ЧМ 26.04

Средни квадратични приближения

Полином на най-добро квадратично приближение

ПНДСКП

Озад $f(x) \in C[a,b]$ $\quad p(x) \in \Pi_n$ т.ч.

$$S(A,B,C) = \int_a^b \mu(x)\,(f(x)-p(x))^2\,dx \to \min$$

$$p(x) = Ax^2 + Bx + C \in \Pi_2$$

НДУ $\quad \frac{\partial S}{\partial A} = \frac{\partial S}{\partial B} = \frac{\partial S}{\partial C} = 0 \Rightarrow A = \_\_,\ B = \_\_,\ C = \_\_$

1) Да се намери ПНДСКП $\in \Pi_2$ за $f(x) = x^4$, $x \in [-1,1]$, $\mu(x) = 1$

Реш

$f(x)$ - четна $\Rightarrow p(x)$ - четна

$\Rightarrow p(x) = Ax^2 + C$

$$S(AC) = \int_{-1}^{1} (x^4 - Ax^2 - C)^2\,dx \to \min$$

(от $S(AC) = \int_{-1}^{1} (f(x)-p(x))^2\,dx \to \min$

$$\begin{cases} \frac{\partial S}{\partial A} = \int_0^1 2\,(x^4 - Ax^2 - C)(-x^2)\,dx = 0 \quad / : (-2) \\ \frac{\partial S}{\partial C} = \int_0^1 2\,(x^4 - Ax^2 - C)(-1)\,dx = 0 \quad / : (-2) \end{cases}$$

$$\left| \begin{aligned} &\int_0^1 x^6 - Ax^4 - Cx^2\,dx = 0 \\ &\int_0^1 x^4 - Ax^2 - C\,dx = 0 \end{aligned} \right\} \Rightarrow$$

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$$\begin{cases} \dfrac{x^7}{7} - \dfrac{Ax^5}{5} - \dfrac{Cx^3}{3}\Big|_0^1 = 0 \\ \dfrac{x^5}{5} - \dfrac{Ax^3}{3} - Cx\Big|_0^1 = 0 \end{cases}$$

$$\begin{cases} \dfrac{1}{7} - \dfrac{A}{5} - \dfrac{C}{3} = 0 \\ \dfrac{1}{5} - \dfrac{A}{3} - C = 0 \end{cases} \Rightarrow \begin{cases} 15 - 21A - 35C = 0 \\ 3 - 5A - 15C = 0 \end{cases}$$

$$C = \frac{3-5A}{15}, \quad A = \frac{4}{3}$$

$$\int_{-1}^{1} (x^6 - Ax^4 - Bx^2 - C)\,dx \to \min$$

② Да се намери ПНБСКП $\in \Pi_1$ в $[-1,1]$ за $f(x) = \frac{1}{2}x^3$, $\mu(x) = x^2$

Реш.

$f(x)$ - нечетна $\Rightarrow p(x)$ - нечетна $\Rightarrow p(x) = Ax \Rightarrow$

$$S(A) = \int_{-1}^{1} x^2 \left(\tfrac{1}{2}x^3 - Ax\right)^2 dx \to \min$$

$$S'_A = 0$$

$$\frac{\partial S}{\partial A} = \int_{-1}^{1} 2x^2\left(\tfrac{1}{2}x^3 - Ax\right)(-x)\,dx = 0 \;\Big|:(-2)$$

③

$$\int_{-1}^{1} x^3(7x^3 - Ax)\,dx = 0$$

$$2\int_0^1 7x^6 - Ax^4\,dx = 0 \;\Big/ :2$$

$$\frac{7x^7}{7} - \frac{Ax^5}{5} = 0 \Big|_0^1 = 0$$

$$1 - \frac{A}{5} = 0 \Rightarrow A = 5 \Rightarrow p(x) = 5x - \text{ПНДСКП}$$

6 зад. $\quad f(x) = \ln x \qquad p(x) = ax + b \in \Pi_1$

$\mu(x) = 1$

$$\int_0^1 (f(x) - p(x))^2\,dx \to \min$$

$$S(a,b) = \int_0^1 (\ln x - ax - b)^2\,dx \to \min$$

$$\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = 0$$

$$\begin{cases} \dfrac{\partial S}{\partial a} = 2\displaystyle\int_0^1 (\ln x - ax - b)(-x)\,dx = 0 \;\Big/ :(-2) \\ \dfrac{\partial S}{\partial b} = 2\displaystyle\int_0^1 (\ln x - ax - b)(-1)\,dx = 0 \;\Big/ :(-2) \end{cases}$$

$$\begin{cases} \displaystyle\int_0^1 x\ln x - ax^2 - bx\,dx = 0 \\ \displaystyle\int_0^1 \ln x - ax - b\,dx = 0 \end{cases}$$

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$$\int \underbrace{\ln x}_{u}\, d\underbrace{x}_{v} \qquad \int u\,dv = uv - \int v\,du$$

$$x\ln x - \int x\,d\ln x = x\ln x - \int \frac{x}{x}dx =$$

$$x\ln x - x$$

$$\int x^2 d\ln x = \int \frac{x^2}{x}dx$$

след интегр. по част

$$\int x \ln x\, dx = \frac{1}{2}\int \ln x\, dx^2 =$$

$$\frac{1}{2}\left(x^2\ln x - \int x^2 d\ln x\right) = \frac{1}{2}x^2\ln x - \frac{x^2}{2} \Rightarrow$$

$$\frac{1}{2}\left(x^2\ln x - \frac{x^2}{2}\right)\Big|_0^1 - \frac{ax^3}{3}\Big|_0^1 - \frac{bx^2}{2}\Big|_0^1 = 0$$

$$\left(x\ln x - x - \frac{ax^2}{2} - bx\right)\Big|_0^1 = 0$$

$$-\frac{1}{4} - \frac{a}{3} - \frac{b}{2} = 0$$

$$-1 - \frac{a}{2} - b = 0 \Rightarrow b = -1 - \frac{a}{2}$$

след заместване горе $\Rightarrow a = \ldots$

Зад 3 $E_n(f)$ - грешка на ПНДРП $\in \Pi_n$

а) $E_n(\lambda f) = |\lambda|\, E_n(f)$

б) $E_n(f+g) \leq E_n(f) + E_n(g)$

в) $E_n(f+g) = E_n(f)$ за $\forall g \in \Pi_n$

г) $E_n(f) \leq \|f\|\; C \in (a, b)$

$P(x) \in \Pi_n$ - ПНДРП за $f(x)$ $\qquad Q(x) \in \Pi_n$ - ПНДРП за $g(x)$

(4)

$$E_n(f) = \max_{x\in[a,b]} |f(x) - P(x)| \to \min$$

$$E_n(g) = \max_{x\in[a,b]} |g(x) - Q(x)| \to \min$$

а) $E_n(\lambda f) = \inf\limits_{R\in \Pi_n} \|\lambda f - R\| = \inf\limits_{R\in \Pi_n} \|\lambda (f - \frac{R}{\lambda})\| =$

$|\lambda| \underbrace{\inf \|f - \frac{R}{\lambda}\|}_{E_n(f)} = |\lambda| E_n(f)$

б) $E_n(f+g) \leq \inf\limits_{P,Q\in \Pi_n} \|f+g-P-Q\| = \inf\limits_{P,Q\in \Pi_n} \|(f-P)+(g-Q)\| \leq$

$\inf\limits_{P\in \Pi_n} \|f-P\| + \inf\limits_{Q\in \Pi_n} \|g-Q\| = E_n(f) + E_n(g)$

в) $E_n(f+g) = \inf\limits_{P\in \Pi_n} \|f+g-P(x)\| = \inf\limits_{P\in \Pi_n} \|f-(\underbrace{P-g}_{\in \Pi_n})\| = E_n(f)$

защото $P-g \in \Pi_n$

г) $E_n(f) \leq \|f - P_0\| = \|f - 0\| = \|f\|_{C[a,b]}$