ЧДУ 11.11.14

① $u_{tt} = a^2 u_{xx}$ $\quad x \in (0, L),\ t > 0$

② $\left| \begin{aligned} u(x,y) &= \varphi_x \in C^3 \\ u_t(x,y) &= \psi_x \in C^2 \end{aligned} \right.$ НУ

③
- I $\quad u(0,t) = u(L,t) = 0$
- II $\quad u(0,t) = u_x(L,t) = 0$
- III $\quad u_x(0,t) = u(L,t) = 0$
- IV $\quad u_x(0,t) = u_x(L,t) = 0$

④ $u(x,t) = X(x)T(t) \neq 0$, $\quad \dfrac{\ddot{T}}{a^2 T} = \dfrac{X''}{X} = -\lambda$

⑤ $X'' + \lambda X = 0$

$$\left.\begin{aligned} &\text{I} \quad X(0) = X(L) = 0 \\ &\text{II} \quad X(0) = X'(L) = 0 \\ &\text{III} \quad X'(0) = X(L) = 0 \\ &\text{IV} \quad X'(0) = X'(L) = 0 \end{aligned}\right\}$$ Задача на Щурм-Лиувил условия

	λ_k	$X_k(x)$
за държ. изпит ⇒ I	$\left(\dfrac{k\pi}{L}\right)^2$	$\sin \dfrac{k\pi x}{L}$
II	$\left[\dfrac{(2k-1)\pi}{2L}\right]^2$	$\sin \dfrac{(2k-1)\pi x}{2L}$
III	$\left[\dfrac{(2k-1)\pi}{2L}\right]^2$	$\cos \dfrac{(2k-1)\pi x}{2L}$
IV	$\left(\dfrac{k\pi}{L}\right)^2$	$\cos \dfrac{k\pi x}{L}$
	0	1

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

1сл) $\lambda = 0$, $X''(x) = 0 \to X(x) = Ax + B$

2сл) $\lambda < 0$, $\mu^2 = -\lambda$, $\mu = \pm\sqrt{-\lambda}$

$\Downarrow$

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$$

3сл) $\lambda > 0$, $\mu^2 = -\lambda$, $\mu = \pm i\sqrt{\lambda}$

$\Downarrow$

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

Разгл сл 1 $X' = A$ - const, $X'(0) = 0 = A = X'(L) \Rightarrow$ $A = 0$, $\lambda_0 = 0$, $X_0 = B\frac{1}{B} = 1$

Разгл сл 2 $X'(x) = C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$

$X'(0) = 0 = \sqrt{-\lambda}(C_1 - C_2)$, $X'(L) = \sqrt{-\lambda}(C_1 e^{\sqrt{-\lambda}L} - C_2 e^{-\sqrt{-\lambda}L})$

$C_1 + C_2 = 0$, $e^{\sqrt{-\lambda}L} = e^{-\sqrt{-\lambda}L} \Rightarrow e^{2\sqrt{-\lambda}L} = 1$

$$X'(x) = -\sqrt{\lambda}\, C_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}\, C_2 \cos\sqrt{\lambda}x$$

$X'(0) = 0 = \sqrt{\lambda}\, C_2 = 0 \Rightarrow C_2 = 0$

$X'(L) = 0 = \sqrt{\lambda}\, C_1 \sin\sqrt{\lambda}x = 0$ но

$\sqrt{\lambda} \neq 0$ и $C_1 \neq 0 \Rightarrow \sqrt{\lambda}x = k\pi \Rightarrow$

$\sqrt{\lambda}L = k\pi$, $\lambda_k = \left(\frac{k\pi}{L}\right)^2$ т.е.

$\ddot{T} + \lambda a^2 T = 0$ но $\lambda \neq 0$ и $a^2 \neq 0$

$$T_k(t) = A_k \cos\sqrt{\lambda_k}at + B_k \sin\sqrt{\lambda_k}at$$

$$u(x,t) = \sum_1^\infty X_k(x) T_k(t) = \sum X_k(x)\left[A_k\cos\sqrt{\lambda_k}at + B_k\sin\sqrt{\lambda_k}at\right]$$

Реш!

Пр 1 $\quad u_{tt} = a^2 u_{xx} \quad , \quad x \in [0, L] \, , \, t > 0$

$$\text{НУ} \left| \begin{array}{l} u(x,0) = 0 \\ u_t(x,0) = \sin \frac{2\pi x}{L} \end{array} \right. \qquad \text{ГУ} \left| \begin{array}{l} u(0,t) = u(L,t) = 0 \\ t > 0 \end{array} \right.$$

$$u(x,t) = \sum_1^\infty \sin \frac{k\pi x}{L} \left[A_k \cos \frac{k\pi a t}{L} + B_k \sin \frac{k\pi a t}{L} \right]$$

$$u(x,0) = 0 = \sum_1^\infty A_k \sin \frac{k\pi x}{L} \rightarrow A_k = 0 \quad \text{т.е.}$$

$$u(x,t) = \sum_1^\infty \sin \frac{k\pi x}{L} \left[B_k \sin \frac{k\pi a t}{L} \right]$$

$$u_t(x,t) = \sum_1^\infty \sin \frac{k\pi x}{L} B_k \frac{k\pi a}{L} \cos \frac{k\pi a t}{L}$$

$$u_t(x,0) = \sin \frac{2\pi x}{L} = \sum_{k=1}^\infty \sin \frac{k\pi x}{L} B_k \frac{k\pi a}{L}$$

$$B_2 \frac{2\pi a}{L} = 1 \quad , \quad B_{k \neq 2} = 0$$

$$u(x,t) = \frac{L}{2\pi a} \sin \frac{2\pi x}{L} \sin \frac{2\pi a t}{L}$$

Разд. III $\quad u_x(0,t) = u_x(L,t) = 0$

$$u_x(x,0) = \varphi'(x) \quad , \quad u_x(0,0) = \varphi'(0) = 0$$

$$u_x(x,0) = \varphi'(x) \quad , \quad u_x(L,0) = \varphi'(L) = 0$$

$$\varphi'(0) = \varphi'(L) = 0$$

$$u_{xt}(0,t) = u_{xt}(L,t) = 0 \, , \quad \psi'(0) = \psi'(L) = 0$$

Зад $u_{tt} = a^2 u_{xx}$ $\quad x \in (0\, L),\ t > 0$

Н.у. $\begin{cases} u(x0) = x(L-x) \\ u_t(x0) = 0 \end{cases}$ $\quad$ Гу $\left| u(0t) = u(Lt) = 0 \right.$

$$u(xt) = X(x)\,T(t)$$

$$\frac{\ddot{T}}{a^2 T} = \frac{X''}{X} = -\lambda \quad , \quad \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad k = 1 \ldots, \quad X_k(x) = \sin\frac{k\pi x}{L}$$

$$\ddot{T}_k + a^2\left(\frac{k\pi}{L}\right)^2 T_k = 0$$

$$T_k(t) = A_k \cos\frac{k\pi a t}{L} + B_k \sin\frac{k\pi a t}{L}$$

$$u(xt) = \sum_1^\infty \sin\frac{k\pi x}{L}\left[A_k \cos\frac{k\pi a t}{L} + B_k \sin\frac{k\pi a t}{L}\right]$$

$$u_t(xt) = \sum_1^\infty \sin\frac{k\pi x}{L}\left[-A_k \frac{k\pi a}{L}\sin\frac{k\pi a t}{L} + B_k \frac{k\pi a}{L}\cos\frac{k\pi a t}{L}\right]$$

$$u_t(x0) = \sum_1^\infty \sin\frac{k\pi x}{L} B_k \frac{k\pi a}{L} = 0 \Rightarrow B_k = 0 \text{ т.е.}$$

$$u(xt) = \sum_{k=1}^\infty \sin\frac{k\pi x}{L} A_k \cos\frac{k\pi a t}{L}$$

$$u(x0) = x(L-x) = \sum_1^\infty A_k \sin\frac{k\pi x}{L}$$

$$A_k = \frac{2}{L}\int_0^L x(L-x)\sin\frac{k\pi x}{L}\,dx\,!$$

$$I = \int x \sin\frac{k\pi x}{L}\,dx \quad , \quad J = \int x^2 \sin\frac{k\pi x}{L}\,dx$$

$$I = -\frac{L}{k\pi}\int x\, d\cos\frac{k\pi x}{L} = -\frac{L}{k\pi}x\cos\frac{k\pi x}{L} + \frac{L}{k\pi}\int\cos\frac{k\pi x}{L}\,dx =$$

$$-\frac{L}{k\pi} x \cos\frac{k\pi x}{L} + \frac{L^2}{(k\pi)^2}\sin\frac{k\pi x}{L}$$

$$J = -\frac{L}{k\pi}\int x^2 \, d\cos\frac{k\pi x}{L} = -\frac{L}{k\pi}x^2\cos\frac{k\pi x}{L} + \frac{L}{k\pi}\int \cos\frac{k\pi x}{L}\, dx =$$

$$-\frac{L}{k\pi}x^2\cos\frac{k\pi x}{L} + \frac{2L^2}{(k\pi)^2}\int x \, d\sin\frac{k\pi x}{L} =$$

$$-\frac{L}{k\pi}x^2\cos\frac{k\pi x}{L} - \frac{2L^2}{(k\pi)^2}\int \sin\frac{k\pi x}{L}\, dx + \frac{2L^2}{(k\pi)^2} x \sin\frac{k\pi x}{L} =$$

$$-\frac{L}{k\pi}x^2\cos\frac{k\pi x}{L} + \frac{2L^2}{(k\pi)^2}x\sin\frac{k\pi x}{L} + \frac{2L^3}{(k\pi)^3}\cos\frac{k\pi x}{L}$$

$$J = \left(-\frac{L}{k\pi}x\cos\frac{k\pi x}{L} + \frac{L^2}{(k\pi)^2}\sin\frac{k\pi x}{L}\right)\Bigg|_0^L = -\frac{L^2}{k\pi}\cos k\pi =$$

$$= -\frac{L^2}{k\pi}(-1)^k$$

$$J = -\frac{L^3}{k\pi}(-1)^k + \frac{2L^3}{(k\pi)^3}\left((-1)^k - 1\right)$$

$$A_k = \cancel{-\frac{2L^2}{k\pi}(-1)^k} - \frac{2}{L}\left[\cancel{-\frac{L^3}{k\pi}(-1)^k} + \frac{2L^3}{(k\pi)^3}\left((-1)^k - 1\right)\right], \quad k \to 2k+1$$

$$A_{2k+1} = \frac{8L^2}{\pi^3(2k+1)^3} \Rightarrow u(x,t) = \frac{8L^2}{\pi^3}\sum_0^\infty \frac{1}{(2k+1)^3}\sin\frac{(2k+1)\pi x}{L}\times\cos\frac{(2k+1)\pi a t}{L}$$

$$u_{xx} \cong \sum_0^\infty \frac{1}{2k+1}\sin\frac{k\pi x}{L}\cos\frac{k\pi a t}{L}$$

Дирихле $\sum_{0}^{\infty} f_k(x) g_k(x)$, $f_k(x) \rightrightarrows 0$

$f_{k+1}(x) \le f_k(x)$

$\sum_{0}^{\infty} f_k(x) \le M \Rightarrow U(x,t)$ е равномерно сходящ

$f_k(y) = \frac{1}{2k+1} \rightrightarrows 0$, $\left| \sum_{0}^{n} \sin \frac{k\pi x}{l} \cos \frac{k\pi a t}{l} \right| \le M$

Изискване само равномерна сходимост