ЧМ 07.05.14

2 зад. По метода на най-малките квадрати намерете правата $P(x) \in \Pi_1$, която най-добре приближава дадените таблично данни

x_i	0	1	2	3	4
y_i	1	2	1	0	4

Реш.

$P(x) = Ax + B$

$S(A,B) = \sum_{i=1}^{5} (y_i - Ax_i - B)^2 \to \min$

НАУ $\frac{\partial S}{\partial A} = \frac{\partial S}{\partial B} = 0$

$$\begin{cases} \frac{\partial S}{\partial A} = 2\sum_{i=1}^{5} (y_i - Ax_i - B)(-x_i) = 0 \ / \cdot (-2) \\ \frac{\partial S}{\partial B} = 2\sum_{i=1}^{5} (y_i - Ax_i - B)(-1) = 0 \ / \cdot (-2) \end{cases}$$

$$\begin{cases} A\sum_{i=1}^{5} x_i^2 + B\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} x_i y_i \\ A\sum_{i=1}^{5} x_i + 5B = \sum_{i=1}^{5} y_i \end{cases}$$

$\sum_i^5 x_i^2 = 30$, $\sum_i^5 x_i = 10$, $\sum_i^5 x_i y_i = 20$, $\sum_i^5 y_i = 8$

$$\begin{cases} 30A + 10B = 20 \\ 10A + 5B = 8 \end{cases} \Rightarrow B = 2 - 3A$$

$10A + 5(2 - 3A) = 8 \Rightarrow A = \frac{2}{5}$, $B = \frac{4}{5}$ $\Rightarrow$

$P(x) = \frac{2x + 4}{5}$

Зад. По метода на най-малките квадрати намерете права $P(x) \in \Pi_1$, която да приближава най-добре дадените данни

x_i	-1	0	1	2
y_i	2	1	0	-1
w_i	1	1	2	1

- тегло

Реш.

$$P(x) = Ax + B$$

$$S(A,B) = \sum_{i=1}^{4} w_i (y_i - Ax_i - B)^2 \to \min$$

$$\frac{\partial S}{\partial A} = \frac{\partial S}{\partial B} = 0$$

$$\left| \begin{array}{l} 2\sum_{i=1}^{4} w_i (y_i - Ax_i - B)(-x_i) = 0 \;/\cdot(-2) \\ 2\sum_{i=1}^{4} w_i (y_i - Ax_i - B)(-1) = 0 \;/\cdot(-2) \end{array} \right.$$

$$\left| \begin{array}{l} A\sum_{i=1}^{4} w_i x_i^2 + B\sum_{i=1}^{4} w_i x_i = \sum_{i=1}^{4} w_i x_i y_i \\ A\sum_{i=1}^{4} w_i x_i + B\sum_{i=1}^{4} w_i = \sum_{i=1}^{4} w_i y_i \end{array} \right.$$

$\sum_i w_i x_i^2 = 7$, $\sum_i w_i x_i = 3$, $\sum_i w_i x_i y_i = -4$, $\sum_i w_i y_i = 2$, $\sum_i w_i = 5$

$$\left| \begin{array}{l} 7A + 3B = -4 \;/\cdot(-5) \\ 3A + 5B = 2 \;/\cdot 3 \end{array} \right. \Rightarrow$$

$$-26A = 26 \Rightarrow A = -1 \Rightarrow B = 1$$

$$P(x) = -x + 1$$

Метод на най-малките квадрати за решаване на предопределена система

Зад. По метода най-малките квадрати да се реши системата

а) $\begin{cases} x - y = 1 \\ x + y = 1 \\ x + y = -1 \\ x - y = -1 \end{cases}$

б) $\begin{cases} x + y + z = 1 \\ x + 2y + z = 2 \\ x + 3y + z = 3 \\ x - 4y + z = 4 \\ x + 5y + z = 4 \end{cases}$

а) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$

$\underbrace{A^t A}_{2\times 2} X = \underbrace{A^t B}_{2\times 1}$ $\quad X = \begin{pmatrix} x \\ y \end{pmatrix}$

$A^t = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$

$A^t A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$

$A^t B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow A^t A X = A^t B$

$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$4x = 0 \Rightarrow x = 0$, $4y = 0 \Rightarrow y = 0$

б) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & -4 & 1 \\ 1 & 5 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 4 \end{pmatrix}$

$$A^t A X = A^t B$$

$$A^t A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & -4 & 1 \\ 1 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 5 \\ 7 & 35 & 7 \\ 5 & 7 & 5 \end{pmatrix}$$

$$A^t B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 18 \\ 14 \end{pmatrix}$$

$$\begin{cases} 5x + 7y + 5z = 14 \\ 7x + 35y + 7z = 18 \\ \sout{5x + 7y + 5z = 14} \end{cases}$$

ФСР – фундаментална сист. решения

Полагаме

$$\begin{cases} 5x + 7y = 14 - 5z & / \cdot 7 \\ 7x + 35y = 18 - 7z & / \cdot (-5) \end{cases}$$

$49y - 175y = 98 - 90$, т.е.

$-126y = 8 \Rightarrow y = -\frac{4}{63} \Rightarrow 5x + 7\left(-\frac{4}{63}\right) = 14 - 5z$

$5x - \frac{4}{9} = 14 - 5z \Rightarrow x = \frac{130}{45} - z = \frac{26}{9} - z$

1 от решенията е

$$(x\ y\ z) = \left(\frac{26}{9} - z,\ -\frac{4}{63},\ z\right)$$

Зад. Да се намери ПНДСКП $\in \Pi_2$ в $[-1,1]$ за

$$f(x) = 5x^4 \quad , \mu(x) = 1$$

Реш.

$$P(x) = Ax^2 + Bx + C \in \Pi_2 \text{ - ПНДСКП}$$

$$S(ABC) = \int_{-1}^{1} \mu(x)\left(f(x) - P(x)\right)^2 dx \to \min$$

$$S(ABC) = \int_{-1}^{1} 1 \left(5x^4 - Ax^2 - Bx - C\right)^2 dx \to \min$$

$f(x)$ - четна $\quad x \in [-1,1]$ - симетрично

$\mu(x)$ - четна $\Rightarrow P(x)$ - четна т.е. $B = 0$

$$P(x) = Ax^2 + C$$

$$S(AC) = \int_{-1}^{1} \left(5x^4 - Ax^2 - C\right)^2 dx \to \min$$

$$\frac{\partial S}{\partial A} = \frac{\partial S}{\partial C} = 0$$

$$\left|\begin{array}{l} \frac{\partial S}{\partial A} = 2\int_{-1}^{1} (5x^4 - Ax^2 - C)(-x^2)\,dx = 0 \quad / : (-2) \\ \frac{\partial S}{\partial C} = 2\int_{-1}^{1} (5x^4 - Ax^2 - C)(-1)\,dx = 0 \quad / : (-2) \end{array}\right\} \Rightarrow$$

$$\left|\begin{array}{l} 2\int_{0}^{1} 5x^6 - Ax^4 - Cx^2\,dx = 0 \quad / : 2 \\ 2\int_{0}^{1} 5x^4 - Ax^2 - C\,dx = 0 \quad / : 2 \end{array}\right.$$

$$\left| \begin{array}{l} \frac{5x^7}{7} - \frac{Ax^5}{5} - \frac{Cx^3}{3} \Big|_0^1 = 0 \\ \frac{5x^5}{5} - \frac{Ax^3}{3} - Cx \Big|_0^1 = 0 \end{array} \right.$$

$$\left| \begin{array}{l} \frac{5}{7} - \frac{A}{5} - \frac{C}{3} = 0 \\ 1 - \frac{A}{3} - C = 0 \Rightarrow C = 1 - \frac{A}{3} \Rightarrow \end{array} \right.$$

$$A = \frac{30}{7} \qquad C = -\frac{3}{7}$$

$$P(x) = \frac{30x^2 - 3}{7}$$

Зад. ДСН ПНОСКП $\in \Pi_1$ за $f(x) = e^x$
$x \in [0,1]$, $M(x) = 1$

Реш.

$$P(x) = ax + b$$

$$S(a,b) = \int_0^1 M(x)\left(f(x) - P(x)\right)^2 dx \to \min$$

$$\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = 0$$

$$\int_0^1 M(x)\left(e^x - ax - b\right)^2 dx \to \min$$

$$\int_0^1 \left(e^x - ax - b\right)^2 dx \to \min$$

$$\frac{\partial S}{\partial a} = 2\int_0^1 (e^x - ax - b)(-x)\,dx = 0 \;/\cdot(-2)$$

$$\frac{\partial S}{\partial b} = 2\int_0^1 (e^x - ax - b)(-1)\,dx = 0 \;/\cdot(-2)$$

$$\begin{cases} \int_0^1 xe^x - ax^2 - bx\,dx = 0 \\ \int_0^1 e^x - ax - b\,dx = 0 \end{cases}$$

$$\int xe^x dx = \int x\,de^x =$$

$$xe^x - \int e^x dx = xe^x - e^x \Rightarrow$$

$$\begin{cases} xe^x - e^x - \frac{ax^3}{3} - \frac{bx^2}{2}\Big|_0^1 = 0 \\ e^x - \frac{ax^2}{2} - bx\Big|_0^1 = 0 \end{cases}$$

$$\begin{cases} e - e - \frac{a}{3} - \frac{b}{2} + 1 = 0 \Rightarrow 6 = 2a + 3b \\ e - \frac{a}{2} - b - 1 = 0 \Rightarrow 2e - a - 2b - 2 = 0 \end{cases}$$

$$a = 2e - 2b - 2$$

$$b = -4e + 10 \qquad a = 18 - 6e$$

$$P(x) = (18 - 6e)x + 10 - 4e$$