ЧДУ лекция 17.11.14

Ф-ла за обръщането

$$F(f)(\xi) = \tilde{f}(\xi) = \int_{-\infty}^{\infty} e^{ix\xi} f(x)\,dx, \quad f(\xi) \in S$$

1) $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\tilde{f}(\xi)\,d\xi$$

Th – Формула 1) е валидна

Док

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\left(\int_{-\infty}^{\infty} e^{-iy\xi} f(y)\,dy\right)d\xi.$$

Сменяме реда на интегриране $\Rightarrow$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\underbrace{\left(\int_{-\infty}^{\infty} e^{i\xi(x-y)}\,d\xi\right)}_{\text{не е сходящ}}dy$$

Идеята на Rie

$g(\xi) \in S$. Разглеждаме

(2) $$J = \iint_{R^2} f(y)g(\xi)e^{i\xi(x-y)}\,dy\,d\xi < \infty$$

$$|J| \le \int_R |f(y)|\,dy \int_R |g(\xi)|\,d\xi \le \|f\|\,\|g\| < \infty$$

$$\left.\begin{aligned} J &= \int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} g(\xi)e^{i\xi(x-y)}\,d\xi\right)dy \to \int_{-\infty}^{\infty} f(y)\tilde{g}(y-x)\,dy \\ J &= \int_{-\infty}^{\infty} g(\xi)\left(\int_{-\infty}^{\infty} f(y)e^{i\xi(x-y)}\,dy\right)d\xi \to \int_{-\infty}^{\infty} g(\xi)e^{i\xi(x-y)}f(\xi)\,d\xi \end{aligned}\right\} \Rightarrow$$

3) $\int_{-\infty}^{\infty} f(y)\tilde{g}(y-x)\,dy = \int_{-\infty}^{\infty} g(\xi) e^{ix\xi} \tilde{f}(\xi)\,d\xi$

Озн. $h_\varepsilon(x) = g(\varepsilon x)$, $g \in S \Rightarrow h_\varepsilon \in S$

$$\tilde{h}_\varepsilon(k) = \int_{-\infty}^{\infty} e^{ikx} h_\varepsilon(x)\,dx = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} e^{\frac{ikx\varepsilon}{\varepsilon}} g(\varepsilon x)\,dx\varepsilon =$$

$$\frac{1}{\varepsilon}\tilde{g}\left(\frac{k}{\varepsilon}\right)$$

В 3) вместо g слагаме h_ε $\Rightarrow$

$$\frac{1}{\varepsilon}\int_{-\infty}^{\infty} f(y)\tilde{g}\left(\frac{y-x}{\varepsilon}\right)dy = \int_{-\infty}^{\infty} g(\varepsilon\xi) e^{ix\xi}\tilde{f}(\xi)\,d\xi$$

Нека $z = \frac{y-x}{\varepsilon} \Rightarrow$ смяна на променливите

4) $\int_{-\infty}^{\infty} f(x+\varepsilon z)\tilde{g}(z)\,dz = \int_{-\infty}^{\infty} g(\varepsilon\xi) e^{ix\xi}\tilde{f}(\xi)\,d\xi$

Нека $\varepsilon \to 0$

$$f(x)\int_{-\infty}^{\infty}\tilde{g}(z)\,dz = g(0)\int_{-\infty}^{\infty} e^{ix\xi}\tilde{f}(\xi)\,d\xi$$

$$\int_{-\infty}^{\infty}\tilde{g}(z)\,dz = \sqrt{\pi}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}\,d\frac{z}{2} = 2\sqrt{\pi}\sqrt{\pi} = 2\pi$$

$$\int e^{-z^2}dz = \sqrt{\pi}$$

$$F^{-1}(\tilde{f})(x) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\tilde{f}(\xi)\,dx, \text{ като } F = \{0\}$$

Следствие

$F: S \to S$ е взаимно еднозначно

$$F^{-1}(f)(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi} f(\xi)\,d\xi = \frac{1}{2\pi}F(f)(-x)$$

Конволюция, $f, g \in L^1$

Деф

$$h(x) = \int_{-\infty}^{\infty} f(y-x) g(y) dy := f * g \quad \text{т.е.} \quad h = f * g$$

Приложение

1) $\Delta u = u_{xx} + u_{yy} = 0$

2) $u/_{y=0} = f(x)$. u_1 е решение на 1) и 2)

$V = u_1 + y$! Ограничено решение в G

$G = \{ x \in R^{\infty}, y > 0 \}$ Предп. $\exists$ решение на 1) и 2) $u(xy)$, $u \in S$ като функция на x

3) $|u(xy)| \leq \frac{M}{(1+|x|)^2} = |u_y(xy)| \leq \frac{M}{(1+|x|)^2}$

$|u_{yy}(xy)| \leq \frac{M}{(1+|x|)^2}$, M - const

Пол $V(\xi y) = \int_{-\infty}^{\infty} e^{-ix\xi} u(xy) dx = F(u)(\xi y)$

$F(u_{xx}) + F(u_{yy}) = 0$, $F(u_{yy}) = \int_{-\infty}^{\infty} e^{-ix\xi} u_{yy} dx$

5) $\boxed{-\xi^2 v(\xi, y) + \frac{d^2}{dy^2}(\xi, y) = 0}$!

$v(y\xi) = C_1(\xi) e^{\xi y} + C_2(\xi) e^{-\xi y}$, $\xi \to +\infty$

$v(y\xi) = C(\xi) e^{-y|\xi|}$, $C(\xi) = \begin{cases} C_2(\xi), & \xi \to \infty \\ C_1(\xi) + C_2(0), & \xi = 0 \\ C_1(\xi) \end{cases}$

$u/_{y=0} = f(x)$, $C(\xi) = \tilde{f}(\xi)$

$v(y\xi) = \tilde{f}(\xi) \cdot \frac{e^{-y|\xi|}}{g(\xi)}$

0

$$F^{-1}(g)(-x) = \frac{1}{2\pi} \frac{2y}{(-x)^2 + y^2}$$

$$v = F(u(xy)) = F(f(x))(\xi) \cdot F\left(\frac{1}{\pi} \frac{y}{x^2+y^2}\right)$$

$$u(xy) = \int_{-\infty}^{\infty} f(t) \frac{y}{\pi(x-t)^2 + y^2} dt \quad \text{т.е.}$$

$$u(xy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2} dt$$

$$|u| = \underbrace{\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{dt}{(x-t)^2 + y^2}}_{\cong 1} \cdot M$$

$x = (x_1 \dots x_n)$, $f(x)$, $x \in R^n$

$\xi = (\xi_1 \dots \xi_n)$

$$\tilde{f}(\xi) = \int_{R^n} e^{-i(\xi, x)} f(x)\, dx$$ Трансформация на Фурие