$\Leftrightarrow \quad + \left| \begin{aligned} &\dot{c}_1 e^{t} + \dot{c}_2 e^{-t} = 0 \\ &\dot{c}_1 e^{t} - \dot{c}_2 e^{-t} = \frac{1}{4} \cdot \left(e^{2t} + e^{-2t} - 2\right) \end{aligned} \right.$$

$$2 \dot{c}_1 e^{t} = \frac{1}{4} \left(e^{2t} + e^{-2t} - 2\right)$$

$$\Rightarrow \dot{c}_1 = \frac{1}{8} \left(e^{t} + e^{-3t} - 2e^{-t}\right)$$

$$\dot{c}_2 = - \dot{c}_1 e^{2t} = - \frac{1}{8} \left(e^{3t} + e^{-t} - 2e^{t}\right)$$

$$c_1 = \frac{1}{8} e^{t} - \frac{1}{24} e^{-3t} + \frac{1}{4} e^{-t}$$

$$c_2 = -\frac{1}{24} e^{3t} + \frac{1}{8} e^{-t} + \frac{1}{4} e^{t}$$

Тогава

$$\tilde{x} = \frac{1}{8} e^{2t} - \frac{1}{24} e^{-2t} + \frac{1}{4} - \frac{1}{24} e^{2t} +$$

$$+ \frac{1}{8} e^{-2t} \ldots$$

$$= \frac{1}{12} e^{2t} + \frac{1}{12} e^{-2t} + \frac{1}{2}$$

$$x_1 = c_1 e^{t} + c_2 e^{-t} + \frac{1}{12} e^{2t} + \frac{1}{12} e^{-2t} + \frac{1}{2}$$

Din $x_1(0) = 0 \Rightarrow c_1 + c_2 + \frac{1}{12} + \frac{1}{12} + \frac{1}{2} = 0$

$$\left| \, c_1 + c_2 = -\frac{2}{3} \right.$$

Din $x_1'(0) = 1$

$$\Rightarrow c_1 - c_2 + \frac{1}{6} - \frac{1}{6} = 1$$

$$\left| \Rightarrow c_1 - c_2 = 1 \right.$$

$$1 + c_2 + c_2 = -\frac{2}{3}$$

$$2c_2 = -\frac{5}{3}$$

$$c_2 = -\frac{5}{6} \qquad c_1 = \frac{1}{6}$$

$$x_1 = \frac{1}{6}e^{t} - \frac{5}{6}e^{-t} + \frac{1}{12}e^{2t} + \frac{1}{12}e^{-2t} + \frac{1}{2}$$

8) $\begin{cases} y' = \mu x + \frac{1}{2y} \\ y(1) = 1 - 2\mu, \quad x > 0 \end{cases}$

Търсим решение $y(x)$ във вида

$$y = y_0(x) + y_1(x)\mu + O(\mu^2)$$

От $y(1) = 1 - 2\mu \Rightarrow$

$$y_0(1) + y_1(1)\mu + O(\mu^2) = 1 - 2\mu$$

$$\begin{array}{l|l} \mu^0: & y_0(1) = 1 \\ \mu^1 & y_1(1) = 1 \end{array}$$

$$\frac{1}{y} = z_0(x) + z_1(x)\mu + O(\mu^2)$$

Ще използваме, че $\frac{1}{y} \cdot y \equiv 1$

$$(z_0 + z_1\mu + O(\mu^2))(y_0 + y_1\mu + O(\mu^2)) \equiv 1$$

$$\mu^0: \quad z_0 \cdot y_0 = 1 \Rightarrow z_0 = \frac{1}{y_0}$$

$$\mu^1: \quad z_0 \cdot y_1 + z_1 y_0 = 0 \Rightarrow z_1 = -\frac{y_1 z_0}{y_0} = \frac{-y_1}{y_0^2}$$

$$\frac{1}{y} = \frac{1}{y_0} - \frac{y_1}{y_0^2}\mu + O(\mu^2)$$

$$y' = y_0' + \mu y_1' + O(\mu^2)$$

$$\mu^0: \left| \begin{array}{l} y_0' = \frac{1}{2} \cdot \frac{1}{y_0} \\ y_0(1) = 1 \end{array} \right.$$

$$y_0' = \frac{1}{2y_0}$$

$$\int 2 y_0 \, dy_0 = \int dx + c$$

$$y_0^2 = x + c$$

От $y_0(1) = 1$

$\Rightarrow 1 = 1 + c \Rightarrow c = 0$

$$\Rightarrow y_0^2 = x$$

$$\Rightarrow \boxed{y_0 = \sqrt{x}}$$

$$\mu^1: \left| \begin{array}{l} y_1' = x - \frac{1}{2} \frac{y_1}{y_0^2} = x - \frac{1}{2} \frac{y_1}{x} \\ y_1(1) = -2 \end{array} \right.$$

$y_1' = -\frac{1}{2x} y_1 + x$ е линейно от първи ред

$$y_1(x) = e^{-\frac{1}{2}\int \frac{dx}{x}} \left\{ c + \int x \cdot e^{\frac{1}{2}\int\frac{dx}{x}} dx \right\}$$

$$= \frac{1}{\sqrt{x}} \left[c + \int x\sqrt{x}\, dx \right]$$

$$y_1 = \frac{1}{\sqrt{x}} \left[c + \frac{2x^{\frac{5}{2}}}{5} \right]$$

$$y_1(1) = 1 \left[c + \frac{2}{5} \right] = -2$$

$$c = -2 - \frac{2}{5} = -\frac{12}{5}$$

$$\boxed{y_1 = \frac{1}{\sqrt{x}} \left[-\frac{12}{5} + \frac{2}{5} x^{5/2} \right]}$$

в) $\left| \begin{array}{l} y' = y - x + \mu x e^{2y} \\ y(1) = 2 - \mu \end{array} \right.$

Търсим решение $y(x)$ във вида

$$y(x) = y_0(x) + \mu y_1(x) + O(\mu^2)$$

$$y'(x) = y_0'(x) + \mu y_1'(x) + O(\mu^2)$$

От $y(1) = 2 - \mu$

$$\Rightarrow \quad y_0(1) + \mu y_1(1) + o(\mu^2) \equiv 2 - \mu$$

$\mu^0:$ $y_0(1) = 2$

$\mu^1:$ $y_1(1) = -1$

$$e^{2y} = e^{2(y_0 + \mu y_1 + o(\mu^2))} =$$

$$= e^{2y_0} + O(\mu)$$

$$\mu^0: \quad \left| \begin{aligned} & y_0' = y_0 - x \\ & y_0(1) = 2 \end{aligned} \right|$$

$y_0' = y_0 - x$ – линейно от 1 ред

$$y_0 = e^{\int dx}\left\{c - \int x \cdot e^{-\int dx}\, dx\right\} =$$

$$= e^x\left\{c - \int x \cdot e^{-x}\, dx\right\} =$$

$$= e^x\left\{c + \int x\, d e^{-x}\right\} = e^x\left\{c + x \cdot e^{-x} - \int e^{-x}\, dx\right\} =$$

$$= e^x \{ c + x \cdot e^{-x} + e^{-x} \}$$

$$y_0 = ce^x + x + 1$$

$$\boxed{y_0 = x + 1} \Leftarrow$$

От $y_0(1) = 2$
$\Rightarrow ce + 1 + 1 = 2$
$\Rightarrow c = 0$

$$\mu': y_1' = y_1 + x e^{2y_0}$$

$$\left| \begin{array}{l} y_1' = y_1 + x e^{2(x+1)} \\ y_1(1) = -1 \end{array} \right.$$

$y_1' = y_1 + x e^{2(x+1)}$ - линейно от I ред

$$y_1(x) = e^x \left[c + \int x e^{2(x+1)} \cdot e^{-x} dx \right] =$$

$$= e^x \left\{ c + \int x e^{x+2} dx \right\} =$$

$$= e^x \left\{ c + \int x \, d e^{x+2} \right] =$$

$$= e^x \left\{ c + x e^{x+2} - e^{x+2} \right]$$

О~~т y_0~~

От $y_1(1) = -1$

$$\Rightarrow e \left[c + e^3 - e^3 \right] = -1$$

$$c = - e^{-1}$$

$$y_1 = e^x \left[-e^{-1} + x e^{x+2} - e^{x+2} \right]$$

<u>реални решения</u>

d) $\begin{cases} \ddot{x} + x = \mu(\sin t + x^2) \\ x(0) = 0 \\ \dot{x}(0) = 0 \end{cases}$ не се развива

Търсим решение $x(t)$ във вида

$$x(t) = x_0(t) + \mu x_1(t) + O(\mu^2)$$

$$\dot{x}(t) = \dot{x}_0(t) + \mu \dot{x}_1(t) + O(\mu^2)$$

$$\ddot{x}(t) = \ddot{x}_0(t) + \mu \ddot{x}_1(t) + O(\mu^2)$$

$$x^2 = x_0^2 + O(\mu)$$

От $x(0) = 0$

$\Rightarrow x_0(0) + \mu x_1(0) + O(\mu^2) \equiv 0$

От $\dot{x}(0) = 0$

$\Rightarrow \dot{x}_0(0) + \mu \dot{x}_1(0) + O(\mu_2) \equiv 0$

$\mu_0 \begin{cases} x_0(0) = 0 \\ \dot{x}_0(0) = 0 \end{cases}$; $\mu^1 \begin{cases} x_1(0) = 0 \\ \dot{x}_1(0) = 0 \end{cases}$

$\mu^0 : \begin{cases} \ddot{x}_0 + x_0 = 0 \\ x_0(0) = 0 \\ \dot{x}_0(0) = 0 \end{cases}$ $x_0 \equiv 0$

$$\mu^1: \left| \begin{array}{l} \ddot{x}_1 + x_1 = \sin t + x_0^2 = \sin t \\ x_1(0) = 0, \ \dot{x}_1(0) = 0 \end{array} \right.$$

$$\ddot{x}_1 + x_1 = \sin t$$

$$x_1(t) = x_h + \tilde{x}$$

$$\ddot{x}_1 + x_1 = 0$$

ХУ: $\lambda^2 + 1 = 0 \Rightarrow \Leftrightarrow \lambda_{1,2} = \pm i$

РФСР: $\{\cos t, \sin t\}$

$$x_1 = c_1 \cos t + c_2 \sin t$$

$$\tilde{x} = c_1(t) \cos t + c_2(t) \sin t$$

$$c_1, c_2 = ?$$

$$\left| \begin{array}{l} \dot{c}_1 \cos t + \dot{c}_2 \sin t = 0 \\ \dot{c}_1 (\cos t)' + \dot{c}_2 (\sin t)' = \sin t \end{array} \right.$$

$$\Leftrightarrow \left| \begin{array}{ll} \dot{c}_1 \cos t + \dot{c}_2 \sin t = 0 & | \cdot \sin t \\ -\dot{c}_1 \sin t + \dot{c}_2 \cos t = \sin t & | \cdot \cos t \end{array} \right.$$

$$\dot{c}_2 = \sin t \cos t$$

$$\dot{c}_1 = -\dot{c}_2 \frac{\sin t}{\cos t} = \boxed{-\sin^2 t - \dot{c}_1}$$

$$C_2 \int \sin t \cos t \, dt = \frac{\sin^2 t}{2}$$

$$C_1 = - \int \sin^2 t \, dt = - \int \frac{1 - \cos 2t}{2} dt =$$

$$= -\frac{1}{2} t + \frac{1}{4} \sin 2t$$

$$\tilde{x} = -\frac{1}{2} t \cos t + \frac{1}{4} \sin 2t \cos t$$

$$+ \frac{1}{2} \sin^3 t =$$

$$= -\frac{1}{2} t \cos t + \frac{1}{2} \sin t$$

$$x_1 = C_1 \cos t + C_2 \sin t - \frac{1}{2} t \cos t + \frac{1}{2} \sin t$$

Om $x_1(0) = 0 \Rightarrow c_1 = 0$

Om $\dot{x}_1(0) = 0 \Rightarrow C_2 - \frac{1}{2} + \frac{1}{2} = 0$

$$\Rightarrow C_2 = 0$$

$$\Rightarrow x_1(t) = -\frac{1}{2} t \cos t + \frac{1}{2} \sin t$$

e) $\begin{cases} \ddot{x} - x = \sin(\dot{x})^2 \\ x(0) = \mu, \\ \dot{x}(0) = \mu^2 \end{cases}$

Търсим решение $x(t)$ във вида

$$x(t) = x_0(t) + x_1(t)\mu + o(\mu^2)$$

$$\dot{x}(t) = \dot{x}_0(t) + \dot{x}_1(t)\mu + o(\mu^2)$$

$$\ddot{x}(t) = \ddot{x}_0(t) + \ddot{x}_1(t)\mu + o(\mu^2)$$

$$(\dot{x})^2 = \dot{x}_0^2 + 2\dot{x}_0\dot{x}_1\mu + o(\mu^2)$$

$$\sin(\dot{x})^2 = \sin(\dot{x}_0^2 + 2\dot{x}_0\dot{x}_1\mu + o(\mu^2)) =$$

$$= \sin\dot{x}_0^2 + 2\dot{x}_0\dot{x}_1 \cdot \cos\dot{x}_0^2 \cdot \mu + o(\mu^2)$$

От $x(0) = \mu$

$$\Rightarrow x_0(0) + x_1(0)\mu + o(\mu^2) \equiv \mu$$

От $\dot{x}(0) = \mu^2$

$$\Rightarrow \dot{x}_0(0) + \dot{x}_1(0)\mu + o(\mu^2) = \mu^2$$

$\mu^0: \begin{cases} x_0(0) = 0 \\ \dot{x}_0(0) = 0 \end{cases}$ $\qquad \mu^1: \begin{cases} x_1(0) = 1 \\ \dot{x}_1(0) = 0 \end{cases}$

μ^0: $\left| \begin{array}{l} \ddot{x}_0 - x_0 = \sin(\dot{x}_0^2) \\ x_0(0) = \dot{x}_0(0) = 0 \end{array} \right.$

$$x_0 \equiv 0$$

μ^1: $\left| \begin{array}{l} \ddot{x}_1 - x_1 = 0 \\ x_1(0) = 1 \\ \dot{x}_1(0) = 0 \end{array} \right.$

$$\ddot{x}_1 - x_1 = 0$$

$\cancel{0}$ xY: $\lambda^2 - 1 = 0$

$$\Rightarrow \lambda_{1,2} = \pm 1$$

$\exists$ CP: $\left\{ e^t, e^{-t} \right\}$

$$x_1(t) = c_1 e^t + c_2 e^{-t}$$

$$x_1(0) = c_1 + c_2 = 1$$

$$\dot{x}(0) = c_1 - c_2 = 0$$

$$x_1(t) = \frac{1}{2}(e^t + e^{-t})$$

$$\underline{x(t+2\pi) = x(t)}$$

15 зад. Да се намери периодично реш. с период, равен на периода на дясната част на уравнението

a) $\ddot{x} + 3x - \mu x^3 = \mu \cos t \qquad o(\mu^2)$

Търсим решение $x(t)$ във вида

$$x(t) = x_0(t) + x_1(t)\mu + O(\mu^2)$$

$$\ddot{x} = \ddot{x}_0(t) + \ddot{x}_1(t)\mu + O(\mu^2)$$

От

$$x(t+2\pi) = x(t)$$

$$\Rightarrow x_0(t+2\pi) = x_0(t)$$

μ_0:

μ^1: $x_1(t+2\pi) = x_1(t)$

$$x^3 = x_0^3 + O(\mu)$$

$$\mu^0: \left| \begin{array}{l} \ddot{x}_0 + 3x_0 = 0 \\ x_0(t+2\pi) = x_0(t) \end{array} \right.$$

$$\ddot{x}_0 + 3x_0 = 0$$

$$\text{ХУ}: \lambda^2 + 3 = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{3}$$

ФСР: $\{\cos\sqrt{3}t, \sin\sqrt{3}t\}$

$$x_0(t) = c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t$$

$x_0 \equiv 0$ (периодите са $\frac{2\pi}{\sqrt{3}}$)

μ^1: $\ddot{x}_1 + 3x_1 = -\mu \underset{0}{\underset{\shortparallel}{x_0^3}} - \cos t$

$$\begin{cases} \ddot{x}_1 + 3x_1 = \cos t \\ x_1(t+2\pi) = x_1(t) \end{cases}$$

$$x_1(t) = C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t + \tilde{x}$$

$\tilde{x}$ - частно реш.

Понеже $\cos t = \mathrm{Re}\, e^{it}$

Търсим частно решение на у-нието

$$\ddot{x}_1 + 3x_1 = e^{it}$$

$$\tilde{x} = Ae^{it}, \quad A = ?$$

$$\dot{\tilde{x}} = Aie^{it}$$

$$\ddot{\tilde{x}} = -Ae^{it}$$

$$-Ae^{it} + 3Ae^{it} = e^{it}$$

$$\Rightarrow \quad 2A = 1 \quad \Rightarrow A = \frac{1}{2}$$

$$\tilde{x} = \frac{1}{2} e^{it} = \frac{1}{2}(\cos t + i \sin t) =$$

$$= \frac{1}{2}\cos t + i\frac{1}{2}\sin t$$

Нашето истинско частно решение е $\operatorname{Re}\tilde{x} = \frac{1}{2}\cos t$

$$\left| \begin{array}{l} x_1(t) = C_1 \cos\sqrt{3}\,t + C_2 \sin\sqrt{3}\,t + \frac{1}{2}\cos t \\ x_1(t+2\pi) = x_1(t) \text{ частно} \end{array} \right.$$

$$\boxed{x_1(t) = \frac{1}{2}\cos t}$$

в) $\ddot{x} + x - \mu x^2 = \sin 2t - \sin 3t$

Търсим решение $x(t)$ във вида

$$x(t) = x_0(t) + x_1(t)\mu + o(\mu^2)$$
$$\ddot{x}(t) = \ddot{x}_0(t) + \ddot{x}_1(t)\mu + o(\mu^2)$$

$$x^2 = x_0^2(t) + o(\mu)$$

μ^0: $x_0(t+2\pi) = x_0(t)$

μ^1: $x_1(t+2\pi) = x_1(t)$

μ°: $\begin{cases} \ddot{x}_0 + x_0 = \sin 2t - \sin 3t \\ x_0(t+2\pi) = x_0(t) \end{cases}$

$\ddot{x}_0 + x_0 = 0$

ХУ: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

Р ФСР: $\{\cos t, \sin t\}$

$x_h(t) = c_1 \cos t + c_2 \sin t$

Търсим частно решение

$\tilde{x}(t) = c_1(t)\cos t + c_2(t)\sin t$

$c_1, c_2 = ?$

$\begin{cases} \dot{c}_1 \cdot \cos t + \dot{c}_2 \sin t = 0 \\ \dot{c}_1 (\dot{\cos t}) + \dot{c}_2 (\dot{\sin t}) = \sin 2t - \sin 3t \end{cases}$

$\begin{cases} \dot{c}_1 \cos t + \dot{c}_2 \sin t = 0 \mid \cdot \sin t & \mid \cos t \\ -\dot{c}_1 \sin t + \dot{c}_2 \cos t = \sin 2t - \sin 3t \end{cases}$ +

$\dot{c}_2 = \sin 2t \cos t - \sin 3t \cos t$

$\dot{c}_1 = -\dot{c}_2 \dfrac{\sin t}{\cos t} = -\sin t \cdot \sin 2t - \sin t \sin 3t$

$\sin 3t = \sin(t + 2t) =$
$= \sin t \cos 2t + \sin 2t \cos t$

$\dot{c}_2 = \sin 2t \cos t - \sin t \cos 2t \cos t$
$- \sin 2t \cos^2 t$

$= 2 \sin t \cos^3 t + \sin t$

$= \sin t \cos^3 t - \sin^3 t \cos t$

$\dot{c}_2 = 2 \sin t \cos^2 t - \sin t \cos^3 t + \sin^3 t \cos t$

$c_2 = -2 \int \cos^2 t \, d\cos t + \int \cos^3 t \, d\cos t$
$+ \int \sin^3 t \, d\sin t =$

$= -\frac{2}{3} \cos^3 t + \frac{1}{4} \cos^4 t + \frac{1}{4} \sin^4 t$

$\dot{c}_1 = -2 \sin^2 t \cos t + \sin(2t + t) \sin t$
$= -2 \sin^2 t \cos t + \sin 2t \cos t \sin t$
$+ \cos 2t \sin^2 t =$
$= - 2 \sin^2 t \cos t + 2 \sin^2 t \cos^2 t +$
$+ \cos^2 t \sin^2 t - \sin^4 t$

$$\text{П}^\circ: \begin{cases} \ddot{x}_0 + x_0 = \sin 2t - \sin 3t \\ x_0(t+2\pi) = x_0(t) \end{cases}$$

$$\ddot{x}_0 + x_0 = 0$$

$$\text{ХУ}: \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

$$\text{РФСР}: \{\cos t, \sin t\}$$

$$x_h(t) = C_1 \cos t + C_2 \sin t$$

Търсим частно решение

$$\tilde{x}(t) = c_1(t)\cos t + c_2(t)\sin t$$

$$c_1, c_2 = ?$$

$$\begin{cases} \dot{c}_1 \cdot \cos t + \dot{c}_2 \sin t = 0 \\ \dot{c}_1 (\cos t)' + \dot{c}_2 (\sin t)' = \sin 2t - \sin 3t \end{cases}$$

$$\begin{cases} \dot{c}_1 \cos t + \dot{c}_2 \sin t = 0 \mid \cdot \sin t \\ -\dot{c}_1 \sin t + \dot{c}_2 \cos t = \sin 2t - \sin 3t \end{cases} \Big| \cos t \quad +$$

$$\dot{c}_2 = \sin 2t \cos t - \sin 3t \cos t$$

$$\dot{c}_1 = -\dot{c}_2 \frac{\sin t}{\cos t} = -\sin t \cdot \sin 2t - \\ - \sin t \sin 3t$$

$$\sin 3t = \sin(t+2t) =$$
$$= \sin t \cos 2t + \sin 2t \cos t$$

$$\dot{C}_2 = \sin 2t \cos t - \sin t \cos 2t \cos t$$
$$- \sin 2t \cos^2 t$$

$$= 2 \sin t \cos^3 t + \sin t$$

$$= \sin t \cos^3 t - \sin^3 t \cos t$$

$$\dot{C}_2 = 2 \sin t \cos^2 t - \sin t \cos^3 t + \sin^3 t \cos t$$

$$C_2 = -2 \int \cos^2 t \, d\cos t + \int \cos^3 t \, d \cos t$$
$$+ \int \sin^3 t \, d \sin t =$$

$$= -\frac{2}{3} \cos^3 t + \frac{1}{4} \cos^4 t + \frac{1}{4} \sin^4 t$$

$$\dot{C}_1 = -2 \sin^2 t \cos t + \sin(2t+t) \sin t$$

$$= -2 \sin^2 t \cos t + \sin 2t \cos t \sin t$$
$$+ \cos 2t \sin^2 t =$$

$$= -2 \sin^2 t \cos t + 2 \sin^2 t \cos^2 t +$$
$$+ \cos^2 t \sin^2 t - \sin^4 t$$

$$= -2\sin^2 t\cos t + 3\sin^2 t\cos^2 t - \sin^4 t$$

$$c_2 = -2\int \sin^2 t \, d\sin t \quad +$$

$$+ \frac{3}{4}\int \sin^2 2t \, dt -$$

$$- \int \sin^4 t \, dt$$

$$= -\frac{2}{3}\sin^3 t$$

$$\cancel{\sin^4 t} =$$

$$\cancel{\sin^4 t} \quad (\sin^4 t) = (\sin^2 t)^2 =$$

$$= \frac{(1-\cos 2t)^2}{4} =$$

$$= 1 - \frac{1}{2}\cos 2t + \frac{\cos^2 2t}{4}$$

$$= -\frac{2}{3}\sin^3 t + \frac{3}{8}t$$

$$c_1 = -\frac{2}{3}\sin^3 t$$

$$c_1 = -\frac{2}{3}\sin^3 t - \frac{1}{8}\sin 4t + \frac{1}{4}\sin 2t$$

$$\tilde{x} = \cancel{C_1} - \frac{2}{3}\sin^3 t\cos t - \frac{1}{8}\sin 4t\cos t + \frac{1}{4}\sin 2t\cos t - \frac{2}{3}\cos^3 t + \left(\frac{1}{4}\cos^4 t + \frac{1}{4}\sin^4 t\right)\sin t$$

$$= -\frac{2}{3}\sin t\cos t - \frac{1}{8}\sin^4 t\cos t + \frac{1}{4}\sin 2t\cos t$$