СМ 23.04.14

$|a \pm b| \le |a| + |b|$

$|a| \ge 0 \quad , \quad a = 0 \iff |a| = 0$

$|a\lambda| = |\lambda||a|$

Задача - схема за построяване на ПНДРП

$f(x) \in C^1[a,b] \quad , \quad f'' \ne 0 \, , \, \forall x \in [a,b]$

(изпъкнала, вдлъбната)

1) $P(x) \in \Pi_1$ т.ч. $g(x)$ минава през точките $(a, f(a))$ и $(b, f(b))$

2) $\exists! \; x_1 \in (a,b) \; : \; f'(x_1) = \frac{f(b) - f(a)}{b-a}$

3) $P(x) = g(x) - \frac{1}{2}\left(g(x_1) - f(x_1)\right)$

Зад Да се намери ПНДРП $\in \Pi_1$ в $[0,1]$ $(x \in [0,1])$ за

а) $f(x) = |x - \frac{1}{3}|$ ; б) $f(x) = e^x$

Реш

б) $f(x) = e^x \Rightarrow f'(x) = f''(x) = e^x \ne 0 \quad (e^x > 0)$

за $\forall x \in [0,1]$, $a = 0$, $b = 1$

$g(x) = \frac{1-x}{1-0} f(0) + \frac{x-0}{1-0} f(1)$ - Лагранж

$= (1-x)1 + x \cdot e^1 = x(e-1) + 1$

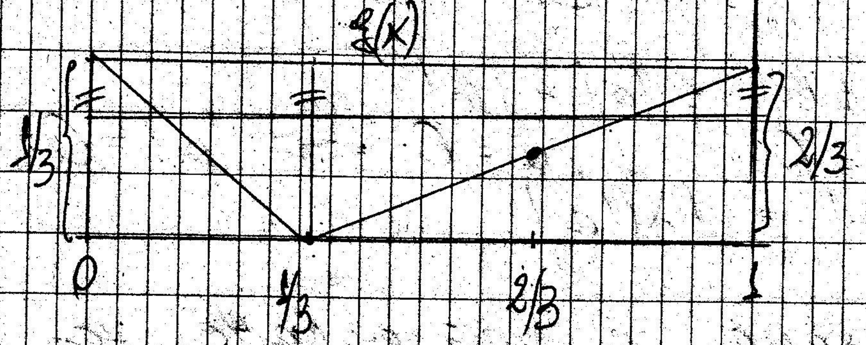
$x_1 = ?$ , $e^{x_1} = \frac{e-1}{1-0} = e - 1$

$x_1 = \ln(e-1) \Rightarrow P(x) = x(e-1) + 1 - \frac{1}{2}\Big((\ln(e-1))(e-1) +$

$+ 1 - e^{\ln(e-1)}\Big) = x(e-1) + 1 - \frac{1}{2}\left[(\ln(e-1))(e-1) - e + 1\right] =$

$x(e-1) + 1 - \frac{1}{2}\left((e-1)(\ln(e-1)) - e + 1\right)$

б)




$f(x) = |x - 1/3|$

1) $g(x)$ : $(0, f(0)) \in g(x)$ , $(1, f(1)) \in g(x)$

2) $x_1 = 1/3$

3) $P(x) = g(x) - \underbrace{\frac{1}{2}\left(g(x_1) - f(x_1)\right)}_{\text{грешка } E_1(f)}$

$$g(x) = \frac{1-x}{1-0} f(0) + \frac{x-0}{1-0} f(1) = (1-x)\frac{1}{3} + \frac{2}{3}x = \frac{2x - x + 1}{3} =$$

$$\frac{x+1}{3} \text{ — лагранж}$$

$$P(x) = \frac{x+1}{3} - \frac{1}{2}\left(\frac{4}{9} - 0\right) = \underline{\underline{\frac{x}{3} + \frac{1}{9}}}$$

$$f(x) = \frac{1}{1+x} = (1+x)^{-1}, \quad f'(x) = -\frac{1}{(1+x)^2}$$

$$f''(x) = \frac{2}{(1+x)^3} > 0 \text{ за } \forall x \in [0,1]$$

$$g(x) = \frac{1-x}{1-0} \cdot 1 + x\frac{1}{2} = \frac{2 - 2x + x}{2} = 1 - \frac{x}{2}$$

$$x_1 = ?, \quad f'(x_1) = \frac{\frac{1}{2} - 1}{1} = -\frac{1}{2}$$

$$-\frac{1}{(1+x)^2} = -\frac{1}{2}$$

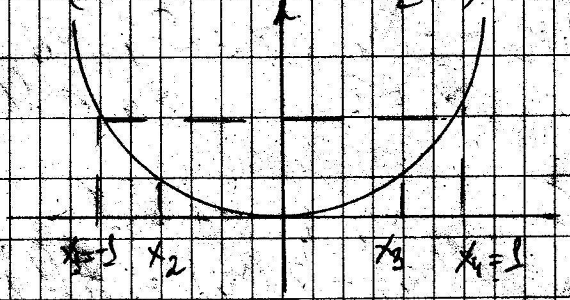
$$(1-x)^2 = 2 \Rightarrow \underbrace{(1-x-\sqrt{2})}_{x_1=\sqrt{2}-1}\underbrace{(1-x+\sqrt{2})}_{x=-1-\sqrt{2}<0}$$

Вътр. реш. на т. на алтернанс

$$P(x) = g(x) - \frac{1}{2}\left(g(x_1) - f(x_1)\right) = 1 - \frac{x}{2} - \frac{1}{2}\left(1 - \frac{\sqrt{2}-1}{2} - \frac{1}{1+\sqrt{2}-1}\right)$$

Зад 4 Да се докаже (намери) ПНДРП $\in \Pi_2$ за

$f(x) = x^4$ в $[-1, 1]$




$P(x) \in \Pi_2$

$P(x)$ – ПНДРП

Но $f(x)$ – четна $\Rightarrow$ $P(x)$ – четна $\Rightarrow$

$P(x) = ax^2 + b$ $\quad a = ?\quad b = ?$

Необходими са ни 4 т. на алтернанс (5 т.)

$$f(x_1) - P(x_1) = P(x_2) - f(x_2) = f(x_3) - P(x_3) = P(x_4) - f(x_4) = \pm L_2(f)$$

от съображения за четност (симетрия) $\Rightarrow$

$x_1 = -x_4$, $x_2 = -x_3$ $\quad (x_4 = -x_1,\ x_3 = -x_2)$

Поне 2 от точките на алтернанс са вътр. ($x_2$ и $x_3$) $\Rightarrow$ $f(x_3) - P(x_3)$ достига локален екстремум в т. $x_3$ $\Rightarrow$ $f'(x_3) - P'(x_3) = 0$

$f'(x_3) = 4x^3$ $\qquad P'(x_3) = 2ax$

$$4x_3^3 - 2ax_3 = 0 \Rightarrow 2x_3(2x_3^2 - a) = 0$$

$x_3 = 0 \qquad x_4 = \sqrt{\frac{a}{2}}$

Предполагаме че $x_4 = 1$ $(x_1 = -1)$ $\Rightarrow$

$$P(1) - f(1) = f(x_3) - P(x_3)$$

$a + b - 1 = a^2 - a^2 - b \Rightarrow a + b - 1 = -b$ т.е.

$a+2b-1=0$

Ако $x_3=0 \Rightarrow$ 3 т. на алтернанс

$f(0)-P(0)=P(1)-f(1)$

$0-b=a+b-1 \Rightarrow a+2b-1=0$

Ако търсим $P(x)$ – ПНБРП $\in \Pi_3$ са ни необходими 5 т. на алтернанс

$f(x)$ - четна $\Rightarrow P(x)$ четна $\Rightarrow P(x)=ax^2+b$

От съображения за симетрия $\Rightarrow$

$x_1=-1,\ x_2=-x_4,\ x_3=0,\ x_4,\ x_5=1$

$x_4=?\qquad x_4\in(0,1)$

Разгл. $x_3=0,\ x_4,\ x_5=1$

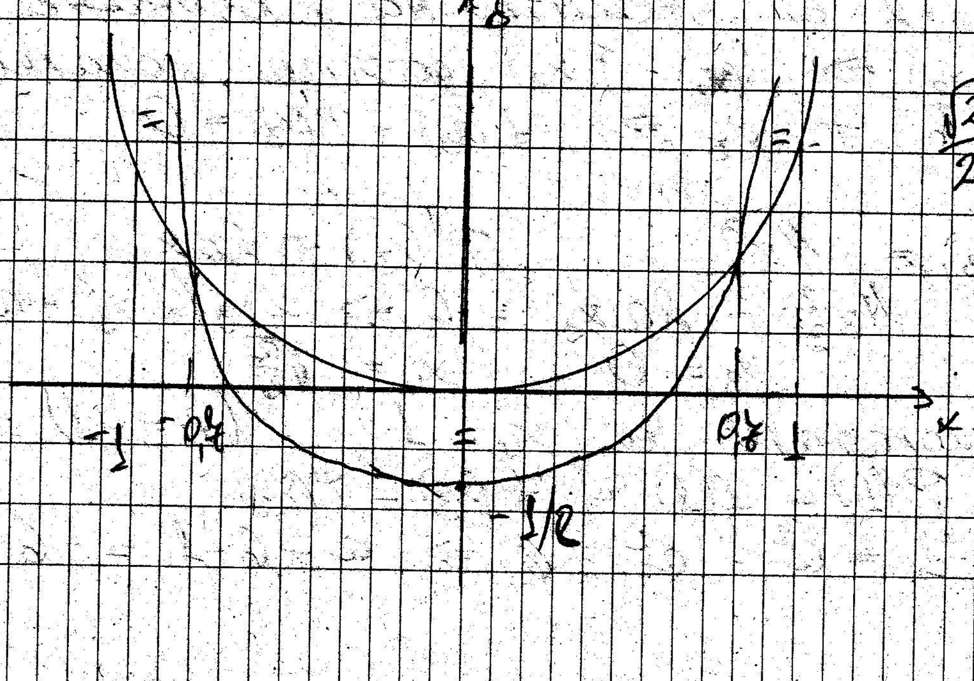
$P(0)-f(0)=f(x_4)-P(x_4)=P(1)-f(1)$

$b=\frac{a^2}{4}-a\left(\frac{a}{2}\right)-b=a+b-1$

$a+b-1=b \Rightarrow a=1$

$b=\frac{1}{4}-\frac{1}{2}-b \Rightarrow 2b=-\frac{1}{4} \Rightarrow b=-\frac{1}{8}$

$$P(x)=x^2-\frac{1}{8}$$




$\frac{\sqrt{2}}{2}\approx 0,7$

① Метод на най-малките квадрати

1) Дадена е таблица от точки

$$\{x_i, y_i\}_{i=1}^{n} \qquad x_1 < x_2 \dots < x_n$$

Д-СЪ, че $\exists!$ права $p(x) = Ax + B$ за която величината $S(A,B) = \sum_{i=1}^{n} (y_i - Ax_i - B)^2 \to \min$

Док.

НДУ за екстремум ($\min$) е $\frac{\partial S}{\partial A} = \frac{\partial S}{\partial B} = 0$

$$\frac{\partial S}{\partial A} = 2\sum_{i=1}^{n} (y_i - Ax_i - B)(-x_i) = 0 \;/ : (-2)$$

$$\frac{\partial S}{\partial B} = 2\sum_{i=1}^{n} (y_i - Ax_i - B)(-1) = 0 \;/ : (-2)$$

$$\left| \begin{aligned} &A\sum_{i=1}^{n} x^2 + B\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i x_i \\ &A\sum_{i=1}^{n} x_i + nB = \sum_{i=1}^{n} y_i \end{aligned} \right.$$

$$\Delta = n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 \neq 0 \quad \text{тк} \quad x_i \neq x_j,\; i \neq j$$

неравенството на Коши-Буняковски $\to$ сист. има единствено решение т.е. $\exists$ единствена права $\square$

Зад

10 а)

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 1 | 2 | 1 | 0 | 4 |

$p(x) = ax + b$

б)

| $x$ | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $y$ | -4 | 15 | -3 | 10 | 7 | 6 |

$p(x) = ax^2 + bx + c$

⑤

Реш

а) $p(x) = ax + b$ , $S(a b) = \sum_{i=1}^{5} (y_i - a x_i - b)^2 \to \min$

$$\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = 0$$

$$\begin{cases} a \sum_{i=1}^{5} x_i^2 + b \sum_{i=1}^{5} x_i = \sum_{i=1}^{5} x_i y_i \\ a \sum_{i=1}^{5} x_i + 5b = \sum_{i=1}^{5} y_i \end{cases}$$

$$\left.\begin{aligned} &\sum x_i^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30 \\ &\sum x_i = 10 \quad , \sum y_i = 8 \quad , \sum y_i x_i = 20 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{cases} 30a + 10b = 20 \\ 10a + 5b = 8 \end{cases}\right\} \Rightarrow a = \frac{2}{5} , b = \frac{4}{5}$$

б) $S(a b c) = \sum_{i=1}^{6} (y_i - a x_i^2 - b x_i - c_i)^2 \to \min$

$$\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = \frac{\partial S}{\partial c} = 0$$

$$\left.\begin{aligned} &\frac{\partial S}{\partial a} = 2 \sum_{i=1}^{6} (y_i - a x_i^2 - b x_i - c)(-x_i^2) = 0 \\ &\frac{\partial S}{\partial b} = 2 \sum_{i=1}^{6} (y_i - a x_i^2 - b x_i - c)(-x_i) = 0 \\ &\frac{\partial S}{\partial c} = 2 \sum_{i=1}^{6} (y_i - a x_i^2 - b x_i - c)(-1) = 0 \end{aligned}\right\} \Rightarrow$$

6

$$\begin{cases} a\sum_{i=1}^{6} x_i^4 + b\sum_{i=1}^{6} x_i^3 + c\sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} x_i^2 y_i \\ b\sum_{i=1}^{6} x_i^3 + b\sum x_i^2 + c\sum x_i = \sum x_i y_i \\ c\sum_{i=1}^{6} x_i^2 + b\sum x_i + 6c = \sum y_i \end{cases}$$

$\sum x_i = 3$ , $\sum x_i^3 = 27$, $\sum x_i^2 = 19$, $\sum x_i^4 = 115$

$\sum y_i = 25$, $\sum y_i x_i = 35$, $\sum y_i x_i^2 = 91$

$$\Rightarrow \begin{cases} 115a + 27b + 19c = 91 \\ 27a + 19b + 3c = 35 \\ 19a + 3b + 6c = 25 \end{cases}$$