23.05.2013г.

четвъртък

Упражнение

Критерий за сходимост

① Даламбер

$\sum a_n \qquad a_n > 0$ произв. от еднотипни

$\boxed{\dfrac{a_{n+1}}{a_n} \le q < 1 \text{ сходящ}}$

$\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \ell \begin{matrix} < 1 & \text{сходящ} \\ \ge 1 & \text{разходящ} \end{matrix}$

$\ell = 1$

$\dfrac{a_{n+1}}{a_n} \ge 1$ за $\forall n$ => разходящ

② Коши :(

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \ell \begin{matrix} < 1 & \text{сходящ} \\ > 1 & \text{разходящ} \end{matrix}$

$= 1$

$\sqrt[n]{a_n} \ge 1$ => разходящ

степени

-1-

$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

$\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$

③ Раабе-Дюамел

$\lim\limits_{n\to\infty} \underbrace{n\left(\dfrac{a_n}{a_{n+1}} - 1\right)}_{\alpha_n} = m \begin{matrix} > 1 & \text{сходящ} \\ < 1 & \text{разходящ} \end{matrix}$

$\alpha_n \le 1$ разходящ
(Даламбер)

$\alpha_n \to m > 0$ разх.
$a_n \to 0$

④ $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ сходящ

$a_n > a_{n+1} > 0$
$a_n \to 0$

Критерий на Лайбниц

$\sum |a_n|$ е сх. => $\sum a_n$ е сх. абс. сходящ

рх

сх - условно сходящ

зад. $\sum_{n=1}^{\infty} n^k q^n$ е сходящ

$-1 < q < 1$

Коши

$$\sqrt[n]{|n^k \cdot q^n|} = \left(\sqrt[n]{n}\right)^k \cdot |q| \to |q| < 1$$

абсолютно сх.

$|q| \geq 1$

$n^k \cdot q^n \not\to 0$

---

зад. $\sum_{n=1}^{\infty} \frac{a^n}{n!}$ $\quad a = 0$ сх.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n}\right| = \left|\frac{a}{n+1}\right| \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ сх.

---

зад. $\sum_{n=1}^{\infty} \frac{2.5.8\ldots[2+3(n-1)]}{1.5.9\ldots[1+4(n-1)]}$

~~1.5.9... [1+4(~~

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{2.5.8\ldots[2+3(n-1)]}\,[2+3n]}{\cancel{1.5.9\ldots}[1+4n]\cdot \cancel{2.5.8\ldots[2+3(n-1)]}} =$$

$$= \frac{3n+2}{4n+1} = \frac{\cancel{n}\left(3+\frac{2}{n}\right)}{\cancel{n}\left(4+\frac{1}{n}\right)} \to \frac{3}{4} < 1$$

$\Rightarrow$ сх.

зад. $\sum_{n=1}^{\infty} \binom{2n}{n}$ $\quad$ ~~(2n)!~~

$$\frac{a_{n+1}}{a_n} = \frac{\binom{2n+2}{n+1}}{\binom{2n}{n}} =$$

$$= \frac{(2n+2)!\,(n!\cdot n!)}{(n+1)!\,(n+1)!\,(2n)!} =$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} =$$

$$= \frac{4n^2 + 2n + 4n + 2}{n^2+2n+1} =$$

$$= \frac{4n^2 + 6n + 2}{n^2+2n+1} \xrightarrow[n\to\infty]{} 4 > 1$$

рх.

**зад.** $\sum \frac{4^n (n!)^2}{(2n)!\,(n+1)}$

$\frac{a_{n+1}}{a_n} = \frac{\cancel{4^{n+1}}((n+1)!)^2\,(2n)!\,(n+1)}{(2n+2)!\,(n+2)\cdot\cancel{4^n}(n!)^2} =$

$= \frac{\cancel{4}^{2}\,\cancel{(n!)^2}\,(n+1)^2\,\cancel{(2n)!}\,\cancel{(n+1)}}{\cancel{2}\,\cancel{(2n+2)}(2n+1)\cdot\cancel{(2n)!}\,(n+2)\,\cancel{(n!)^2}} =$

$= \frac{2(n+1)^2}{2n^2+4n+n+2} = \frac{2(n+1)^2}{2n^2+5n+2} \xrightarrow[n\to\infty]{} 1$

$n\left(\frac{2n^2+5n+2}{2(n+1)^2} - 1\right) =$

$= n\left(\frac{\cancel{2n^2}+5n+\cancel{2}-\cancel{2n^2}-4n-\cancel{2}}{2(n+1)^2}\right) =$

$= \frac{n^2}{2(n+1)^2} = \frac{n^2}{2n^2+4n+2} \to \frac{1}{2} < 1$

р.х.

$\frac{2n^2+4n+2}{2n^2+5n+2} < 1$ лошо

$\sum_{n=1}^{\infty} \frac{(-4)^n (n!)^2}{(2n)!\,(n+1)}$ $(-1)^n$

сх. Лайбниц

$\frac{a_{n+1}}{a_n} < 1$

$a_{n+1} < a_n$

$\Rightarrow$ усл. сх.

**зад.** $\sum \left(\frac{n}{e}\right)^n \cdot \frac{1}{n!}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{\cancel{e^{n+1}}} \cdot \frac{1}{\cancel{(n+1)!}} \cdot \frac{\cancel{e^n}\,\cancel{n!}}{n^n} =$

$= \frac{(n+1)^n}{e \cdot n^n} = \frac{\left(1+\frac{1}{n}\right)^n}{e} \xrightarrow[n\to\infty]{} 1$

$\left(1+\frac{1}{n}\right)^n \to e$
расте

$1+\frac{1}{n} < e$ $x \to 0$

$n\left(\frac{e\,n^n}{(n+1)^n} - 1\right) = \frac{e\cdot n^{n+1} - n(n+1)^n}{(n+1)^n}$ $n = 1/x$

Зад. $\sum_{n=1}^{\infty} n^k q^n$ е сходящ

$-1 < q < 1$

Коши

$$\sqrt[n]{|n^k \cdot q^n|} = \left(\sqrt[n]{n}\right)^k \cdot |q| \to |q| < 1$$

абсолютно сх.

$|q| \geq 1$

$n^k \cdot q^n \not\to 0$

---

Зад. $\sum_{n=1}^{\infty} \frac{a^n}{n!}$ $\qquad a=0$ сх.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n}\right| = \left|\frac{a}{n+1}\right| \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ сх.

---

Зад. $\sum_{n=1}^{\infty} \frac{2.5.8\ldots[2+3(n-1)]}{1.5.9\ldots[1+4(n-1)]}$

$\cancel{1.5.9\ldots[1+4(\ldots}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{2.5.8\ldots[2+3(n-1)]}[2+3n]}{\cancel{1.5.9\ldots}[1+4n]\cdot\cancel{2.5.8\ldots[2+3(n-1)]}} =$$

$$= \frac{3n+2}{4n+1} = \frac{\cancel{n}(3+\frac{2}{n})}{\cancel{n}(4+\frac{1}{n})} \to \frac{3}{4} < 1$$

$\Rightarrow$ сх.

Зад. $\sum_{n=1}^{\infty} \binom{2n}{n}$

$$\frac{a_{n+1}}{a_n} = \frac{\binom{2n+2}{n+1}}{\binom{2n}{n}} =$$

$$= \frac{(2n+2)! \cdot n! \cdot n!}{(n+1)!\,(n+1)!\,(2n)!} =$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} =$$

$$= \frac{4n^2+2n+4n+2}{n^2+2n+1} =$$

$$= \frac{4n^2+6n+2}{n^2+2n+1} \xrightarrow[n\to\infty]{} 4 > 1$$

pt.

$$= \frac{1}{x}\, e$$

$$f(x) = \frac{\dfrac{e \cdot 1}{x^{\frac{1}{x}+1}} - \dfrac{1}{x}\cdot\left(\dfrac{1}{x}+1\right)^{\frac{1}{x}}}{\left(\dfrac{1}{x}+1\right)^{\frac{1}{x}}} =$$

$$= \frac{1}{(1+x)^{\frac{1}{x}}} \cdot \frac{e - (1+x)^{\frac{1}{x}}}{x}$$

$$\frac{1}{(1+x)^{\frac{1}{x}}} \to e$$

Λοπιταλ

$$g(+)$$

$$(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)}$$

$$\Rightarrow \frac{-e^{\frac{1}{x}\ln(1+x)} \cdot \left(-\dfrac{1}{x^2}\ln(1+x) + \dfrac{1}{x}\cdot\dfrac{1}{1+x}\right)}{1} =$$

$$= -(1+x)^{\frac{1}{x}} \cdot \left(\frac{-\ln(1+x)(x+1) + x}{x^2(x+1)}\right) =$$

$$= (1+x)^{\frac{1}{x}} \frac{(x+1)\ln(1+x) - x}{x^2(x+1)}$$

$$\frac{x - (x+1)\ln(x+1)}{x^3 + x^2}$$

$$\frac{\cancel{1} - \ln(x+1) - \cancel{(x+1)} \cdot \dfrac{1}{\cancel{x+1}}}{3x^2 + 2x} =$$

$$= \frac{\ln(x+1)}{x(3x+2)}$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$= \frac{x - (x+1)\ln(x+1)}{x^2} =$$

$$= \frac{x - (x+1)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^2)\right)}{x^2} =$$

$$= \frac{1}{2}\frac{x^2 + o(x^2)}{x^2} = \frac{1}{2}\left(1 + \frac{o(x^2)}{x^2}\right) \to \frac{1}{2}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

<u>Триг. сист.</u>

$$\frac{1}{\sqrt{2}}, \cos x, \sin x, \cos 2x, \sin 2x, \dots$$

$$c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\,dx = \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{\sqrt{2}}$$

$c_1 = a_1$

$c_2 = b_1 = \int_{-\pi}^{\pi} f(x)\sin x\,dx$

$c_3 = a_2$

$c_4 = b_2$ и т.н.

$\cancel{S_n \gamma_x}$

$$S_n(x) = \sum_{i=1}^{n} \gamma_i \varphi_i(x)$$ – произволни

$$[f(x) - S_n(x)]^2 = f^2(x) - 2\sum_{i=1}^{n} f(x)\gamma_i\varphi_i(x) + \sum_{i,k=1}^{n} \gamma_i\gamma_k\varphi_i(x)\varphi_k(x)$$

$$0 \le \Delta_n = \int_{-\pi}^{\pi} (f(x) - S_n(x))^2\,dx =$$

$$= \int_{-\pi}^{\pi} f^2(x)\,dx - 2\sum_{i=1}^{n}\gamma_i \underbrace{\int_{-\pi}^{\pi} f(x)\varphi_i(x)\,dx}_{\pi c_i} + \sum_{i=1}^{n}\gamma_i^2\int_{-\pi}^{\pi}\varphi_i^2(x)\,dx =$$

$$\int \varphi_i\varphi_k(x) = 0 \quad i \neq k$$

$$= \int_{-\pi}^{\pi} f^2(x)\,dx - 2\sum_{i=1}^{n}\pi\gamma_i c_i + \pi\sum_{i=1}^{n}\gamma_i^2 + \pi\sum_{i=1}^{n} c_i^2 - \pi\sum_{i=1}^{n} c_i^2 =$$

$$= \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\sum_{i=1}^{n} c_i^2 + \pi\sum_{i=1}^{n}(\gamma_i - c_i)^2$$

Ако изберем $\gamma_i = c_i$

$$\int_{-\pi}^{\pi} f^2(x)\,dx - \pi\sum_{i=1}^{n} c_i^2 \ge 0$$

$$\sum_{i=1}^{n} c_i^2 \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

<u>Неравенство на Бесел</u>

при триг. с-ма

$$\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \le \int_{-\pi}^{\pi} f^2(x)\,dx$$

$n \to \infty$

$$\boxed{\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx}$$

Ясно е, че $\Delta_n$ достига НМС

$\gamma_i = c_i$, т.е. когато

$S_n(x)$ отг. на Реда на Фурие

$\Delta_n^{\Phi} \le \Delta_n$

Нека $f$ е непр. $f(-\pi) = f(\pi)$

По Th Вайерщрас по задад. $\varepsilon$ може да се намери тригонометричен полином $T$, $|f(x) - T(x)| < \sqrt{\frac{\varepsilon}{2\pi}}$

$\forall x \in [-\pi, \pi]$

$T(x)$ е полином от вида

$\sum_{1}^{n} \gamma_i \varphi_i(x)$:

$$\Delta_n^{T} = \int_{-\pi}^{\pi}[f(x) - T_n(x)]^2\,dx$$

$$\int_{-\pi}^{\pi}\left(\sqrt{\frac{\varepsilon}{2\pi}}\right)^2 dx = \varepsilon \text{ (д. големи } n)$$

$$\Rightarrow \Delta_n^{\Phi} < \varepsilon \Rightarrow \Delta_n^{\Phi} \to 0$$

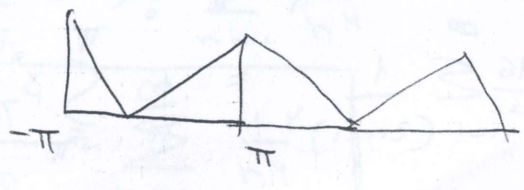
$$\Delta_n^{\Phi} = \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\sum_{i=1}^{n} c_i^2 \xrightarrow[n\to\infty]{} 0$$

гр. преход

$$\boxed{\sum_{i=1}^{\infty} c_i^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx}$$

<u>равенство на Парсевал</u>

$f(x) = |x| \ne \frac{\pi}{2}$ $[-\pi, \pi]$



$$f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$x = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 0}{(2n-1)^2}$$

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots + = \frac{\pi^2}{8}$$

Равенство на Парсевал

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi} \int_0^{\pi} x\,dx =$$

$$= \frac{2}{\pi} \frac{x^2}{2} \Big|_0^{\pi} = \pi$$

четна
няма
развитие по sin

$$\frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\,dx =$$

$$= \frac{1}{\pi} \frac{x^3}{3} \Big|_{-\pi}^{\pi} = \frac{1}{\pi} \left( \frac{\pi^3 - (-\pi)^3}{3} \right) = \frac{2}{3}\pi^2$$

$$\frac{2}{3}\pi^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

$$\frac{\pi^2}{6} = \frac{16}{\pi^2} \sum$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{81} + \frac{1}{625} + \dots$$

$$\boxed{1 + \left( \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots \right.}$$

$f(x) = x^2$ четна

$b_n = 0$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx\,dx =$$

$$= \frac{1}{n\pi} \left[ \int_{-\pi}^{\pi} x^2\,d\sin nx = \right.$$

по части

$$= \frac{1}{n\pi}\left[ x^2 \sin nx \Big|_{-\pi}^{\pi} - 2\int \sin nx . x\, dx \right] =$$

$$= \frac{\cancel{1}+2}{n^2\pi}\int_{-\pi}^{\pi} x\, d\cos nx = \frac{2}{n^2\pi}\left[ x\cos nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos nx\, dx \right] =$$

$$= \frac{2}{n^2\pi}\left[ \pi\cos n\pi - (-\pi)\cos n\pi - \frac{1}{n}\sin nx \Big|_{-\pi}^{\pi} \right] =$$

$$= \frac{2}{n^2\pi}\left[ 2\pi\cos n\pi - 0 \right] =$$

$$= \frac{4}{n^2}\cos n\pi \; (-1)^n$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2}{\pi}\frac{x^3}{3}\Big|_0^{\pi} = \cancel{\pi}$$

$$= \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

$$\boxed{x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx}$$

$x = 0$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}}$$

$x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\underbrace{(-1)^n}_{\cos n\pi}$$

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}} \; !$$

<u>Парсевал</u>

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\, dx = \frac{1}{\pi}\frac{x^5}{5}\Big|_{-\pi}^{\pi} = \frac{\pi^5 + \pi^5}{5\pi} =$$

$$= \frac{2\pi^4}{5}$$

$$\frac{2}{5}\pi^4 = \frac{2\pi^4}{9} + 4^2\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{18-10}{45}$$

$$\frac{\cancel{8}\pi^4}{45} = \cancel{16}^{\,2}\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\boxed{\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}}$$

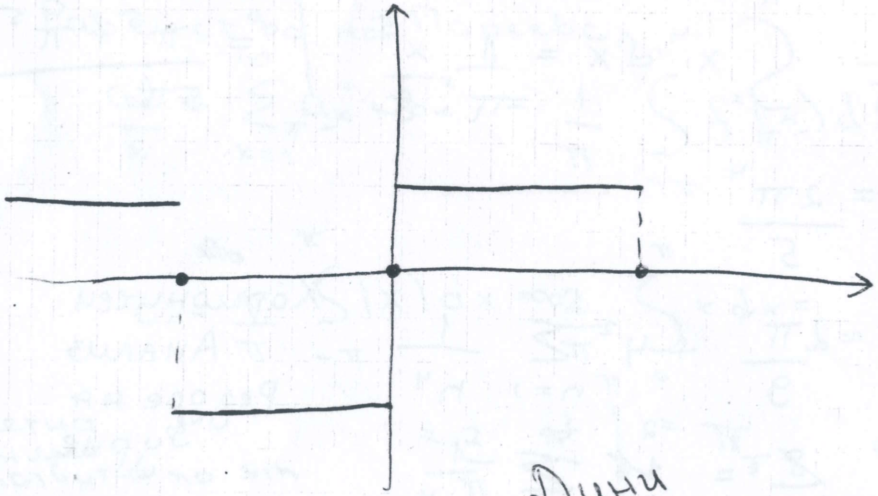
Хармоничен
Анализ
Редове на
Фурие

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \hat{f}(x)}$$

преобразуване на Фурие

Явление на Гибс

$$f(x) = \begin{cases} \frac{\pi}{2} & 0 < x < \pi \\ 0 & x = 0, \pm\pi \\ -\frac{\pi}{2} & -\pi < x < 0 \end{cases}$$



Дини

нечетна само по sin

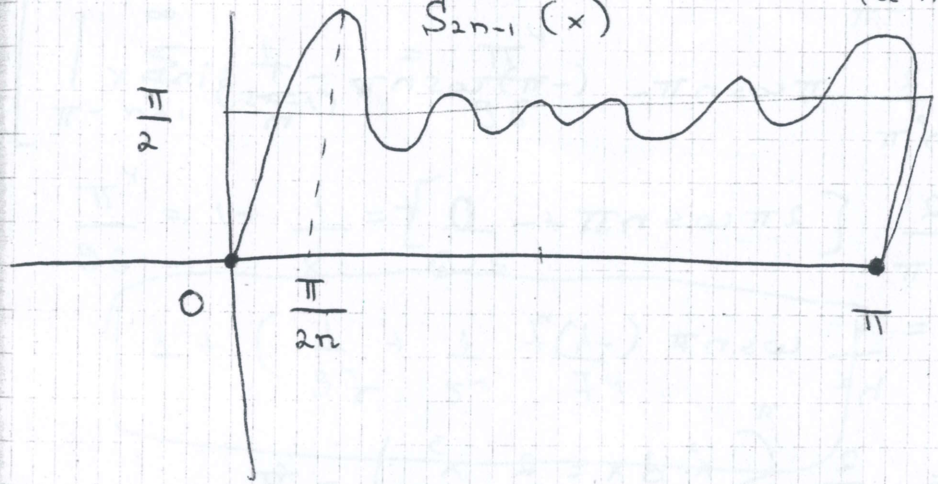
$$f(x) = 2\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} =$$

$$= 2\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right.$$

парциални $\Sigma$

$S_{2n} = S_{2n-1}$

нечетни п. суми




максимумите намаляват при $x \to \frac{\pi}{2}$