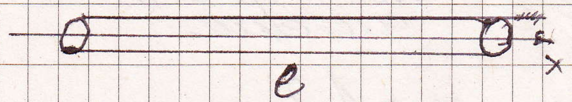
ЧДУ Христов 27.10.14

3) Смесена задача за уравнението на топлопроводимостта



$u(x,t)$

(1) $u_t = a^2 u_{xx}$, $x \in (0,l)$, $t \geq 0$

(2) $u|_{t=0} = f(x)$, $x \in [0,l]$ НУ

(3) $u(0,t) = \psi_1(t)$; $u(l,t) = \psi_2(t)$ ГУ

(1),(2),(3) - задача на Дирихле

① $u_t = a^2 u_{xx}$

② $u|_{t=0} = f(x) \in C^1$

③ $u(0,t) = \underset{\text{хомогенно}}{u(l,t) = 0}$ $\longrightarrow f(0) = f(l) = 0$

4) $u(x,t) = X(x)T(t)$ - заместваме в (1) $\Rightarrow$

$X(x)\dot{T}(t) = a^2 X''(x) T(t)$

$$\frac{\dot{T}(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda \text{ - const}$$

(5) $X'' + \lambda X = 0$, заместваме (4) в (3)

$X(0)T(t) = X(l)T(t) = 0 \Rightarrow$

(5) $\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$

$\lambda_k = \left(\frac{k\pi}{l}\right)^2$, $k = 1,2..$; $X_k(x) = \sin\frac{k\pi}{l}$, $k = 1,2..$

(6) $\dot{T}_k + \lambda_k a^2 T_k = 0$ - у-е с разделящи се пром

$$\frac{dT_k}{T_k} = -\frac{a^2 k^2 \pi^2}{l^2} dt \Rightarrow T_k(t) = A_k e^{-\frac{a^2 k^2 \pi^2}{l^2} t}$$

Решението може да се представи в безкраен ред т.е.

$$u(x,t) = \sum_{k=0}^{\infty}{}' X_k(x) T_k(t) = \sum_{k=0}^{\infty}{}' \sin\frac{k\pi}{\ell}x \cdot A_k \cdot e^{-\frac{a^2 k^2 \pi^2}{\ell^2}t}$$

$$u(x,0) = f(x) = \sum_{k=0}^{\infty}{}' A_k \sin\frac{k\pi}{\ell}x$$ пълна ортогонална система

$$A_k = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{k\pi}{\ell}x\,dx, \quad k = 1, 2 \ldots$$

$|u| \le \sum{}' |A_k| < \infty$, Сега интегриране по части

$$A_k = -\frac{2}{\ell}\frac{\ell}{k\pi}\int_0^\ell f(x)\,d\cos\frac{k\pi x}{\ell} =$$

$$= \underbrace{-\frac{2}{\ell}\frac{\ell}{k\pi} f(x)\frac{k\pi x}{\ell}\Big|_0^\ell}_{0} + \underbrace{\frac{\ell}{k\pi}\frac{2}{\ell}\int_0^\ell f'(x)\cos\frac{k\pi x}{\ell}\,dx}_{A_k'}$$

$$A_k \cong \frac{1}{k}A_k', \quad |A_k| \cong \left|A_k'\frac{1}{k}\right| \le \frac{1}{2}\left(|A_k'|^2 + \frac{1}{k^2}\right)$$

$$\sum{}' |A_k| \le \frac{1}{2}\sum{}' |A_k'|^2 + \frac{1}{2}\sum{}' \frac{1}{k^2} < \infty !!!$$

u е равномерно сходящ $\longrightarrow u \in C$

$$u_t = \sum_{k=1}^{\infty}{}' \sin\frac{k\pi x}{\ell} A_k e^{-\frac{k^2\pi^2 a^2}{\ell^2}t}\left(-\frac{k^2\pi^2 a^2}{\ell^2}\right)$$

$$|u_t| \lessapprox \sum{}' |A_k|\, k^2 e^{-\frac{k^2\pi^2 a^2}{\ell^2}t}, \quad t \ge \delta > 0$$

$$k^2 e^{-\frac{k^2 a^2 \pi^2}{\ell^2}\delta} < 1 \xrightarrow{\text{мажорира}} |u_t| \le \sum{}' |A_k| < \infty$$

Th Задачата на Дирихле (1), (2), (3) има единствено C^2 решение при условие $f(0) = f(\ell) = 0$, $f \in C^1$

(1) $u_t = a^2 u_{xx}$, $u_{xx} = 0$

Стационарно решение $u_s = A_1 x + A_2$

Помага да се реши следната задача

(2) $u|_{t=0} = f(x)$

(3) $u(0,t) = T_1$, $u(l,t) = T_2$

метод на разделяне на променливите по Фурие

Iстъпка $u_t = a^2 u_{xx}$, стационарни решения

$u(0,t) = T_1$, $u(l,t) = T_2$, $u = u_s$, $u_{xx} = 0$

$u_s = A_1 x + A_2$, $u_s(0) = T_1 \Rightarrow T_1 = A_2$

$u_s(l) = T_2 \Rightarrow A_1 l + T_1 = T_2 \Rightarrow A_1 = \frac{T_2 - T_1}{l}$

$$u_s(x) = \frac{T_2 - T_1}{l} x + T_1$$

$v = u - u_s$, $v_t = a^2 v_{xx}$

$v|_{t=0} = \underline{f(x) - u_s(x)}$ свежда задачата до хомогенна

$v(0,t) = 0$, $v(l,t) = 0$

Пример $u_t = u_{xx}$, $x \in (0, \pi)$, $t > 0$

Начално условие

$$u(x,0) = \begin{cases} x, & x \in (0, \frac{\pi}{2}) \\ (\pi - x), & x \in [\frac{\pi}{2}, \pi) \end{cases}$$

$u(0,t) = 100$, $u(\pi, t) = 50$

Реш

$u_{xx} = 0 \longrightarrow u = A_1 x + A_2$

$u(0,t) = 100$, $u(\pi, t) = 50$

$u(0) = 100 = A_2$, $u = A_1 \pi + 100 = 50 \rightarrow A_1 = -\frac{50}{\pi}$

$u_s = 50(2 - \frac{x}{\pi})$

$v = u - u_s$, $v_t = v_{xx}$, $x \in (0, \pi)$

$$v(x,0) = \begin{cases} x - 50(2 - \frac{x}{\pi}), & x \in (0, \frac{\pi}{2}) \\ \pi - x - 50(2 - \frac{x}{\pi}), & x \in [\frac{\pi}{2}, \pi) \end{cases}$$

$$V(0,t) = V(\pi,t) = 0$$

$$V(t,x) = \sum_{k=1}^{\infty} A_k \sin kx \cdot e^{-k^2 t}$$

$$A_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx\,dx = \frac{2}{\pi}\int_0^{\pi/2}\left(\left(1+\frac{50}{\pi}\right)x-100\right)\sin kx\,dx$$

$$+\frac{2}{\pi}\int_{\pi/2}^{\pi}\left(\left(\frac{50}{\pi}-1\right)x-100+\pi\right)\sin kx\,dx$$

$$\int \sin kx\,dx = -\frac{1}{k}\cos kx$$

$$\int x\sin kx\,dx = -\frac{1}{x}\int x\,d\cos kx = \frac{x\cos kx}{k}+\int\frac{\cos kx}{k}dx$$

$$= -\frac{1}{k}x\cos kx + \frac{1}{k^2}\sin kx \Rightarrow$$

$$A_k = \frac{2}{\pi}\left(1+\frac{50}{\pi}\right)\left(-\frac{1}{k}x\cos kx+\frac{1}{k^2}\sin kx\right)\Big|_0^{\pi/2} +$$

$$\frac{200}{k\pi}\cos k\pi\Big|_0^{\pi/2} + \frac{2}{\pi}\left(\frac{50}{\pi}-1\right)\left(-\frac{1}{k}\cos kx+\frac{1}{k^2}\sin kx\right)\Big|_{\pi/2}^{\pi} -$$

$$\left(-\frac{1}{k}\,\frac{2}{\pi}(\pi-100)\cos kx\right)\Big|_{\pi/2}^{\pi} =$$

$$\frac{2}{\pi}\left(1+\frac{50}{\pi}\right)\left(-\frac{\pi}{2k}\cos k\frac{\pi}{2}+\frac{1}{k^2}\sin\frac{k\pi}{2}\right)+\frac{200}{k\pi}\left(\cos k\frac{\pi}{2}-1\right)+$$

$$\frac{2}{\pi}\left(\frac{50}{5}-1\right)\left(-\frac{\pi}{k}\cos k\pi\right)-\frac{2}{\pi}\left(\frac{50}{\pi}-1\right)\left(-\frac{\pi}{2k}\cos\frac{k\pi}{2}+\frac{1}{k^2}\sin\frac{k\pi}{2}\right)$$

$$+\frac{2}{k\pi}(\pi-100)\left(\cos k\pi-\cos\frac{k\pi}{2}\right) =$$

$$= \frac{4}{\pi}\left(-\frac{\pi}{2k}\cos\frac{k\pi}{2} + \frac{1}{k^2}\sin\frac{k\pi}{2}\right) +$$

$$\frac{2}{k}\cos\frac{k\pi}{2} + \frac{2}{\pi}\left(\frac{50}{\pi}-1\right) - \frac{\pi}{k}(-1)^k\left(\frac{2}{\pi}\right)\left(\frac{50}{\pi}-1\right) -$$

$$-\frac{200}{k\pi} - \frac{2}{k\pi}(\pi-100)(-1)^k$$

$$(-1)^k\left[-\left(\frac{100}{\pi}-\frac{2}{k}\right)\frac{2}{k}+\frac{2}{k\pi}(100-\pi)\right] - \frac{200}{k\pi} + \frac{4}{\pi k^2}\sin\frac{k\pi}{2}$$

$$? \frac{6}{\pi}(-1)^k - \frac{200}{k\pi} + \frac{4}{\pi}\sin\frac{k\pi}{2} = A$$

$$v(x,t) = \frac{1}{\pi}\sum_{l=1}^{\infty}\sin(2l-1)x\left[6+\frac{200}{(2l-1)}+\frac{4}{(2l-1)^3}\right]e^{-(2l-1)^2 t}$$

$u = u_{st} v$!

$$u = 50\left(2-\frac{x}{\pi}\right) - \frac{1}{\pi}\sum\sin(2l-1)x\left[6+\frac{200}{2l-1}+\frac{200}{(2l-1)^2}\right]e^{-(2l-1)^2 t}$$