МО 05.05.2015

Двойнственост. Правила

(1)

$$\min_P z = \sum_{j=1}^{n} c_j x_j, \quad (y_i \geq 0) \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1 .. m_1$$

$$y_i \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = m_{1+1} .. m, \quad x_j \geq 0, \quad j = 1 .. l$$

(2)

$$\max_Q V = \sum_{i=1}^{m} b_i y_i$$

Ограничения

$$(x_j \geq 0) \sum_{i=1}^{m} a_{ij} x_j \leq c_j \quad j = 1 .. l$$

$$x_j \quad \sum_{i=1}^{m} a_{ij} y_i = c_j, \quad y_i \geq 0 \quad i = 1 .. m_1, \quad j = l+1 .. n$$

Зад

$$\max z = 2x_1 - x_2 + 5x_3 \qquad \min V = 2y_1 + y_2 - 3y_3$$

(2)

$$\begin{array}{l|l} y_1 \geq 0 & -x_1 + 2x_2 - x_3 \leq 2 \\ y_2 & x_1 \quad - 2x_3 = 1 \\ y_3 \geq 0 & 2x_1 - x_2 + x_3 \geq 3 \quad / \cdot (-1) \\ & x_1, x_2 \geq 0 \end{array} \Rightarrow \begin{array}{l|l} (x_1 \geq 0) & -y_1 + y_2 - 2y_3 \geq 2 \\ (x_2 \geq 0) & 2y_1 + y_3 \geq -1 \\ x_3 & -y_1 - 2y_2 - y_3 = 5 \\ & y_1, y_3 \geq 0 \end{array}$$

1. $x^*(1)\ y^*(2) \geq (x^*) = V(y^*)$
2. $\min_P z = -\infty \Rightarrow Q = \emptyset$

   $\max z = \infty \Rightarrow P = \emptyset$
3. $P = \emptyset$; $\emptyset = Q$

Две допустими решения на двете задачи са оптимални $\Longleftrightarrow$ са изпълнени следните условия за допустимост за $x^* \in P$, $y^* \in Q$

$$y_i^* \left(\sum_{j=1}^{n} a_{ij} x_j^* - b_i\right) = 0 \qquad i = 1 .. m_1$$

$$x_j^* \left(\sum_{i=1}^{m} a_{ij} y_i^* - c_j\right) = 0 \qquad j = 1 .. l$$

Зад $\max z = 9x_1 + 5x_2 + 5x_3$ Двойнствена? Оптимални ли са решенията $\bar{x} = (2, 0, 9)$, $\bar{y} = (0, 1, 1)$

$y_1 \geq 0$ $\quad x_2 \leq 5$

y_2 $\quad 9x_1 + 4x_2 + 4x_3 = 54$

$(y_3 \geq 0)$ $\quad -x_2 - x_3 \geq -9 \; | \cdot (-1)$

$x_1, x_2 \geq 0$

$\bar{x} \in P$

$\min v = 5y_1 + 54y_2 + 9y_3$ $\quad \bar{y} \in Q$

$(x_1 \geq 0)$ $\quad 9y_2 \geq 9$

$(x_2 \geq 0)$ $\quad y_1 + 4y_2 + y_3 \geq 5$ $\quad$ Трябва да проверим дали

x_3 $\quad 4y_2 + y_3 = 5$ $\quad z(\bar{x}) = v(\bar{y})$

$y_1, y_3 \geq 0$ $\quad 63 \quad 63$

Оттук директно заместване векторите в у-та на $\max z$ и $\min v$

Зад

$\max z = 2x_1 + 3x_2 + 4x_3$

$(y_1 \geq 0)$ $\quad 2x_1 + 3x_2 + 2x_3 \leq 6$

$(y_2 \geq 0)$ $\quad x_1 + x_2 + 4x_3 \leq 8$

$x_1, x_2, x_3 \geq 0$

$\min v = 6y_1 + 8y_2$

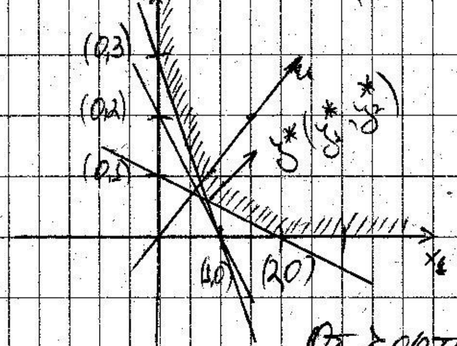
$(x_1 \geq 0)$ $\quad 2y_1 + y_2 \geq 2$ $\quad (1,0), (0,2)$

$\Rightarrow (x_2 \geq 0)$ $\quad 3y_1 + y_2 \geq 3$ $\quad (1,0), (0,3)$

$(x_3 \geq 0)$ $\quad 2y_1 + 4y_2 \geq 4$ $\quad (0,1), (2,0)$

$y_1, y_2 \geq 0$

$u = (6, 8) = (3, 4) = (1.5, 2)$



Щом двойнствената има решение $\Rightarrow$ другата задача също има решение

От чертежа виждаме че $y_1^* > 0$ и $y_2^* > 0$.

Тогава $\exists x^*$, т.ч. $2x_1^* + 3x_2^* + 2x_3^* = 6$ от $y_1^* > 0$ и

$x_1^* + x_2^* + 4x_3^* = 8$ от $y_2^* > 0$

но $2y_1^* + y_2^* > 0 \Rightarrow x_1^* = 0$ т.е. решението е

$x^*(0, 4/5, 9/5)$ $\quad y^*(4/5, 3/5)$

$z(x^*) = 12/5 + 36/5 = 48/5$ $\quad v(y^*) = 24/5 + 24/5 = 48/5$

допустимо

Зад $\bar{x} = (3,0,1,3) \in P$

$$\max z = x_1 - 2x_2 + 3x_3 - x_4$$

y_1: $x_1 + x_2 - 3x_3 - x_4 = -3$

y_2: $x_1 + 3x_4 = 12$

y_3: $x_2 + x_3 = 1$

$x_j \geq 0, \ j = 1..4$

$\Rightarrow$

$$\min V = -3y_1 + 12y_2 + y_3$$

$(x_1 \geq 0)$ $y_1 + y_2 \geq 1$

$(x_2 \geq 0)$ $y_1 + y_3 \geq -2$

$(x_3 \geq 0)$ $-3y_1 + y_3 \geq 3$

$(x_4 \geq 0)$ $-y_1 + 3y_2 \geq -1$

$\bar{x}_1 > 0 \Rightarrow \exists \bar{y}$ т.е. $\bar{y}_1 + y_2 = 1 \Rightarrow \bar{y} = (1,0,6) \in Q$ доп. реш.

Понеже $\max z(\bar{x}) = 3$ търсим реш. при $\min V(\bar{y}) = 3$

$\bar{x}_3 > 0 \Rightarrow \exists \bar{y}$ т.е. $-3\bar{y}_1 + \bar{y}_3 = 3$

$\bar{x}_4 > 0 \Rightarrow \exists \bar{y}$ т.е. $-\bar{y}_1 + 3\bar{y}_2 = -1$

$z(\bar{x}) = 3, \ V(\bar{y}) = 3$

Опт. реш. за двойнственост

Зад $\min z = -x_1 + 2x_2 + 3x_3$

$(y_1 \geq 0)$ $-x_1 - 2x_2 + x_3 - x_4 = -1$

$(y_2 \geq 0)$ $x_1 - x_2 + 2x_4 \leq -4$ $(\cdot(-1))$

$x_j \geq 0, \ j = 1..4$

$\Rightarrow$

$$\max V = -y_1 + 4y_2$$

$(x_1 \geq 0)$ $-y_1 - y_2 \leq -1 \Rightarrow x_1 = 0$

$(x_2 \geq 0)$ $-2y_1 + y_2 = 2$ $(-1,0), (0,2)$

$(x_3 \geq 0)$ $y_1 = 3$ $(-2,1), (2,-1)$

$(x_4 \geq 0)$ $-y_1 - 2y_2 \leq 0 \Rightarrow (x_4 = 0)$

$y_j \geq 0, \quad u = (-1,4)$

$x^* = (0,4,7,0) \ / \ y^* = (3,8)$