Partial Differential Equations (Week 3+4)
Cauchy-Kovalevskaya and Holmgren

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1 Introduction

Recall the Cauchy-Kovalevskaya warm-up exercise from last week. There you showed that the transport equation $\partial_t u + cu_x = 0$ admits an analytic solution in a neighborhood of $(0,0)$ provided data $h(x)$ which are *analytic* near $x = 0$ are prescribed at $t = 0$. We also saw that for non-analytic data the series did not have to converge to a solution. Finally, we observed that if data were prescribed on a characteristic curve, we could not determine all partial derivatives of the solution on the curve, which was a necessary requirement to construct the local power series.

This week we will look at this circle of ideas from a more general perspective. We will state and prove the Cauchy-Kovalevskaya (CK) theorem, which provides the general setting when analytic Cauchy data yield an analytic solution.

A beautiful application of the CK-theorem is Holmgren's uniqueness theorem, which – quite surprisingly – will allow us to make a uniqueness assertion for *linear* equations with analytic coefficients *in the smooth category.* A typical setting where this becomes useful is the following. Consider a region $\Omega = \Omega_1 \cup \Omega_2$ separated by a curve Γ. Suppose u satisfies a linear PDE $Pu = 0$ in Ω, and suppose also that you know that $u = 0$ in Ω_1. Does this mean that $u = 0$ in all of Ω? In the analytical class, this is the well-known unique-continuation principle. But this does not prevent *smooth* non-zero solutions to exist! You will see examples in the exercises.

2 The Cauchy-Kovalevskaya theorem

Recall the index notation of Laurent Schwartz:

$$\begin{aligned}
\alpha &= (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \\
\partial^\alpha &= (\partial_{x_1})^{\alpha_1} \ldots (\partial_{x_n})^{\alpha_n} \\
x^\alpha &= x_1^{\alpha_1} \ldots x_n^{\alpha_n} \\
\alpha! &= \alpha_1! \cdot \ldots \cdot \alpha_n! \\
|\alpha| &= \alpha_1 + \cdots \alpha_n
\end{aligned}$$

Hence the most general *linear* partial differential operator can be written

$$P_{lin} = \sum_{|\alpha|\leq m} a_\alpha (x)\, \partial^\alpha \, .$$

We first investigate the CK theorem in the simple setting where data are prescribed on $t = 0$ and where the PDE is already solved for the highest t derivative:

$$\begin{aligned} \partial_t^m u &= G\left(t, x, \partial_t^j \partial_x^\alpha u; j \leq m-1\right) \\ \partial_t^\nu u\,(0,x) &= g_\nu (x) \qquad \nu = 0,\ldots,m-1 \ \text{ near } x = 0 \end{aligned} \tag{1}$$

and with G and g_ν smooth functions. (Examples: $u_{tt} = \Delta u + u^3 (\partial_x u)^2$ or $u_t = \Delta u$.) To even construct a *formal* power series for the solution, we should be able to determine all partial derivatives:

Proposition 2.1. *If u is a smooth solution of (1), then all derivatives of $u\,(t,x)$ are determined at $(0,0)$.*

Proof. We can compute $\partial_x^\alpha \partial_t^\nu u\,(0,0) = \partial_x^\alpha g_\nu (0)$ from the data for $\nu = 0,1,\ldots,m-1$. For the higher t-derivatives, suppose that $k \geq m$ and that $\partial_t^\nu \partial_x^\alpha u\,(0,x)$ are known for $\nu \leq k-1$ and all α. Then, differentiating the PDE we have

$$\partial_t^k \partial_x^\alpha u = \partial_t^{k-m} \partial_x^\alpha \left[G\left(t, x, \partial_t^j \partial_x^\alpha u; j \leq m-1\right)\right]$$

After applying the chain rule, the function on the right hand side only involves terms with at most $k-1$ t-derivatives, which are known by the induction assumption. □

Hence the basic requirement that all partial derivatives are determined is met for (1). We also know that we will have to assume g_ν and G in (1) to be analytic functions to have a chance of finding an analytic solution.

The crucial issue is of course the convergence. The following exercise shows that there is an obstruction for the associated Taylor series to converge, i.e. that the class (1) is still "too general":

Exercise 2.2. *Consider the one-dimensional heat equation $u_t = u_{xx}$ with data $u\,(0,x) = \frac{1}{1+x^2}$ (analytic near $x = 0$) prescribed at $t = 0$. Show that while all partial derivatives are determined from the data, the associated Taylor series at the origin does not converge for any (t,x) with $t \neq 0$.*

It is likely that the problem is caused by the fact that the one-dimensional heat equation is actually not highest order in t. Hence we are lead to consider the class of PDEs

$$\begin{aligned} \partial_t^m u &= G\left(t, x, \partial_t^j \partial_x^\alpha u; j \leq m-1\ ,\ j + |\alpha| \leq m\right) \\ \partial_t^\nu u\,(0,x) &= g_\nu (x) \qquad \nu = 0,\ldots,m-1 \ \text{ near } x = \bar{x} \end{aligned} \tag{2}$$

with G and g_ν analytic functions.

Theorem 2.3. *Suppose g_j is real analytic on a neighborhood of $\bar{x} \in \mathbb{R}^d$ and that G is real analytic on a neighborhood of $(0, \bar{x}, \partial_x^\alpha g_j(\bar{x})\,;\ j \leq m-1, j+|\alpha| \leq m)$. Then, there exists a real analytic solution of (2) defined on a neighborhood of $(0, \bar{x}) \subset \mathbb{R}_t \times \mathbb{R}^d$. The solution is unique is the class of analytic solutions, i.e. two analytic solutions u and v of (2) defined on a neighborhood of $(0, \bar{x})$ have to agree.*

This is a basic version of the Cauchy-Kovalevskaya theorem for the class of PDEs given by (2) and with Cauchy data prescribed at $t = 0$. We will later generalize to fully non-linear equations and arbitrary initial data hypersurfaces.

The uniqueness part of Theorem 2.3 is easy to prove: By Proposition 2.1, two functions which satisfy (2) must have the same Taylor series at $(0, \bar{x})$ and hence must agree on an entire neighborhood if they are assumed to be analytic.

For the existence part we have to work harder, namely we need to establish that the Taylor series which we can compute formally, actually converges. In the following sections, I will give at least a sketch of the proof.

2.1 Preliminaries

Definition 2.4. *A function $f : \mathbb{R}^d \to \mathbb{R}$ is called real analytic near x_0 if there exists an $r > 0$ and constants f_α such that*

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha \qquad \text{for } |x - x_0| < r.$$

Note that for a real analytic function one has $f_\alpha = \frac{D^\alpha f(x_0)}{\alpha!}$.
The following example is key:

Example 2.5. *For $r > 0$ set*

$$f(x) = \frac{r}{r - (x_1 + \ldots + x_d)} \qquad \text{for } |x| < \frac{r}{\sqrt{d}}$$

Then

$$\begin{aligned} f(x) = \frac{1}{1 - \frac{x_1 + \ldots + x_d}{r}} &= \sum_{k=0}^\infty \left(\frac{x_1 + \ldots + x_d}{r} \right)^k = \sum_{k=0}^\infty \frac{1}{r^k} (x_1 + \ldots + x_d)^k \\ &= \sum_{k=0}^\infty \frac{1}{r^k} \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} x^\alpha = \sum_\alpha \frac{1}{r^{|\alpha|}} \frac{|\alpha|!}{\alpha!} x^\alpha \end{aligned} \tag{3}$$

where we have used the multinomial identity (cf. Exercise 1 below).

Definition 2.6. *Let $f = \sum_\alpha f_\alpha x^\alpha$ and $g = \sum_\alpha g_\alpha x^\alpha$ be two power series. We say that g majorizes f, written $g \gg f$, provided $g_\alpha \geq |f_\alpha|$ for all α.*

Lemma 2.7. *If $g \gg f$ and g converges for $|x| < r$, then f also converges for $|x| < r$.*

Proof. Exercise. □

Lemma 2.8. *Let $f = \sum_\alpha f_\alpha x^\alpha$ converge for $|x| < r$. Then, for s satisfying $0 < s\sqrt{d} < r$, the function f can be majorized by (4) for $|x| < \frac{s}{\sqrt{d}}$.*

Remark 2.9. *The main point of this Lemma is that there is a simple, explicit majorant which will be constructed in the proof.*

Proof. Let $0 < s\sqrt{d} < r$. Set $y = s(1, \ldots, 1)$ so that $|y| = s\sqrt{d} < r$. This implies that $\sum_\alpha f_\alpha y^\alpha$ converges which is only possible if $|f_\alpha y^\alpha| < C$ for all α and a uniform C. Hence $|f_\alpha| \leq \frac{C}{s^{|\alpha|}} \leq C\frac{|\alpha|!}{s^{|\alpha|}\alpha!}$. But on the other hand, by Example 2.5 above, we have

$$g(x) := \frac{Cs}{s - (x_1 + \ldots + x_d)} = \sum_\alpha C\frac{|\alpha|!}{s^{|\alpha|}\alpha!}x^\alpha \tag{4}$$

and so g indeed majorizes f for $|x| < \frac{s}{\sqrt{d}}$ as claimed. □

Finally, note that we can generalize all this to vector valued $\underline{f} = (f_1, \ldots, f_n)$ and $\underline{g} = (g_1, \ldots, g_n)$ with each component f_i, g_i given by a power series. We will say $\underline{g}$ majorizes $\underline{f}$ and write $\underline{g} \gg \underline{f}$ if $g_i \gg f_i$ holds for all i from 0 to n.

2.2 Sketch of the proof of the CK-theorem

To get the idea, we will do the proof for a second oder quasi-linear equation of the form (with $\alpha = (\alpha_1, ..., \alpha_d, \alpha_t)$, G_α, $\tilde{G}$ analytic)

$$\begin{aligned} \partial_t^2 u &= \sum_{|\alpha|=2, \alpha_t \leq 1} G_\alpha(t, x, u, \partial_t u, \partial_{x_1} u, ..., \partial_{x_n} u)\, \partial^\alpha u + \tilde{G}(t, x, u, \partial u) \\ u(x, 0) &= g_0(x) \\ \partial_t u(x, 0) &= g_1(x) \end{aligned} \tag{5}$$

and attempt to find a solution near $(0, 0)$ (note we can always reduce to $x = 0$ by a translation).

Step 0. Transformation to zero Cauchy data, formulation as first order system Let us subtract the (analytic) Cauchy data and state the PDE satisfied by $u - g_0(x) - tg_1(x)$. This allows us to wlog consider the problem (5) for vanishing Cauchy data (and, of course now different analytic functions G_α and $\tilde{G}$).

Next, we want to write the system (5) as a first order system. We set

$$\underline{u} = (u, u_{x_1}, \ldots, u_{x_d}, u_t)$$

to be the vector containing u and all its partial derivatives. It has $m = d + 2$ components and satisfies $\underline{u} = 0$ at $t = 0$ by the previous considerations. Considering the vector $\underline{u}_t$, it is clear that its components u_t^j for $j = 1, 2, \ldots, m-1$ are determined by the vectors $\left(\underline{u}_{x_i}\right)_{i=1}^d$ (and $\underline{u}$ itself). The missing component,

$\underline{u}_t^m = u_{tt}$ is determined by the same quantities in view of the PDE (5). Given the quasi-linear structure, we hence obtain the first order system

$$\underline{u}_t = \sum_{j=1}^{d} \underline{B}_j\left(\underline{u}, x\right) \underline{u}_{x_j} + \underline{c}\left(\underline{u}, x\right)$$
$$\underline{u} = 0 \qquad \text{for } |x| < r \text{ at } t = 0\,. \tag{6}$$

Here

$$\underline{B}_j : \mathbb{R}^m \times \mathbb{R}^d \to Mat\left(m \times m\right) \qquad \text{for } j = 1, \dots, d$$
$$\underline{c} : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^m \tag{7}$$

are a matrix with components $\underline{B}_j = \left(b_j^{nl}\right)$ and a vector with components $\underline{c} = \left(c^1, \dots, c^m\right)$, both depending on $\underline{u}$ and x.

Remark 2.10. *Without loss of generality we have assumed $\underline{B}_j$ and $\underline{c}$ not to depend on t. We can always achieve this, by adding an additional component u^{m+1} to the vector $\underline{u}$ and demand the equation $\partial_t u^{m+1} = 1$ (hence $u^{m+1} = t$).*

In components, our PDE reads

$$u_t^n = \sum_{j=1}^{d} \sum_{l=1}^{m} b_j^{nl}\left(\underline{u}, x\right) u_{x_j}^l + c^n\left(\underline{u}, x\right) \tag{8}$$

for $n = 1, \dots m$ with zero data imposed for each u^n.

Step 1: Compute the Taylor series.
We expect that we locally have

$$u^n\left(x, t\right) = \sum_{\alpha} u_\alpha^n \mathbf{x}^\alpha = \sum_{\alpha} \left[\frac{\partial^\alpha u^n}{\alpha!}\left(0, 0\right)\right] x^{\alpha'} t^{\alpha_t} \tag{9}$$

with $\mathbf{x} = (x, t)$ and $\alpha = (\alpha', \alpha_t) = (\alpha_1, ..., \alpha_d, \alpha_t)$ and that this power series converges. In view of the analyticity assumption on $\underline{B}_j$ and $\underline{c}$, we have that

$$\underline{B}_j\left(z, x\right) = \sum_{\gamma,\delta} \underline{B}_{j,\gamma,\delta} z^\gamma x^\delta \qquad \text{for } j = 1, \dots, d$$
$$\underline{c}\left(z, x\right) = \sum_{\gamma,\delta} \underline{c}_{\gamma,\delta} z^\gamma x^\delta \tag{10}$$

are convergent power series for $|z| + |x| < s$ for some small s where

$$\underline{B}_{j,\gamma,\delta} = \frac{\partial_z^\gamma \partial_x^\delta \underline{B}_j}{(\gamma+\delta)!}\left(0, 0\right) \qquad , \qquad \underline{c}_{\gamma,\delta} = \frac{\partial_z^\gamma \partial_x^\delta \underline{c}}{(\gamma+\delta)!}\left(0, 0\right)\,. \tag{11}$$

To compute the $u_\alpha^n = \frac{\partial^\alpha u^n}{\alpha!}(0, 0)$ we note that for α of the form $\alpha = (\alpha', 0)$ we have $u_\alpha^n = 0$ for all $n = 1, ..., m$ (why?). For α of the form $(\alpha', 1)$ we have, from equation (8), $u^n_\alpha = \partial^{\alpha'} c^n_\alpha(0,0)$ (why?). For $\alpha = (\alpha', 2)$ we obtain (differentiating (8), using the chain rule and the fact that we evaluate at $(0,0)$) that $u^n_\alpha = \partial^{\alpha'} \left(\sum_{j=1}^d \underline{B}_j(\underline{u}, x)\, \underline{u}_{x_j t} + \sum_{j=1}^m c^n_{z_j} \underline{u}^j_t \right) \Big|_{(\underline{u},x)=(0,0)}$. Pushing the derivative through and evaluating at $(0,0)$ it is clear that for any fixed α' we are going to see an expression polynomial in the (components of finitely many) Taylor "coefficients" $\underline{B}_{j,\gamma,\delta}$, $\underline{c}_{\gamma,\delta}$ and the Taylor "coefficients" $\underline{u}_\beta$ where $\beta = (\beta', \beta_t)$ with $\beta_t \leq 1$ and $|\beta'| \leq |\alpha'| + 1$. Moreover, the polynomial has only positive integer coefficients, as this is all the chain and product rule can produce.

Continuing in this way it is easy to see that for general $\alpha = (\alpha', \alpha_t)$ one will obtain

$$u^k_\alpha = q^k_\alpha\left(..., \underline{B}_{j,\gamma,\delta}, ..., \underline{c}_{\gamma,\delta}, ..., \underline{u}_\beta, ...\right) \tag{12}$$

where q^k_α is a polynomial with *non-negative* coefficients and $\beta_t \leq \alpha_t - 1$, $|\beta'| \leq |\alpha'| + 1$.

Step 2: Use the method of majorants to establish convergence.
Now that we have computed the formal Taylor series (i.e. the u^k_α), we would like to show that it converges for $|x| + |t| < r$ with r sufficiently small. Suppose that we had majorizing $\underline{B}^\star_j \gg \underline{B}_j$ and $\underline{c}^\star \gg \underline{c}$, i.e.

$$0 \leq |\underline{B}_{j,\gamma,\delta}| \leq \underline{B}^\star_{j,\gamma,\delta} \qquad , \qquad 0 \leq |\underline{c}_{\gamma,\delta}| \leq \underline{c}^\star_{\gamma,\delta} \tag{13}$$

(in the sense that it holds for all components and $j = 1, ..., d$) for power series

$$\underline{B}^\star_j = \sum_{\gamma,\delta} \underline{B}^\star_{j,\gamma,\delta} z^\gamma x^\delta \qquad \underline{c}^\star = \sum_{\gamma,\delta} \underline{c}^\star_{\gamma,\delta} z^\gamma x^\delta \,. \tag{14}$$

Given these majorants, we can consider the PDE system

$$\begin{aligned} &\underline{u}^\star_t = \sum_{j=1}^d \underline{B}^\star_j(\underline{u}^\star, x)\, \underline{u}^\star_{x_j} + \underline{c}^\star(\underline{u}^\star, x) \qquad \text{for } |x| + |t| < r \\ &\underline{u}^\star = 0 \qquad \text{for } |x| < r \text{ on } t = 0\,. \end{aligned} \tag{15}$$

If we can find a convergent power series (for sufficiently small r) for $\underline{u}^\star$ which solves (15), then we have in particular that $\underline{u}^\star \gg \underline{u}$. This follows from

$$\begin{aligned} |u^k_\alpha| = |q^k_\alpha\left(..., \underline{B}_{j,\gamma,\delta}, ..., \underline{c}_{\gamma,\delta}, ..., \underline{u}_\beta, ...\right)| &\leq q^k_\alpha\left(..., |\underline{B}_{j,\gamma,\delta}|, ..., |\underline{c}_{\gamma,\delta}|, ..., |\underline{u}_\beta|, ...\right) \\ &\leq q^k_\alpha\left(..., \underline{B}^\star_{j,\gamma,\delta}, ..., \underline{c}^\star_{\gamma,\delta}, ..., \underline{u}^\star_\beta, ...\right) = \left(u^k_\alpha\right)^\star\,. \end{aligned}$$

But this implies (cf. Lemma 2.7) that our power series for $\underline{u}$ also converges. Therefore, once we have proven the existence of a majorizing $\underline{u}^\star$, we are done: Indeed, the power series of the left hand side and the right hand side of the PDE (6) agree by construction at $(0,0)$ and they also converge. Hence the left hand side and the right hand side of the PDE have to agree on a neighborhood of $(0,0)$.

The missing bit is to find the majorizing $\underline{u}^\star$. Note that by Lemma 2.8 we have a simple majorant for both $\underline{B}_j$ and $\underline{c}$: The expressions

$$\underline{B}_j^\star = \frac{Cr}{r - (x_1 + \dots x_d) - (z_1 + \dots z_m)} \begin{pmatrix} 1 & 1 & \dots \\ 1 & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{16}$$

$$\underline{c}^\star = \frac{Cr}{r - (x_1 + \dots x_d) - (z_1 + \dots z_m)} (1, \dots, 1) \tag{17}$$

will majorize $\underline{B}_j$ and $\underline{c}$ respectively in $|x| + |z| < r$ provided C is chosen sufficiently large and r sufficiently small.

The point is that we can solve explicitly the resulting PDE (15) for $\underline{u}^\star$. Indeed, one easily sees that each component satisfies

$$(u^\star)^k_t = \frac{Cr}{r - (x_1 + \dots x_d) - ((u^\star)^1 + \dots (u^\star)^m)} \left(\sum_{j,l} (\underline{u}^\star)^l_{x_j} + 1 \right), \tag{18}$$

the right hand side being independent of the particular components k considered. This suggests setting $u^\star(x,t) = v^\star(1, \dots, 1)$ with $v^\star$ satisfying

$$v^\star_t = \frac{Cr}{r - (x_1 + \dots x_d) - m \cdot v^\star} \left[m \sum_{i=1}^d v^\star_{x_i} + 1 \right]. \tag{19}$$

The ansatz $v^\star(t, x_1, \dots x_d) = v^\star(t, s = x_1 + \dots + x_d)$, i.e. $v^\star$ a function of t and the sum of the x_i only, leads to

$$v^\star_t = \frac{Cr}{r - s - m \cdot v^\star} \left[md \cdot v^\star_s + 1 \right] \tag{20}$$

for a function $v^\star(t,s)$ with $v^\star(0,s) = 0$ initially. This can be solved using the method of characteristics (Exercise) to find

$$v^\star(x,t) = \frac{r - (x_1 + \dots x_d) - \sqrt{(r - (x_1 + \dots x_d))^2 - 2m(d+1) \cdot C \cdot r \cdot t}}{m(d+1)}. \tag{21}$$

This is analytic for $|x| + |t| < \tilde{r}$ provided $\tilde{r} \leq r$ is chosen sufficiently small.

2.3 CK-theorem for non-linear equations and data at $t = 0$

We would like to generalize our result to *fully non-linear equations* and *arbitrary surfaces*. If you recall our discussion of first order fully non-linear PDE, you will remember that it did not suffice to specify data along a curve Γ but that one has to specify the highest derivatives at *one point* of Γ, consistent with the PDE and the geometry of Γ. A "non-degeneracy" condition allowed one to apply the implicit function theorem to complete the data on Γ and to make the Cauchy problem well-posed.

Consider the fully non-linear PDE

$$F\left(t, x, \partial_t^m u, \partial_t^j \partial_x^\alpha u \ ; \ j \leq m-1, j+|\alpha| \leq m\right) = 0 \tag{22}$$

with prescribed data

$$\partial_t^j u\,(0,x) = g_j\,(x) \qquad 0 \leq j \leq m-1\,. \tag{23}$$

Then, we can locally solve (22) uniquely for $\partial_t^m u$ provided that we have a solution at a point and the non-degeneracy condition, i.e. if for some $\gamma \in \mathbb{R}$ the conditions

$$\begin{gathered} F\,(0, \bar{x}, \gamma, \ \partial_x^\alpha g_j\,(\bar{x})) = 0 \\ \frac{\partial}{\partial s} F\,(0, \bar{x}, s, \partial_x^\alpha g_j\,(\bar{x}))\Big|_{s=\gamma} \neq 0 \end{gathered} \tag{24}$$

hold, then we locally have

$$\partial_t^m u = G\left(t, x, \partial_t^j \partial_x^\alpha u \ ; \ j \leq m-1, j+|\alpha| \leq m\right)$$

with G real analytic if F is.[1] This reduces (22) to our previously studied case and we immediately obtain

Theorem 2.11. *Consider the non-linear PDE (22) with data (23) imposed at* $t=0$ *and with* F *real analytic near* $(0, \bar{x}, \gamma, \partial_x^\alpha g_j\,(\bar{x}))$ *and* g_j *analytic near* $\bar{x}$ *for some* $\gamma \in \mathbb{R}$ *such that also (24) holds. Then, there exists an analytic solution* u *of (22) realizing the given data at* $t=0$*. The solution is unique within the class of analytic solutions.*

Let us investigate the non-degeneracy condition (24) a bit more.

Definition 2.12. *Consider the PDE (22). We will say that the surface* $t=0$ *is non-characteristic at* $(0,x)$ *on the solution* u *of the PDE (22) provided that*

$$\frac{\partial F}{\partial\,(\partial_t^m u)}\left(0, x, \partial_t^m u\,(0,x)\,, \partial_t^j \partial_x^\alpha u\,(0,x)\right) \neq 0$$

holds at $t=0$*.*

Example 2.13. *For a* linear *partial differential operator*

$$F = \sum_{|\alpha|+j \leq m} a_{j,\alpha}\,(t,x)\,\partial_x^\alpha \partial_j^t u - f\,(t,x)$$

we need $a_{m,0}\,(0,x) \neq 0$*. For a linear operator the non-characteristic condition does not depend on the solution. No implicit function theorem is needed.*

[1] There is an analytic version of the implicit function theorem. See for instance, Exercise 6 of Section 3.3b) in Fritz John's book.

We can carry the idea of being non-characteristic further by relating it to the linearization of a PDE. Suppose we have a solution u of a PDE

$$F\left(x, \partial^\beta u\right) = 0 \tag{25}$$

Suppose further that we want to consider small perturbations of u, i.e. we think about

$$F\left(x, \partial^\beta \left(u + \epsilon \cdot v\right)\right) = 0 \tag{26}$$

Then, to linear order in ϵ we can obtain v by solving the linear equation

$$a_\alpha\left(x\right)\partial^\alpha v = 0 \qquad \text{where} \qquad a_\alpha\left(x\right) := \frac{\partial F}{\partial\left(\partial^\alpha u\right)}\left(x, \partial^\beta u\right) . \tag{27}$$

This suggests:

Definition 2.14. *If $F\left(x, \partial^\beta u \; ; \; |\beta| \leq m\right) = 0$, the linearization of F at u is the linear partial differential operator*

$$P\left(x, \partial\right) = \sum a_\alpha\left(x\right)\partial^\alpha \quad \textit{where} \quad a_\alpha\left(x\right) := \frac{\partial F}{\partial\left(\partial^\alpha u\right)}\left(x, \partial^\beta u\right) . \tag{28}$$

Exercise 2.15. *Show that the linearization of Burger's equation at a solution u is given by $v_t + uv_x + vu_x = 0$. Note that linearization at the constant solution $u = c$ yields the transport equation!*

Theorem 2.16. *The following are equivalent*

1. *$t = 0$ is non-characteristic at $(0, \bar{x})$ for the solution u to $F\left(x, \partial^\beta u \; ; \; |\beta| \leq m\right) = 0$.*

2. *$t = 0$ is non-characteristic at $(0, \bar{x})$ for the linearization of F at u.*

3. *For any smooth function $\psi\left(t, x\right)$ with $\psi\left(0, x\right) = 0$ for x near $\bar{x}$ and $\psi_t\left(0, \bar{x}\right) \neq 0$, the linearization $P\left(x, \partial\right)$ satisfies $P\left(\psi^m\right) \neq 0$ at $(0, \bar{x})$.*

Proof. By Definition 2.12, Statement (1) means $\frac{\partial F}{\partial(\partial^\alpha u)}\left(u\left(0, \bar{x}\right), \partial^\beta u\left(0, \bar{x}\right)\right) \neq 0$ for α being the multi-index $(0, 0, ..., m)$. Statement (2) means $\alpha_{(0,m)} \neq 0$ holds for the linearization with a_α defined in (28). The two expressions are equivalent by definition of the linearization. To show the equivalence of (2) and (3) note that

$$P\left(\psi^m\right)\Big|_{x=\bar{x}, t=0} = m! \cdot a_{(m,0)}\left(\psi_t\left(0, \bar{x}\right)\right)^m$$

is the only term surviving the evaluation at $(0, \bar{x})$. □

Observe that if (3) holds for one such ψ it holds for any such (Exercise). The point of including the statement (3) is that we will be able to give a *coordinate independent* formulation of it in the next section, which will be useful to define the notion of being non-characteristic for arbitrary surfaces.

2.4 The CK theorem for arbitrary (analytic) surfaces

We now tackle the most general case. We'll assume that Σ is an analytic hypersurface in $\mathbb{R}^{d+1}$ given locally as the graph of an analytic function.

The first problem we face is how to *formulate* the Cauchy problem. It would seem that we are free to describe all $m-1$ partial derivatives on Σ. Then all but one m^{th} derivatives will also be determined by differentiating along Σ, while the PDE should allow us to solve for the missing m^{th} order "normal" derivative.[2] However, when specifying the $m-1$ partial derivatives on Σ, there are clearly compatibility conditions between the various partial derivatives, and it is a-priori not clear what is "free" to be prescribed and what can be determined.

An easy way around this is to formulate the problem as follows. For $u : \mathbb{R}^{d+1} \supset \Omega \to \mathbb{R}$ consider the PDE

$$\begin{aligned} F\left(x, \partial^{\alpha} u \ ; \ |\alpha| \leq m\right) &= 0 \\ \partial^{\alpha} u = \partial^{\alpha} v \qquad & \text{on } \Sigma \text{ for all } |\alpha| \leq m-1 \end{aligned} \tag{29}$$

for a *given* smooth function v defined *on a neighborhood* of Σ.

For instance, for our old $t = const$ problem we could choose $v(t,x) = \sum_{j=0}^{m-1} \frac{1}{j!} t^j \cdot g_j(x)$.

The general idea will be to introduce an analytic diffeomorphism Φ mapping a neighborhood in $\mathbb{R}^{d+1}_x$ of $\bar{x} \in \Sigma$ to neighborhood of the origin in $\mathbb{R}^{d+1}_y$ such that $\Phi(U \cap \Sigma) \subset \{y_{d+1} = 0\} \subset \mathbb{R}^{d+1}_y$, i.e. such that Σ becomes $y_{d+1} = 0$ in the new coordinates. Then the problem reduces to the problem studied previously.

All we need to achieve this is a notion of non-characteristic which does not depend on any choice of coordinates and moreover reduces to the old definition in appropriate coordinates.

Definition 2.17. *If $F\left(\bar{x}, \partial^{\beta} u(\bar{x}) \ ; \ |\beta| \leq m\right) = 0$, then the hypersurface Σ is non-characteristic for F on u at $\bar{x}$, if for any real-valued C^{∞} function ψ defined on a neighborhood of $\bar{x}$ with*

$$\psi\Big|_{\Sigma} = 0 \qquad , \qquad d\psi\Big|_{\Sigma} \neq 0 \tag{30}$$

we have $P(\psi^m)(\bar{x}) \neq 0$.

Note that

- this definition is coordinate invariant (check!)
- it reduces to the old definition if Σ is given by $t = 0$
- if the condition holds for one ψ satisfying (30), it holds for any such ψ.

[2] Formulating the problem purely geometrically in terms of normal- and tangential derivatives is tricky, however, because it will generally involve taking normal derivatives of the normal, which requires (and depends) on the extension of the normal of the surface. See the discussion in Rauch's book.

Example 2.18. *Let $H(t,x,y,z) := t - \phi(x,y,z) = 0$ define a smooth hypersurface in $\mathbb{R}^4$. The differential of H is non-vanishing (it has components $(1, \nabla_x \phi)$ with ∇_x being the spatial gradient). For the linear wave equation, the hypersurface H is non-characteristic at points $p \in H$, for which*

$$\left(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right) H^2 \Big|_\Sigma = 2\left(1 - |\nabla\phi|^2\right) \neq 0$$

holds. Hence we recover the eikonal equation as the equation for characteristic surfaces of the wave equation. As an exercise, you may want to find all characteristic hyperplanes.

Exercise 2.19. *Repeat the previous example with the Laplace equation $\Delta u = 0$. Show that all surfaces are non-characteristic.*

Remark 2.20. *A linear PDE $Lu = 0$ is called elliptic if no hypersurfaces are characteristic for L.*

Exercise 2.21. *Some of you mentioned a "simple" criterion for ellipticity for equations of the form*

$$Pu = au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y + fu = 0\,, \tag{31}$$

with a, b, c, d, e, f function of x and y. Show that P is elliptic at (x,y) if

$$b^2 < ac$$

holds at (x,y). [Hint: Let $H(x,y) = y - f(x) = 0$ describe a curve in the xy-plane....]

With the above, we finally obtain the (so far) most general version of the Cauchy-Kovalevskaya theorem (see the book of Rauch).

Theorem 2.22. *Suppose that*

1. *$\bar{x} \in \Sigma \subset \mathbb{R}^{d+1}_x$ and Σ is a real analytic hypersurface*
2. *$v : \mathbb{R}^{d+1}_x \to \mathbb{R}$ is real analytic in a neighborhood of $\bar{x}$ and $F\left(x, \partial^\beta v(x)\right) = 0$ for x in Σ.*
3. *Σ is non-characteristic for F on v at $\bar{x}$.*
4. *F is real analytic on a neighborhood Ω of $\left(\bar{x}, \partial^\beta v(\bar{x})\right)$*

Then, there is a neighborhood Ω of $\bar{x}$ and a $u : \mathbb{R}^{d+1}_x \supset \Omega \to \mathbb{R}$, real analytic on Ω such that

- $F(x, \partial^\alpha u(x)) = 0$ *in* Ω
- $\partial^\alpha u\Big|_{\Sigma\cap\Omega} = \partial^\alpha v\Big|_{\Sigma\cap\Omega}$ *for all* $|\alpha| \leq m-1$
- $\partial^\alpha u(\bar{x}) = \partial^\alpha v(\bar{x})$ *for all* $|\alpha| = m$.

Moreover, the solution is unique in the class of analytic solutions.

3 Holmgren's uniqueness theorem

The setting of Holmgren's theorem is the following. You have an m^{th}-order, *linear* partial differential operator, whose coefficients are *analytic* functions:

$$P = \sum_{|\alpha| \le m} a_\alpha (x)\, \partial^\alpha . \tag{32}$$

You pose the Cauchy-problem on a *non-characteristic* analytic surface Σ:

$$\begin{aligned} Pu &= f(x) \\ \partial^\alpha u &= \partial^\alpha v \qquad \text{on } \Sigma \text{ for all } |\alpha| \le m-1 \end{aligned} \tag{33}$$

for v a given smooth function near Σ (the data). We know that if the data (and f) are analytic near a point of Σ, then there exists a unique analytic solution of (33) near that point. However, this does not prevent other (smooth) solutions to exist. Also, at the moment we cannot say anything about existence and uniqueness if the data are merely assumed to be smooth.

Holmgren's theorem makes a statement about the *uniqueness* of solutions to the problem (33) in the class of C^m functions. Suppose you have two classical (i.e. m times continuously differentiable) solutions u_1, u_2 of (33) near a point of Σ. Then their difference $u = u_1 - u_2$ satisfies the homogeneous PDE

$$\begin{aligned} Pu &= 0 \\ \partial^\alpha u &= 0 \qquad \text{on } \Sigma \text{ for all } |\alpha| \le m-1 \end{aligned} \tag{34}$$

Holmgren's theorem states that the only solution to the above which is m times continuously differentiable is identically zero and hence $u_1 = u_2$.

Theorem 3.1. *[Holmgren] Let $P(x, \partial)$ be a linear partial differential operator of order m whose coefficients are analytic in a neighborhood of a point $\bar{x} \in \mathbb{R}^d$. Let Σ be an analytic hypersurface which is non-characteristic at $\bar{x}$. Then the following statement holds. If u is a C^m solution of (34) in a neighborhood of $\bar{x}$, then u vanishes on a neighborhood of $\bar{x} \in \mathbb{R}^d$.*

You should convince yourself immediately why *non-characteristic* is an essential requirement, by revisiting the example of the transport equation, for instance. The necessity of the coefficients being analytic is harder to see (but true: there are linear equations with C^∞ coefficients exhibiting non-uniqueness). Finally, you will relax the assumption of the hypersurface being analytic (to being $C^2 \cap C^m$) in one of the Exercises below.

3.1 The idea of the proof and preliminaries

A priori it may seem strange that the CK Theorem (which is a statement in the analytic world) can be used to prove Holmgren's theorem. The idea is following. Let X, Y be normed linear spaces and $T : X \to Y$ be a continuous linear map. The transpose of T is defined as a map $T^t : Y' \to X'$ by

$$\langle T^t y', x \rangle = \langle y', Tx \rangle \qquad \text{for all } y' \in Y \text{ and all } x \in X.$$

Proposition 3.2. *If the range of T^t is dense in X', then* $\ker T = \{0\}$*, i.e.* $Tu = 0$ *implies* $u = 0$.

Proof. Assume $u \in X$ satisfies $Tu = 0$. Then $\langle y', Tu \rangle = 0$ for all y', which is equivalent to $\langle T^t y', u \rangle = 0$ for all y'. Since T^t has dense range we obtain $\langle x', u \rangle = 0$ for all $x' \in X'$. The Hahn-Banach theorem implies $u = 0$. □

Therefore, the strategy to show that $Pu = 0$ has trivial kernel, will be to solve the "transpose" equation $P^t v = g$ for a dense set of right hand sides. In particular, by the Weierstrass approximation theorem, it suffices to solve this for polynomial (=analytic!) g's and this is how the CK theorem will come into play.

To see how Proposition 3.2 is useful, we need to define a notion of the transpose of P. Let Ω be a connected, open set of $\mathbb{R}^n$ and let $u \in C^m(\Omega)$ satisfy $Pu = 0$ in Ω. Clearly, then, for any test-function $v \in C_0^\infty(\Omega)$ we have

$$\int_\Omega (Pu)\, v dx = 0$$

The idea is to pass all the derivatives onto the v. In view of the identity

$$\int_\Omega a_\alpha(x)\, \partial^\alpha u \cdot v = (-1)^{|\alpha|} \int_\Omega u \cdot \partial^\alpha (a_\alpha(x)\, v)$$

which holds for all α with $|\alpha| \leq m$, we obtain the identity

$$\int_\Omega Pu \cdot v dx = \int_\Omega P^t v \cdot u dx\,,$$

provided we define the transpose of P, denoted P^t, by

$$P^t(x, \partial)\, v := \sum_\alpha (-1)^{|\alpha|} \partial^\alpha (a_\alpha(x)\, v) \tag{35}$$

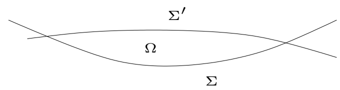
Note that if Σ is non-characteristic for P at x, then it is also non-characteristic for P^t at x. Now do the same computation with $u, v \in C^m(\bar{\Omega})$ and the boundary $\partial\Omega$ sufficiently regular such that Stokes' theorem holds to establish the identity:

$$\int_\Omega \left[Pu \cdot v - P^t v \cdot u \right] dx = \sum_{|\beta| + |\gamma| \leq m-1} a_{\beta\gamma}(x)\, \partial^\beta u \partial^\gamma v d\sigma\,. \tag{36}$$

Lemma 3.3. *Let $u, v \in C^m\left(\overline{\Omega}\right)$ and for each $x \in \partial\Omega$ and any γ with $|\gamma| \leq m-1$ either $\partial^\gamma u(x) = 0$ OR $\partial^\gamma v(x) = 0$ holds. Then*

$$\int_\Omega \left[Pu \cdot v - P^t v \cdot u \right] dx = 0\,. \tag{37}$$

Now the idea becomes more clear: Given the analytic, non-characteristic hypersurface Σ, we will find a hypersurface Σ' nearby, which is also non-characteristic and together with Σ encloses a “lens-shaped” region Ω.



We know that $Pu = 0$ in Ω and $\partial^\alpha u = 0$ on Σ for all $|\alpha| \leq m-1$ by assumption. Now, if we can solve the transpose problem $P^t v = g$ in Ω with $\partial^\alpha v = 0$ on Σ' for all $|\alpha| \leq m-1$ (such that, say, $v \in C^m\left(\bar{\Omega}\right)$), then for this g we have

$$\int_\Omega g \cdot u \; dx = 0\,.$$

If we could obtain such a solution v of the transpose problem in Ω for *any polynomial* (analytic!) g, we would be done because then we would approximate u *uniformly* in $\overline{\Omega}$ by polynomials g_n, $g_n \to u$, and use

$$0 = \lim_{n\to\infty} \int_\Omega g_n \cdot u \; dx = \int_\Omega |u|^2 dx$$

to conclude $u = 0$ in Ω.

3.2 The proof of Theorem 3.1

Choose analytic coordinates so that $\Sigma = \{x_1 = 0\}$, $\bar{x} = (0, 0, ..., 0)$. Introduce $(t, y = (y_1, ...y_d))$ coordinates via

$$t = x_1 + x_2^2 + \ldots x_d^2 \qquad y_2 = x_2, \ldots, y_d = x_d\,. \tag{38}$$

In (t, y) coordinates, $\Sigma = \{t = |y|^2\}$. For $\epsilon > 0$, define $\tilde{\Sigma}_\epsilon = \{t = \epsilon\}$ and $\omega_\epsilon = \{|y|^2 < t < \epsilon\}$.

Σ is non-characteristic for P at the origin, so with $P = \sum_{j+|\beta|\leq m} a_{j,\beta}(x)\, \partial_t^j \partial_y^\beta$ we have $a_{m,0}(0, \ldots 0) \neq 0$ (note ∂_t is normal to Σ_t at the origin). We choose $r_1 > 0$ such that $a_{m,0}(t, y) \neq 0$ holds for all (t, y) with $|t| + |y| \leq r_1$ and such that the coefficients of P are real analytic for $|t| + |y| < 2r_1$.

It follows that there are constants C, B such that the coefficients of the transpose P^t satisfy

$$\frac{|\partial_{t,y}^\alpha a_{j,\beta}|}{\alpha!} \leq C B^{|\alpha|} \qquad \text{in } |t| + |y| \leq r_1$$

see Exercise 1. We choose $\epsilon_0 > 0$ so that $\omega_{\epsilon_0} \subset \{|t| + |y| \leq r_1\}$. Note that with this choice, for any $\epsilon \in (0, \epsilon_0)$ the slice $\Sigma_\epsilon \cap \{t \geq |y|^2\}$ is non-characteristic and that we have uniform estimates for the coefficients in the associated set ω_ϵ depending only on B and C, which we regard now as having been fixed.

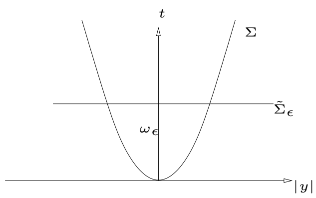
For any $t = \epsilon \leq \epsilon_0$ we consider the following Cauchy problem

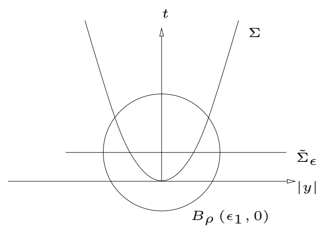
$$\begin{aligned} P^t v &= g\,(t, y) \\ \partial^\alpha v|_{t=\epsilon} &= 0 \quad \text{for all } |\alpha| \leq m-1 \end{aligned} \tag{39}$$

with $g\,(t, y)$ real analytic at $(\epsilon, 0)$. In particular, for any g we have an estimate

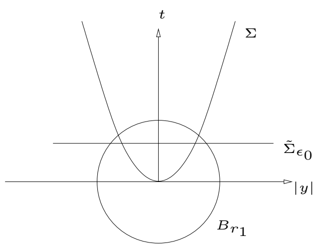
$$\frac{|\partial^\alpha_{t,y} g|}{\alpha!} \leq \underline{C}\,\underline{B}^{|\alpha|} \tag{40}$$

in a small ball around $(\epsilon, 0)$. By the CK-Theorem, (39) has a unique analytic solution around $(\epsilon, 0)$. The domain of convergence of this solution depends on B,C (which were *fixed* above) and on $\underline{C}$, $\underline{B}$. This is a consequence of the methods of majorants used in the proof of the CK theorem. We would like to make sure that – at least for g a polynomial – the domain of convergence does not depend on the particular g. To see this note first that (39) it *linear*. This implies that the domain of convergence for the solution does not change if we multiply g by a constant. This in turn means that the size of the domain of convergence does not depend on $\underline{C}$. On the other hand, for g a polynomial, the estimate (40) holds for *any* $\underline{B} < \infty$ (related to the fact that the radius of convergence in infinite). Therefore, we obtain a *uniform* domain of convergence for any $\epsilon \in [0, \epsilon_0]$. In other words, for any $\epsilon \in [0, \epsilon_0]$ and for polynomial g, there exists an analytic solution of (39) in $|t - \epsilon| + |y| < \rho$, with ρ *independent* of the polynomial g chosen.





$\omega_{\epsilon_1} \subset\subset \{|t - \epsilon_1| + |y| \leq \rho\}$



$\omega_{\epsilon_0} \subset \{|t| + |y| \leq r_1\}$

Finally, we want to choose $\epsilon_1\,(0, \epsilon_0]$ such that $\omega_{\epsilon_1} \subset\subset \{|t - \epsilon_1| + |y| \leq \rho\}$. Now we can apply Lemma 3.3 in ω_{ϵ_1}: We find a sequence of polynomials g_n

which approximates u uniformly on $\overline{\omega_{\epsilon_1}}$, apply the Lemma for each g_n and take the limit to conclude $u = 0$ in ω_{ϵ_1}.

The argument show that $u = 0$ to one side of Σ. The argument for the other side is of course identical (replace x_1 by $-x_1$ in the definition of the coordinate t).

3.3 A Global Holmgren Theorem (F. John, 1948)

It is of course natural to ask: How large is the neighborhood on which u is forced to vanish, if the Cauchy for u on a (piece of) non-characteristic hypersurface Σ vanish and u satisfies $Pu = 0$.

Let Ω be an open region of $\mathbb{R}^d$. Let P be a linear partial differential operator of order m with analytic coefficients and $\Sigma \subset \Omega$ be an embedded non-characteristic hypersurface. Moreover, suppose we know that $u \in C^m(\Omega)$, $Pu = 0$ in Ω with $\partial^\alpha u = 0$ on Σ for all $|\alpha| \leq m - 1$.

Under these conditions, we already know that the solution has to vanish in a neighborhood of Σ. The idea is to sweep out a region by a one-parameter family σ_λ of (sufficiently) smooth embedded non-characteristic hypersurfaces.

Suppose $\mathcal{O}$ is a bounded open set of $\mathbb{R}^{d-1}$ (in many applications: a disc) and that

1. $\sigma : [0,1] \times cl(\mathcal{O}) \to \Omega \subset \mathbb{R}^d$ is continuous
2. For each $\lambda \in [0,1]$, $\sigma_\lambda : \mathcal{O} \to \Omega \subset \mathbb{R}^d$ is a C^m embedding of a non-characteristic hypersurface Σ_λ.
3. $\Sigma_0 \subset \Sigma$
4. $\sigma([0,1] \times \partial\mathcal{O}) \subset \Sigma$ (edge remains in Σ)

Theorem 3.4. *Let Ω be an open region of $\mathbb{R}^d$. Let P be a linear partial differential operator of order m with analytic coefficients and $\Sigma \subset \Omega$ an embedded non-characteristic hypersurface for P. Suppose that $u \in C^m(\Omega)$ satifies $Pu = 0$ in Ω and $\partial^\alpha u = 0$ on Σ for all $|\alpha| \leq m - 1$. Then, we have $\partial^\alpha u = 0$ for $|\alpha| \leq m - 1$ on all of $\sigma([0,1] \times cl(\mathcal{O}))$.*

We will not prove this theorem in detail but it is fairly clear how the proof is going to go. One defines a subset of $[0,1]$:

$$\mathcal{A} = \{\lambda \in [0,1] \quad | \quad \partial^\alpha u = 0 \text{ for all } |\alpha| \leq m-1 \text{ in } \sigma([0,\lambda] \times cl(\mathcal{O}))\} \tag{41}$$

It is a closed set by the continuity of $\partial^\alpha u$, it is non-empty by our local theorem and it is also open (hardest part) because we can redo the local statement on each Σ_λ for $\lambda \in \mathcal{A}$.

3.4 Applications

We give two main applications: One is the unique continuation for (linear) elliptic PDEs with analytic coefficients. The other is to determine the domain of dependence and domain of influence for the linear wave equation.

3.4.1 Unique Continuation for elliptic PDEs

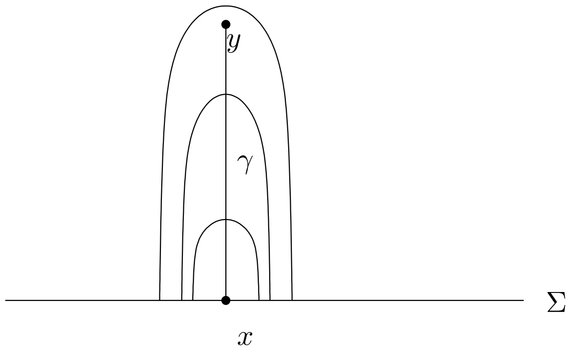
Theorem 3.5. *Let $P(x, \partial)$ be a linear, m^{th} order, elliptic partial differential operator with analytic coefficients in an open connected set $\Omega \subset \mathbb{R}^n$ and let Σ be a piece of C^m hypersurface in Ω. Then, the following statement holds: If*

- *$u \in C^m(\Omega)$ satisfies $P(x, \partial) u = 0$ in Ω and*
- *for all α with $|\alpha| \leq m-1$ we have $\partial^\alpha u = 0$ on Σ*

then $u = 0$ in all of Ω.

Note that we only have to assume that the hypersurface is C^m in view of Exercise 3 below.

The theorem states that for linear elliptic PDEs with analytic coefficients, vanishing Cauchy data on (however small!) a piece of hypersurface in Ω already implies vanishing of u in all of Ω. This is a very strong statement, reminiscent of analyticity! Indeed, one can actually prove that for a $P(x, \partial)$ as in Theorem 3.5, all solutions of $Pu = 0$ have to be real analytic. We'll get back to this once we discuss regularity theory for elliptic equations.



To prove this theorem we are only going to draw a picture (leaving the details to you). To show that the solution vanishes in a neighborhood of an arbitrary point $y \in \Omega$, pick a smooth embedded curve γ connecting a point $x \in \Sigma$ with y. Choose a small tubular neighborhood $\mathcal{T}$ around γ. Then choose a one-parameter family of hypersurfaces contained in $\mathcal{T}$ with edges ending on Σ which foliate $\mathcal{T}$. They are all non-characteristic by the definition of elliptic, so applying global Holmgren yields $u = 0$ in $\mathcal{T}$.

3.4.2 The wave equation: Domain of dependence and influence

Consider the linear wave equation, $\Box u = \left[-\partial_t^2 + c^2\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\right] u = 0$.

Theorem 3.6. *If $u \in C^2\left(\mathbb{R}_t \times \mathbb{R}_x^d\right)$ satisfies*

$$\Box u = 0 \quad \text{and} \quad u|_{t=0} = u_t|_{t=0} = 0 \text{ in } |x| < R \tag{42}$$

then $u = 0$ in $\{(t, x) \mid |x| < R - c|t|\}$.

The set in which u is claimed to vanish is a double-cone of revolution. To prove the theorem (exercise) one sweeps out the cone out by a one parameter family of hyperboloids (which are seen to be non-characteristic) and applies global Holmgren.

We also have

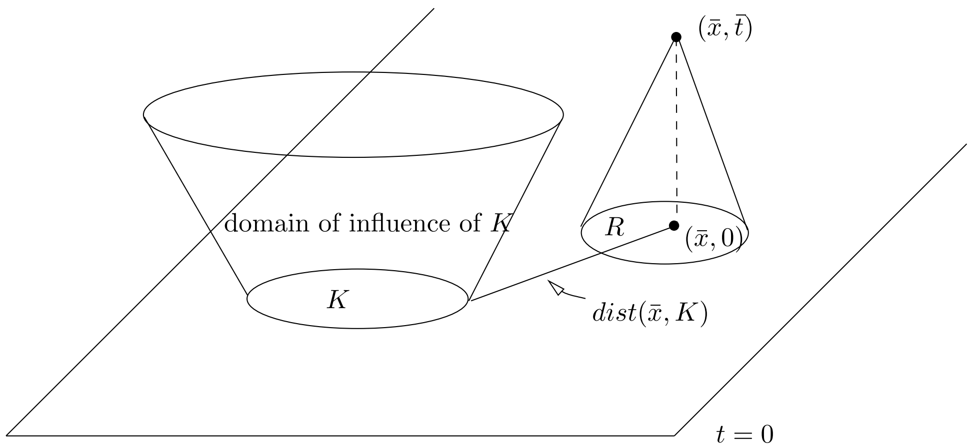
Corollary 3.7. *Suppose that $u \in C^2\left(\mathbb{R}^{1+d}\right)$ is a solution of $\Box u = 0$ and*

$$K = supp\ u_t|_{t=0} \cup supp\ u|_{t=0} \subset \mathbb{R}^d_x$$

is the support of the Cauchy data on the hypersurface $t = 0$. Then,

$$supp\ u \subset \{(t,x)\ \mid dist\,(x,K) \leq c|t|\}\,.$$

Proof. Pick a point $(\bar{x}, t) \in \mathbb{R}^{d+1}$ for which $dist\,(\bar{x}, K) > c|\bar{t}|$. We want to show u vanishes there. Choose R such that $dist\,(\bar{x}, K) > R > c|\bar{t}|$. Then u vanishes inside the double-cone of all (t,x) with $|x - \bar{x}| < R - c|t|$ by Theorem 3.6. Clearly, $(\bar{x}, t)$ lies inside that cone as $0 < R - c|t|$ by our choice of R. □



3.4.3 d'Alembert's formula

Finally, we would like to derive a general formula for solutions to the wave equation in $1+1$ dimensions, the famous d'Alembert's formula. This will reveal that the estimates obtained for the domain of dependence and the domain of influence are in fact sharp (in the sense that it can happen that the solution is supported everywhere on the domain of influence). While the formula is restricted to $1 + 1$ dimensions; we'll show later that the result is true in all dimensions.

We consider the Cauchy problem

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{with data} \quad u\,(0,x) = f\,(x) \quad u_t\,(0,x) = g\,(x) \tag{43}$$

It is easily checked that $u = \varphi(x+ct) + \psi(x-ct)$ solves $u_{tt} - c^2 u_{xx} = 0$ for any C^2 functions φ and ψ. To realize the given data we need

$$\psi(x) + \varphi(x) = f(x) \tag{44}$$

$$-c\psi'(x) + c\varphi'(x) = g(x) \tag{45}$$

Differentiating (44) and inverting the linear system we obtain

$$\psi' = \frac{f'}{2} - \frac{g}{2c} \quad \text{and} \quad \varphi' = \frac{f'}{2} + \frac{g}{2c} \tag{46}$$

Choosing a G with $G' = g$ we obtain the expressions

$$\psi = \frac{f}{2} - \frac{G}{2c} + a \quad \text{and} \quad \varphi = \frac{f}{2} + \frac{G}{2c} + b \tag{47}$$

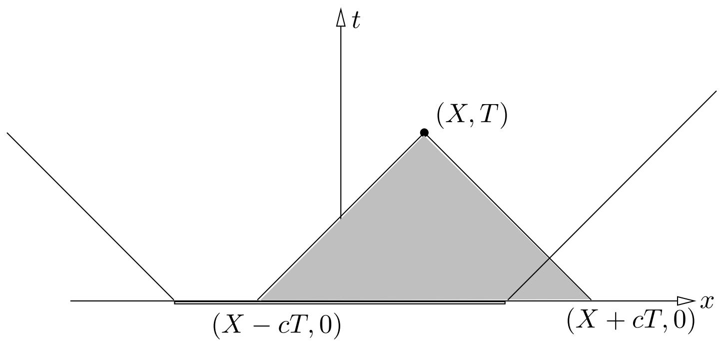
On the other hand, we have $a + b = 0$ by (44) such that

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\left(G(x+ct) - G(x-ct)\right)$$

which we can write as

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds \tag{48}$$

From this formula one easily reads off what we obtained for the domain of dependence and the domain of influence. In particular, choosing $f = 0$ and g to be non-positive and compactly supported one sees that u will be supported in the entire domain of influence.



We also obtain that for any given $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, there is a unique solution $u \in C^2(\mathbb{R}_t \times \mathbb{R}_x)$ of the Cauchy problem (43). Indeed, a solution is given by (48) and it is unique by global Holmgren.

4 Exercises

1. In this question you will prove some useful identities and estimates which have been used (explicitly or implicitly) in the text.

(a) The multinomial identity: For any integer m and $x = (x_1, ..., x_n)$ we have $(x_1 + \ldots x_n)^m = \sum_{|\alpha|=m} = \frac{m!}{\alpha!} x^\alpha$.

(b) For $\alpha, \beta \in \mathbb{N}^d$, $x \in \mathbb{R}^n$, $|x_i| < 1$ for all i we have

$$\sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha - \beta)!} x^{\alpha - \beta} = D^\beta \left(\frac{1}{1 - x_1} \cdot \frac{1}{1 - x_2} \cdot \ldots \cdot \frac{1}{1 - x_n} \right)$$

$$= \frac{\beta!}{(1 - x_1)^{1+\beta_1} \cdot \ldots \cdot (1 - x_n)^{1+\beta_n}} \tag{49}$$

Here $\alpha \geq \beta$ means that $\alpha_i \geq \beta_i$ for all $i = 1, ..., n$.

(c) Let f be a function which is real analytic at $x_0 \in \mathbb{R}^n$. Show that there exists a neighborhood $\mathcal{N}$ around x_0 and positive numbers M, r such that for all $x \in \mathcal{N}$ we have

$$|D^\beta f(x)| \leq M \cdot \beta! \cdot r^{-\beta} \qquad \text{for all } \beta \in \mathbb{N}^n$$

Hint: Differentiate the power series for f term-wise and prove convergence using b).

2. (Rauch, 1.8, Problem 3) Show by explicit computation that

$$u(t, x) = (4\pi t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4t} \right)$$

is a smooth solution of the one-dimensional heat equation $u_t = u_{xx}$ in $t > 0$. Extend u to vanish for $t \leq 0$. Prove that the resulting function is $C^\infty\left(\mathbb{R}^2 \setminus \{0\}\right)$ and satisfies the heat equation in $\mathbb{R}^2 \setminus \{0\}$. In addition, show that u does not vanish in a neighborhood of any point $(0, \bar{x})$ with $\bar{x} \neq 0$. Why does this not violate Holmgren's theorem?

3. State and prove Holmgren's theorem for surfaces which have only regularity $C^m \cap C^2$.

4. (Evans, Chapter 4, Problem 2) Consider Laplace's equation $\Delta u = 0$ in $\mathbb{R}^2$, taken with the Cauchy data

$$u = 0 \quad \text{and} \quad u_{x_2} = \frac{1}{n} \sin(n x_1) \qquad \text{on } \{x_2 = 0\} \tag{50}$$

Use separation of variable to derive the solution

$$u(x_1, x_2) = \frac{1}{n^2} \sin(n x_1) \sinh(n x_2) \ .$$

Discuss whether the Cauchy problem for Laplace's equation is well-posed. [Hint: Does the solution depend continuously on the data?]

5. Do Exercises 2.2 and 2.19 if you haven't already done so.