СДУ Кръстов 27.11.14

Задача на Дирихле за уравнението на Лаплас в кръгови области

(1) $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Зад 1 Напишете уравнението на Лаплас (1) в полярни координати

Реш

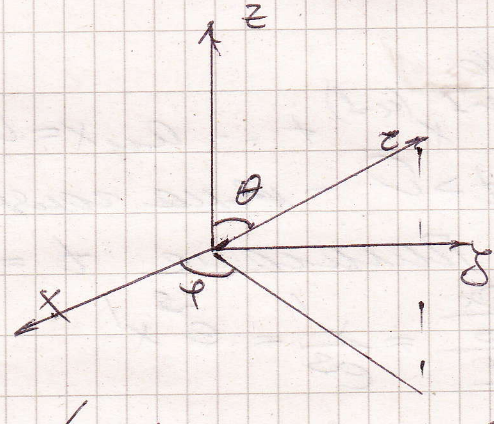
$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0$$

$x = r\cos\varphi$, $y = r\sin\varphi$, $r \in (0, +\infty)$, $\varphi \in [0, 2\pi]$

Оттук веднага следва, че (1) в цилиндрични координати има уравнение

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$$



Зад 2 Запишете (1) в сферични координати

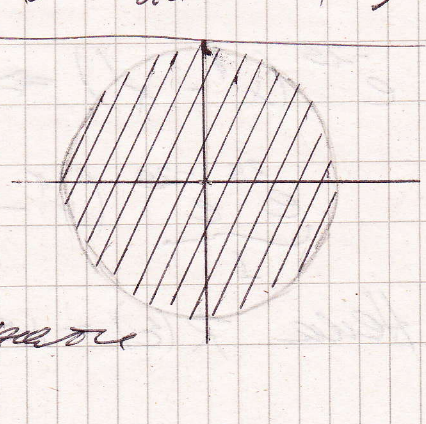
Реш

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\right)$$

A) Задача на Дирихле за кръг

1) $\Delta u = 0$, $x^2+y^2 < a^2$

2) $u|_{x^2+y^2=a} = f(x,y)$



Записване (1) и (2) в полярни координати

(1') $\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2}$

(2') $U|_{r=a} = f(\varphi)$ {2π периодична}

3) $U(r,\varphi) = R(r)\Phi(\varphi) \neq 0$. Заместваме (3) в (1') $\Rightarrow$

$R''(r)\Phi(\varphi) + \frac{1}{r}R'(r)\Phi(\varphi) + \frac{1}{r^2}R(r)\Phi''(\varphi) = 0 \quad / \cdot r^2, : R, : \Phi$

$$\frac{r^2 R'' + r R'}{R(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = n^2 \geq 0$$

и задължително е по-голямо от 0, const.
Изискването е заради периода на φ

(4) $r^2 R'' + r R' - n^2 R = 0$ - Уравнение на Ойлер
(5) $\Phi'' + n^2 \Phi = 0$

$\Phi_n(\varphi) = A_n \cos \varphi + B_n \sin \varphi$!!!

Припомняне
$t^n x^{(n)} + a_1 t^{n-1} x^{(n-1)} + \ldots a_n x = 0$ - Уравнение на Ойлер
$t < 0$ или $t > 0$ - няма голяма разлика
Нека $t > 0$. Полагаме $t = e^s$

$\dot{x} = \frac{\partial x}{\partial t} = \frac{\frac{\partial x}{\partial s}}{\frac{\partial t}{\partial s}} = \frac{x'}{e^s} = e^{-s}x'$

$\ddot{x} = \frac{\partial}{\partial t}(\dot{x}) = \frac{\partial}{\partial t}(e^{-s}x') = \frac{\frac{\partial}{\partial s}(x'e^{-s})}{\frac{\partial}{\partial s}t} = \frac{x''e^{-s} - x'e^{-s}}{e^s} =$

$e^{-2s}(x'' - x') \Rightarrow t^2\ddot{x} + a_1 t\dot{x} + a_2 x = 0$

$$\underbrace{e^{2s}e^{-2s}}_{1}(x'' - x') + \underbrace{a_1 e^{s} e^{-s}}_{a_1 \cdot 1} x' + a_2 x = 0$$

Нека $R(r) = r^\alpha$. Заместваме в уравнението

$r^2 \alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^{\alpha} = 0$

$\alpha^2 - \alpha + \alpha - n^2 = 0 \Rightarrow \alpha = \pm n \neq 0$

$$\boxed{R_n(r) = C_1 r^n + C_2 r^{-n}}$$

За да намерим ограничено решение в 0 трябва

$C_2 = 0$ т.е. $R_n(r) = C_1 r^n$

Заместваме (3) в (2')

$R(a)\Phi(\varphi) = f(\varphi)$. Избираме $R(a) = 1$, $R_n(r) = \left(\frac{r}{a}\right)^n$

$\Rightarrow R_n(a) = C_1 a^n = 1 \Rightarrow C_1 = \frac{1}{a^n}$

При $n = 0 \Rightarrow r^2 R'' + r R' = 0$, $\quad k = R'$, $\frac{k'}{k} = -\frac{1}{r}$

$k = \frac{c}{r}$, $R' = \frac{c}{r} \Rightarrow R_0 = c \ln r + D$.

За да намерим ограниченост в 0 изискваме $c = 0$

$\Rightarrow$ (5) $\quad u(r,\varphi) = A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos \varphi + B_n \sin \varphi)$

От (2') при $r = a$

$u|_{r=a} = f(\varphi) = A_0 + \sum_1^{\infty} A_n \cos \varphi + B_n \sin \varphi$

Това е редът на Фурие на f в $[0, 2\pi]$ по пълната система $\{1, \cos n\varphi, \sin n\varphi\}$. Тогава

(6) $\quad A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$; $A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$

$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$

$|u| \le \sum_1^{\infty} (|A_n| + |B_n|)$, $\quad f \in C^1$, $\quad f(0) = f(2\pi)$

Стандартна сметка с неравенството на Бесел показва

$$\sum |A_n| + |B_n| < \infty$$

(5) е равномерно сходящ

③

$n^k \alpha^n (|A_n| + |B_n|)$, $\alpha = \frac{r}{a} < 1$

$\lim\limits_{n\to\infty} \frac{n^k}{p^n} = 0$, $p > 1$. За всяка фиксирана производна трябва оценка

(6) заместваме в (5)

$$(7)\quad u = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta + \frac{1}{\pi}\left(\frac{r}{a}\right)^n \sum_{n=1}^{\infty}\int_0^{2\pi} f(\theta)[\cos n\theta \cos n\varphi + \sin n\theta \sin n\varphi]\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta + \frac{1}{\pi}\left(\frac{r}{a}\right)^n \sum_{n=1}^{\infty}\int_0^{2\pi} f(\theta)\cos n(\varphi-\theta)\, d\theta =$$

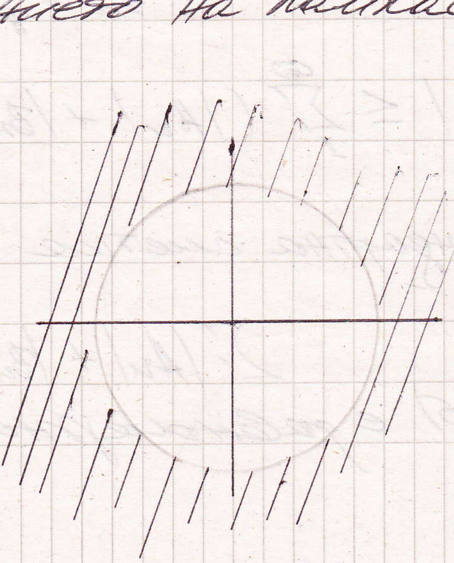
$$\frac{1}{2\pi}\left(\frac{r}{a}\right)^n \int_0^{2\pi} f(\theta)\left[1 + 2\sum_1^{\infty}\left(\frac{r}{a}\right)^n \cos n(\varphi-\theta)\right] d\theta$$

$$\boxed{z = \left(\frac{r}{a}\right)^n e^{in(\varphi-\theta)}} \;!!!$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, \mathrm{Re}\left[1 + 2\sum_1^{\infty} z^n\right] d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{1-|z|^2}{|1-z|^2}\, d\theta =$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{a^2 - r^2}{a^2 - 2ar\cos(\varphi-\theta) + r^2}\, d\theta = u$$ – Интеграл на Поасон

Б) Задача на Дирихле за уравнението на Лаплас извън кръгът



$$\begin{cases} \Delta u = 0 & x^2+y^2 > a \\ u|_{r=a} = f(\varphi) \end{cases}$$

$\Phi_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi$

$R_n(r) = C_1 r^n + C_2 r^{-n}$, $R_0(r) = C_1 \ln r + C_2$

$$(8)\quad \underline{u(r,\varphi) = A_0 + \sum_1^{\infty} \left(\frac{a}{r}\right)^n (A_n \cos n\varphi + B_n \sin n\varphi)} \;!!!$$

3) Задача на Дирихле за пръстен

$$\begin{cases} \Delta u = 0 \quad в \quad a^2 < x^2 + y^2 < b^2 := A \\ u|_{r=a} = f_1(\varphi) \\ u|_{r=b} = f_2(\varphi) \end{cases}$$

$R_0 = a_0 \ln r + b_0 \quad , R_n = C_1 r^n + C_2 r^{-n}$

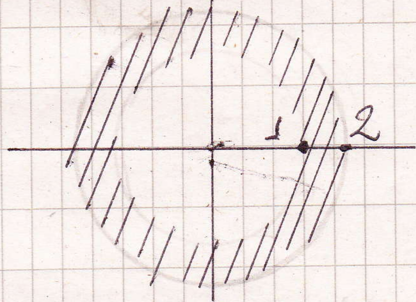
$\Phi_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi , \quad n = 0, 1 \dots$

$$u = R_0 \Phi_0 + \sum_1^{\infty} R_n(r) \Phi_n(\varphi) = a_0 \ln r + b_0 + \sum (C_1 r^n + C_2 r^{-n})(A_n \cos n\varphi + B_n \sin n\varphi)$$

$$= a_0 \ln r + b_0 + \sum_1^{\infty} \left(a_n r^n + \frac{c_n}{r^n}\right)\cos n\varphi + \left(b_n r^n + \frac{d_n}{r^n}\right)\sin n\varphi = u(r,\varphi) \quad (9)$$

Пример - Задача на Дирихле за пръстен

$$\begin{cases} \Delta u = 0 \\ 1 < x^2 + y^2 < 4 \\ u|_{r=1} = \sin\varphi \\ u|_{r=2} = \sin^3\varphi = \frac{3}{4}\sin\varphi - \frac{1}{4}\sin 3\varphi \end{cases}$$



$$u = a_0 \ln r + b_0 + \sum \left(a_n r^n + \frac{c_n}{r^n}\right)\cos n\varphi + \left(b_n r^n + \frac{d_n}{r^n}\right)\sin n\varphi$$

$$u|_{r=1} = \sin\varphi = 0 + \overset{0}{b_0} + \sum_1^{\infty} \overbrace{(a_n + c_n)}^{0} \cos n\varphi + (b_n + d_n)\sin n\varphi$$

$$b_n \mp d_n \quad , n \geq 2 \quad , b_1 + d_1 = 1$$

$$u|_{r=2} = \frac{3}{4}\sin\varphi - \frac{1}{4}\sin 3\varphi = \underbrace{a_0 \ln 2}_{0} + \sum \left(a_n 2^n + \frac{c_n}{2^n}\right)\cos n\varphi +$$

$$\left(b_n 2^n + \frac{d_n}{2^n}\right)\sin n\varphi$$

$$a_n 2^n + \frac{c_n}{2^n} = 0 \quad \forall n$$

$$b_n 2^n + \frac{d_n}{2^n} = 0, \quad \forall n \neq 1,3$$

$$b_1 2 + \frac{d_1}{2} = \frac{3}{4}, \qquad b_3 8 + \frac{d_3}{8} = -\frac{1}{4}$$

$$a_n + c_n = 0 \qquad\qquad b_n + d_n = 0$$

$$a_n 2^n + \frac{c_n}{2^n} = 0 \qquad\qquad b_n 2^n + \frac{d_n}{2^n} = 0 \text{ за } n \neq 1,3$$

$$b_1 + d_1 = 1 \qquad\qquad b_3 + d_3 = 0$$

$$2b_1 + \frac{d_1}{2} = 3/4 \qquad\qquad 8b_3 + \frac{d_3}{8} = -1/4$$