КА 02.10.14

Комплексни числа

$\mathbb{C}$ - мн-вото на комплексните числа

Ако $z \in \mathbb{C} \Rightarrow z = x + iy$ - алгебричен вид на z

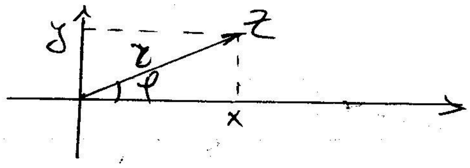
$x, y \in \mathbb{R}$, $i \in \mathbb{C}$ $(i^2 = -1)$

x - реална част на z $\rightarrow \operatorname{Re} z$

y - имагинерна част на z $\rightarrow \operatorname{Im} z$

Ако в равнината фиксираме правоъгълна координатна система, то м/у $\mathbb{C}$ и мн-то от точки в равнината можем да установим взаимно еднозначно съответствие

$$z = x + iy \Leftrightarrow (x, y)$$



$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \Rightarrow$$

$z = r(\cos\varphi + i\sin\varphi)$ - тригонометричен вид

r - модул на z - $|z| = \sqrt{x^2 + y^2}$

$\varphi + 2k\pi$ - аргумент на z - $\arg z$

$e^{i\varphi} \overset{def}{=} \cos\varphi + i\sin\varphi$, $\varphi \in \mathbb{R}$ т.е. $\underline{\underline{z = e^{i\varphi} r}}$

Векторно се проверява че $e^{i(\varphi_1 + \varphi_2)} = e^{i\varphi_1} e^{i\varphi_2}$ и

$$e^{i(\varphi_1 - \varphi_2)} = \frac{e^{i\varphi_1}}{e^{i\varphi_2}} \qquad \varphi_1, \varphi_2 \in \mathbb{R}$$

Следователно ако $z_1 = r_1 e^{i\varphi_1}$ а $z_2 = r_2 e^{i\varphi_2}$

то $z_1 z_2 = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$ и $z_1/z_2 = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}$

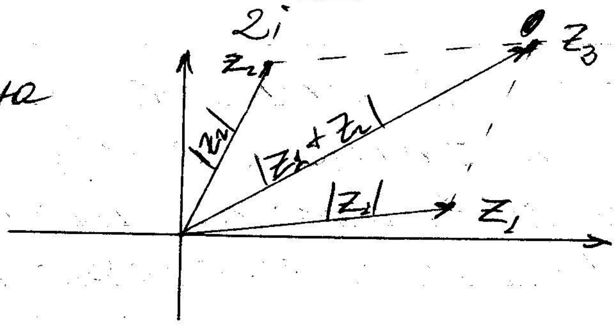
$e^{i\varphi} = \cos\varphi + i\sin\varphi$ (1)

$e^{-i\varphi} = \cos\varphi - i\sin\varphi$ (2)

$(1)+(2) \Rightarrow \cos\varphi = \dfrac{e^{i\varphi}+e^{-i\varphi}}{2}$

$(1)-(2) \Rightarrow \sin\varphi = \dfrac{e^{i\varphi}-e^{-i\varphi}}{2i}$

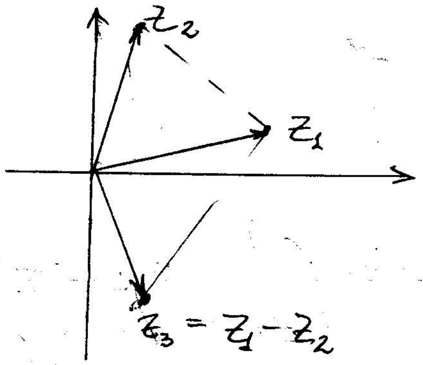
• по правилото на успоредника



сбор

$$\big||z_1|-|z_2|\big| \le |z_1+z_2| \le |z_1|+|z_2|$$

неравенството на триъгълника



разлика

Ако $z = x+iy$, $\bar{z} = x-iy$ - спрегнатото на z

$z+\bar{z} = 2\operatorname{Re} z$, $z-\bar{z} = 2\operatorname{Im} z$, $z\bar{z} = |z|^2$

$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$, $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

Зад (1.1,б) Да се запише в алгебричен вид

числото $z = \dfrac{3+i}{1-i}$

Реш: Умножаваме с $1+i$ (спрягане знаменателя)

$$\Rightarrow \quad \frac{(3+i)(1+i)}{(1+i)(1-i)} = \frac{2+4i}{2} = \underline{\underline{1+2i}}$$

(1.2) Нека $a = 2+i$, $b = 1-2i$, да се пресметне
a^2+b^2, $|b|$, $a\bar{b}$, a/b

$$a^2+b^2 = 4+4i-1+1-4i-4 = 0$$

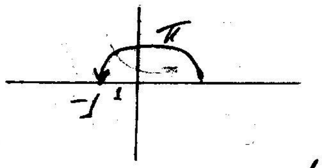
$$|b| = \sqrt{1^2+(-2)^2} = \sqrt{5}$$

$$a\bar{b} = (2+i)(1+2i) = 2+i+4i-2 = 5i$$

$$\frac{a}{b} = \frac{2+i}{1-2i}\cdot\frac{1+2i}{1+2i} = \frac{(2+i)(1+2i)}{(1-2i)(1+2i)} = \frac{2+i+4i-2}{5} = i$$

1.3. Да се запише в тригонометричен вид

а) -1

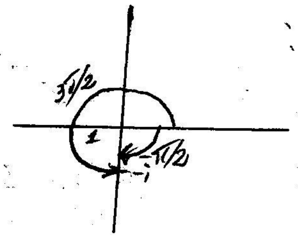


$|-1| = 1$

$\arg(-1) = \pi$

$-1 = 1\cdot e^{i\pi}$

б) $-i$

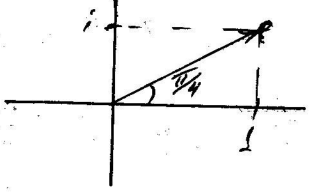


$|-i| = 1$

$\arg(-i) = -\pi/2$

$-i = 1\, e^{-i\pi/2}$

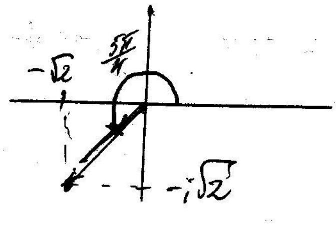
в) $1+i$



$|1+i| = \sqrt{1+1} = \sqrt{2}$

$1+i = \sqrt{2}\, e^{i\pi/4}$

г) $-\sqrt{2}-i\sqrt{2}$



$\sqrt{\sqrt{2}^2+(-i\sqrt{2})^2} = \sqrt{4} = 2$

$\arg(-\sqrt{2}-i\sqrt{2}) = \frac{5\pi}{4}$

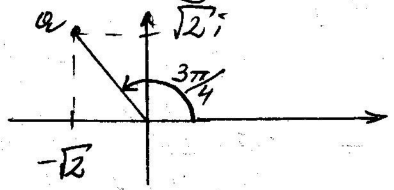
$-\sqrt{2}-i\sqrt{2} = 2e^{i\frac{5\pi}{4}}$

(1.4) Нека $a = -\sqrt{2} + i\sqrt{2}$, $b = 1 + i\sqrt{3}$. Да се запишат в тригонометричен вид чрез главен аргумент (?) числата $a, \bar{a}, b, \bar{b}, ab, a/b, \frac{a^7}{b^6}, a+b$

Забележка

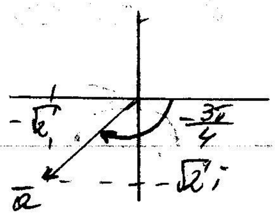
Всяко различно от 0 комплексно число z има безкрайно много аргументи, но само един от тях лежи в интервала $(-\pi, \pi]$; той се нарича главен аргумент на $z \to \arg_0 z$

$a)$



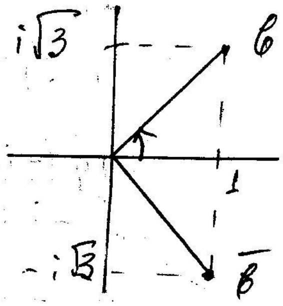
$$|a| = 2$$
$$a = 2e^{i\frac{3\pi}{4}}$$

$\bar{a})$



$$|\bar{a}| = 2$$
$$\bar{a} = 2e^{i\left(-\frac{3\pi}{4}\right)}$$

$b)$



$$|b| = 2 = |\bar{b}|$$
$$\arg_0 b = \pi/3$$
$$b = 2e^{i\pi/3}$$
$$\bar{b} = 2e^{-i\pi/3}$$

$$ab = 2e^{i\pi/3} \cdot 2e^{i\frac{3\pi}{4}} = 4e^{i\frac{13\pi}{12}} \Rightarrow 4e^{i\left(\frac{13\pi}{12} - 2\pi\right)} = \underbrace{4e^{-i\frac{11\pi}{12}}}_{\arg_0}$$

$$\frac{a}{b} = \frac{2e^{i\frac{3\pi}{4}}}{2e^{i\pi/3}} = e^{i\pi(3/4 - 1/3)} = e^{i\pi\frac{5}{12}}$$

$$\frac{a^7}{b^6} = \frac{2e^{i7\frac{3\pi}{4}}}{e^{i6\frac{\pi}{3}}} = 2e^{i\pi\left(\frac{21}{4} - 2\pi\right)} = 2e^{i\pi\frac{13}{4}} = 2e^{-i\pi\frac{3}{4}}$$

$$a+b = 2e^{i\frac{3\pi}{4}} + 2e^{i\frac{\pi}{3}} = 2e^{i\frac{\frac{3\pi}{4}+\frac{\pi}{3}}{2}}\left(e^{i\frac{\frac{3\pi}{4}-\frac{\pi}{3}}{2}} + e^{-i\frac{\frac{3\pi}{4}-\frac{\pi}{3}}{2}}\right)$$

$$= 2e^{i\frac{13\pi}{24}}\left(e^{i\frac{5\pi}{24}} + e^{-i\frac{5\pi}{24}}\right) = 2e^{i\frac{13\pi}{4}} \cdot 2\cos\frac{5\pi}{4} = \underbrace{4\cos\frac{5\pi}{4}}_{>0} e^{i\frac{13\pi}{4}}$$