ЕМ упр

Разделени разлики - ф-ла на Нютон

$f[x_i] = f(x_i)$

$f[x_0 \dots x_k] = \dfrac{f[x_1 \dots x_k] - f[x_0 \dots x_{k-1}]}{x_k - x_0}$

$a \le x_0 < \dots < x_n \le b$

Нютон

$L_n(f,x) = f[x_0] + f[x_0,x_1](x-x_0) + \dots + f[x_0 \dots x_n](x - x_0)\dots(x - x_{n-1})$

$L_n(f;x) = \sum_{k=0}^{n} f[x_0 \dots x_k](x-x_0)\dots(x-x_{k-1})$

$f[x_0 \dots x_n] =$ коеф пред x^n в $L_n(f,x)$

Друга ф-ла за разделени разлики

$f[x_0 \dots x_n] = \sum_{k=0}^{n} \dfrac{f(x_k)}{\omega'(x_k)}$

$\omega(x) = (x-x_0)\dots(x-x_n) \in \pi_{n+1}$

Задачи

1) Докажете тъждествата

а) $f[x_0 \dots x_n] = 0, \quad \forall f(x) \in \pi_{n-1}$

б) $f[x_0 \dots x_n] = 1, \quad f(x) = x^n$

Док

а) $f(x) \in \pi_{n-1} \Rightarrow f(x) = L_n(f,x) \Rightarrow$ коеф пред $x^n = 0$
$\Rightarrow f[x_0 \dots x_n] = 0$

б) $f(x) = x^n \in \pi_n \Rightarrow f(x) \equiv L_n(f,x) \Rightarrow f[x_0 \dots x_n] = 1$

2 зад Да се намери

$$\sum_{k=0}^{n} \frac{x_k \omega''(x_k)}{\omega'(x_k)}$$

Реш $\sum_{k=0}^{n}{}' \frac{x_k \omega''(x_k)}{\omega'(x_k)} = f[x_0 \dots x_n]$ за $f(x_k) = x_k \omega''(x_k) \Rightarrow$

$f(x) = x\omega''(x) \Rightarrow f(x) \in \Pi_n$

$$\left.\begin{array}{l} \omega(x) = x^{n+1} + \dots \\ \omega'(x) = (n+1)x^n + \dots \\ \omega''(x) = n(n+1)x^{n-1} + \dots \end{array}\right\} \Rightarrow x\omega''(x) = \underbrace{n(n+1)}_{\text{реш}} x^n$$

$x\omega''(x) = n(n+1) \Rightarrow \sum_{k=0}^{n}{}' \frac{x_k \omega''(x_k)}{\omega'(x_k)} = n(n+1)$

3 зад Да се намери $\sum_{k=0}^{n}{}' \frac{\omega''(x_k)}{\omega'(x_k)}$

Реш

$\omega(x) \in \Pi_{n+1} \Rightarrow \omega''(x) \in \Pi_{n-1} \Rightarrow$

$\omega''(x) \equiv L_n(\omega'', x) \Rightarrow$

$\sum_{k=0}^{n}{}' \frac{\omega''(x_k)}{\omega'(x_k)} = \omega''[x_0 \dots x_n] \Rightarrow \sum_{k=0}^{n}{}' \frac{\omega''(x_k)}{\omega'(x_k)} = 0$

т.к. коеф пред $x^n = 0$, понеже $\omega''(x) \in \Pi_{n-1}$

4 зад ДСЧ $\sum_{k=0}^{n}{}' \frac{x_k^{n+1}}{\omega'(x_k)} = \sum_{k=0}^{n}{}' x_k$

Док $\sum_{k=0}^{n}{}' \frac{x_k^{n+1}}{\omega'(x_k)} = x^{n+1}[x_0 \dots x_n]$ ва в $f(x) = x^{n+1}$

$f(x) - L_n(f, x) \in \Pi_{n+1}$ със старши коеф 1

$f(x_k) = L_n(f, x_k)$, $k = 0 \dots n$; $f(x_k) - L_n(f, x_k) = 0 \Rightarrow$

$f(x) - L_n(f, x) = \omega(x)$

$x^{n+1} - L_n(x^{n+1}, x) = (x-x_0)\dots(x-x_n)$

Приравняваме коеф-те пред x^n от двете страни на равенството $\Rightarrow$

$-x^{n+1}[x_0 \dots x_n] = -x_0 - x_1 - \dots - x_n \Rightarrow$

$x^{n+1}[x_0 \dots x_n] = \sum_{k=0}^{n} x_k$ II

5 зад. Да се намери $P_3(x) = ?$ за който

x	-1	0	1	2
f	3	1	-1	3

Реш.

X	$P[x_i]$	$P[x_i, x_2]$	$P[x_1 x_2 x_3]$	$P[\dots]$
-1	3	$\frac{-2}{1} = -2$	$\frac{2}{2} = 0$	$\frac{2-0}{3} = \frac{2}{3}$
0	1	$\frac{-2}{1} = -2$	$\frac{4}{2} = 2$	
1	-1	$\frac{2}{1} = 2$		
2	3			

$P(x) = 3 - 2(x+1) + 0(x+1)(x-0) + \frac{2}{3}x(x^2-1)$

6 з. Да се намери $P_3(x) = ?$, който

$p(-1) = 3$, $p(0) = 1$, $p(2) = 3$, $p(4) = 53$

Решение

x	$P[x]$	$P[x_1, x_2]$	$P[x_1 x_2, x_3]$	$P[\dots]$
-1	3	-2	1	1
0	1	1	6	
2	3	25		
4	53			

$P(x) = 3 + (-2)(x+1) + (x+1)x + (x+1)x(x-2) = x^3 - 3x + 1$

(3)

13. С интерполационен полином на Лагранж

1 св-во

а) $\sum_{k=0}^{n}{}' (-1)^{n-k}\binom{n}{k}\binom{m}{n}\frac{1}{m-k} = \frac{1}{m-n}, \ \forall m > n \geq 0$

б) $\sum_{k=0}^{n}{}' (-1)^{n-k}\binom{n}{k}\binom{m}{n}\frac{k}{m-k} = \frac{m}{m-n}, \ \forall m > n \geq 0$

Док.

Нека $x_k = k$, $k = 0 \div n$, $\omega(x) = x(x-1)(x-2)\dots(x-n)$

$$L_n(f,x) = \sum_{k=0}^{n}{}' \frac{f(x_k)\,\omega(x)}{(x-x_k)\,\omega'(x_k)} = \sum_{k=0}^{n}{}' \frac{f(k)\,\omega(x)}{(x-k)\,\omega'(k)}$$

$$\omega'(k) = \underbrace{k(k-1)(k-2)\dots 1}_{k!}\underbrace{(-1)(-2)\dots[-(n-k)]}_{(-1)^{n-k}(n-k)!} \Rightarrow$$

$$\Rightarrow \sum_{k=0}^{n}{}' \frac{f(k)(x-1)\dots(x-n)}{(x-k)\,k!\,(-1)^{n-k}(n-k)!} \Rightarrow$$

$$L_n(f,m) = \sum_{k=0}^{n}{}' (-1)^{n-k}\frac{f(k)\,m(m-1)\dots(m-n)}{(m-k)\,k!\,(n-k)!}$$

делим на $n!$ $\Rightarrow$

$$\sum_{k=0}^{n}{}' (-1)^{n-k}\frac{m(m-1)\dots(m-n+1)}{n!}\cdot\frac{n!}{k!\,(n-k)!}\,f(k)\frac{m-n}{m-k} =$$

$$= \sum_{k=0}^{n}{}' (-1)^{n-k}\binom{n}{k}\binom{m}{n}\frac{m-n}{m-k}f(k)$$

а) $f(x) = 1 \in \Pi_0 \subset \Pi_n \Rightarrow f(m) \equiv L_n(f, m)$

$$1 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{m}{n} \frac{m-n}{m-k} \quad \Big| : (m-n) \Rightarrow \square$$

б) $f(x) = x \in \Pi_1 \subset \Pi_n \Rightarrow f(m) = L_n(f, m) \Rightarrow$

$$m = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{m}{n} \frac{m-n}{m-k} \cdot k \quad \Big| : (m-n) \Rightarrow \square$$