ЧДУ 04.11.14

① Покажете че
$$\begin{cases} u_{xy} = 0, & x \in (a,b),\ y \in (c,d) \\ u|_{y=0} = \varphi(x) = u_0(x) \in C^2(a;b) \end{cases}$$

$$u_y|_{y=\varphi(x)} = u_1(x) \in C^1(a,b)$$

има единствено решение

$$u_{xy} = u_0(x) + \int_x^{\varphi^{-1}(y)} u_1(\xi)\,\varphi'(\xi)\,d\xi$$

$c = \inf \varphi$, $d = \sup \varphi$

② $u_0 \in C^2(-1,1)$, $u_1 \in C^1(-1,1)$
$$\begin{cases} u_{xx} - u_{yy} = -1 \\ u|_{y=0} = u_0(x), \quad u_y|_{y=0} = u_1(x) \end{cases}$$
? кога има единствено решение ?

③ $u_{xy} + u_x = x$, $x > 0$, $y > 0$
$u|_{x=0} = y^2$, $u|_{y=0} = x^2$

④ $x^2 u_{xx} - 2x u_{xy} + u_{yy} - x u_x = 0$
$u|_{x=1} = e^{2y}$, $u_x|_{x=1} = 0$

⑤ $x u_{xx} + (x+y) u_{xy} + y u_{yy} = 0$
$u|_{y=\frac{1}{x}} = x^3$, $u_x|_{y=\frac{1}{x}} = 2x^2$

Реш ①

$u_{xy} = 0$, $u(x,y) = f(x) + g(y)$

$$u|_{y=\varphi(x)} = u_0(x) = f(x) + g(\varphi(x)), \quad u_y = g'(y)$$

$$u_y|_{y=\varphi_x} = g'(\varphi(x)) = u_1(x). \text{ Нека } t = \varphi(x) \Rightarrow$$

$$x = \varphi^{-1}(t) \Rightarrow g'(t) = u_1(\varphi^{-1}(t)); \quad g(t) = \int_{t_0}^{t} u_1(\varphi^{-1}(\xi))\,d\xi$$

полагаме $\xi = \varphi(\eta) \Rightarrow g(t) = \int_{x_0}^{\varphi^{-1}(t)} u_1(\eta)\varphi'(\eta)\,d\eta$

$$f(x) = u_0(x) - \int_{x_0}^{\varphi^{-1}(\varphi(x))} u_1(\eta)\varphi'(\eta)\,d\eta$$

$$f(x) = u_0(x) - \int_{x_0}^{x} u_1(\eta)\varphi'(\eta)\,d\eta \Rightarrow$$

$$u(x,y) = u_0(x) - \int_{x_0}^{x} u_1(\eta)\varphi'(\eta)\,d\eta + \int_{x_0}^{\varphi^{-1}(y)} u_1(\eta)\varphi'(\eta)\,d\eta \Rightarrow$$

$$u(x,y) = u_0(x) + \int_{x}^{\varphi^{-1}(y)} u_1(\eta)\varphi'(\eta)\,d\eta$$

$$c = \inf \varphi(x), \quad d = \sup \varphi(x), \quad x \in [a, b]$$

Реш ②

$$u_{xx} - u_{yy} = 1, \quad u|_{y=0} = u_0, \quad u_y|_{y=0} = u_1$$

представяме $u = v + u_1$, $u_1 = -\frac{y^2}{2}$

$$v_{xx} - v_{yy} = 0$$

$$v|_{y=0} = u_0(x), \quad v_y|_{y=0} = u_1(x), \quad x \in (-1, 1)$$

②

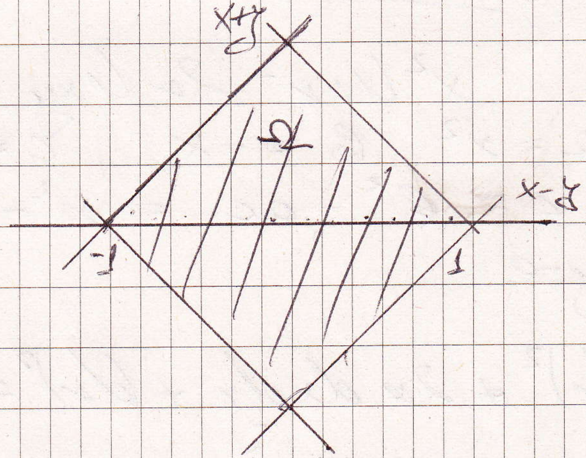
$\xi = x+y$, $\eta = x-y$ (заради диференцирането на производна $\neq 0$)

$v_{\xi\eta} = 0$, $v(\xi\eta) = F(\xi) + G(\eta)$

$v(xy) = F(x+y) + G(x-y)$
$v(x,0) = u_0(x) = F(x) + G(x)$
$v_y = F'(x+y) - G'(x-y)$
$v_y/_{y=0} = u_1(x) = F'(x) + G'(x)$

$v(xy) = \frac{1}{2}(u_0(x+y) + u_0(x-y)) + \frac{1}{2}\int_{x-y}^{x+y} u_1(\lambda)\,d\lambda$ Даламбер

$u(xy) = v(xy) - \frac{b^2}{2}$?



$u_x - x \Rightarrow u = \frac{x^2}{2} = u_1$

$u = v + u_1$
$v_{xy} = -v_x$
$v/_{x=0} = y^2$, $v/_{y=0} = x^2$

Нека $v_x = k$, $k_y = -k$
$\frac{\partial k}{k} = -\partial y \Rightarrow \ln|k| = -y + C(x)$
$k = C(x)e^{-y}$
$v_x = e^{-y}C(x)$, $v = e^{-y}\int C(x)dx + D(y)$

$v(xy) = e^{-y}\tilde{C}(x) + D(y)$ $x = const$
$y = const$

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$v(0,y) = y^2, \quad v(x,0) = x^2$

$e^{-y}\tilde{C}(0) + D(y), \quad \tilde{C}(x) + D(0)$

$\tilde{C}(x) = x^2 - D(0) \rightarrow x = 0 \Rightarrow \tilde{C}(0) = -D(0)$

$D(y) = y^2 + e^{-y}D(0)$

$v(xy) = e^{-y}(x^2 \cancel{- D(0)}) + y^2 \cancel{+ e^{-y}D(0)}$

$v(xy) = y^2 + x^2 e^{-y}$

$$\underline{\underline{u = v + u_1 = y^2 + x^2 e^{-y} + \frac{x^2}{2}}}$$

Реш ① $\quad x^2 u_{xx} - 2x u_{xy} + u_{yy} - x u_y = 0$

$a = x^2, \; b = -x, \; c = 1$

$\Delta = b^2 - ac = x^2 - x^2 = 0$ параболично уравнение

хар. у-е

$x^2(dy)^2 + 2x\,dx\,dy + (dx)^2 = 0 \quad / : (dx)^2$

$x^2\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} + 1 = 0$

$\frac{dy}{dx} = \frac{-2x \pm \sqrt{4x^2 - 4x^2}}{2x^2} = -\frac{1}{x}\; ; \; dy = -\frac{dx}{x}$

$y = -\ln x + C, \quad xe^{y} = C_1$

$\xi = xe^{y}, \quad \eta = y$

Избираме η от Якобиана, като гледаме при избора си самият Якобиан да е различен от 0

$\xi_x = e^y$, $\xi_y = xe^y$, $\xi_{xx} = 0$, $\xi_{yy} = xe^y$, $\xi_{xy} = e^y$

$\eta_x = 0$, $\eta_y = 1$, $\eta_{xx} = 0$, $\eta_{yy} = 0$, $\eta_{xy} = 0$

$u_x = u_\xi \cdot \xi_x + u_\eta \eta_x = u_\xi e^y$

$u_{xx} = u_{\xi\xi} e^{2y}$, $u_{yy} = u_{\xi\xi}(xe^y)^2 + 2u_{\xi\eta} xe^y + u_{\eta\eta} + u_\xi xe^y$

$u_{xy} = u_{\xi\xi} e^y + u_{\xi\eta} e^y + u_{\xi\xi} xe^y + u_\xi e^y$

$x^2 u_{\xi\xi} e^{2y} - 2x(x u_{\xi\xi} e^y + u_{\xi\eta} e^y + u_{\xi\xi} xe^y + u_{\eta\eta} + u_\xi e^y)$

$+ u_{\xi\xi} x^2 e^{2y} + 2u_{\xi\eta} xe^y + u_\xi xe^y - u_\xi xe^y$

$-2xe^y u_\xi + u_{\eta\eta} + u_\xi xe^y - u_\eta xe^y = 0$

$\emptyset$