ЧМ 19.03.14

$$f[x_0 \dots x_k] = \begin{cases} \dfrac{f[x_1 \dots x_k] - f[x_0 \dots x_{k-1}]}{x_k - x_0} & \text{при } x_k \neq x_0 \\ \dfrac{f^{(k)}_{x_0}}{k!} & x_k = x_0 \end{cases}$$

Зад 11

x_i	-1	0	1
$f(x_i)$	1	0	1
$f'(x_i)$	0	*	4
$f''(x_i)$	*	*	16

=>

x_i	$f(x_i)$	$f[,]$	$f[,,]$	$f[,,,]$	$f[,,,,]$	$f[,,,,,]$
-1	1	$\frac{f'(-1)}{1!}$ 0	-1	1	0	1
-1	1	-1	1	1	2	
0	0	1	3	$\frac{8-3}{1}=5$		
1	1	$\frac{f'(1)}{1!}$	$\frac{f''(1)}{2!}=8$			
1	1	4				
1	1					

$$P(x) = 1 + 0(x+1) + (-1)(x+1)^2 + 1(x+1)^2 x + 0(x+1)^2 x(x-1)$$
$$+ 1(x+1)^2 x(x-1)^2$$

Зад 12 $T_n(x) = \cos(n \arccos x)$, $y_k = \cos\frac{(2k-1)\pi}{2n}$, $k = 1 \ldots n$

$\sum_{k=1}^{n} y_k = ?$ $\prod_{k=1}^{n} y_k = ?$

Реш

$T_n(x) = 2^{n-1}x^n + a_1 x^{n-1} + \ldots + a_n$; $a_0 = 2^{n-1}$

$y_1 + y_2 + \ldots + y_n = \frac{-a_1}{a_0}$; $y_1 y_2 \ldots y_n = \frac{a_n}{a_0}(-1)^n$

$a_n = T_n(0) = \cos(n \arccos 0) = \cos\frac{n\pi}{2} \Rightarrow \prod_{k=1}^{n} y_k = \frac{(-1)^n \cos\frac{n\pi}{2}}{2^{n-1}}$

$T_n(x) = T_{n-1}(x) \overset{?}{-} T_{n-2}(x)$

$y_1 + y_2 + \ldots + y_n = \frac{-a_1}{a_0} = 0$ защото $T_n(x)$ е четен или нечетен в зав от n

Зад $L_n(f, x) = \sum_{k=0}^{n} \frac{f(x_k)\,\omega(x)}{(x - x_k)\,\omega'(x_k)}$

x_i	0	1	2	3
$f(x_i) = y$	1	2	3	4

$\omega(x) = x(x-1)(x-2)(x-3)$

$\omega'(0) = (0-1)(0-2)(0-3) = -6$

$\omega'(1) = (1-2)(1-3)1 = 2$

$\omega'(2) = 2(2-1)(2-3) = -2$

$\omega'(3) = 3(3-1)(3-2) = 6 \Rightarrow$

$\Rightarrow f(x) = x + 1$

$f(0)\frac{(x-1)(x-2)(x-3)}{-6} + f(1)\frac{x(x-2)(x-3)}{2} + \ldots + f(3)\frac{x(x-1)(x-2)}{6} =$

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Зад 3 $\sum_{k=0}^{n}{}' l_k(x) = 1$

Лагранж

$\sum_{k=0}^{n}{}' f(x_k)\, l_k(x) = L_n(f; x) \Rightarrow f(x) = 1 \in \Pi_0 \subseteq \Pi_n \Rightarrow$

$f \equiv L_n \Rightarrow 1 = \sum_{k=0}^{n}{}' l_k$

Зад 4 $f[x_0 \ldots x_n] =$ коеф пред x^n в $L_n(f, x)$

$L_n(f, x_i) = f(x_i)$ за $i = 1 \ldots n$

$L_n(f, x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$

$$\left|\begin{array}{l} a_0 x_0^n + \ldots + a_n = f(x_0) \\ \vdots \\ a_0 x_n^n + \ldots + a_n = f(x_n) \end{array}\right\} \Rightarrow a_0 = \frac{\Delta_0}{\Delta} =$$

$$= \det \begin{vmatrix} f(x_0) \ldots x_0^{n-1} & \ldots & 1 \\ \vdots & & 1 \\ f(x_n) \ldots x_n^{n-1} & \ldots & 1 \end{vmatrix} \Big/ \begin{vmatrix} x_0^n & \ldots & 1 \\ \vdots & & \vdots \\ x_n^n & \ldots & 1 \end{vmatrix}$$

Зад 10 $f(x) = \int_0^x \operatorname{arctg} t\, dt = t \operatorname{arctg} t \Big|_0^x - \int_0^x t\, d\operatorname{arctg} t =$

$x \operatorname{arctg} t - \int_0^x \frac{t}{1+t^2}\, dt = \underline{x \operatorname{arctg} x} + \frac{1}{2} \ln(1+t^2) \Big|_0^x$

$= x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) \Rightarrow P(0) = 0 = f(0)$

$P'(x) = f'(x) = \operatorname{arctg} x \Rightarrow P'(0) = \operatorname{arctg} 0 = 0$

$f''(x) = \frac{1}{1+x^2} = f''(0) = 1 = P''(0)$ т.е.

(3)

x	0	1
$f(x_i)$	0	$0{,}439$
$f'(x_i)$	0	$*$
$f''(x_i)$	1	$*$

	$f[x_i]$	$f[x_1, x_2]$
0	0	
0	0	
0	0	
1	$0{,}439$	

За грешка! – $\left|f(x) - L_n(f,x)\right| = \max \left|\frac{f^{(n+1)}(x)}{n!}\right| |\omega(x)|$

Лема на Поповичу – Стефансон

$$(f.g)[x_0 \dots x_n] = \sum_{k=0}^{n} f[x_0 \dots x_k]\, g[x_k \dots x_n]$$

Зад. Да се намери $\frac{1}{x}[x_0 \dots x_n] = ?$

Реш.

$$\frac{1}{x} = f(x), \quad g(x) = x \Rightarrow f(x)g(x) = \frac{1}{x}x = 1 \in \pi_0 \subseteq \pi_n$$

$1[x_0 \dots x_n] = 0$ за $\forall\, n = 1 \dots n$

$$\underbrace{(f.g)[x_0 \dots x_n]}_{0} = \sum_{k=0}^{n} g[x_0 \dots x_k]\, f[x_k \dots x_n]$$

$$\sum_{k=0}^{n} x[x_0 \dots x_k]\, \frac{1}{x}[x_k \dots x_n] = 0 = x[x_0]\, \frac{1}{x}[x_0 \dots x_n] +$$

$$+\, x[x_0, x_1]\, \frac{1}{x}[x_1 \dots x_n] = 0$$

$$x_0 \underbrace{\frac{1}{x}[x_0 \dots x_n]}_{\Downarrow} + 1\, \frac{1}{x}[x_1 \dots x_n] = 0$$

$$\frac{1}{x}[x_0 \dots x_n] = -\frac{1}{x_0}\frac{1}{x}[x_1 \dots x_n] = \dots = \frac{(-1)^n}{x_0 x_1 \dots x_n}$$

Зад 6 $\frac{1}{x+a}$ - Реш аналогично с горната но умн с $(x+a)=g(x)$

Крайни разлики и полином на Нютон

$x_k = x_0 + kh$, $k=\overline{0,n}$, h - стъпка

$h = x_{i+1} - x_i$

$\Delta^n f_0 = \Delta^{n-1} f_1 - \Delta^{n-1} f_0$

$f_i = f(x_i) = f(x_0+ih)$

$\Delta f_0 = f_1 - f_0$, $\Delta f_i = f_{i+1} - f_i$

$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = f_2 - f_1 - (f_1 - f_0) = f_2 - 2f_1 + f_0$

$$\Delta^n f_0 = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} f_k$$

x_i	$f(x_i)$	Δf_i	Нютон назад
x_0	f_0	f_1-f_0	$f_2-2f_1+f_0$
x_1	f_1	f_2-f_1	
$\vdots$	$\vdots$	$\vdots$	
x_n	f_n	f_n-f_{n-1}	Нютон напред

$$\boxed{f[x_0 \ldots x_n] = \frac{\Delta^n f_0}{n!\, h^n}}$$

Ф-ла на Нютон

$$L_n(f,x) = \sum_{k=0}^{n} f[x_0 \ldots x_k](x-x_0)\ldots(x-x_{k-1}) =$$

$$= \sum_{k=0}^{n} \binom{t+k-1}{k} \Delta^k f_{n-k} = \sum_{k=0}^{n} \binom{t}{k} \Delta^k f_0$$

Зад ДСД а) $\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^j = 0,\ j = 0 \dots n-1$

б) $\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^n = n!$

Док а) $\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^j = \Delta^n f_0$ за $f_k = f(k) = k^j$

$\Rightarrow f(x) = x^j \qquad j = 0 \dots n-1$

$x_k = k \qquad k = 0 \dots n \qquad h = 1$

$f(x) \in \Pi_{n-1} \subseteq \Pi_n \Rightarrow f(x) = L_n(f, x) \Rightarrow$

коеф пред x^n в $L_n(f, x) = 0 \Rightarrow f[x_0 \dots x_n] = 0 \Rightarrow$

$\Delta^n f_0 = h^n n! f[x_0 \dots x_n] = 1^n n! \, 0 = 0$

б) $\Delta^n f_0 = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^n$ за $f_j = j^n \Rightarrow$

$f(x) = x^n \in \Pi_n$ със старши коеф $1 \Rightarrow$

$f[x_0 \dots x_n] = 1 \Rightarrow \Delta^n f_0 = h^n n! \, 1 = n!$

Зад ДСД $\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{m+j}{k} = 0,\ m \in \mathbb{N}$

Док $\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{m+j}{k} = \Delta^n f_0$ за $f_j = \binom{m+j}{k}$

$x_j = j,\ j = 0 \div n,\ h = 1,\ f_j = \binom{m+j}{k} \Rightarrow$

$f(x) = \binom{m+x}{k} = \dfrac{(m+x)(m+x-1) \dots (m+x-k+1)}{k!}$

$f(x) \in \Pi_k \subseteq \Pi_{n-1} \Rightarrow \Delta^n f_0 = 0$ т. к. коеф пред x^n е 0 (от предишната задача