14. V. 2013г. ДИС - лекция   началото на телефона

Тh на Абел

$\sum_{n=0}^{\infty} a_n x^n$ е сх. за $x = R$, то сумата на реда е непрекъсната в т. $x = R$

$f(x)$ е в интервала $(-R, R)$ и е вярно че $f(x) = \sum_{n=0} a_n x^n$

нека $f(x)$ е непрекъсната в $x = R$. Тогава според Th на Абел можем да кажем, че $f(R) = \sum_{n=0}^{\infty} a_n R^n$ (гр. преход).

т.е. развитието на $f(x)$ в степенен ред е валидно и за $x = R$

$\sum a_n x^n = \sum_{n=0}^{\infty} a_n R^n \left(\frac{x}{R}\right)^n$ изп. критерие на Абел за равн. сходимост. $\sum a_n R^n$ е равномерно сх. в $(-R, R]$

$\left(\frac{x}{R}\right)^n$ е мон. нам. и огр. $\Rightarrow$ редът е равн. сх $(-R, R]$ $\Rightarrow$ сумата му е непрек. функция;

Пример:

1. $f(x) = \operatorname{arctg} x$ използване: $\frac{1}{1+y} = 1 + y + y^2 + \dots + y^n + \dots \quad |y| < 1$

$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots$

$(-1, 1)$ $f(x) = \int f'(x)dx + C = \underbrace{C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots (-1)^n \frac{x^{2n+1}}{2n+1} + \dots}_{\operatorname{arctg} x}$

полагаме

$f(0) = \operatorname{arctg} 0 = C + 0 + 0 + \dots$ $C = 0$

$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

крит. на Лайбниц $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, $a_n \searrow 0$ сх по Лайбниц.

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ $\frac{1}{2n+1} \searrow 0$ $x = 1$ - сх по Лайбниц

$x = -1$ $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ сх по Лайбниц.

$\frac{1}{2n+1} \searrow 0$

$$\operatorname{arctg} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

за $x \in [-1; 1]$

$x = 1$ $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \frac{(-1)^n}{2n+1} + \ldots$

~~ред на Тейлър в степенен ред~~

Пример 2: $f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \ldots (-1)^n x^n + \ldots$$

$$\lg(1+x) = C + \int f(x)dx = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots + \frac{(-1)^n x^{n+1}}{n+1} + \ldots$$

$\frac{(-1)^{n-1} x^n}{n}$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^n x^{n+1}}{n+1} + \ldots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

при $x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ - разх.

при $x = 1$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ ред на Лайбниц $a_n = \frac{1}{n} \searrow 0$

$\Rightarrow$ сх.

полагаме $x = 1$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{(-1)^{n-1}}{n} + \dots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + R_n(x)$$

$$R_n(x) \to 0$$

$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, x^{2n-1}}{(2n-1)!}$$

аналогично за:

$\cos x =$

$e^x =$

пример 3:

$$f(x) = (1+x)^{\alpha} = \binom{\alpha}{0} + \binom{\alpha}{1}x + \dots \binom{\alpha}{n}x^n + \dots$$

$$u_n = \binom{\alpha}{n}x^n \qquad \frac{|u_{n+1}|}{|u_n|} = |x|\frac{n!\,\alpha(\alpha-1)\dots(\alpha-n+1)(\alpha-n-1+1)}{\alpha(\alpha-1)\dots(\alpha-n+1)(n+1)!} =$$

ясно е че е интересно при големи числа

$$= \frac{|x|\,|\alpha-n|}{n+1} = |x|\frac{n-\alpha}{n+1} \to |x|$$

$|x| < 1$ - сх.

$|x| > 1$ - рх.

рх. -1 сх 1 рх.

зависимост от α, ако

1сл. $\alpha > 0$

$x = 1$ $\quad \sum_{n=1}^{\infty}\binom{\alpha}{n}$

$$n\left(\frac{|u_n(1)|}{|u_{n+1}(1)|} - 1\right) = n\left(\frac{n+1}{n-\alpha} - 1\right) = n\,\frac{n+1-n+\alpha}{n-\alpha} = \frac{n(1+\alpha)}{(n-\alpha)} \to (1+\alpha)$$

$\alpha > 0 \Rightarrow$ редът е сх. по Раабе-Дюамбел.

при $x = -1$ същото, заради $|\ |$

$\Rightarrow$ при $\alpha > 0$ редът е сх. и се развива в степенен ред.

при $\alpha = 0$ не е интересно

(II) етап

~~$\alpha \in (-1, 0)$~~

1. $\alpha > 0$ $\quad f(x) = \sum$ за $x \in [-1, 1]$

2. $-1 < \alpha < 0$

$$\binom{\alpha}{n+1} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)(\alpha-n)}{n!\,(n+1)}$$

$-1 < \alpha < 0$ $\quad x = -1$ $\quad \sum (-1)^n \binom{\alpha}{n}$

$$n\left(\frac{|u_n|}{|u_{n+1}|} - 1\right) = n\left(\frac{n-\alpha}{n+1} - 1\right) = \longrightarrow 1+\alpha, \quad 1+\alpha < 1 \text{ ред. е рх.}$$

$|u_n| \downarrow 0$

$\sum \binom{\alpha}{n}$ $\quad$ т.к. $n\left(\frac{|u_n|}{|u_{n+1}|} + 1\right)$ $\quad 1+\alpha > 0 \to |u_n| \downarrow 0$ – сх

$\alpha \in (-1, 0)$

$(-1, 1]$ – интерв. на сх. –

$\alpha \leq -1$

$$|u_n| = \frac{|\alpha||\alpha-1|\dots|\alpha-n+1|}{1.2\dots n} = \frac{|-\alpha||-\alpha-1|\dots|-\alpha+n-1|}{1.2\dots n} \geq 1$$

$|u_n| \not\to 0$ – рх.

$$\sum_{n=1}^{\infty} \frac{17^n |x-5|^n}{4n-3}$$

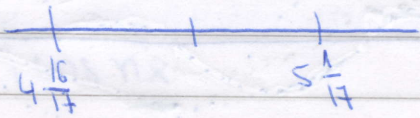
$$R = \frac{1}{\lim \sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{17}{\sqrt[n]{4n-3}}} = \frac{1}{17}$$

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim \frac{17^{n+1}|x-5|^{n+1}(4n-3)}{(4n+1)\,17^n\,|x-5|^n} = 7\lim |x-5| \frac{4n-3}{4n+1} =$$

$$= 17|x-5| < 1$$

$$-\frac{1}{17} < x-5 < \frac{1}{17}$$

$$4\frac{16}{17} < x < 5\frac{1}{17}$$



$x = 5\frac{1}{17}$ $\qquad x-5 = \frac{1}{17}$

$$\sum_{n=1}^{\infty} \frac{1}{4n-3} \sim \sum \frac{1}{n} \text{ - разх.}$$

$x = 4\frac{16}{17}$ $\qquad x-5 = -\frac{1}{17}$

$$\sum \frac{17^n(-1)^n}{(4n-3)\,17^n} = \sum \frac{(-1)^n}{4n-3} \qquad \frac{1}{4n-3} \searrow 0$$

Редове на Фурие

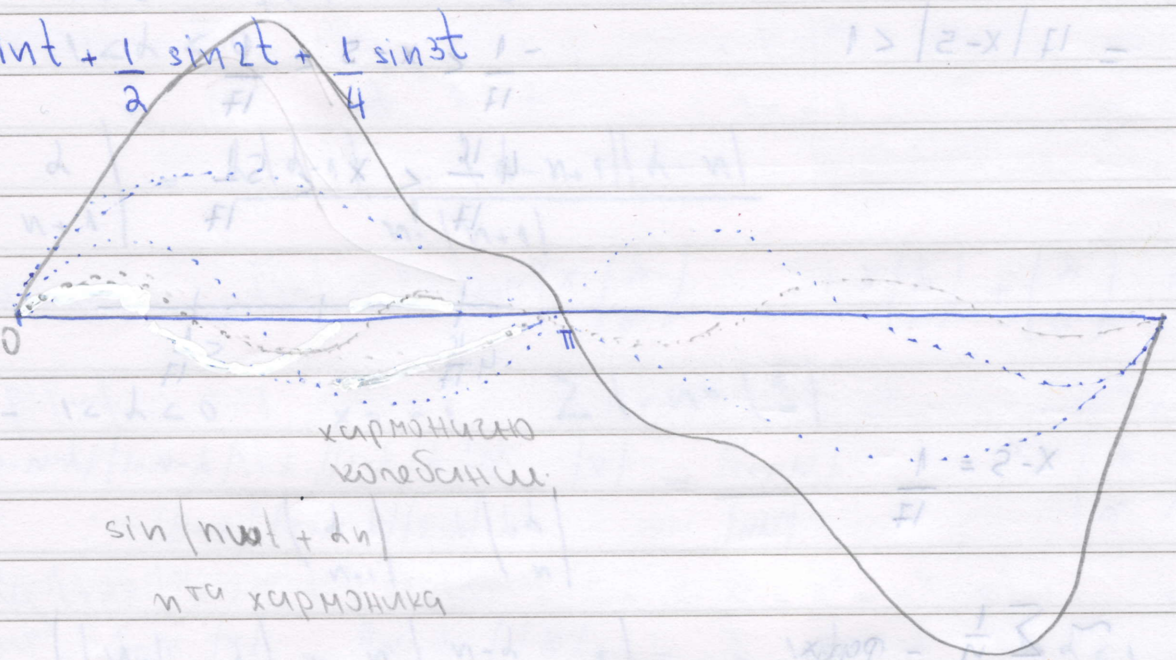
$y(t) = A\sin(\omega t + \alpha)$ ω - честота $y(t + T) = y(t)$

$\omega = \frac{2\pi}{T}$

$y_0 = A_0$, $y_1 = A_1 \sin(\omega t + \alpha_1)$

$y_2 = A_2 \sin(2\omega t + \alpha_2)$ ....

$\sin t + \frac{1}{2}\sin 2t + \frac{1}{4}\sin 3t$



хармонични колебания

$\sin(n\omega t + \alpha_n)$

$n^{\text{та}}$ хармоника

Нека φ е 2π периодична

$\varphi(t) = A_0 + A_1 \sin(\omega t + \alpha_1) + A_2 \sin(2\omega t + \alpha_2) + \dots =$

(сложно колебание ?)

$x = \omega t$

$\sin(x + \alpha_1) = \sin x . \cos\alpha_1 + \cos x \sin\alpha_1$

$\sin(2x + \alpha_2) = \sin 2x \cos\alpha_2 + \cos 2x . \sin\alpha_2$

$f(x) = a_0 + (a_1 \cos x + b \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots + (a_n \cos nx + b_n \sin nx)$

$z = e^{ix}$

$e^{inx} = \cos nx + i\sin nx$

Тв. Нека $f(x)$ е непрекъсната (интегруема в $[-\pi;\pi]$) искаме да видим, дали можем да я разложим

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{a_n \cos nx + b_n \sin nx}$$

Ɪ е в сила разлагането и е допустимо посл. интегр.

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} 1\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\,dx\right) =$$

$$= \frac{a_0}{2}x\Big|_{-\pi}^{\pi} + \sum\left(\frac{a_n}{n}\overset{=0}{\sin nx}\Big|_{-\pi}^{\pi} - \frac{b_n}{n}\sin x\Big|_{-\pi}^{\pi}\right) = \frac{a_0}{2}(\pi+\pi) = a_0\pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad | \cos nx$$

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\,\cancel{\cos}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\cos nx \cdot \sin nx\,dx\right)$$

$m \neq n$

$$\cos mx \cdot \cos nx = \frac{1}{2}\left(\cos(m+n)x - \cos(m-n)x\right)$$

$$\int_{-\pi}^{\pi}\cos mx \cos nx\,dx = \frac{1}{2}\frac{1}{m+n}\sin(m+n)x\Big|_{-\pi}^{\pi} - \frac{1}{2(m-n)}\sin(m-n)x\Big|_{-\pi}^{\pi} = 0$$

$m = n$

$$a_m\int_{-\pi}^{\pi}\cos^2 mx\,dx = \frac{a_m}{2}\int_{-\pi}^{\pi}(1+\cos 2mx)\,dx = \frac{a_m}{2\pi}\left(x\Big|_{-\pi}^{\pi} + \frac{1}{2m}\sin 2mx\Big|_{-\pi}^{\pi}\right) =$$

$$= \frac{a_m}{2}\left[\pi - (-\pi) + 0\right] = \pi a_m$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$$

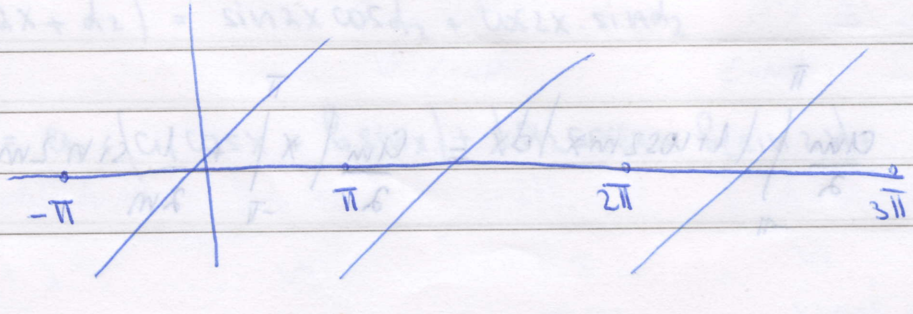
разбити само по $\sin nx$ $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{\underbrace{f(x)\cos nx}_{\text{четна}}}^{\text{нечетна}}\,dx = 0$

т.в. в разб. на нечетна функция участва само $\sin nx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{f(x)\sin nx}^{\text{четна}}\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx \qquad a_n = 0$$

ако f е четна $b_n = 0$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$f(x) = x$



$a_n = 0$ $\qquad b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dnx =$

$= -\frac{2}{n\pi}\int_0^{\pi} x \, d\cos nx =$

$= -\frac{2}{n\pi}\left[x\cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx \, dx \right] =$

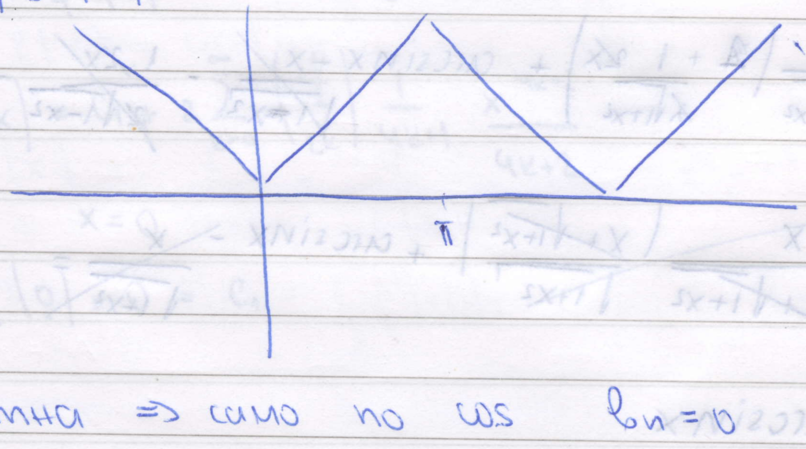
$= -\frac{2}{n\pi}\left(\pi \cos n\pi - 0 - \frac{1}{n}\sin nx \Big|_0^{\pi} \right) =$

$= -\frac{2}{n\pi}\left(\pi(-1)^n - \frac{1}{n}\sin n\pi + \frac{1}{n}\sin 0 \right) =$

$= \frac{2(-1)^{n-1}}{n}$

$$x \to \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \sin nx$$

$f(x) = f(|x|)$



четна $\Rightarrow$ само по cos $\quad b_n = 0$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi^2 - 0}{\pi} = \pi$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx =$$

$$a_n = \frac{2}{n\pi}\int_0^\pi x\,\cancel{\sin}\cos nx\,dnx = \frac{2}{n\pi}\int_0^\pi x\,d(\sin x) =$$

$$= \frac{2}{n\pi}\left(x\sin nx\Big|_0^\pi - \int_0^\pi \sin nx\,dx\right) = \frac{2}{n\pi}\left(\pi\sin n\pi - 0 - \frac{1}{n}\int_0^\pi \sin nx\,dnx\right) =$$

$$= +\frac{2}{n\pi}\cos n\cancel{x}\Big|_0^\pi = \frac{2}{n\pi}(\cos n\pi - \cos 0) = \frac{2}{n^2\pi}\left((-1)^n - 1\right) =$$

$$= \frac{2(-2)}{(2k-1)^2\pi} = \frac{-4}{(2k-1)^2\pi} \qquad |x| \longrightarrow -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}$$

задача:

$$f(x) = x\ln\left(x+\sqrt{1+x^2}\right) + x\arcsin x + \sqrt{1+x^2} - \sqrt{1+x^2}$$

$$\cancel{f'(x) = \ln\left(x+\sqrt{1+x^2}\right) + \arcsin x + \sqrt{1+x^2} - \sqrt{1+x^2}) + x}$$

$$f'(x) = \ln\left(x+\sqrt{1+x^2}\right) + \frac{x}{x+\sqrt{1+x^2}}\left(1 + \frac{1\cdot 2x}{2\sqrt{1+x^2}}\right) + \arcsin x \cancel{+ \frac{x}{\sqrt{1-x^2}}} \cancel{- \frac{1\cdot 2x}{2\sqrt{1-x^2}}} - \frac{2x}{2\sqrt{1+x^2}}$$

$$= \ln\left(x+\sqrt{1+x^2}\right) + \frac{x}{\cancel{x+\sqrt{1+x^2}}}\cdot\frac{\cancel{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}} + \arcsin x \cancel{- \frac{x}{\sqrt{1+x^2}}} =$$

$$= \ln\left(x+\sqrt{1+x^2}\right) + \arcsin x$$

$f''(x) = \ln(x + \sqrt{1+x^2})$

$$\frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1 \cdot 2x}{2\sqrt{1+x^2}}\right) + \frac{1}{\sqrt{1-x^2}} = \frac{x+\sqrt{1+x^2}}{(x+\sqrt{1+x^2})\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}} =$$

$$= \frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}} = (1+x^2)^{-\frac{1}{2}} + (1-x)^{-\frac{1}{2}}$$

$$(1+y)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} y^n$$

$$f''(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n} + \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^{2n} =$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (n + (-1)^n) \cdot x^{2n} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{2k+1} 2x^{4k+2} \quad n = 2k+1$$

интегрираме

$$f'(x) = C + \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{2k} \frac{x^{4k+1}}{4k+1} \qquad x = 0$$

$0 = f'(0) = C \Rightarrow C = 0$

$$f(x) = C_1 + 2\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{2k} \frac{1}{4k+1} \frac{x^{4k+2}}{4k+2}$$

$x = 0$

$f(0) = 0 = C_1$