ЧМ 21.05.14

Зад. Да се намери квадратурна формула на Гаус с 2 възела за приближеното пресмятане на $\int_{-1}^{1} x^2 f(x)\,dx$

Реш.

$x \in [-1,1]$, $\mu(x) = x^2$

$\int_{-1}^{1} \mu(x) f(x)\,dx \approx A f(x_1) + B f(x_2)$ където

x_1, x_2 са корени на полином $\in \Pi_2$ със старши коефициент 1-ца, който е ортогонален на Π_0 и Π_1 (1 и x) при тегло $\mu(x)$

$p(x) = x^2 + ax + b$

$$\left| \begin{array}{l} \int_{-1}^{1} x^2 \cdot 1 \cdot p(x)\,dx = 0 \\ \int_{-1}^{1} x^2 x\, p(x)\,dx = 0 \end{array} \right\} \Rightarrow \left| \begin{array}{l} \int_{-1}^{1} x^4 + ax^3 + bx^2\,dx = 0 \\ \int_{-1}^{1} x^5 + ax^4 + bx^3\,dx = 0 \end{array} \right.$$

$$\left| \begin{array}{l} \frac{x^5}{5} + \frac{ax^4}{4} + \frac{bx^3}{3} \Big|_0^1 = 0 \\ \frac{x^6}{6} + \frac{ax^5}{5} + \frac{bx^4}{4} \Big|_0^1 = 0 \end{array} \right\} \Rightarrow \left| \begin{array}{l} \frac{1}{5} + \frac{a}{4} + \frac{b}{3} = 0 \\ \frac{1}{6} + \frac{a}{5} + \frac{b}{4} = 0 \end{array} \right.$$

$\frac{ax^4}{4} = 0 \Rightarrow a = 0$ (заради четността) $\Rightarrow$

$\frac{1}{5} + \frac{b}{3} = 0 \Rightarrow b = -\frac{3}{5} \Rightarrow p(x) = x^2 - \frac{3}{5}$

$x_{1,2} = \pm\sqrt{\frac{3}{5}} \Rightarrow$

$$\int_{-1}^{1} x^2 f(x)\,dx \approx A f\left(-\sqrt{\tfrac{3}{5}}\right) + B f\left(\sqrt{\tfrac{3}{5}}\right) -$$

точна за $f(x)=1$ и $f(x)=x$

$$\int_{-1}^{1} x^2 1\,dx = A1 + B1$$

$$\underbrace{\int_{-1}^{1} x^2 x\,dx}_{0} = A\left(-\sqrt{\tfrac{3}{5}}\right) + B\left(\sqrt{\tfrac{3}{5}}\right)$$

0 - четна $\left(\Rightarrow x^3 dx = 0\right.$ - нечетност$\left.\right)$

$$0 = \sqrt{\tfrac{3}{5}}(-A+B) \Rightarrow A = B \Rightarrow$$

от $\left.\frac{x^3}{3}\right|_{-1}^{1} = A + B \Rightarrow \frac{2}{3} = A + A \Rightarrow A = B = \frac{1}{3}$

$$\int_{-1}^{1} x^2 f(x)\,dx \approx \frac{1}{3}\left(f\left(-\sqrt{\tfrac{3}{5}}\right) + f\left(\sqrt{\tfrac{3}{5}}\right)\right)$$

Зад Да се намери лява квадратурна формула на Радо с 2 възела в $[0,1]$, $\mu(x) \equiv 1$

Реш.
$$\int_0^1 \mu(x) f(x)\,dx \approx A f(0) + B f(x_1)$$

x_1 - корен на полином от Π_1 със старши коефициент 1-ца, който е ортогонален на Π_0 (1) при <u>тегло $\mu(x)(x-0)$!!!</u>

$$p(x) = x - x_1 \Rightarrow \int_0^1 1(x-0)1(x-x_1)\,dx = 0$$

②

$$\int_0^1 x^2 - x x_1 \, dx = 0 \Rightarrow \frac{x^3}{3} - \frac{x^2}{2} x_1 \Big|_0^1 = 0$$

$$\frac{1}{3} - \frac{x_1}{2} = 0 \Rightarrow x_1 = \frac{2}{3} \in (0,1) \Rightarrow$$

$$\int_0^1 f(x)\,dx \approx A\, f(0) + B f\left(\frac{2}{3}\right)$$

точна за $f(x) = 1$ и $f(x) = x$

$$\left| \begin{array}{l} \int_0^1 1\,dx = A + B \\ \int_0^1 x\,dx = A \cdot 0 + B \frac{2}{3} \end{array} \right\} \Rightarrow \left| \begin{array}{l} 1 = A + B \\ \frac{x^2}{2} \Big|_0^1 = B \frac{2}{3} \Rightarrow B = \frac{3}{4} \Rightarrow A = \frac{1}{4} \end{array} \right.$$

$$\int_0^1 f(x)\,dx \approx \frac{1}{4} f(0) + \frac{3}{4} f\left(\frac{2}{3}\right)$$

Зад. Да се намери лява квадратурна формула на Радо с 2 възела за пресмятане на
$$\int_0^1 (1-x) f(x)\,dx$$

Реш.

$\mu(x) = 1 - x$ $\qquad \int_0^1 f(x)\,dx \approx A f(0) + B f(x_1)$

x_1 - корен на $p(x) = x - x_1$ - ортогонален на $П_0$ с тегло $\mu(x)(x-0) = (1-x)x$

$$\int_0^1 (1-x)x \; p(x)\, 1 \, dx = 0$$

$$\int_0^1 (1-x)x(x-x_1)\,dx = 0 \Rightarrow \int_0^1 x^2 - xx_1 - x^3 + x^2x_1\,dx = 0$$

$$\left.\frac{x^3}{3} - \frac{x^2}{2}x_1 - \frac{x^4}{4} + \frac{x^3}{3}x_1\right|_0^1 = 0$$

$$\frac{1}{3} - \frac{x_1}{2} - \frac{1}{4} + \frac{x_1}{3} = 0 \Rightarrow 2x_1 = 1 \Rightarrow x_1 = \frac{1}{2} \in (0,1)$$

$$\int_0^1 (1-x)f(x)\,dx \approx Af(0) + Bf\left(\frac{1}{2}\right)$$

точна за $f(x)=1$ и $f(x)=x$

$$\left|\begin{aligned} &\int_0^1 (1-x)1\,dx = A+B \\ &\int_0^1 (1-x)x\,dx = A.0 + B\frac{1}{2} \end{aligned}\right\} \Rightarrow \left|\begin{aligned} &\left.\left(x-\frac{x^2}{2}\right)\right|_0^1 = A+B \\ &\left.\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\right|_0^1 = B\frac{1}{2} = \frac{1}{6} \Rightarrow \end{aligned}\right.$$

$$B = \frac{1}{3} \quad ; \quad A = \frac{1}{2} - B = \frac{1}{6}$$

$$\int_0^1 (1-x)f(x)\,dx \approx \frac{1}{6}f(0) + \frac{1}{3}f\left(\frac{1}{2}\right)$$

Зад Да се намери дясна квадратурна формула на Радо с 2 възела за пресмятане на

$$\int_0^1 x\,f(x)\,dx$$

Реш $\mu(x) = x$; $\int_0^1 x f(x)\,dx \approx Af(x_1) + Bf(1)$

(4)

x_1 - корен на полином $p(x) \in \Pi_1$ със старши коефициент 1-ца ортогонален на Π_0 с тегло $\mu(x)(1-x) \geq 0$ и $p(x) = x - x_1$

$$\int_0^1 x(1-x)(x-x_1)1\,dx = 0$$

$$\int_0^1 x^2 - x x_1 - x^3 + x^2 x_1\,dx = 0$$

$$\left.\frac{x^3}{3} - \frac{x^2}{2}x_1 - \frac{x^4}{4} + \frac{x^3}{3}x_1 \right|_0^1 = 0 \Rightarrow 4 - 6x_1 - 3 + 4x_1 = 0$$

т.е. $x_1 = \frac{1}{2} \Rightarrow$

$$\int_0^1 x f(x)\,dx \approx A f\left(\frac{1}{2}\right) + B f(1)$$

точна за $f(x) = 1$ и $f(x) = x$

$$\left|\begin{array}{l} \int_0^1 x\,dx = A + B \\ \int_0^1 x^2\,dx = \frac{A}{2} + B \end{array}\right\} \Rightarrow \left|\begin{array}{l} \frac{1}{2} = A + B \\ \frac{1}{3} = \frac{A}{2} + B \end{array}\right\} \Rightarrow \begin{array}{l} A = \frac{1}{3} \\ B = \frac{1}{6} \end{array}$$

$$\int_0^1 x f(x)\,dx \approx \frac{1}{3} f\left(\frac{1}{2}\right) + \frac{1}{6} f(1)$$

Зад. Да се намери квадратурна формула на Лобато с три възела за $[0,1]$ при $\mu(x) = 1 - x$

Реш.

$$\int_0^1 (1-x)f(x)dx \approx Af(0)+Bf(x_1)+Cf(1)$$

x_1 - корен на полином от Π_1 т.е. $p(x)=x-x_1$ със старши коефициент 1-ца, ортогонален на Π_0 с тегло $\mu(x)(x-0)(1-x)$

$$\int_0^1 (1-x)^2(x-0)\,p(x)\,1\,dx=0$$

$$\int_0^1 (x-2x^2+x^3)(x-x_1)\,dx=0$$

$$\int_0^1 x^2-xx_1+2x^3+2x^2x_1+x^4\,x^3x_1\,dx=0$$

$$\frac{x^3}{3}-\frac{x^2}{2}x_1-\frac{2x^4}{4}+\frac{2x^3}{3}x_1+\frac{x^5}{5}-\frac{x^4}{4}x_1\Big|_0^1=0$$

$20-30x_1-30+40x_1+12-15x_1=0 \Rightarrow 5x_1=2$ т.е.

$$x_1=\frac{2}{5}\in(0,1)$$

$$\int_0^1 (1-x)f(x)dx \approx Af(0)+Bf\left(\tfrac{2}{5}\right)+Cf(1)$$

точна за $f(x)=1$, $f(x)=x$, $f(x)=x^2$

$$\left|\begin{array}{l}\int_0^1 (1-x)dx=A+B+C \Rightarrow A+B+C=\frac{1}{2}\\ \int_0^1 (1-x)x\,dx=A.0+B\frac{2}{5}+C.1 \Rightarrow \frac{2B}{5}+C=\frac{1}{6}\\ \int_0^1 (1-x)x^2dx=A.0^2+B\frac{4}{25}+C.1^2 \Rightarrow B=\frac{25}{72}\ \left(\text{от } \frac{4B}{25}+C=\frac{1}{12}\right)\end{array}\right.$$

$$\frac{1}{12} = \left(\frac{2}{5} - \frac{4}{25}\right)B \Rightarrow \frac{1}{2} = \frac{6}{25}B \Rightarrow B = \frac{25}{72}$$

$$C = \frac{1}{16} - \frac{2B}{25} = \frac{1}{36}, \quad A = \frac{1}{2} - B - C = \frac{1}{8}$$

$$\int_0^1 (1-x)f(x)\,dx \approx \frac{1}{8}f(0) + \frac{25}{72}f\left(\frac{2}{5}\right) + \frac{1}{36}f(1)$$

Зад. Да се намери квадратурна формула на Лобато с три възела за $[0,1]$ при $\mu(x)=1$

Реш.

$$\int_0^1 f(x)\,dx \approx Af(0) + Bf(x_1) + Cf(1)$$

x_1: $p(x) = x - x_1 \perp \Pi_0$ с тегло $\mu(x) = x(1-x) \Rightarrow$

$$\int_0^1 x(1-x)(x-x_1)\cdot 1\,dx = 0$$

$$\int_0^1 x^2 - x^3 - x x_1 + x^2 x_1\,dx = 0 \Rightarrow \frac{x^3}{3} - \frac{x^2}{2}x_1 - \frac{x^4}{4} + \frac{x^3}{3}x_1 \Big|_0^1 = 0$$

$$\frac{1}{3} - \frac{x_1}{2} - \frac{1}{4} + \frac{x_1}{3} = 0 \Rightarrow x_1 = \frac{1}{2} \in (0,1)$$

$$\int_0^1 f(x)\,dx \approx Af(0) + Bf\left(\frac{1}{2}\right) + Cf(1)$$

точна за полиноми $\in \Pi_2$

$$\begin{cases} \int_0^1 dx = A+B+C \Rightarrow A+B+C = 1 \\ \int_0^1 x\,dx = A.0 + B\frac{1}{2} + C1 \Rightarrow \frac{1}{2} = C + \frac{B}{2} \\ \int_0^1 x^2\,dx = A0^2 + B\frac{1}{4} + C1^2 \Rightarrow C + \frac{B}{4} = \frac{1}{3} \end{cases}$$

$$\frac{1}{6} = \frac{B}{4} \Rightarrow B = \frac{2}{3}$$

$$\frac{1}{2} - \frac{1}{3} = C = \frac{1}{6} \Rightarrow A = \frac{1}{6}$$

$$\int_0^1 f(x)\,dx \approx \frac{1}{6} f(0) + \frac{2}{3} f\left(\frac{1}{2}\right) + \frac{1}{6} f(1)$$