On the uniqueness theorem of Holmgren

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The Cauchy-Kovalevskaya theorem for the Laplacian

We consider the planar Laplacian

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in the plane $\mathbb{C} \cong \mathbb{R}^2$, supplied with the complex coordinate $z = x + \mathrm{i}y$. The bilaplacian is $\Delta^2 = \Delta\Delta$. The laplacian is the simplest elliptic operator of order 2, and the bilaplacian is the simplest elliptic operator of order 4.

Let Ω be a bounded simply connected domain whose boundary $\partial\Omega$ is a real-analytically smooth Jordan curve. Moreover, let I be a closed arc of $\partial\Omega$, such that both I and $\partial\Omega \setminus I$ are non-trivial arcs.

Theorem CK1 (Cauchy-Kovalevskaya for Δ)

If f_1, f_2 are real-analytic on the closed arc I, then there exists a solution u locally near I of the following Cauchy problem: $\Delta u = 0$ and

$$u|_I = f_1, \quad \partial_n u|_I = f_2.$$

The Cauchy-Kovalevskaya theorem for the bilaplacian

Theorem CK2 (Cauchy-Kovalevskaya for Δ^2)

If f_j are all real-analytic on the closed arc I for $j = 1, 2, 3, 4$, then there exists a solution u locally near I of the following Cauchy problem: $\Delta^2 u = 0$ and

$$\partial_n^{j-1} u|_I = f_j, \qquad j = 1, 2, 3, 4.$$

Remark

We should compare Theorems CK1 and CK2 with the the corresponding Dirichlet problems. In the context of Theorem CK1, the Dirichlet problem would ask for a solution in Ω with just $u|_{\partial\Omega}$ given. In the context of Theorem CK2, the Dirichlet problem would ask for a solution in Ω with just $u|_{\partial\Omega}$ and $\partial_n u|_{\partial\Omega}$ given.

The theorem of Holmgren

We apply the uniqueness theorem of Holmgren to this elementary setting. As a matter of fact, the Cauchy-Kovalevskaya theorem usually includes a statement that the solution is unique among real-analytic solutions. Holmgren's theorem gives uniqueness generally.

Theorem H1 (Holmgren)

The local solution u of Theorem CK1 is unique. Moreover, if u extends as a solution throughout Ω, then u is unique throughout Ω.

Theorem H2 (Holmgren)

The local solution u of Theorem CK2 is unique. Moreover, if u extends as a solution throughout Ω, then u is unique throughout Ω.

Remark

As a local problem, the Cauchy-Kovalevskaya theorem and Holmgren's theorem combine to supply the final answer. Here, we shall consider instead the *non-local problem*.

The non-local Holmgren uniqueness problem 1

Problem NLH1

Let I be an arc of $\partial\Omega$. Suppose u solves $\Delta u = 0$ on Ω, and that u extends C^1-smoothly to $\Omega \cup I$ with local boundary data

(1) $u|_I = 0$ OR

(2) $u|_I = 0$ and $\partial_n u|_I = 0$.

Does it follow that $u = 0$ identically?

Remark

It is clear that under condition (2), we have uniqueness, by Holmgren's theorem. Similarly, under condition (1) we have non-uniqueness, as we may consider the Dirichlet problem with data which vanish on I but not on the remaining arc $\partial\Omega \setminus I$. So the non-local Holmgren uniqueness problem is uninteresting for Δ.

The non-local Holmgren uniqueness problem 2

Problem NLH2

Let I be an arc of $\partial\Omega$. Suppose u solves $\Delta^2 u = 0$ on Ω, and that u extends C^3-smoothly to $\Omega \cup I$ with local boundary data

(1) $u|_I = \partial_n u|_I = 0$, OR

(2) $u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0$, OR

(3) $u|_I = \partial_n u|_I = \partial_n^2 u|_I = \partial_n^3 u|_I = 0$.

Does it follow that $u = 0$ identically?

Remark

It is clear that under condition (3), we have uniqueness, by Holmgren's theorem. Similarly, under condition (1) we have non-uniqueness, as we may consider the Dirichlet problem with data which vanish on I but not on the remaining arc $\partial\Omega \setminus I$. However, it is not clear at all what happens under condition (2).

An illuminating example (Problem NLH2)

We consider the function

$$u(z) = \frac{(1-|z|^2)^3}{|1-z|^4}$$

on the unit disk $\Omega = \mathbb{D} = \{z : |z| < 1\}$.

Proposition

The function u is non-trivial and solves Problem NLH2 with boundary data variant (2) for $\Omega = \mathbb{D}$, for any closed arc of $\partial\Omega$ which does not contain the point 1.

Proof

Direct calculation gives that $\Delta^2 u = 0$ on $\mathbb{D}$. The boundary property follows by direct inspection.

Remark

This example might lead us to believe that Problem NLH2 with boundary data variant (2) never gives uniqueness. However, this is far from the truth. Instead, the above example is very special and not at all generic.

The local Schwarz function

Definition

The (local) Schwarz function S_I is the holomorphic function in a neighborhood of I with $S(z) = \bar{z}$ on I.

Remark

The local Schwarz function exists and is unique, provided that I is a real-analytically smooth arc.

The main theorem

Theorem

Suppose that the local Schwarz function S_I does not extend to a meromorphic function on Ω. Then Problem NLH2 with condition (2) obtains a unique solution. That is, if $\Delta^2 u = 0$ on Ω extends to a C^2-smooth function on $\Omega \cup I$, then

$$u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0 \implies u(z) \equiv 0.$$

Remark

(i) In the case of the unit disk $\Omega = \mathbb{D}$, the local Schwarz function is $S(z) = 1/z$, which is rational and hence meromorphic in $\mathbb{D}$.

(ii) If the local Schwarz function S_I extends to a Schwarz function for the whole boundary $\partial\Omega$, then Ω is a so-called *quadrature domain*.

Ellipses

Corollary

Suppose Ω is the interior of an ellipse $\partial\Omega$ which is not a circle. Then Problem NLH2 with condition (2) obtains a unique solution. That is, if $\Delta^2 u = 0$ on Ω extends to a C^2-smooth function on $\Omega \cup I$, then

$$u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0 \implies u(z) \equiv 0.$$

Proof

This follows from the theorem and the well-known fact that the Schwarz function for an ellipse develops branch points at the foci.

Proof of the main theorem (1)

We introduce the standard complex differentiation operators

$$\partial_z := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y), \quad \bar{\partial}_z := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y),$$

so that

$$\Delta = 4\partial_z\bar{\partial}_z.$$

We will prove the contrapositive statement, namely that: *If a non-trivial solution to Problem NLH2 with condition (2) exists, then the local Schwarz function S_I must extend meromorphically to Ω.*

We consider the function $F := \partial_z^2 u$, which then solves $\bar{\partial}_z^2 F = 0$ on Ω. As such, it may be decomposed according to Almansi: $F(z) = F_1(z) + \bar{z}F_2(z)$. From the boundary data, we see that $F(z) = 0$ on I. Since $S_I(z) = \bar{z}$ on I, we conclude that $F_1(z) + S(z)F_2(z) = 0$ on I. By the uniqueness of holomorphic functions, we get that $F_1 + SF_2 = 0$ near I in Ω. Unless both F_1, F_2 vanish identically, the ratio $\tilde{S} := -F_1/F_2$ defines a meromorphic extension of S_I to Ω. Note here that if F_2 vanishes identically, then F_1 does too.

Proof of the main theorem (2)

It is still possible that F_1, F_2 both vanish simultaneously. What should we do then? In this case, we have that F vanishes, so that $\partial_z^2 u = 0$. We then consider the function $G := \bar{u}$, which solves $\bar{\partial}_z^2 G = 0$ and we proceed as in the case of F. The function G has a decomposition $G = G_1 + \bar{z} G_2$, where G_1, G_2 are holomorphic in Ω, and the non-triviality of u forces both G_1, G_2 to be non-trivial, so that $\tilde{S} := -G_1/G_2$ defines the meromorphic extension of S_I.

The non-local Holmgren uniqueness problem for the N-laplacian

We turn to the general case of Δ^N.

Problem NLH-Nn

Let I be an arc of $\partial\Omega$. Suppose u solves $\Delta^N u = 0$ on Ω, and that u extends C^{2N-1}-smoothly to $\Omega \cup I$ with local boundary data

$\partial_n^j u|_I = 0$ for $j = 0, \ldots, n-1$.

Here, $1 \leq n \leq 2N$ is given. Does it follow that $u = 0$ identically?

Remark

It is clear that if $n \leq N$ we can supply additional Dirichlet data and obtain a smooth non-trivial function u with the given boundary data. On the other hand, we have uniqueness from Holmgren's theorem for $n = 2N$.

The general version of the main theorem

We now supply the main theorem for Δ^N.

Theorem

Let I be an arc of $\partial\Omega$. Suppose u solves $\Delta^N u = 0$ on Ω, and that u extends C^{2N-1}-smoothly to $\Omega \cup I$ with local boundary data

$$\partial_n^j u|_I = 0 \quad \text{for} \quad j = 0, \dots, n-1,$$

where $N+1 \leq n \leq 2N$. If u is non-trivial, then $w = S_I(z)$ solves the system of equations

$$\partial_z^i \partial_w^j Q(z,w) = 0, \qquad 0 \leq i + j \leq n - 1 - N.$$

Here, $Q(z,w)$ is a non-trivial function of the type

$$Q(z,w) := \phi_N(z)\, w^{N-1} + \phi_{N-1}(z)\, w^{N-2} + \cdots + \phi_1(z) = 0,$$

where the functions ϕ_j are all holomorphic in Ω.

Observations

Remark

The theorem is already interesting for $n = N + 1$, in which case $w = S_I(z)$ solves the equation

$$\phi_N(z)\, w^{N-1} + \phi_{N-1}(z)\, w^{N-2} + \cdots + \phi_1(z) = 0.$$

We might want to call such solutions algebraico-meromorphic.

Remark

For $n = 2N$, the theorem can be used to derive Holmgren's theorem in this special case. Indeed, assuming that a non-trivial solution exists, we obtain that $\partial_w^{N-1} Q(z, w) = (N-1)!\phi_N(z) = 0$ holds on the surface $w = S_I(z)$, so that $\phi_N(z) = 0$ identically. Proceeding inductively, we obtain that $Q(z, w) = 0$ identically, so that the theorem tells us that no non-trivial solution u can exist.

Bibliography

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