

06. III. 2013г.

Висша алгебра — упражнения
Константин Пандяков

Евклидовы пространства

V -линейн. простран. над $\mathbb{R}$

$c, a, b \in V$

$$(a, b) = \lambda, \lambda \in \mathbb{R} \quad \varphi: V \times V \rightarrow \mathbb{R}$$

$$1) (a, b) = (b, a)$$

$$2) (a+b, c) = (a, c) + (b, c)$$

$$3) (\lambda a, b) = \lambda (a, b)$$

$$4) (a, a) \geq 0 \quad (a, a) = 0 \Leftrightarrow a = \mathbf{0}$$

$e_1, e_2, \dots, e_n$ — базис на V

$$a = (\xi_1, \xi_2, \dots, \xi_n)$$

$$b = (\eta_1, \eta_2, \dots, \eta_n)$$

$$(a, b) = \sum_{i,j=1}^n \xi_i \eta_j (e_i, e_j)$$

$$|a| = \sqrt{(a, a)} \geq 0$$

Нормиране: $a^* = \frac{1}{|a|} \cdot a$
нормиран
вектор

Ортогонализиране $(a, b) = 0$

$(e_i, e_j) = \delta_{ij}$ - символ на Кронекер $\begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ (ортонормални)

$$(a, b) = |a| \cdot |b| \cdot \cos \varphi(a, b)$$

① $a = (2, 1, 3, 2)$ $\varphi(a, b) = ?$
 $b = (1, 2, -2, 1)$

$$\cos \varphi(a, b) = \frac{(a, b)}{|a| \cdot |b|} = 0$$

$$|a| = \sqrt{(a, a)} = \sqrt{2^2 + 1^2 + 3^2 + 2^2} = \sqrt{18} = 3\sqrt{2}$$

$$|b| = \sqrt{(b, b)} = \sqrt{1^2 + 2^2 + (-2)^2 + 1^2} = \sqrt{10}$$

$$(a, b) = 2 + 2 - 6 + 2 = 0$$

$$\varphi(a, b) = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z} \quad k = 0, 1, \dots$$

Метод на Грам-Шмид за ортогонализация на n -ма вектори

$$l(a_1, a_2, \dots, a_n) = V$$

$e_1, e_2, \dots, e_n$ - ортонормален базис на V

$$l(a_1, a_2, \dots, a_n) = l(e_1, e_2, \dots, e_n)$$

$$e_k = a_k + \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_{k-1} e_{k-1}$$

$$\begin{aligned} \textcircled{2} \quad a_1 &= (1, -1, 1, 1) \\ a_2 &= (3, -1, 2, -2) \\ a_3 &= (0, 2, -5, 3) \\ a_4 &= (-2, 6, -3, 9) \end{aligned}$$

База?

Ортонормализация по Г-Ш?

$$\begin{pmatrix} \textcircled{1} & -1 & 1 & 1 \\ 3 & -1 & 2 & -2 \\ 0 & 2 & -5 & 3 \\ -2 & 6 & -3 & 9 \end{pmatrix} \begin{matrix} / \cdot 2 \\ \\ + \\ \end{matrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & \textcircled{2} & -1 & -5 \\ 0 & 2 & -5 & 3 \\ 0 & 4 & -1 & 11 \end{pmatrix} \begin{matrix} / \cdot (-3) \\ \\ / \cdot (-2) \\ + \end{matrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & -5 \\ 0 & 0 & -4 & 8 \\ 0 & 0 & 1 & 21 \end{pmatrix} \begin{matrix} \\ \\ / \cdot (-4) \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 21 \end{pmatrix} \rightarrow \sim$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 23 \end{pmatrix}$$

$\Rightarrow$ База (a_1, a_2, a_3, a_4) , линейно
св. ЛНБ

$$l(e_1, e_2, e_3, e_4) = l(a_1, a_2, a_3, a_4)$$

$$(e_i, e_j) = \delta_{ij}$$

$$e_1^* = a_1 = (1, -1, 1, 1)$$

$$e_2^* = a_2 + \lambda_1 e_1^* \quad / \cdot e_1^*$$

$$0 = (e_1^*, e_2^*) = \underbrace{(e_1^*, a_2)} + \lambda_1 \underbrace{(e_1^*, e_1^*)}$$

$$\lambda_1 = \frac{-(e_1^*, e_2^*)}{(e_1^*, e_1^*)} = -1$$

$$e_2^* = a_2 - a_1 = (2, 0, 1, -3)$$

$$e_3^* = a_3 + \mu_1 e_1^* + \mu_2 e_2^* \quad | \cdot e_1^* \quad | \cdot e_2^*$$

$$0 = (e_1^*, e_3^*) = (e_1^*, a_3) + \mu_1 (e_1^*, e_1^*) + \mu_2 (e_2^*, e_1^*)$$

0

$$\mu_1 = - \frac{(e_1^*, a_3)}{(e_1^*, e_1^*)} = \frac{4}{4} = 1$$

$$0 = (e_2^*, e_3^*) = (e_2^*, a_3) + \mu_1 (e_1^*, e_2^*) + \mu_2 (e_2^*, e_2^*)$$

0

$$\mu_2 = - \frac{(e_2^*, a_3)}{(e_2^*, e_2^*)} = \frac{14}{14} = 1$$

$$e_3^* = a_3 + e_1^* + e_2^* = (3, 1, -3, 1)$$

$$e_4^* = a_4 + \eta_1 e_1^* + \eta_2 e_2^* + \eta_3 e_3^* \quad | \cdot e_1^* \quad | \cdot e_2^* \quad | \cdot e_3^*$$

$$0 = (e_1^*, e_4^*) = (e_1^*, a_4) + \eta_1 (e_1^*, e_1^*)$$

$$\eta_1 = - \frac{(e_1^*, a_4)}{(e_1^*, e_1^*)} = \frac{2}{4} = \frac{1}{2}$$

$$0 = (e_2^*, e_4^*) = (e_2^*, a_4) + \eta_2 (e_2^*, e_2^*)$$

$$\eta_2 = - \frac{(e_2^*, a_4)}{(e_2^*, e_2^*)} = \frac{34}{14} = \frac{17}{7}$$

$$0 = (e_3^*, e_4^*) = (e_3^*, a_4) + \eta_3 (e_3^*, e_3^*) \quad \eta_3 = - \frac{(e_3^*, a_4)}{(e_3^*, e_3^*)} = - \frac{18}{20} = - \frac{9}{10}$$

VII. 2.4

$$e_4^* = a_4 + \frac{1}{2} e_1^* + \frac{17}{7} e_2^* - \frac{9}{10} e_3^* =$$

$$= (-2, 6, -3, 9) + \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \left(\frac{37}{7}, 0, \frac{17}{7}, -\frac{51}{7}\right) - \left(\frac{27}{10}, \frac{9}{10}, -\frac{27}{10}, \frac{9}{10}\right) = \frac{46}{70} (1, 7, 4, 2)$$

$$e_1 = \frac{1}{|e_1^*|} e_1^* = \frac{1}{\sqrt{2}} (1, -1, 1, 1)$$

$$e_2 = \frac{1}{|e_2^*|} e_2^* = \frac{1}{\sqrt{14}} (2, 0, 1, -3)$$

$$e_3 = \frac{1}{\sqrt{5}} (3, 1, -3, 1)$$

$$e_4 = \frac{1}{\sqrt{70}} (1, 7, 4, 2)$$

$U^\perp$ - ортогональное дополнение
 V - евклидово пространство
 $U \subseteq V$

$$U^\perp \oplus U = V$$

$$(a, U) = 0$$

$$U^\perp = \{a \in V \mid (a, U) = 0\}$$

$$\textcircled{B} \begin{aligned} a_1 &= (1, 2, 0, 1) \\ a_2 &= (3, 2, 1, 2) \\ a_3 &= (1, -2, 1, 0) \end{aligned}$$

$U = \ell(a_1, a_2, a_3)$
 ортонормирован базис на $U^\perp$?

$$U^\perp: \begin{cases} x_1 + 2x_2 + x_4 = 0 \\ 3x_1 + 2x_2 + x_3 + 2x_4 = 0 \\ x_1 - 2x_2 + x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & -2 & 1 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & -4 & 0 & 2 \end{pmatrix} \xrightarrow{1/2} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + x_4 = 0 \\ 3x_1 + 2x_2 + x_3 + 2x_4 = 0 \end{cases}$$

$$\operatorname{rg}(U^\perp) = 2$$

	x_1	x_2	x_3	x_4
b_1	-2	1	4	0
b_2	-1	0	1	1

$$U^\perp = \ell(b_1, b_2) = \ell(e_1, e_2)$$

$$e_1^* = b_1 = (-2, 1, 4, 0)$$

$$e_2^* = b_2 + \lambda e_1^* \quad | \cdot e_1^*$$

$$\ell(e_1^*, e_2^*) = (e_1^*, b_2) + \lambda (e_1^*, e_1^*)$$

$$\lambda = - \frac{(e_1^*, b_2)}{(e_1^*, e_1^*)} = - \frac{6}{21} = - \frac{2}{7}$$

$$e_2^* = (-1, 0, 1, 1) + \left(\frac{4}{7}, -\frac{2}{7}, -\frac{8}{7}, 0 \right)$$

$$= \left(-\frac{3}{7}, -\frac{2}{7}, -\frac{1}{7}, 1 \right)$$

$$e_1 = \frac{1}{|e_1^*|} \cdot e_1^* = \frac{1}{\sqrt{21}} \cdot (-2, 1, 4, 0)$$

$$e_2 = \frac{1}{|e_2^*|} \cdot e_2^* = \frac{1}{3\sqrt{7}} \cdot \left(-3, -2, -1, 7 \right)$$

$$\ell(e_1, e_2) = U^\perp$$

ортонормиран базис

III. 2.6

$$V = U \oplus U^\perp$$

$$\forall a \in V:$$

$$a = a_U + h \quad \begin{array}{l} \text{перпендикуляр к } U \\ \text{ортогональна проекция} \end{array}$$

Ортогональная проекция

$$\textcircled{4} U = l(a_1, a_2, a_3), \quad a \in V$$

$$a_1 = (1, 1, 1, 1)$$

$$a_2 = (1, 2, 2, -1)$$

$$a_3 = (1, 0, 0, 3)$$

ортогональная проекция a_0 ?

перпендикуляр h к U ?

$$a = (4, -1, -3, 4)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 1 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{(-2)} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -3 \\ 1 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{+} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$l(a_1, a_3) = l(a_1, a_2, a_3) = U$$

$$a_1 = e_1 = (1, 1, 1, 1)$$

$$a_3 = e_2 = (e_1, e_2) \neq 0$$

$$e_2 = a_3 + \lambda \cdot e_1 \perp e_1$$

$$\lambda = - \frac{(e_1, a_3)}{(e_1, e_1)} = - \frac{4}{4} = -1$$

$$e_2 = \cancel{(1, 2, 2, -1)} + a_3 - e_1 = (0, -1, -1, 2)$$

$$e_1, e_2 \quad l(e_1, e_2) = l(a_1, a_3) = U$$

ортогонален

$$a_0 = \lambda_1 e_1 + \lambda_2 e_2$$

$$h = a - a_0 \perp U$$

$$(h, e_1) = (h, e_2) = 0$$

$$(a - a_0, e_1) = (a - a_0, e_2) = 0$$

$$(a, e_1) - (a_0, e_1) = 0$$

$$(a, e_2) - (a_0, e_2) = 0$$

$$(a, e_1) - \lambda_1 (e_1, e_1) = 0$$

$$(a, e_2) - \lambda_2 (e_2, e_2) = 0$$

$$4 - 4\lambda_1 = 0 \Rightarrow \lambda_1 = 1$$

$$12 - 6\lambda_2 = 0 \Rightarrow \lambda_2 = 2$$

$$a_0 = (1, 1, 1, 1) + (0, -2, -2, 4) = (1, -1, -1, 5)$$

$$h = a - a_0 = (3, 0, -2, -1)$$

$$a = a_0 + h$$

2. P. $a = (10, -10, 8, 11)$

$$a_1 = (1, -1, 2, 3)$$

$$a_2 = (2, -3, 1, 2)$$

$$a_3 = (2, -1, 7, 10)$$

$$(5) \quad U = \begin{cases} 3x_1 + 2x_2 + x_3 - 2x_4 = 0 \\ 5x_1 + 4x_2 + 3x_3 + 2x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 10x_4 = 0 \end{cases}$$

$$a = (-3, 0, -5, 9)$$

$$a_0 \in U$$

$$h \in U^\perp$$

$$a = a_0 + h$$

$$A = \begin{pmatrix} 3 & 2 & 1 & -2 \\ 5 & 4 & 3 & 2 \\ \textcircled{1} & 2 & 3 & 10 \end{pmatrix} \xrightarrow{\substack{1.(-5) \\ 1.(-3)}} \begin{pmatrix} 1 & 2 & 3 & 10 \\ 0 & -4 & -8 & -32 \\ 0 & -6 & -12 & -48 \end{pmatrix} \xrightarrow{\substack{1.(-1) \\ 1.(-6)}} \begin{pmatrix} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & 2 & 8 \end{pmatrix} \xrightarrow{-} \begin{pmatrix} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{rg}(A) = 2$$

$$l(a_2, a_3) = l(a_1, a_2, a_3) = U$$

$$a_3 = e_1 = (1, 2, 3, 10)$$

$$(e_1, e_2) = 0$$

$$e_2 = a_2 + \lambda e_1 \quad | \cdot e_1$$

$$e_2 = a_2 - \frac{44}{57} e_1$$

$$\lambda = - \frac{(e_1, a_2)}{(e_1, e_1)} = - \frac{88}{114} = - \frac{44}{57}$$

$$= (0, 1, 2, 8) - \frac{44}{57} (1, 2, 3, 10) = \left(-\frac{44}{57}, -\frac{101}{57}, -\frac{101}{57}, -\frac{158}{57} \right) =$$

$$= -\frac{1}{57} (44, 101, 101, 158)$$

$$l(e_1, e_2) = l(a_3, a_2) = U$$

ортogonalен

$$a_0 = \lambda_1 e_1 + \lambda_2 e_2$$

$$h = a - a_0 \perp U$$

$$(h, e_1) = (h, e_2) = 0$$

$$(a - a_0, e_1) = (a - a_0, e_2) = 0$$

$$\begin{cases} (a, e_1) - (a_0, e_1) = 0 \\ (a, e_2) - (a_0, e_2) = 0 \end{cases} \quad \begin{cases} (a, e_1) - \lambda_1 (e_1, e_1) = 0 \\ (a, e_2) - \lambda_2 (e_2, e_2) = 0 \end{cases}$$

$$72 - 88 \lambda_1 = 0 \Rightarrow \lambda_1 = \frac{9}{11}$$

$$-\frac{758}{57} + \frac{47302}{3279} \lambda_2 = 0 \Rightarrow \lambda_2 = \frac{2462242}{2696214}$$

$$a_0 = \underline{\hspace{2cm}}$$

$$h = \underline{\hspace{2cm}}$$

$$a \approx a_0 + h$$

VII.2.10

13. II - 2013₂

Симетричен оператор

V-евклидово пространство над R

$$\varphi: V \rightarrow V$$

$$u, v \in V$$

$$(\varphi(u), v) = (u, \varphi(v)) \Rightarrow \varphi \text{ е симетричен}$$

$$\varphi \rightarrow A A^t \text{ (симетрична матрица)}$$

$$D = T^{-1} A T \rightarrow \text{ортогонален}$$

$$(\varphi(u), \varphi(v)) = (u, v)$$

$$A \cdot A^t = E$$

$$\textcircled{1} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$f_A(\lambda) = \det(A - \lambda E) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \lambda^2(1-\lambda) - (1-\lambda) =$$

$$= (1-\lambda)(\lambda^2 - 1) = -(1+\lambda)^2(1-\lambda) = 0$$

$$\lambda_{1,2} = 1 \quad \lambda_3 = -1 \quad - \text{собственные значения}$$

$$\lambda_{1,2} = 1$$

$$(A - \lambda_{1,2} E) u_{1,2} = 0$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\cancel{x_1 - x_2 = 0} \quad -x_1 + x_2 = 0$$

$$u_1 \perp u_2$$

$$(u_1, u_2) = 0$$

	x_1	x_2	x_3
u_1	1	1	0
u_2	0	0	1

VII. 2.11

$$v_1 = \frac{1}{|u_1|} \cdot u_1 = \frac{1}{\sqrt{2}} (1, 1, 0)$$

$$v_2 = \frac{1}{|u_2|} \cdot u_2 = (0, 0, 1)$$

$$\lambda_3 = -1 \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$x_1 + x_2 = 0$$

$$x_3 = 0$$

$$u_3 = (1, -1, 0)$$

$$v_3 = \frac{1}{|u_3|} \cdot u_3 = \frac{1}{\sqrt{2}} (1, -1, 0)$$

$$D = T^{-1} A T$$

matrix of coefficients

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

$$f_A(\lambda) = \det(A - \lambda E) =$$

$$= \begin{vmatrix} 2-\lambda & -2 & 0 \\ -2 & 3-\lambda & -2 \\ 0 & -2 & 4-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(4-\lambda) - 4(4-\lambda) - 2(2-\lambda) =$$

$$= (4-\lambda)((2-\lambda)(3-\lambda) - 4) - 2(2-\lambda) =$$

$$\begin{aligned}
 &= (6 - \lambda^2 - 5\lambda)(4 - \lambda) - \cancel{16} + 4\lambda - 4 + 2\lambda = \\
 &= -4\lambda^2 - 20\lambda + 24 + \lambda^3 + 5\lambda^2 - \cancel{6\lambda} - 16 + 4\lambda - 4 + 2\lambda = \\
 &= \lambda^3 + \lambda^2 - 20\lambda + 4 = 0
 \end{aligned}$$

$$\lambda^2(\lambda + 1) - 4(5\lambda + 1) = 0$$

$$= (2 - \lambda)(\lambda^2 - 7\lambda + 12) - 4(4 - \lambda) =$$

$$= (2 - \lambda)(\lambda^2 - 7\lambda + 12) - 4(4 - \lambda) =$$

$$= (2 - \lambda)(\lambda^2 - 7\lambda + 12) - 4(4 - \lambda) =$$

$$= -\lambda^3 + 7\lambda + 12\lambda + 2\lambda^2 - 16 + 4\lambda - 16 + 4\lambda =$$

$$= -\lambda^3 + 2\lambda^2 - 15\lambda + 4 = 0$$

$\lambda_1 = 6$

$$\begin{pmatrix} -4 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -2 & -3 & -2 \\ -4 & -2 & 0 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{R_1 \cdot (-1/2)} \begin{pmatrix} 1 & 1.5 & 1 \\ -4 & -2 & 0 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{R_2 + 4R_1} \begin{pmatrix} 1 & 1.5 & 1 \\ 0 & 4 & 4 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{R_2 \cdot 1/4} \begin{pmatrix} 1 & 1.5 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{R_3 + 2R_2} \begin{pmatrix} 1 & 1.5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline u_1 & -1 & 2 & -2 \end{array}$$

$\lambda_2 = 3$

$$\begin{pmatrix} -1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{R_1 \cdot (-1)} \begin{pmatrix} 1 & 2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{R_2 \cdot 1/4} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/2 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{R_3 + 2R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline u_2 & -2 & 1 & -2 \end{array}$$

$$\lambda_3 = 0$$

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 4 \\ -2 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 2 \\ 4 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$v_1 = \frac{1}{3} (1, 2, -2)$$

$$v_2 = \frac{1}{3} (2, -1, -2)$$

$$v_3 = \frac{1}{3} (2, 2, 1)$$

$$T = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(2) \quad A = \begin{pmatrix} -1 & -8 & 4 \\ -8 & -1 & -4 \\ 4 & -4 & -7 \end{pmatrix}$$

$$f_A(\lambda) = A - \det |A - \lambda E| =$$

$$= \begin{vmatrix} -1-\lambda & -8 & 4 \\ -8 & -1-\lambda & -4 \\ 4 & -4 & -7-\lambda \end{vmatrix} = (-1-\lambda)^2 (-7-\lambda) + 128 + 128 -$$

$$- 16(-1-\lambda) - 16(-1-\lambda) - 64(-7-\lambda) =$$

$$= (1-\lambda)((-1-\lambda)(-7-\lambda) - 32) + 256 - 64(-7-\lambda) =$$

$$= (-1-\lambda)(7+\lambda+\lambda^2-32) + 256 + 64 \cdot 7 + 64\lambda =$$

$$= -(1+\lambda)(\lambda^2 + 8\lambda - 25) + 256 + 448 + 64\lambda =$$

$$= -\lambda^3 - 8\lambda + 25 - \lambda^3 - 8\lambda^2 + 25\lambda + 704 + 64\lambda =$$

$$= -\lambda^3 - 8\lambda^2 + 81\lambda + 729 = 0$$

$$\lambda^3 + 8\lambda^2 - 81\lambda - 729 = 0$$

$$(\lambda - 9)(\lambda^2 + 9\lambda + 81) + 9\lambda(\lambda - 9) = 0$$

$$(\lambda - 9)(\lambda + 9)^2 = 0$$

$$\lambda_1 = 9 \quad \lambda_{2,3} = -9$$

$$\lambda_1 = 9$$

$$\begin{pmatrix} -10 & -8 & 4 \\ -8 & -10 & -4 \\ 4 & -4 & -16 \end{pmatrix} \begin{matrix} | :(-2) \\ | :(-2) \\ | :4 \end{matrix} \sim \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 1 & -1 & -4 \end{pmatrix} \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} \sim \begin{pmatrix} 5 & 4 & 2 \\ 0 & 9 & 18 \\ 1 & -1 & -4 \end{pmatrix} \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} \sim$$

$$\sim \begin{pmatrix} 5 & 4 & 2 \\ 0 & 9 & 18 \\ 1 & -1 & -4 \end{pmatrix} \begin{matrix} | :9 \\ | :9 \end{matrix} \sim \begin{pmatrix} 5 & 4 & 2 \\ 0 & 1 & 2 \\ 1 & -1 & -4 \end{pmatrix} \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} \sim \begin{pmatrix} 5 & 3 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & -4 \end{pmatrix} \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & -1 & -4 \\ 0 & 8 & 20 \\ 0 & 1 & 2 \end{pmatrix} \begin{matrix} | :4 \\ | :4 \end{matrix} \sim \begin{pmatrix} 1 & -1 & -4 \\ 0 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix} \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} \sim \begin{pmatrix} 1 & -1 & -4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} \sim$$

$$\sim \begin{pmatrix} 0 & -9 & -18 \\ 0 & -9 & -18 \\ 1 & -1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} | x_1 & x_2 & x_3 \\ | 2 & -2 & 1 \end{matrix}$$

$$v_1(2, -2, 1)$$

$$u_1 = \frac{1}{|v_1|} \cdot v_1 = \frac{1}{3} (2, -2, 1)$$

$\lambda_{2,3} = 0$

$$\begin{pmatrix} 8 & -8 & 4 \\ -8 & 8 & -4 \\ 4 & -4 & 2 \end{pmatrix} \xrightarrow{(-2) \cdot \frac{1}{2}} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

	x_1	x_2	x_3
u_2	1	1	0
u_3	-1	0	2

$$e_1 = v_2$$

$$e_2 = v_3 + \lambda e_1$$

$$(e_1, e_2) = 0 \Leftrightarrow v_2(v_3 + \lambda e_1)$$

$$(1, 1, 0) \cdot (-1, 0, 2) + \lambda(1, 1, 0) =$$

$$= (1, 1, 0) \cdot (-1 + \lambda, \lambda, 2) = 0$$

$$-1 + \lambda + \lambda = 0$$

$$2\lambda = 1$$

$$\lambda = \frac{1}{2}$$

$$e_2 = (-1, 0, 2) + \left(\frac{1}{2}, \frac{1}{2}, 0\right) =$$

$$= \left(-\frac{1}{2}, \frac{1}{2}, 2\right)$$

$$u_2 = \frac{1}{|e_1|} \cdot e_1 = \frac{1}{\sqrt{2}} (1, 1, 0)$$

$$u_3 = \frac{1}{|e_2|} \cdot e_2 = \frac{\sqrt{2}}{3} \left(-\frac{1}{2}, \frac{1}{2}, 2\right)$$

$$T = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{3} \\ \frac{1}{3} & 0 & \frac{\sqrt{2}}{3} \end{pmatrix}$$

$$D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

VII - 2.16

③ V е n -мерно евклидово пространство
 $a, b \in V$

$$|a| = |b| = 1, \quad (a, b) = 0$$

$$\varphi: V \rightarrow V$$

$$\varphi(x) = (a, x)b + (b, x)a \quad \text{за } \forall x \in V$$

а) φ -симметричен оператор

б) $\varphi^3 = \varphi$
в) $\varphi^{2000} = ?$

г) ортонормиран базис на V , в който φ има ортогонална матрица D , $D = ?$

$$\forall x, y \in V$$

$$(\varphi(x), y) = (x, \varphi(y))$$

$$(\varphi(x), y) = ((a, x)b + (b, x)a, y) =$$

$$= (a, x)(b, y) + (b, x)(a, y) =$$

$$= (a, x)(b, y) + (b, y)(a, x) =$$

$$(x, \varphi(y)) = (x, (a, y)b + (b, y)a) = (a, y)(b, x) + (b, y)(a, x)$$

$$d) \quad \varphi(a) = (a, a)b + (b, a)a = b$$

$$\varphi(b) = (b, a)b + (b, b)a = a$$

$$\begin{aligned} \varphi^2(\varphi(x)) &= \varphi((a, x)b + (b, x)a) = (a, x)\varphi(b) + (b, x)\varphi(a) = \\ &= (a, x)a + (b, x)b \end{aligned}$$

$$\begin{aligned} \varphi^3(\varphi(\varphi(x))) &= (a, x)\varphi(a) + (b, x)\varphi(b) = (a, x)b + (b, x)a = \\ &= \varphi(x) \Rightarrow \varphi^3 = \varphi \end{aligned}$$

$$\begin{aligned} b) \quad \varphi^{2000} &= \varphi^{1998} \cdot \varphi^2 = (\varphi^3)^{666} \cdot \varphi^2 = \varphi^2 = (\varphi^3)^{222} \cdot \varphi^2 = \\ &= (\varphi^3)^{74} \cdot \varphi^2 = (\varphi^3)^{72} \cdot \varphi^2 \cdot \varphi^2 = (\varphi^3)^{24} \cdot \varphi^2 = \\ &= (\varphi^3)^8 \cdot \varphi^2 = \varphi^6 \cdot \varphi^2 = \varphi^2 \cdot \varphi^2 \cdot \varphi^2 = \varphi^2 \end{aligned}$$

$$2) \quad a, b, e_3, e_4, \dots, e_n$$

$$\varphi(e_i) = (a, e_i)b + (b, e_i)a \neq 0$$

$$\begin{aligned} e_1 &= \frac{1}{2}(a+b) \\ e_2 &= \frac{1}{2}(a-b) \end{aligned}$$

$$\varphi(e_1) = (a, \frac{1}{2}(a+b))b + (b, \frac{1}{2}(a+b))a =$$

$$\begin{aligned} &= \frac{1}{2}(a, a)b + \frac{1}{2}(b, a)b + \frac{1}{2}(a, b)a + \frac{1}{2}(b, b)a \\ &= \frac{1}{2}(a+b) = e_1 \end{aligned}$$

$$\varphi(e_2) = e_2$$

$$\varphi(e_i) = e_i$$

$$i=3, \dots, n$$

$$(e_i, e_1) = 0$$

$$(e_i, e_2) = 0$$

$$(e_i, \frac{1}{2}(a+b)) = \frac{1}{2}(e_i, a) + \frac{1}{2}(e_i, b) = 0$$

$$(e_i, a) = -(e_i, b)$$

$$(e_i, a) = (e_i, b) = 0$$

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

$$a, b, e_3, e_4, \dots, e_n$$

$$\varphi(a) = b \quad \varphi(b) = a$$

~~$$(e_i, a) = 0$$~~
~~$$(e_i, b) = 0$$~~
~~$$\varphi(e_i) =$$~~

$$e_2 = \frac{1}{2}(a+b)$$

$$e_2 = \frac{1}{2}(a-b)$$

$$\varphi(e_1) = e_1$$

$$\varphi(e_1) = e_1$$

$$\varphi(e_2) = e_2$$

$$\varphi(e_2) = e_2$$

$$D = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ 0 & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

Числовые сравнения

$$a, b \in \mathbb{Z}, m \in \mathbb{N}$$

$$a \equiv b \pmod{m} \Leftrightarrow m \mid (a-b)$$

$$a, b \in \mathbb{Z} \quad a \mid b \Leftrightarrow \exists c \in \mathbb{Z} : b = ac$$

$$5 \equiv 3 \pmod{2}$$

$$1) \quad a \equiv a \pmod{m}$$

$$m \mid (a-a)$$

$$2) \quad a \equiv b \pmod{m} \Leftrightarrow b \equiv a \pmod{m}$$

$$m \mid (a-b) \Leftrightarrow m \mid (b-a)$$

$$3) \begin{matrix} a \equiv b \pmod{m} \\ b \equiv c \pmod{m} \end{matrix} \Rightarrow a \equiv c \pmod{m} \quad \text{transitivitt}$$

$$\begin{matrix} m | (a-b) \\ m | (b-c) \end{matrix} \Rightarrow m | (a-b) + (b-c) \Rightarrow m | (a-c)$$

$$4) \begin{matrix} a_1 \equiv b_1 \pmod{m} \\ a_2 \equiv b_2 \pmod{m} \\ \vdots \\ a_n \equiv b_n \pmod{m} \end{matrix} \Rightarrow a_1 + a_2 + \dots + a_n \equiv b_1 + b_2 + \dots + b_n \pmod{m}$$

$$\begin{matrix} m | (a_1 - b_1) \\ \vdots \\ m | (a_n - b_n) \end{matrix} \Rightarrow m | (a_1 - b_1) + \dots + (a_n - b_n)$$

$$5) a + b \equiv c \pmod{m} \Rightarrow a \equiv c - b \pmod{m}$$

$$m | (a + b) - c$$

$$m | (a - (c - b))$$

$$6) \begin{matrix} a \equiv b \pmod{m} \\ a + c \equiv b + c \pmod{m} \end{matrix}$$

$$m | (a - b)$$

$$7) a_1 \equiv b_1 \pmod{m}$$

$$a_n \equiv b_n \pmod{m}$$

$$a_1 \dots a_n \equiv b_1 \dots b_n \pmod{m}$$

$$m \mid (a_1 - b_1)$$

$$m \mid (a_2 - b_2)$$

$$m \mid (a_n - b_n)$$

$$m \mid (a_1 a_2 + b_1 b_2 - a_1 b_2 - a_2 b_1)$$

$$8) a \equiv b \pmod{m}$$

$$ac \equiv bc \pmod{m}$$

$$m \mid c(a-b)$$

$$\rightarrow m \mid (a_1 a_2 + b_1 (b_2 - a_2) - a_1 b_2) =$$

$$= m \mid (a_1 a_2 + b_2 (b_1 (1 - \frac{a_2}{b_2}) - a_1)) =$$

$$m \mid a_1 - b_1$$

$$\Rightarrow m \mid a_1 a_2 - b_1 a_2 \Rightarrow m \mid a_2 - b_2 \Rightarrow m \mid a_2 b_1 - b_1 b_2$$

$$m \mid a_1 a_2 - b_1 a_2 + b_1 a_2 - b_1 b_2 = m \mid a_1 a_2 - b_1 b_2$$

$$\exists n \in \mathbb{Z} \text{ o.k. } \text{Ass } n-1 \in \text{o.k.}, \text{ no } n \in \text{o.k.}$$

$$9) a \equiv b \pmod{m}$$

$$a^n \equiv b^n \pmod{m}$$

$$\text{no } 7)$$

$$10) a \equiv b \pmod{m}$$

$$f(a) \equiv f(b) \pmod{m}$$

$$f(x) \in \mathbb{Z}[x] \quad m \mid (a-b)$$

$$a_1 \equiv b_1 + b_1 m$$

$$a_2 \equiv b_2 + b_2 m$$

$$\vdots$$

$$a_n \equiv b_n + b_n m$$

$$a_1 a_2 \dots a_n \equiv b_1 b_2 \dots b_n \pmod{m}$$

20.11.2013.

$$(1) 2^{70} + 3^{70} = 13m + k$$

$$2^4 \equiv 3 \pmod{13}$$

$$2^{12} = (2^4)^3 \equiv 3^3 \pmod{13} \equiv 1 \pmod{13}$$

$$(2^{12})^5 = 2^{60} \equiv 1^5 \pmod{13}$$

$$2^{10} = (2^4) \cdot 2^2 = 3^2 \cdot 3^2 \pmod{13} \equiv -3 \pmod{13}$$

$$3^{70}$$

$$3^{69} = (3^3)^{23} \equiv 1 \pmod{13}$$

$$3^{70} \equiv 3^{69} \cdot 3 \equiv 1 \cdot 3 \pmod{13}$$

$$-2^{70} + 3^{70} \equiv -3 + 3 \pmod{13} \equiv 0 \pmod{13}$$

$$(2) (85^{74} + 19^{32})^{40} \equiv k \pmod{13}$$

$$85 \equiv 7 \pmod{13}$$

$$85^2 \equiv 7^2 \pmod{13} \equiv -3 \pmod{13}$$

$$85^3 \equiv -3 \cdot 7 \pmod{13} \equiv -1 \pmod{13}$$

$$85^6 \equiv 5^2 \pmod{13} \equiv -1 \pmod{13}$$

$$(85^6)^{12} \equiv (-1)^{12} \pmod{13} \equiv 1$$

$$85^{72} \cdot 85^2 \equiv 3 \cdot 1 \pmod{13}$$

$$19^{32}$$

$$19 \equiv 6 \pmod{13}$$

$$19^2 \equiv -3 \pmod{13}$$

$$19^3 \equiv (-3)^3 \pmod{13} \equiv (-1) \pmod{13}$$

$$(19^3)^5 \equiv -1^5 \pmod{13}$$

VII. 2.22

$$19^{32} = -1(-3) \pmod{13} = 3 \pmod{13}$$

$$\begin{aligned} (85^{74} + 19^{32}) &\equiv 0 \pmod{13} \\ (85^{74} + 19^{32})^{40} &\equiv 0 \pmod{13} \end{aligned}$$

Сравнение от първа степен

$$a, b \in \mathbb{Z}, m \in \mathbb{N}$$

$$ax \equiv b \pmod{m}$$

$$\mathbb{Z}_m = \{\overline{0}, \overline{1}, \dots, \overline{m-1}\}$$

$(a, m) \mid d = d$ Безусловно, следва от това, че
в рамките $\square$.

$$\exists u, v \in \mathbb{Z} : au + mv = d$$

$$(u, v) = 1$$

Евклид:

$$a = mq_1 + r_1$$

$$m = r_1 q_2 + r_2$$

$$r_1 = r_2 q_3 + r_3$$

$$0 < r_1 < m$$

$$0 < r_2 < |r_1|$$

$$r_{k-2} = r_{k-1} q_k + r_k$$

$$r_{k-1} = r_k q_{k+1} + r_{k+1}$$

$$r_k = r_{k+1} q_{k+2}$$

r

$$(r_{k+1}, r_k) = r_{k+1}(a, m) = r_{k+1}$$

$$(r_k, r_{k+1}) = r_{k+1}$$

$$r_{k+1} = r_{k+1} - r_k q_{k+1} = r_{k+1} - (r_{k-2} - r_{k-1} q_k) q_{k+1}$$

Безусловно $\Rightarrow \exists u, v \in \mathbb{Z} :$

$$au + mv = 1$$

$$u \neq 0$$

$$a = mq_1 + r_1$$

$$uax \equiv ub \pmod{m}$$

$$x \equiv ub \pmod{m}$$

$$r_{k+1} = r_k q_{k+1} + 1$$

VII. 2.23

$$\textcircled{3} \quad 37x \equiv 25 \pmod{117}$$

$$(37, 117) \equiv 1$$

$$\exists u, v \in \mathbb{Z} : 37u + 117v = 1$$

$$117 = 3 \cdot 37 + 6$$

$$3 \cdot 37 = 6 \cdot 6 + 1$$

$$1 = 37 - 6 \cdot 6 = 37 - 6(117 - 3 \cdot 37) = 19 \cdot 37 - 6 \cdot 117$$

$$19 \cdot 37x \equiv 19 \cdot 25 \pmod{117}$$

$$x \equiv 475 \pmod{117} \equiv 7 \pmod{117}$$

$$\textcircled{4} \quad 29x + 3 \equiv 0 \pmod{12}$$

$$29x \equiv -3 \pmod{12}$$

$$(29, 12) = 1 \quad \exists u, v \in \mathbb{Z} : 29u + 12v = 1$$

$$29 = 12 \cdot 2 + 5$$

$$12 = 5 \cdot 2 + 2$$

$$5 = 2 \cdot 2 + 1$$

$$1 = 5 - 2 \cdot 2 =$$

$$= 5 - 2(12 - 5 \cdot 2) =$$

$$= 5 \cdot 5 - 12 \cdot 2 = 5(29 - 12 \cdot 2) - 12 \cdot 2$$

$$= 5 \cdot 29 - 12 \cdot 12$$

"u"
"v"

$$\underbrace{5 \cdot 29}_1 \cdot x \equiv 5(-3) \pmod{12}$$

$$x \equiv 9 \pmod{12}$$

$$ax \equiv b \pmod{m}$$

$$(a, m) = d \neq 1$$

d не имеет общих делителей с m
или d не делит b

не имеет решения, так как d не делит b

$$m/a x \equiv b$$

$$\exists a_1, m_1, b_1 \in \mathbb{Z}:$$

$$\begin{aligned} a &= a_1 d \\ b &= b_1 d \\ m &= m_1 d \end{aligned}$$

$$\begin{aligned} m_1 p_1 x_1 &= b_1 \\ a_1 x &\equiv p_0 \pmod{m_1} \\ (a_1, m_1) &= 1 \end{aligned}$$

$$x_0, x_0 + m_1, x_0 + 2m_1, \dots, x_0 + (d_1 - 1)m_1$$

$$\textcircled{5} \quad 117x \equiv 42 \pmod{87} \quad | :3$$

$$(117, 87) = 3$$

$$\begin{aligned} 39x &\equiv 14 \pmod{29} \\ (39, 29) &= 1 \end{aligned}$$

$$\exists u, v \in \mathbb{Z} : 39u + 29v = 1$$

$$\begin{aligned} 39 &= 29 \cdot 1 + 10 \\ 29 &= 10 \cdot 2 + 9 \\ 10 &= 9 \cdot 1 + 1 \end{aligned}$$

$$\begin{aligned} 1 &= 10 - 9 \cdot 1 = 10 - (29 - 10 \cdot 2) = 10 \cdot 3 - 29 = \\ &= (9 - 29 \cdot 1) \cdot 3 - 29 = 39 \cdot 3 - 29 \cdot 4 \end{aligned}$$

$$u = 3 \quad v = -4$$

$$\begin{aligned} x &\equiv 3 \cdot 14 \pmod{29} \equiv 42 \pmod{29} \text{ после сокращения} \\ x &\equiv 13, 42, 71 \pmod{87} \end{aligned}$$

$$\text{D.P.} \quad 10x \equiv 25 \pmod{35}$$

⑥ K3M $423 \overline{xyz} / 7, 8, 9$

$$423 \cdot xyz \equiv 0 \pmod{7, 8, 9}$$

$$423 \overbrace{xyz}^m \equiv 0 \pmod{504}$$

$$423 + 10^3 + m \equiv 0 \pmod{504}$$

$$(-81)(-8) + m \equiv 0 \pmod{504}$$

$$648 + m \equiv 0 \pmod{504}$$

$$m \equiv 360 \pmod{504}$$

$$m \equiv 504k + 360$$

$$k \geq 0, 1;$$

$$m \equiv 360, 864 \pmod{504}$$

⑦ K3M $xy32 / 3, 7$

$$xy32 \equiv 0 \pmod{21}$$

$$10^2 m + 32 \equiv 0 \pmod{21}$$

$$(-5)m + 11 \equiv 0 \pmod{21}$$

$$(-5)m \equiv -11 \pmod{21}$$

$$5m \equiv 11 \pmod{21}$$

$$(5, 21) = 1$$

$$1 = 21 - 4 \cdot 5$$

$$m \equiv -44 \pmod{21}$$

$$m \equiv 19 \pmod{21}$$

$$m \equiv 19, 40, 61, 82, \pmod{21}$$

$$\textcircled{8} \quad \begin{cases} 3x \equiv 1 \pmod{10} \\ 4x \equiv 3 \pmod{5} \\ 5x \equiv 7 \pmod{9} \end{cases}$$

$$3x \equiv 1 \pmod{10}$$

$$(3, 10) = 1$$

$$10 \equiv 3 \cdot 3 + 1$$

$$10 - 3 \cdot 3 = 1$$

$$\boxed{\begin{cases} x \equiv -3 \pmod{10} \\ x \equiv 7 \pmod{10} \end{cases}}$$

$$4x \equiv 3 \pmod{5}$$

$$(4, 5) = 1$$

$$5 \equiv 4 \cdot 1 + 1$$

$$1 \equiv 5 - 1 \cdot 4$$

$$\boxed{x \equiv 2 \pmod{5}}$$

$$2x \equiv 7 \pmod{9}$$

$$(2, 9) = 1$$

$$1 \equiv 9 - 4 \cdot 2$$

$$\boxed{x \equiv +28 \pmod{9} \equiv 8 \pmod{9}}$$

$$x \equiv 2 \pmod{50}$$

$$x \equiv 17 \pmod{90}$$

$$a_1 \equiv b_1 \pmod{m_1}$$

$$a_2 \equiv b_2 \pmod{m_2}$$

$$\vdots$$

$$a_n \equiv b_n \pmod{m_n}$$

$$\Rightarrow a_1 \equiv b_1 \pmod{m_1, m_2, \dots, m_n}$$

$$\text{quo } \text{ca } (m_1, m_2, \dots, m_n) = 1$$

$$a_1 \equiv b_1 \pmod{m_n}$$

$$\Rightarrow a_1 \equiv b_1 \pmod{(m_1, m_2, \dots, m_n)}$$

$$\text{quo } (m_1, m_2, \dots, m_n) \neq 1$$

Ферма

$a \in \mathbb{Z}$ p -простое

$$a^p \equiv a \pmod{p}$$

$$(a, p) = 1$$

$$a^{p-1} \equiv 1 \pmod{p}$$

Ойлер

$a \in \mathbb{Z}$ $n \in \mathbb{N}$ $(a, n) = 1$

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

$$\boxed{\varphi(p) = p-1}$$

$$n = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$$

p_i -простое

$\varphi(n)$ — функция на Ойлер

$\varphi(n)$ — количество чисел меньше n и взаимно простых с n

$$\varphi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_n}\right)$$

$$\varphi(n) = p_1^{k_1-1} p_2^{k_2-1} \dots p_n^{k_n-1} (p_1-1)(p_2-1)\dots(p_n-1)$$

$$\varphi(75) = 75 \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) = 5 \cdot 8 = 40$$

$$(a, b) = 1$$

$$\varphi(a)\varphi(b) = \varphi(ab)$$

9) $\forall p$ -простое?

$$7^p \equiv 13 \pmod{p}$$

$$(7, p) = 1$$

$$7^p \equiv 7 \pmod{p}$$

$$7 \equiv 13 \pmod{p}$$

$$6 \equiv 0 \pmod{p}$$

$$p|6 \Rightarrow p = 2, 3;$$

$$7^2 \equiv 13 \pmod{3}$$

$$7^3 \equiv 13 \pmod{3}$$

10) Последните 2 цифри, в десетична запис на 137^{42}

$$137^{42} \equiv 2 \pmod{100}$$

$$100 = 2^2 \cdot 5^2$$

$$\varphi(100) \equiv 100 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{5}\right) = 40$$

$$137^{\varphi(100)} = 137^{40} \equiv 1 \pmod{100} \equiv 69 \pmod{100}$$

от Ойлер

11) $(x+y)^p \equiv x^p + y^p \pmod{p}$

$$x^p + \binom{p}{1} x^{p-1} y + \binom{p}{2} x^{p-2} y^2 + \dots +$$

$$+ \binom{p}{p-1} x y^{p-1} + y^p \equiv x^p + y^p \pmod{p}$$

$$\binom{p}{k} = \frac{p(p-1) \dots (p-k+1)}{k!} \equiv 0 \pmod{p}$$

Теорема на Уинсх

Ако p -просто, то

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}$$

$$p-1 \equiv -1 \pmod{p}$$

$$\binom{p-1}{k} = \frac{(p-1) \dots (p-k)}{k!}$$

$$p-2 \equiv -2 \pmod{p}$$

$$p-k \equiv -k \pmod{p}$$

$$(p-1) \dots (p-k) \equiv (-1)^k k! \pmod{p}$$

12) Да се покаже $2^{11 \cdot 31} \equiv 2 \pmod{11, 31}$

$$2^{11 \cdot 31} \equiv 2 \pmod{11}$$

$$2^{11 \cdot 31} \equiv 2 \pmod{31}$$

$$(11, 31) = 1$$

Ом дѡерма

$$(2, 11) = 1$$

$$2^{10} \equiv 1 \pmod{11}$$

$$2^{11 \cdot 31} \equiv 2^{31} \pmod{11}$$

$$2^{30} \cdot 2^1 \pmod{11} =$$

$$(2, 31) = 1$$

дѡерма

$$2^{30} \equiv 1 \pmod{31}$$

$$(2^{31})^{11} \equiv 2^{11} \pmod{31}$$

$$(2^{31})^{11} \equiv 2^{11} \pmod{31}$$

$$2^2 \equiv 1 \pmod{31}$$

$$2^{10} \equiv 1 \pmod{31}$$

$$2^{11} \equiv 2 \pmod{31}$$

13) Поегнугне 2 чюдорч ~~на~~ 6 гесетухе 3агуча

$$2^{2013} \equiv 2 \pmod{100}$$

$$2^{2013} \equiv 2 \pmod{4}$$

$$2^{2013} \equiv 2 \pmod{25}$$

$$(2, 25)$$

$$\varphi(25) = 25 \left(1 - \frac{1}{5}\right) = 20$$

$$100 \equiv 4 \pmod{25}$$

$$(4, 25) = 1$$

Оутер

$$2^{\varphi(25)} \equiv 2^{20} \equiv 1 \pmod{25}$$

$$2^{2000} \equiv 1 \pmod{25}$$

$$2^{10} \equiv -1 \pmod{25}$$

$$2^{2013} \equiv 17 \pmod{25} = 42, 67, 92$$

$$2^{2003} \equiv 92 \pmod{100}$$

VII. 2.30

27.III.2013.

Групи

$G \neq \{\emptyset\}$ непразно множество $*, +$

0) $\forall a, b \in G$
 $ab \in G, a+b \in G$

1) $(ab)c = a(bc) = abc, (a+b)+c = a+(b+c) = a+b+c$

2) $\exists! e \in G: ae = ea = a$ (единичен елемент)

$0 \in G: 0+a = a+0 = a$

3) $\exists a^{-1} \in G: aa^{-1} = a^{-1}a = e$

$-a \in G: a+(-a) = (-a)+a = 0$

0, 1, 2, 3 $\Rightarrow$ ~~е~~ група

4) $ab = ba$
 $a+b = b+a$ $\rightarrow$ комутативен закон

0, 1, 2, 3, 4 $\Rightarrow$ абелева група

Подгрупа: $H \neq \{\emptyset\} \subseteq G$

$h_1, h_2 \in H$

1) $h_1 h_2 \in H$

2) $\exists h^{-1} \in H: h h^{-1} = h^{-1} h = e \Rightarrow H \leq G$

$F^* = F \setminus \{0\}$

H е подгрупа

$\mathbb{N}$	$\otimes$	$\otimes$
$\mathbb{Z}$	$\oplus$	$\otimes$
$\mathbb{Q}$	$\oplus$	$\otimes$
$\mathbb{R}$	$\oplus$	$\otimes$
$\mathbb{C}$	$\oplus$	$\otimes$

$\mathbb{R}^*, \mathbb{C}^*$

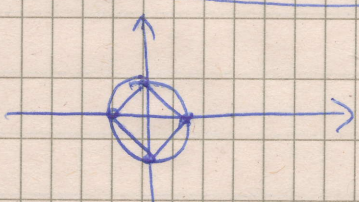
$n \in \mathbb{N}$

$n\mathbb{Z} = \{0, \pm n, \pm 2n, \dots\}$

$n\mathbb{Z} < \mathbb{Z}$

$C_n = \{z \in \mathbb{C} \mid z^n = 1\} \rightarrow$ Група относно $\otimes$
 $n \in \mathbb{N}$
 фиксирано

$w_k = \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n} \quad k=0, 1, 2, \dots, n-1$



$w_1^k = w_k$

$w_k^n = 1$

$U = \{z \in \mathbb{C} \mid |z| = 1\}$ - группа относительно умножения.

$$|z^{-1}| = 1$$

$$z = 1(\cos 0 + i \sin 0)$$

$$\frac{1}{z}$$

$$n \geq 2$$

$$GL_n(F)$$

$$\det \neq 0$$

$$F \neq \mathbb{Z}$$

general
linear
group

общая
линейная
группа

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

$$SL_n(F)$$

$$\det = 1$$

special
linear
group

специальная
линейная
группа

$$\det A \cdot \det B = \det(AB)$$

$$\begin{matrix} \parallel & \parallel & \parallel \\ 1 & 1 & 1 \end{matrix}$$

$$\det A \cdot \det A^{-1} = \det(AA^{-1}) = \det E$$

$$\begin{matrix} \parallel & \parallel & \parallel \\ 1 & 1 & 1 \end{matrix}$$

$$O_n(\mathbb{R})$$

general
orthogonal
group

общая
ортогональная
группа

$$A^{-1} = A^T$$

$$\det A = \pm 1$$

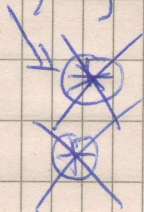
$SO_n(\mathbb{R})$

$$\det = 1$$

special
orthogonal
group

спеціальна
ортогональна
група

① $\{-1, 1\}$; $\{-1, 0, 1\}$
група
относно
⊗



② Симметричні матриці ст. n з елем. $\mathbb{R}$ раціональні числа на основі $n \times n$ з $\mathbb{R}$

$A = A^T$ члени
относно ⊗

$$A, B \in S^*$$

$$A = A^T$$

$$\det A \neq 0$$

$$B = B^T$$

$$\det B \neq 0$$

$$AB \in S: (AB)^T = AB$$

$$BA = B^T A^T = AB$$

G, H -групи
($G, *$)
($H, \circ$)

$$a, b \in G$$

$$\varphi: G \rightarrow H$$

$$\varphi(a) \circ \varphi(b) = \varphi(a * b)$$

φ - гомоморфізм на групи
 φ - бієкція

індукція
стерекція

$$a \neq b$$

$$\varphi(a) \neq \varphi(b)$$

$$\varphi(a) \neq \varphi(b)$$

$$\forall a$$

$\Rightarrow \varphi$ — изоморфизм на групи $G \cong H$

③ $\mathbb{R}^+$ — положительные реальные числа

$(\mathbb{R}^+, *)$ — группа?

$$(\mathbb{R}^+, *) \cong (\mathbb{R}, +)$$

$(\mathbb{R}^+, *)$ — группа

$$\{\emptyset\} \neq \mathbb{R}^+ \subset \mathbb{R}^+$$

$$1) a, b \in \mathbb{R}^+ \\ ab \in \mathbb{R}^+ - \text{закрытость}$$

$$2) \exists a^{-1} \in \mathbb{R}^+ : aa^{-1} = a^{-1}a = 1 \\ \rightarrow \text{га}$$

$$\varphi: \mathbb{R}^+ \rightarrow \mathbb{R} \quad \varphi(a) = \ln a$$

$$\varphi(a) + \varphi(b) = \varphi(ab)$$

$$\lg a + \lg b = \lg ab \Rightarrow \text{гомоморфизм на группы}$$

Инъекция?

$$a \neq b \quad \ln a \neq \ln b \quad \swarrow \searrow$$

$$\textcircled{1} \mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$\mathbb{Q}_8$ - е кватернарна група
(група на кватернионите)

$\ast, 1$ - единичен елемент

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

$\mathbb{Q}_8 \neq \{\emptyset\}$ затворена относно $\ast$ по дефиниция.

1) $(ijk = i(jk))$ - асоциативност
 $k^2 = i^2 = -1$

2) по деф. 1-единичен елемент

3) $\exists x \in \mathbb{Q}_8 : ix = xi = 1$
 $i\bar{i} = 1$
 $(-i)\bar{i} = 1 = i \cdot (-i)$

$\mathbb{Q}_8$ е група

4) по дефиницията $ij = -ji \Rightarrow \mathbb{Q}_8$ е кватернарна група

$$\textcircled{5} \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$i \in \mathbb{C}$$

мнимая единица

$$\{ \pm E, \pm I, \pm J, \pm K \} \cong Q_8$$

$$|G| \leq n \quad |G| < \infty$$

$$SL_2(\mathbb{C})$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = K$$

$$\downarrow$$

$$\downarrow$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\downarrow$$

$$\downarrow$$

$$\exists x \in G : KX = XK$$

$$X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -K$$

*	E	-E	I	-I	J	-J	K	-K
E								
-E								
I								
-I								
J								
-J								
K								
-K								

D_n -группа зрота
 $n \in \mathbb{Z}, n \in \mathbb{N}$

$$A = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D_n \leq GL_2(\mathbb{R})$$

$$\langle A, B \rangle \cong D_n$$

$$a) A^n = E, B^2 = E, \boxed{B^{-1}AB = A^{-1}}$$

$$B \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \cdot A = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos \frac{4\pi}{n} & -\sin \frac{4\pi}{n} \\ \sin \frac{4\pi}{n} & \cos \frac{4\pi}{n} \end{pmatrix}$$

по индукции

$$BAB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos \frac{2\pi}{n} & \sin \frac{2\pi}{n} \\ -\sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} = A^{-1}$$

$$d) D_n = \{ A^i B^j \mid i=0, 1, \dots, n-1; j=0, 1 \}$$

$$A^i B^j A^k B^l = A^{i+k(-1)^j} B^{j+l}$$

$$A^i B^j A^k B^l = A^i B^j A^k B^l \quad j=1, l=1$$

$$A^{i+k} B^l \quad j=0; l=1$$

$$A^{i+k} B^l \quad j=0, l=0$$

$$A^{-k} = (BAB)^k = \underbrace{BAB}_{E} \underbrace{BAB}_{E} \dots =$$

$$A^i B A^k B = A^i B A^k B = A^i B B A^{-k} B = A^{i-k} B$$

$$BAB = A^{-1} \rightarrow A = B A^{-1} B$$

$$A^k = B A^{-k} B$$

6) D_n е неабелева група от ред $2n$

$$AB = BA^{-1}$$

G -група

$$g \in G$$

$$g^2 = 1$$

$$|g| = 2 \rightarrow \text{наималко}$$

$$z(g) = 2 \rightarrow \text{еквивалентни}$$

ред на елемента g
 2 -най-малкото ест. число,
 за което $g^2 = 1$

7) G -grupa, $a \in G$, $|a| = z$

a) $a^n = 1 \ (n \in \mathbb{Z}) \Leftrightarrow z | n$

$\Rightarrow$ $n = zq + r$ $0 < r < z$

$a^n = a^{zq+r} = (a^z)^q \cdot a^r = 1^q \cdot a^r = a^r = 1$, $\text{którego } r < z$
 $\Rightarrow r = 0$
 $\Rightarrow z | n$

$\Leftarrow$ $z | n \quad \exists q \in \mathbb{Z} : n = zq$

$a^n = a^{zq} = (a^z)^q = 1^q = 1$

b) $a^m = a^n \ (m, n \in \mathbb{Z}) \Leftrightarrow m \equiv n \pmod{z}$

$\Rightarrow$ $a^m = a^n \Rightarrow a^{m-n} = 1$ $\text{om a)} \Rightarrow z | m-n$
 $m \equiv n \pmod{z}$

$\Leftarrow$ $m \equiv n \pmod{z}$ $\frac{z}{z} | m-n$ $\exists q \in \mathbb{Z} : m-n = zq$

$a^{m-n} = a^{zq} = 1 \Rightarrow a^m = a^n$

b) $a^k \ (k \in \mathbb{Z})$

$|a^k| = \frac{z}{(k, z)} = z_1 \quad (k, z) = d$

$\frac{d}{d} | k$ $\exists k_1, z_1 : k = dk_1$
 $\frac{z}{d} | z$ $z = dz_1$

$|a^k| = z_1$

$$(k_1, z_1) \geq 1$$

згачно нросту

$$(a^k)^g = a^{kg} = a^{dk_1g} = 1 = \cancel{a^{dk_1g}} = \cancel{a^{dk_1g}}$$

$$z \mid dk_1g$$

$$dk_1g \equiv 0 \pmod{z}$$

$$k_1g \equiv 0 \pmod{z_1}$$

$$\Rightarrow g \equiv 0 \pmod{z_1}$$

$$z_1 \mid g$$

$$\begin{matrix} z \mid d \\ z \mid g \\ d \mid z \end{matrix}$$

$$\begin{aligned} a^{kz_1} &= a^{dk_1z_1} = a^{dz_1} = 1 \\ (a^k)^{z_1} &= 1 \Rightarrow g \mid z_1 \\ |a^k| &= g \end{aligned}$$

8) G -група $a, b \in G$
а) ? a и $g^{-1}ag$ ($g \in G$) имат един и същи ред

$$|a| = z$$

$$z \mid g$$

$$|g^{-1}ag| = g$$

$$\begin{matrix} z \mid g \\ g \mid z \end{matrix}$$

$$a^g = (gg^{-1}agg^{-1})^g =$$

$$= \underbrace{gg^{-1}agg^{-1} \cdot gg^{-1}agg^{-1} \cdot \dots \cdot gg^{-1}agg^{-1}}_{g\text{-нест}} =$$

$$= g(g^{-1}ag)^g \cdot g^{-1} = g \cdot g^{-1} = 1 \Rightarrow z \mid g$$

$$\cancel{g} (g^{-1} a g)^z = \underbrace{g^{-1} a g \cdot g^{-1} a g}_{z\text{-крат}} \cdots \underbrace{g^{-1} a g}_1 =$$

$$= g^{-1} a^z g = 1 \Rightarrow g | z$$

$$\Rightarrow z \equiv g$$

б) ab и ba имеют один и тот же делитель g .

$$\begin{aligned} |ab| &= z \\ |ba| &= g \end{aligned} \quad z/g, g | z \Rightarrow z = zg$$

$$(a, b) = g = (aba a^{-1})^g =$$

$$= \underbrace{aba a^{-1} \cdot aba a^{-1} \cdots aba a^{-1}}_{g\text{-крат}} \cdot a^g =$$

$$= a (ba)^g a^{-1} = 1 \quad z/g$$

$$(ba)^z = (a^{-1} a b a)^z = \cdots = 1 \quad g | z$$