ЕДУ Христов 24.11.14

Уравнение на Лаплас

(1) $$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u = x - y$, $u = x^2 - y^2$, $u = \frac{x}{x^2+y^2}$, $u = \ln(x^2+y^2)$, $e^{x}\sin y$

$\Delta u = 0 \to$ хармонични

$f(z) = u + iv \to$ холоморфна $\Leftrightarrow$ $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

тогава $\Delta u = 0$ и $\Delta v = 0$

А) Задачата на Коши за уравнението на Лаплас не е коректно

Редица от решения $u_n(x,y) = \frac{e^{-\sqrt{n}}}{n} \sin nx \, \mathrm{sh}\, ny$

$\mathrm{sh}\, x = \frac{e^{x} - e^{-x}}{2}$, $\mathrm{ch}\, x = \frac{e^{x} + e^{-x}}{2}$, $\mathrm{ch}\, x^2 - \mathrm{sh}\, x^2 = -1$

$u_n|_{y=0} = 0$ (от $\mathrm{sh}\, ny$)

$$\frac{\partial(u_n)}{\partial y}\Big|_{y=0} = e^{-\sqrt{n}} \sin nx \cdot 1 \xrightarrow[\text{сх}]{\text{равн}} 0$$

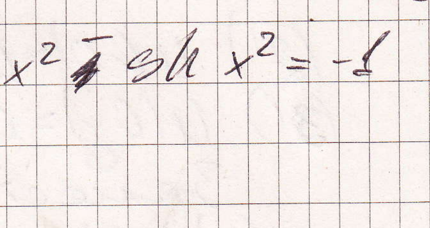
При $y \neq 0 \Rightarrow \frac{\partial u_n}{\partial y} = e^{-\sqrt{n}} \sin nx \, \mathrm{ch}\, ny \not\to 0$

Задача на Дирихле за уравнение на Лаплас в правоъгълник

(1) $\Delta u = 0$, $x \in (0, a)$, $y \in (0, b)$

(2) $u(x, 0) = f_1(x)$, $u(x, b) = f_2(x)$

(3) $u(0, y) = g_1(y)$, $u(a, y) = g_2(y)$



Припомняне принципа на максимума

Ако (4) $m \le f_1, f_2 \le M$, $m \le g_1, g_2 \le M$, то (5) $m \le u(x,y) \le M$

Следствие: Th за единственост задача (1)(3) има единствено решение

Док

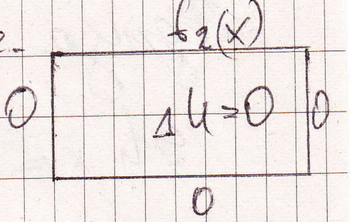
Допускаме че u_1 и u_2 са решения на задачата - тогава $u = u_1 - u_2$ и ще $= \mu = 0 \Rightarrow u \equiv 0$

(6) $u(x,y) = X(x)Y(y) \neq 0$. Заместваме (6) в (1)

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda, \quad X'' + \lambda X = 0$$

Предполагаме $f_1 = g_1 = g_2 = 0$ т.е.



(2') $u(x,0)=0$, $u(x,b)=f_2(x)$

(3') $u(0,y)=0$, $u(a,y)=0$

Заместваме (6) в (3')

$$X(0)\cancel{Y(y)} = X(a)\cancel{Y(y)} = 0$$

$$\left| \begin{array}{l} X'' + \lambda X = 0 \\ X(0) = X(a) = 0 \end{array} \right. \Rightarrow \lambda_k = \left(\frac{k\pi}{a}\right)^2, \quad X_k(x) = \sin\frac{k\pi x}{a}$$

$$Y_k''(y) - \left(\frac{k\pi}{a}\right)^2 Y_k(y) = 0$$

$Y_k(y) = C_1 e^{\frac{k\pi}{a}y} + C_2 e^{-\frac{k\pi}{a}y}$ което е еквивалентно

на $\boxed{Y_k(y) = C_1 \operatorname{ch}\frac{k\pi}{a}y + C_2 \operatorname{sh}\frac{k\pi}{a}y}$

От първото условие на (2') $Y(0) = 0 \Rightarrow C_1 = 0$

$$\Rightarrow \boxed{Y_k(y) = A_k \operatorname{sh}\frac{k\pi}{a}y}$$

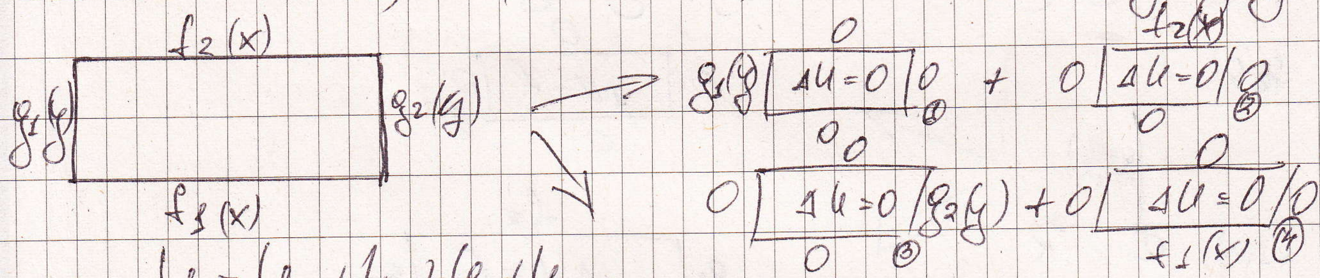
(7) $u(x,y) = \sum_1^\infty A_k \sin\frac{k\pi x}{a} \operatorname{sh}\frac{k\pi y}{a}$

от второго условие (2') $u(x,b) = f_2(x) =$

$\sum_1^\infty \underbrace{A_k \operatorname{sh}\frac{k\pi b}{a}}_{A_k^*} \sin\frac{k\pi x}{a}$, $A_k \operatorname{sh}\frac{k\pi b}{a} = \frac{2}{a}\int_0^a f_2(x)\sin\frac{k\pi x}{a}dx$

т.е. (8) $A_k = \frac{1}{\operatorname{sh}\frac{k\pi b}{a}}\frac{2}{a}\int_0^a f_2(x)\sin\frac{k\pi x}{a}dx$

$\Delta u = 0$, $u(x,0)=f_1(x)$, $u(x,b)=f_2(x)$, $u(0,y)=g_1(y)$, $u(a,y)=g_2(y)$



$u = u_1+u_2+u_3+u_4$

Зад 1 $\Delta u = 0$

$u(x,0)=0$, $u(x,b)=0$

$u(0,y)=g_1(y)$, $u(a,y)=0$

$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$, $Y''+\lambda Y=0$, $Y(0)=Y(b)=0$

$Y_k(y) = \sin\frac{k\pi y}{b}$, $A_k = \left(\frac{k\pi}{b}\right)^2$

$X_k = A_k \operatorname{ch}\frac{k\pi}{b}x + B_k \operatorname{sh}\frac{k\pi}{b}x$

$X_k(0)=0$, $A_k \operatorname{ch}\frac{k\pi(a-x)}{b} + B_k\operatorname{sh}\frac{k\pi(a-x)}{b}$

$$X_k(x) = B_k \,\mathrm{sh}\, \frac{k\pi(a-x)}{b}$$

$$u_2 = \sum_1^{\infty} \sin\frac{k\pi y}{b}\,\mathrm{sh}\,\frac{k\pi(a-x)}{b}\,B_k$$

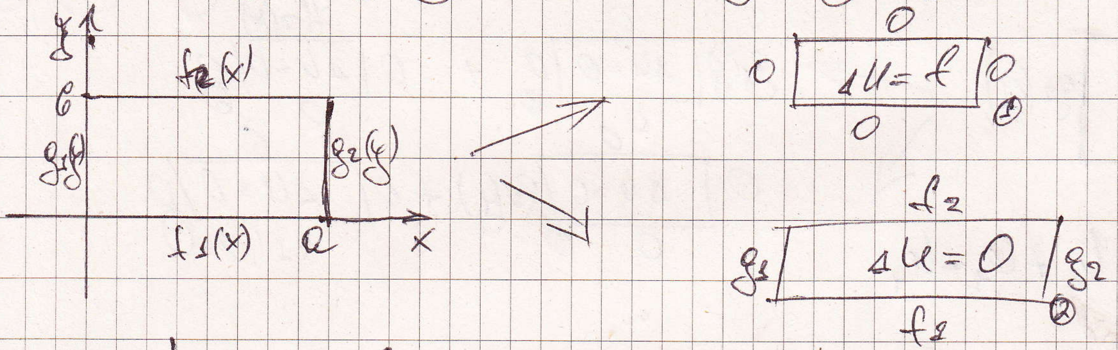
$$B_k = \frac{2}{b}\,\frac{1}{\mathrm{sh}\,\frac{k\pi a}{b}}\int_0^b g_1(y)\sin\frac{k\pi y}{a}\,dy$$

Уравнение на Поасон

(1) $\Delta u = f(x,y)$, $x\in(0,a)$, $y\in(0,b)$

(2) $u(x,0) = f_1(x)$, $u(x,b) = f_2(x)$

(3) $u(0,y) = g_1(y)$, $u(a,y) = g_2(y)$



$$\begin{cases}\Delta u = f\\ u|_{\Gamma} = 0\end{cases},\qquad u(x,y) = \sum_1^{\infty} E_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$u_{xx}+u_{yy} = \sum E_{mn}\left[-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = f(x,y)$$

$$E_{mn}\lambda_{mn} = \frac{2}{a}\,\frac{2}{b}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$E_{mn} = \frac{1}{\lambda_{mn}}\,\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

Метод на собствените функции

Нека $\left| \begin{array}{l} \Delta u = f = 1 \\ u/_{\Gamma} = 0 \end{array} \right.$, $a = b = \pi$

Тогава

$$u(x,y) = \sum_{mn=0}^{\infty} E_{mn} \sin mx \sin ny$$

$$\lambda_{mn} = -m^2 - n^2$$

$$E_{mn} = \frac{-4}{\pi^2(m^2+n^2)} \int_0^{\pi}\int_0^{\pi} \sin mx \sin ny \, dx \, dy$$

$$E_{mn} = -\frac{4}{\pi^2(m^2+n^2)} \int_0^{\pi} \sin mx \, dx \int_0^{\pi} \sin ny \, dy$$

$$\frac{-4}{\pi^2(m^2+n^2)} \cos mx \Big|_0^{\pi} \cos ny \Big|_0^{\pi} = \frac{-4}{\pi^2(m^2+n^2)} \left((-1)^{n}-1\right)\left((-1)^{n}-1\right)$$

$$E_{2m+1,2n+1} = \frac{1}{\pi^2(2m+1)(2n+1)} \frac{1}{\left[(2m+1)^2+(2n+1)^2\right]}$$

$$u(x,y) = -\frac{16}{\pi^2} \sum_{mn=0}^{\infty} \frac{\sin(2m+1)x \sin(2n+1)y}{(2m+1)(2n+1)\left[(2m+1)^2+(2n+1)^2\right]}$$