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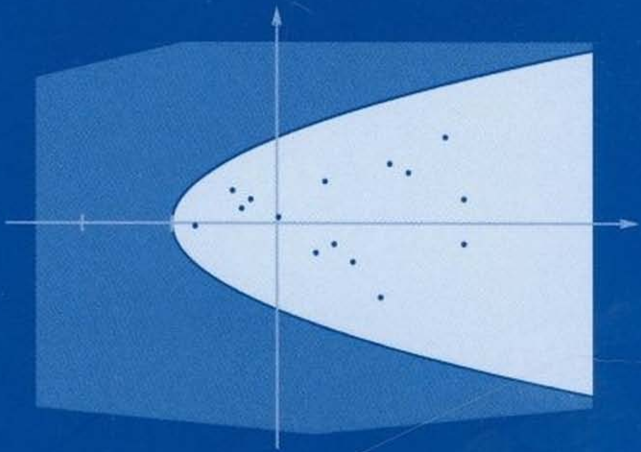
Dorothee D. Haroske
Hans Triebel

# Distributions, Sobolev Spaces, Elliptic Equations



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# Preface

This book grew out of two-semester courses given by the second-named author in 1996/97 and 1999/2000, and the first-named author in 2005/06. These lectures were directed to graduate students and PhD students having a working knowledge in calculus, measure theory and in basic elements of functional analysis (as usually covered by undergraduate courses).

It is one of the main aims of this book to develop at an accessible, moderate level an $L_2$ theory for elliptic differential operators of second order,

$$Au = -\sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{l=1}^{n} a_l(x) \frac{\partial u}{\partial x_l} + a(x)u, \qquad (*)$$

on bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$, including a priori estimates for homogeneous and inhomogeneous boundary value problems in terms of (fractional) Sobolev spaces on $\Omega$ and on its boundary $\partial\Omega$, and a related spectral theory. This will be complemented by a few $L_p$ assertions mostly connected with degenerate elliptic equations.

This book has 7 chapters. The first chapter deals with the well-known classical theory for the Laplace–Poisson equation and harmonic functions. It may also serve as an introduction to the typical questions related to boundary value problems. Chapter 2 collects the basic ingredients of the theory of distributions, including tempered distributions and Fourier transforms. In Chapters 3 and 4 we introduce Sobolev spaces on $\mathbb{R}^n$ and in domains, including embeddings, extensions and traces. The heart of the book is Chapter 5 where we develop an $L_2$ theory for elliptic operators of type $(*)$. Chapter 6 deals with some specific problems of the spectral theory in Hilbert spaces and Banach spaces on an abstract level, including approximation numbers, entropy numbers, and the Birman–Schwinger principle. This will be applied in Chapter 7 to elliptic operators of type $(*)$ and their degenerate perturbations. Finally we collect in Appendices A–D some basic material needed in the text, in particular some elements of operator theory in Hilbert spaces.

The book is addressed to graduate students and mathematicians seeking for an accessible introduction to some aspects of the theory of function spaces and its applications to elliptic equations. However it is not a comprehensive monograph, but it can be used (so we hope) for one-semester or two-semester courses (as we did). For that purpose we interspersed some *Exercises* throughout the text, especially in the first chapters. Furthermore each chapter ends with *Notes* where we collect some references and comments. We hint in these Notes also at some more advanced topics, mostly related to the recent theory of function spaces, and the corresponding literature. For this reason we collect in Appendix E a few relevant assertions.

In addition to the bibliography, there are corresponding indexes for (cited) *authors*, *notation*, and *subjects* at the end, as well as a *list of figures* and *selected solutions* of those exercises which are marked by a $*$ in the text. References are ordered by names, not by labels, which roughly coincides, but may occasionally cause minor deviations.

It is a pleasure to acknowledge the great help we have received from our colleagues David Edmunds (Brighton) and Erich Novak (Jena) who made valuable suggestions which have been incorporated in the text.

Jena, Fall 2007

Dorothee D. Haroske
Hans Triebel

# Contents

# Chapter 1
# The Laplace–Poisson equation

## 1.1 Introduction, basic definitions, and plan of the book

Many questions in mathematical physics can be reduced to elliptic differential equations of second order, their boundary value problems and related spectral assertions. This book deals with some aspects of the underlying recent mathematical theory on a moderate level. We use basic notation according to Sections A.1, A.2 in Appendix A without further explanations.

**Definition 1.1.** Let $\Omega$ be an (arbitrary) domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$. Let

$$\{a_{jk}\}_{j,k=1}^n \subset C^{\mathrm{loc}}(\Omega), \quad \{a_l\}_{l=1}^n \subset C^{\mathrm{loc}}(\Omega), \quad a \in C^{\mathrm{loc}}(\Omega) \tag{1.1}$$

with

$$a_{jk}(x) = a_{kj}(x) \in \mathbb{R}, \quad x \in \Omega, \; j,k = 1,\dots,n. \tag{1.2}$$

Then the differential expression $A$,

$$(Au)(x) = -\sum_{j,k=1}^n a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{l=1}^n a_l(x) \frac{\partial u}{\partial x_l}(x) + a(x)u(x), \tag{1.3}$$

of second order is called *elliptic* if there is a constant $E > 0$ such that for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$ the *ellipticity condition*

$$\sum_{j,k=1}^n a_{jk}(x)\xi_j \xi_k \;\geq\; E|\xi|^2 \tag{1.4}$$

is satisfied.

**Remark 1.2.** As usual, $A$ is called an *elliptic operator*, sometimes also denoted as *uniformly elliptic* operator since $E$ in (1.4) is independent of $x \in \Omega$. One may think of $u \in C^{2,\mathrm{loc}}(\Omega)$ in (1.3). But later on $u$ might be also an element of more general Sobolev spaces.

**Example 1.3.** The most distinguished example is

$$-\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}, \tag{1.5}$$

where $\Delta$ is the *Laplace operator*. We shall adopt the usual convention to call (1.5) simply *Laplacian*.

**Remark 1.4.** The following observation will be of some use. Let

$$\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n \quad \text{with } \xi_j = \operatorname{Re}\xi_j + i\operatorname{Im}\xi_j = \eta_j + i\zeta_j, \tag{1.6}$$

where $\eta_j, \zeta_j \in \mathbb{R}$, $j = 1, \dots, n$. Then

$$\overline{\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\overline{\xi_k}} = \sum_{j,k=1}^{n} \overline{a_{jk}}(x)\overline{\xi_j}\xi_k = \sum_{j,k=1}^{n} a_{kj}(x)\xi_k\overline{\xi_j} \tag{1.7}$$

is real and one thus obtains

$$\begin{aligned}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\overline{\xi_k} &= \sum_{j,k=1}^{n} a_{jk}(x)(\eta_j + i\zeta_j)(\eta_k - i\zeta_k)\\ &= \sum_{j,k=1}^{n} a_{jk}(x)(\eta_j\eta_k + \zeta_j\zeta_k)\\ &\geq E|\xi|^2. \end{aligned}\tag{1.8}$$

Hence the ellipticity condition (1.4) with $\overline{\xi_j}$ in place of $\xi_j$ applies to $\xi$ with (1.6), too.

**Exercise* 1.5.** In which domains $\Omega \subset \mathbb{R}^2$ is

$$(Au)(x_1, x_2) = -\frac{\partial^2 u}{\partial x_1^2}(x_1, x_2) - x_2\frac{\partial^2 u}{\partial x_2^2}(x_1, x_2)$$

elliptic?

### Typical problems, plan of the book

Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3, and let $A$ be an elliptic operator, say, the Laplacian (1.5). Let $f$ be a function in $\Omega$ and $\varphi$ be a function on the boundary $\partial\Omega$. In the typical *boundary value problem* we are dealing with one asks for functions in $\overline{\Omega}$ such that the *Dirichlet problem*,

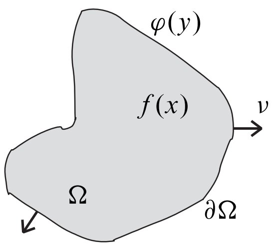



Figure 1.1

$$\begin{cases} Au = f & \text{in } \Omega,\\ u|_{\partial\Omega} = \varphi, \end{cases} \tag{1.9}$$

and the *Neumann problem*,

$$\begin{cases} Au = f & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \varphi, \end{cases} \tag{1.10}$$

can be solved.

Furthermore of interest are *eigenvalues* $\lambda \in \mathbb{C}$ and (non-trivial) *eigenfunctions* $u$ such that, for example,

$$Au = \lambda u \text{ in } \Omega \quad \text{and} \quad u_{|\partial\Omega} = 0 \text{ on } \partial\Omega. \tag{1.11}$$

It is the main aim of this book to develop an $L_2$ theory of these problems, subject to Chapters 5 and 7. The other chapters are not merely a (minimised) preparation to reach these goals, but self-contained introductions to

- the classical theory of the Laplace–Poisson equation (Chapter 1),
- the theory of distributions (Chapter 2),
- Sobolev spaces in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ (Chapter 3) and on domains (Chapter 4),
- the abstract spectral theory in Hilbert spaces and Banach spaces (Chapter 6).

## 1.2 Fundamental solutions and integral representations

Let $n \geq 2$, and $r = r(x) = |x| = \sqrt{\sum_{j=1}^n x_j^2}$ be the usual distance of a point $x \in \mathbb{R}^n$ to the origin. We ask for radially symmetric solutions $u(x) = v(r)$ of the Laplace equation

$$\Delta u(x) = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}(x) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}. \tag{1.12}$$

Inserting

$$\frac{\partial u}{\partial x_j}(x) = \frac{\mathrm{d}v}{\mathrm{d}r}(r)\frac{\partial r}{\partial x_j}(x) = \frac{\mathrm{d}v}{\mathrm{d}r}(r)\,\frac{x_j}{r}, \quad j = 1, \ldots, n, \tag{1.13}$$

and

$$\frac{\partial^2 u}{\partial x_j^2}(x) = \frac{\mathrm{d}^2 v}{\mathrm{d}r^2}(r)\,\frac{x_j^2}{r^2} + \frac{\mathrm{d}v}{\mathrm{d}r}(r)\left(\frac{1}{r} - \frac{x_j^2}{r^3}\right) \tag{1.14}$$

in (1.12) one gets

$$\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \frac{n-1}{r}\,\frac{\mathrm{d}v}{\mathrm{d}r} = 0, \quad r > 0. \tag{1.15}$$

If $n \geq 3$, then $v(r) = c_1\, r^{2-n} + c_2$ is the solution, which must be modified by $v(r) = c_1 + c_2 \ln r$ when $n = 2$, where $c_1, c_2 \in \mathbb{C}$.

Let $|\omega_n|$ be the volume of the unit sphere $\omega_n = \{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$. Otherwise we refer for notation to Sections A.1, A.2.

**Definition 1.6.** Let $n \geq 2$, and $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^n$ according to Definition A.3 (ii). Let $\Phi \in C^2(\Omega)$ with $\Delta\Phi(x) = 0$, $x \in \Omega$, and $x^0 \in \Omega$. Then

$$\gamma_{x^0}(x) = \begin{cases} -\dfrac{1}{2\pi} \ln|x - x^0| + \Phi(x), & n = 2, \\ \dfrac{1}{(n-2)|\omega_n|} \dfrac{1}{|x - x^0|^{n-2}} + \Phi(x), & n \geq 3, \end{cases} \tag{1.16}$$

is called a *fundamental solution* for $\Delta$ and $x^0 \in \Omega$.

**Remark 1.7.** By the above considerations we have

$$\Delta\gamma_{x^0}(x) = 0 \quad \text{in } \Omega \setminus \{x^0\}. \tag{1.17}$$

Recall that $|\omega_n|$ can be expressed in terms of the $\Gamma$-function as

$$|\omega_n| = \frac{2\sqrt{\pi}^{\,n}}{\Gamma(\frac{n}{2})} \tag{1.18}$$

with the well-known special cases $|\omega_2| = 2\pi$, $|\omega_3| = 4\pi$. But (1.18) will not be needed in the sequel. A proof may be found in [Cou36, Chapter IV, Appendix 3, p. 303].

Again we refer for notation to the Appendix A. In particular, $\nu$ stands for the outer normal on $\partial\Omega$ according to (A.13), and the related normal derivative is given by (A.14).

**Theorem 1.8** (Green's representation formula). *Let $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^n$ where $n \geq 2$. Let $u \in C^2(\Omega)$ and $\Delta u(x) = f(x)$, $x \in \Omega$. Let $x^0 \in \Omega$, and $\gamma_{x^0}(x)$ be a fundamental solution according to Definition 1.6. Then*

$$u(x^0) = \int_{\partial\Omega} \left[\gamma_{x^0}(\sigma)\frac{\partial u}{\partial\nu}(\sigma) - u(\sigma)\frac{\partial\gamma_{x^0}}{\partial\nu}(\sigma)\right] \mathrm{d}\sigma - \int_{\Omega} \gamma_{x^0}(x)\, f(x)\, \mathrm{d}x. \tag{1.19}$$

*Proof. Step* 1. By (A.17) with $u$ and $\Phi$ in place of $f$ and $g$, respectively, one obtains

$$\int_{\Omega} \Phi(x)\, \Delta u(x)\, \mathrm{d}x = \int_{\partial\Omega} \left[\Phi(\sigma)\frac{\partial u}{\partial\nu}(\sigma) - u(\sigma)\frac{\partial\Phi}{\partial\nu}(\sigma)\right] \mathrm{d}\sigma, \tag{1.20}$$

due to $\Delta\Phi = 0$. Hence it is sufficient to prove (1.19) for $\Phi \equiv 0$. Furthermore we may assume $x^0 = 0 \in \Omega$.

*Step* 2. Let $n \geq 3$. In view of $\Phi \equiv 0$, $x^0 = 0$, (1.16) reads as

$$\gamma_0(x) = \frac{1}{(n-2)|\omega_n||x|^{n-2}}. \tag{1.21}$$

Let $\varepsilon > 0$ such that

$$K_\varepsilon = K_\varepsilon(0) = \{x \in \mathbb{R}^n : |x| < \varepsilon\} \subset \Omega.$$

We apply (A.17) with $u$ in place of $g$ and $\gamma_0$ in place of $f$ to the $C^1$ domain $\Omega \setminus K_\varepsilon$ and get by (1.17) and $\Delta u = f$ that

$$\begin{aligned} -\int_{\Omega\setminus K_\varepsilon} f(x)\gamma_0(x)\mathrm{d}x &= \int_{\partial\Omega} \left[u(\sigma)\frac{\partial\gamma_0}{\partial\nu}(\sigma) - \gamma_0(\sigma)\frac{\partial u}{\partial\nu}(\sigma)\right]\mathrm{d}\sigma \\ &\quad - \int_{\partial K_\varepsilon} \left[u(\sigma)\frac{\partial\gamma_0}{\partial\nu}(\sigma) - \gamma_0(\sigma)\frac{\partial u}{\partial\nu}(\sigma)\right]\mathrm{d}\sigma, \end{aligned} \tag{1.22}$$

where $\nu$ in the last term is now the outer normal with respect to the ball $K_\varepsilon$ in Figure 1.2. Lebesgue's bounded convergence theorem implies

$$\left|\int_{K_\varepsilon} f(x)\gamma_0(x)\mathrm{d}x\right| \le c\,\varepsilon^2,$$

such that the left-hand side of (1.22) tends to the last term in (1.19) when $\varepsilon \to 0$. Comparing (1.19) and (1.22) it remains to prove that

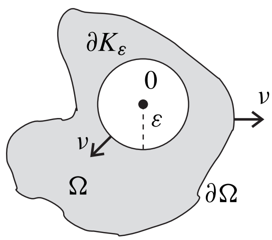



Figure 1.2

$$\lim_{\varepsilon\to 0} \int_{\partial K_\varepsilon} \left[\gamma_0(\sigma)\frac{\partial u}{\partial\nu}(\sigma) - u(\sigma)\frac{\partial\gamma_0}{\partial\nu}(\sigma)\right]\mathrm{d}\sigma = u(0). \tag{1.23}$$

By (1.21) it follows that

$$\begin{aligned} \left|\int_{\partial K_\varepsilon} \gamma_0(\sigma)\frac{\partial u}{\partial\nu}(\sigma)\,\mathrm{d}\sigma\right| &\le \frac{1}{(n-2)|\omega_n|\varepsilon^{n-2}} \int_{|\sigma|=\varepsilon} \left|\frac{\partial u}{\partial\nu}(\sigma)\right| \mathrm{d}\sigma \\ &\le \frac{c}{|\omega_n|\varepsilon^{n-2}} \int_{|\sigma|=\varepsilon} \mathrm{d}\sigma = c\,\varepsilon \longrightarrow 0 \end{aligned} \tag{1.24}$$

for $\varepsilon \to 0$. Furthermore, with

$$\frac{\partial\gamma_0}{\partial\nu}(\sigma) = -\frac{1}{|\omega_n||\sigma|^{n-1}} = -\frac{1}{|\omega_n|\varepsilon^{n-1}}, \quad \sigma \in \partial K_\varepsilon,$$

we can continue,

$$-\int\limits_{\partial K_\varepsilon} u(\sigma)\frac{\partial \gamma_0}{\partial \nu}(\sigma)\,\mathrm{d}\sigma = \frac{1}{|\omega_n|\varepsilon^{n-1}}\int\limits_{|\sigma|=\varepsilon} u(0)\,\mathrm{d}\sigma + \frac{1}{|\omega_n|\varepsilon^{n-1}}\int\limits_{|\sigma|=\varepsilon}(u(\sigma)-u(0))\,\mathrm{d}\sigma$$
$$= u(0) + \frac{1}{|\omega_n|\varepsilon^{n-1}}\int\limits_{|\sigma|=\varepsilon}(u(\sigma)-u(0))\,\mathrm{d}\sigma. \tag{1.25}$$

Since $u$ is continuous the absolute value of the last term can be estimated from above by

$$\frac{1}{|\omega_n|\varepsilon^{n-1}}\sup_{|\sigma|=\varepsilon}|u(\sigma)-u(0)|\left|\int\limits_{|\sigma|=\varepsilon}\mathrm{d}\sigma\right| = \sup_{|\sigma|=\varepsilon}|u(\sigma)-u(0)| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0. \tag{1.26}$$

Now (1.24)–(1.26) prove (1.23). □

**Exercise 1.9.** Prove (1.19) for $n = 2$ using (1.16).

## 1.3 Green's functions

Boundary value problems of type (1.9), (1.10) are at heart of the theory of elliptic differential equations of second order. The *Representation Theorem* 1.8 gives a first impression how $u$ in $\Omega$ may look like if $\Delta u = f$ in $\Omega$ and the behaviour of $u$ at the boundary $\partial\Omega$ is known. However, compared with (1.9), (1.10) where $A = -\Delta$, it seems to be desirable to simplify the first integral on the right-hand side of (1.19), i.e., to have only the term with $u(\sigma)$ on $\partial\Omega$, or only the term with $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$ in (1.19). In case of the Dirichlet problem (1.9) this would mean to eliminate the first term in the integral over $\partial\Omega$ in (1.19) by choosing $\Phi$ in Definition 1.6 in such a way that one has $\gamma_{x^0}(\sigma) = 0$ for $\sigma \in \partial\Omega$. This is the basic motivation for Green's functions.

**Definition 1.10.** Let $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^n$, $n \geq 2$, and let $x^0 \in \Omega$. Then $g(x^0, x)$ is called a *Green's function* if

(i) $\gamma_{x^0}(x) = g(x^0, x)$ is a fundamental solution according to Definition 1.6,

(ii) $g(x^0, \sigma) = 0$, $\sigma \in \partial\Omega$.

**Remark 1.11.** As we shall see later on there are good reasons to look at $g(x^0, x)$ as a function of $2n$ variables in $\Omega \times \overline{\Omega}$. This may explain the different notation of $g(x^0, x)$ (with the separately indicated off-point $x^0 \in \Omega$) compared with $\gamma_{x^0}(x)$. It is one of the major problems to prove the existence of Green's functions which results in the solution of the Dirichlet problem

$$\Delta\Phi = 0 \text{ in } \Omega, \quad \text{and} \quad \Phi(\sigma) = \frac{-1}{(n-2)|\omega_n|}\frac{1}{|\sigma - x^0|^{n-2}} \quad \text{if } \sigma \in \partial\Omega \tag{1.27}$$

for the function $\Phi$ in Definition 1.6 (for $n \geq 3$, with obvious modification for $n = 2$). If a Green's function $g(x^0, x)$ exists, then (1.19) reduces to

$$u(x^0) = -\int_{\partial\Omega} \varphi(\sigma) \frac{\partial g(x^0, \sigma)}{\partial \nu}\, \mathrm{d}\sigma - \int_{\Omega} g(x^0, x)\, f(x)\, \mathrm{d}x \tag{1.28}$$

with

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi. \end{cases} \tag{1.29}$$

This coincides with the Dirichlet problem (1.9) replacing $A$ by $\Delta$. But even if $g(x^0, x)$ exists, then (1.28) does not mean automatically that $u(x^0)$ solves (1.29) for given $f$ and $\varphi$. This must be checked in detail and the conditions for $f$ and $\varphi$ have to be specified. This will be done below in case of balls in $\mathbb{R}^n$,

$$K_R = K_R(0) = \{x \in \mathbb{R}^n : |x| < R\}, \quad R > 0, \tag{1.30}$$

where we are first going to construct $g(x^0, x)$ explicitly.

**Theorem 1.12.** *Let $n \geq 3$, $R > 0$, $\Omega = K_R \subset \mathbb{R}^n$ given by (1.30), and $x^0 \in \Omega$. Then*

$$g(x^0, x) = \frac{1}{(n-2)|\omega_n|} \begin{cases} \dfrac{1}{|x - x^0|^{n-2}} - \left(\dfrac{R}{|x^0|}\right)^{n-2} \dfrac{1}{|x - x^0_*|^{n-2}}, & x^0 \neq 0, \\ \dfrac{1}{|x|^{n-2}} - \dfrac{1}{R^{n-2}}, & x^0 = 0, \end{cases} \tag{1.31}$$

*is a Green's function in $\Omega$, where $x^0_* = x^0 \dfrac{R^2}{|x^0|^2}$ for $x^0 \neq 0$.*

*Proof.* Let $0 \neq x_0 \in \Omega$; since $|x^0||x^0_*| = R^2$ for $0 < |x^0| < R$, it follows from Definitions 1.6 and 1.10 that $\gamma_{x^0}(x) = g(x^0, x)$ is a fundamental solution. It remains to check Definition 1.10 (ii), i.e., $g(x^0, x) = 0$ for $|x| = R$, or, equivalently,

$$\frac{R^2}{|x^0|^2}\, |x - x^0|^2 = |x - x^0_*|^2, \quad |x| = R,\ 0 < |x^0| < R. \tag{1.32}$$

Let $|x| = R$; by the definition of $x^0_*$, see also Figure 1.3,

$$|x - x^0_*|^2 = |x|^2 - 2\langle x, x^0_*\rangle + |x^0_*|^2 = R^2 - \frac{2R^2}{|x^0|^2}\langle x, x^0\rangle + \frac{R^4}{|x^0|^2}, \tag{1.33}$$

such that

$$\frac{R^2}{|x^0|^2}\, |x^0 - x|^2 = \frac{R^2}{|x^0|^2}\left(|x^0|^2 - 2\langle x, x^0\rangle + R^2\right) = |x - x^0_*|^2. \tag{1.34}$$

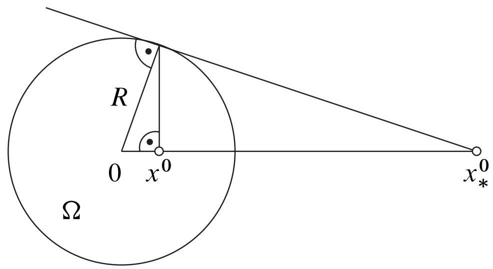



Figure 1.3

The case $x^0 = 0$ is obvious. □

**Exercise 1.13.** Let $n = 2$, $R > 0$, and $\Omega = K_R$ be given by (1.30).

(a) Justify Figure 1.3 for the *reflection point* $x^0_* = x^0 \dfrac{R^2}{|x^0|^2}$ of $x^0 \in \Omega \setminus \{0\}$.

(b) Let $x^0 \in \Omega$, and $x^0_* = x^0 \dfrac{R^2}{|x^0|^2}$ for $x^0 \neq 0$. Prove that

$$g(x^0, x) = -\frac{1}{2\pi} \begin{cases} \ln|x - x^0| - \ln|x - x^0_*| + \ln \dfrac{R}{|x^0|}, & x^0 \neq 0, \\ \ln|x| - \ln R, & x^0 = 0, \end{cases} \tag{1.35}$$

is a Green's function in the circle $\Omega$.

**Theorem 1.14.** *Let $n \geq 2$, $R > 0$, and $\Omega = K_R \subset \mathbb{R}^n$ given by* (1.30). *Let $u \in C^2(\Omega)$ and $\Delta u(x) = 0$, $x \in \Omega$. Then for $x^0 \in \Omega$,*

$$u(x^0) = \frac{R^2 - |x^0|^2}{R\,|\omega_n|} \int\limits_{|\sigma|=R} \frac{u(\sigma)}{|\sigma - x^0|^n}\, \mathrm{d}\sigma. \tag{1.36}$$

*Proof.* Let $n \geq 3$. In view of (1.28) with $g(x^0, x)$ given by (1.31) and $\Delta u = f = 0$ we have

$$u(x^0) = -\int\limits_{|\sigma|=R} u(\sigma) \frac{\partial g}{\partial \nu}(x^0, \sigma)\mathrm{d}\sigma. \tag{1.37}$$

Let $x^0 \neq 0$. Then one obtains by (1.31) that

$$\frac{\partial g}{\partial x_j}(x^0, x) = -\frac{1}{|\omega_n|}\left[\frac{x_j - x^0_j}{|x - x^0|^n} - \left(\frac{R}{|x^0|}\right)^{n-2} \frac{x_j - (x^0_*)_j}{|x - x^0_*|^n}\right]; \tag{1.38}$$

thus (1.32) leads for $|x| = R$, $x^0_* = x^0 \frac{R^2}{|x^0|^2}$, to

$$\frac{\partial g}{\partial x_j}(x^0, x) = -\frac{1}{|\omega_n|\,|x - x^0|^n}\left(x_j - x^0_j - \frac{|x^0|^2}{R^2}(x_j - (x^0_*)_j)\right)$$
$$= -\frac{x_j}{|x - x^0|^n}\,\frac{R^2 - |x^0|^2}{R^2|\omega_n|}. \tag{1.39}$$

Now (1.36) follows from (1.37) and

$$\frac{\partial g}{\partial \nu}(x^0, x) = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(x^0, x)\frac{x_j}{R} = -\frac{R^2 - |x^0|^2}{R|\omega_n|}\,\frac{1}{|x - x^0|^n}. \tag{1.40}$$

□

**Exercise 1.15.** Check the case $x^0 = 0$. Prove (1.36) for $n = 2$.

**Remark 1.16.** Rewriting the right-hand side of (1.36) as an integral operator (of $u$), the function

$$K(x, y) = \frac{R^2 - |x|^2}{R|\omega_n|}\,\frac{1}{|x - y|^n}$$

is sometimes called *Poisson's kernel* for the ball $K_R$, $R > 0$, and $n \geq 2$.

**Corollary 1.17.** *Let $n \geq 2$, $R > 0$, and $\Omega = K_R$ according to* (1.30). *Then*

$$\frac{R^2 - |x^0|^2}{R\,|\omega_n|} \int\limits_{|\sigma| = R} \frac{\mathrm{d}\sigma}{|\sigma - x^0|^n} = 1, \quad |x^0| < R. \tag{1.41}$$

*Proof.* We apply (1.36) to $u \equiv 1$. □

**Exercise* 1.18.** (a) One can extend the notion of Green's function from a bounded domain $\Omega$ in $\mathbb{R}^n$ to, say, the half-space

$$\Omega = \mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

Let $\gamma_{x^0}(x)$ be a fundamental solution according to (1.16) with $\Phi \in C^2(\mathbb{R}^n_+)$, $\Delta\Phi(x) = 0$, $x \in \mathbb{R}^n_+$. Then $g(x^0, x) = \gamma_{x^0}(x)$ is a Green's function for $\Omega = \mathbb{R}^n_+$ if $g(x^0, \sigma) = 0$ for $\sigma \in \partial\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_n = 0\}$.

Determine $\Phi$ appropriately, formulate and prove the counterpart of Theorem 1.14 for $\Omega = \mathbb{R}^n_+$.

*Hint:* Use a suitable reflection idea similar to the proof of Theorem 1.12.

(b) Let $\Omega = K^+_R = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < R,\ x_3 > 0\}$, $R > 0$. Determine a Green's function for $K^+_R$.

*Hint:* Combine ideas from (a) and Theorem 1.12.

## 1.4 Harmonic functions

Again we refer for notation to Appendix A.

**Definition 1.19.** Let $\Omega$ be a connected domain in $\mathbb{R}^n$. Then $u$ is called a *harmonic function* in $\Omega$ if $u \in C^{2,\text{loc}}(\Omega)$ and

$$\Delta u(x) = 0 \quad \text{for } x \in \Omega. \tag{1.42}$$

**Remark 1.20.** Obviously the real part and the imaginary part of a harmonic function are also harmonic functions.

In case of $n = 1$ the connected domain $\Omega$ is an interval and $f$ is harmonic in $\Omega$ if, and only if, it is linear, $u(x) = ax + b$, $a, b \in \mathbb{C}$, and $x \in \Omega$. For $n \geq 2$ there are many harmonic functions. In the plane $\mathbb{R}^2$ all polynomials $1, x, y, xy, x^2 - y^2, x^3 - 3xy^2, \ldots$, but also $e^{ax}\sin(ay)$, $e^{ax}\cos(ay)$ with $a \in \mathbb{R}$ are real harmonic functions.

**Definition 1.21.** Let $\Omega$ be a connected domain in $\mathbb{R}^n$, and let $u$ be continuous in $\overline{\Omega}$. Then $u$ is said to have the *mean value property* if

$$u(x^0) = \frac{1}{R^{n-1}|\omega_n|} \int\limits_{|\sigma - x^0| = R} u(\sigma)\mathrm{d}\sigma \tag{1.43}$$

for any $x^0 \in \Omega$ and any ball

$$K_R(x^0) = \{x \in \mathbb{R}^n : |x - x^0| < R\} \subset \Omega. \tag{1.44}$$

**Remark 1.22.** Since $u$ is continuous in $\overline{\Omega}$, the right-hand side of (1.43) makes sense even if $\partial K_R(x^0) \cap \partial\Omega \neq \emptyset$. Obviously, $|\partial K_R(x^0)| = R^{n-1}|\omega_n|$ what explains to call (1.43) a mean value. Recall that a (real or complex) function $u$ in a domain $\Omega$ is called *analytic* if it can be expanded at any point $x^0 \in \Omega$ into a Taylor series

$$u(x) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\mathrm{D}^\alpha u(x^0)}{\alpha!}(x - x^0)^\alpha, \quad 0 \leq |x - x^0| < r, \tag{1.45}$$

for some $r = r(x^0) > 0$. Obviously an analytic function is a $C^\infty$ function.

**Theorem 1.23.** *Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$ and let $u$ be a real function which is continuous in $\overline{\Omega}$ and harmonic in $\Omega$ according to Definition* 1.19.

(i) *Then $u$ has the mean value property according to Definition* 1.21.

(ii) *Furthermore $u$ is an analytic (and hence $C^\infty$) function in $\Omega$.*

(iii) (Maximum–Minimum principle) *There are points* $x^1 \in \partial\Omega$ *and* $x^2 \in \partial\Omega$ *such that*

$$u(x^1) = \max_{x\in\bar{\Omega}} u(x) \quad \textit{and} \quad u(x^2) = \min_{x\in\bar{\Omega}} u(x). \tag{1.46}$$

*Proof. Step* 1. We prove (i) and assume $0 = x^0 \in \Omega$. Then $K_R(0)$ in (1.44) coincides with $K_R$ in (1.30), and application of Theorem 1.14 gives the desired result,

$$u(0) = \frac{R^2}{R\,|\omega_n|} \int_{|\sigma|=R} \frac{u(\sigma)}{R^n}\,\mathrm{d}\sigma = \frac{1}{R^{n-1}\,|\omega_n|} \int_{|\sigma|=R} u(\sigma)\,\mathrm{d}\sigma, \tag{1.47}$$

where one has to apply an additional continuity argument if $\overline{K_R(0)} \cap \partial\Omega \neq \emptyset$.

*Step* 2. To prove (ii), note first that the kernel $|\sigma - x^0|^{-n}$ of the integral (1.36) is a $C^\infty$ function on $K_R \times \partial K_R$. Hence $u$ is $C^\infty$ in $K_R$. Furthermore, the coefficients of the Taylor series in $\mathbb{C}^n$ of

$$|\sigma - z|^{-n} = \Big( \sum_{j=1}^{n} (\sigma_j - z_j)^2 \Big)^{-n/2}, \quad z \in \mathbb{C}^n,\ |z| < \varepsilon, \tag{1.48}$$

can be estimated uniformly with respect to $|\sigma| = R$ if $\varepsilon > 0$ is chosen sufficiently small. This results in

$$|\mathrm{D}^\alpha u(0)| \le c\,\alpha!\,\tau^{-|\alpha|} \sup_{|\sigma|=R} |u(\sigma)|, \quad \alpha \in \mathbb{N}_0^n, \tag{1.49}$$

for some $c > 0$ and $0 < \tau < 1$, independent of $\alpha$. Here $u$ is given by (1.36). This can be proved by elementary reasoning. But it may also be found in [Tri97, (14.22), 14.5, pp. 95, 97] with a reference to the $\mathbb{C}^n$-version of Cauchy's formula, [Tri97, (14.62), (14.63), p. 103]. Thus (1.45) with $x^0 = 0$ converges if $|x| < \tau$.

*Step* 3. We prove assertion (iii) and restrict ourselves to the maximum. Since the real function $u$ is continuous on the compact set $\bar{\Omega}$ one finds points $x^1 \in \bar{\Omega}$ with (1.46).

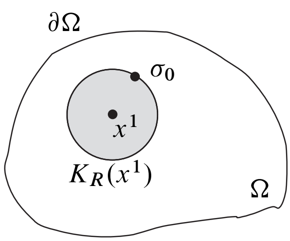


Figure 1.4

We assume that there is some $x^1 \in \Omega$ with this property and choose $R > 0$ such that $K_R(x^1) \subset \bar{\Omega}$, see Figure 1.4 aside. Plainly, $u(\sigma) \le u(x^1)$ for all $\sigma \in \partial K_R(x^1)$. If there was some $\sigma_0 \in \partial K_R(x^1)$ with $u(\sigma_0) < u(x^1)$, then by continuity one has $u(\sigma) < u(x^1)$ in a neighbourhood of $\sigma_0$, and we get a contradiction if we apply (1.43) with $x^0 = x^1$,

$$u(x^1) = \frac{1}{R^{n-1}|\omega_n|} \int\limits_{\partial K_R(x^1)} u(\sigma) \mathrm{d}\sigma < \frac{1}{|\partial K_R(x^1)|} \int\limits_{\partial K_R(x^1)} u(x^1) \mathrm{d}\sigma = u(x^1). \quad (1.50)$$

Hence $u(x) = u(x^1)$ for all $x \in \overline{K_R(x^1)}$. This argument applies in particular to balls with $\partial K_R(x^1) \cap \partial\Omega \neq \emptyset$, recall Remark 1.22. Hence there are points $x^1 \in \partial\Omega$ with (1.46). □

**Exercise* 1.24.** (a) Prove the estimate (1.49) directly.

*Hint:* Expand (1.48) for $z \in \mathbb{R}^n$ with $|z| < \varepsilon$.

(b) Let $u(x_1, x_2) = x_1^2 - x_2^2$, $(x_1, x_2) \in \mathbb{R}^2$, and $K_R \subset \mathbb{R}^2$ given by (1.30), $R > 0$. Determine

$$\sup_{(x_1,x_2)\in K_R} u(x_1, x_2) \quad \text{and} \quad \inf_{(x_1,x_2)\in K_R} u(x_1, x_2).$$

(c) Why is the function

$$f(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2 - 1)\, e^{\sin(x_1+x_2+x_3)}$$

not harmonic in the unit ball $K_1 \subset \mathbb{R}^3$ according to (1.30)?

**Exercise 1.25.** (a) Let $n \geq 2$, $R > 0$, and $\Omega = K_R \subset \mathbb{R}^n$ according to (1.30). Let $u \in C^2(\Omega)$ be harmonic in $\Omega$ and let $u(x) \geq 0$, $x \in \Omega$. Prove *Harnack's inequality*,

$$R^{n-2} \frac{R - |x|}{(R + |x|)^{n-1}}\, u(0) \leq u(x) \leq R^{n-2} \frac{R + |x|}{(R - |x|)^{n-1}}\, u(0), \quad x \in K_R.$$

*Hint:* Use the Theorems 1.14 and 1.23 (i).

(b) Prove another *Harnack's inequality:* Let $\Omega$ be a connected bounded domain in $\mathbb{R}^n$ and $K$ a compact subset of $\Omega$. Let $u$ be harmonic and $u \geq 0$ in $\Omega$. Then there exists a constant $c > 0$ depending only on $K$ and $\Omega$ such that for all $x, y \in K$,

$$c^{-1}\, u(x) \leq u(y) \leq c\, u(x).$$

*Hint:* Cover $K$ with finitely many balls $K_r(x_j)$ according to (1.44) and use (a).

**Corollary 1.26.** *Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$ and let $u \in C(\Omega)$ be a real harmonic function in $\Omega$ such that*

$$u(x^1) = \max_{x\in\overline{\Omega}} u(x) \quad \left(\text{or } u(x^2) = \min_{x\in\overline{\Omega}} u(x)\right) \quad (1.51)$$

*for some $x^1 \in \Omega$ (or some $x^2 \in \Omega$). Then $u$ is constant in $\overline{\Omega}$.*

*Proof.* Any point $x \in \Omega$ can be connected with $x^1 \in \Omega$ having the property (1.51) by a smooth path in $\Omega$. Applying the arguments from Step 3 of the proof of Theorem 1.23 to a suitable finite sequence of balls one gets $u(x) = u(x^1)$. Similarly for the minimum. □

There is a converse of Theorem 1.23 (i). The spaces $C(\Omega)$ and $C^{2,\mathrm{loc}}(\Omega)$ have the same meaning as in Section A.1.

**Corollary 1.27.** *Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$ and let $u \in C(\Omega) \cap C^{2,\mathrm{loc}}(\Omega)$ be a real function satisfying the mean value property according to Definition 1.21. Then $u$ is harmonic.*

*Proof.* We conclude from (1.43) with $x^0 = 0$ by straightforward calculation that

$$|\omega_n| u(0) = \int\limits_{|\sigma|=1} u(r\sigma)\mathrm{d}\sigma, \quad 0 < r \le r_0, \tag{1.52}$$

for some suitably chosen number $r_0 > 0$. Taking the derivative with respect to $r$ one gets (after re-transformation),

$$0 = \int\limits_{|\sigma|=r} \frac{\partial u}{\partial \nu}(\sigma)\mathrm{d}\sigma = \int\limits_{|x|<r} \Delta u(x)\mathrm{d}x, \tag{1.53}$$

for small $r > 0$, where the latter equality comes from (A.17). Then $\Delta u(0) = 0$. An additional translation argument gives the desired assertion $\Delta u(x) = 0$ for all $x \in \Omega$. □

**Corollary 1.28.** *Let $n \ge 3$, and $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^n$ according to Definition A.3 (ii). Let $x^0 \in \Omega$ and $g(x^0, x)$ be a real Green's function according to Definition 1.10. Then for all $x^1, x^2 \in \Omega$, $x^1 \ne x^2$,*

$$0 < g(x^1, x^2) < \frac{1}{(n-2)|\omega_n|} \frac{1}{|x^1 - x^2|^{n-2}}, \tag{1.54}$$

*and*

$$g(x^1, x^2) = g(x^2, x^1). \tag{1.55}$$

*Proof. Step* 1. We show (1.54). Let $x^1 \in \Omega$ be fixed, then the real harmonic function $\Phi$ in (1.16) is negative on $\partial\Omega$, and as a consequence of Theorem 1.23 (iii), $\Phi$ is negative on $\overline{\Omega}$. This proves the right-hand side of (1.54). Concerning the left-hand side we remark first that $g(x^1, \cdot)$ is positive in $K_\delta(x^1) \subset \Omega$ for sufficiently small $\delta > 0$, since $\Phi$ is bounded. Here $K_\delta(x^1)$ is given by (1.44). Moreover, $g(x^1, \cdot)$ is positive on $\partial\Omega \cup \partial K_\delta(x^1)$, harmonic in $\Omega \setminus K_\delta(x^1)$, such that Theorem 1.23 (iii) implies the left-hand side of (1.54).

*Step* 2. Let $x^1, x^2 \in \Omega$, $x^1 \neq x^2$, and

$$u_1(x) = g(x^1, x), \quad u_2(x) = g(x^2, x),$$

such that it remains to show

$$u_1(x^2) = u_2(x^1).$$

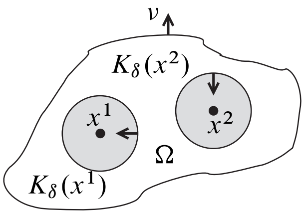



Figure 1.5

We apply (A.17) to the harmonic functions $u_1$ and $u_2$ in the domain $\Omega_\delta = \Omega \backslash \{K_\delta(x^1) \cup K_\delta(x^2)\}$ with sufficiently small $\delta > 0$, $K_\delta(x^1) \cap K_\delta(x^2) = \emptyset$. Since $g$ is a Green's function,

$$\int_{\Omega_\delta} (u_1(x)\Delta u_2(x) - u_2(x)\Delta u_1(x))\mathrm{d}x$$

$$= 0 = \int_{\partial\Omega} \left(u_1(\sigma)\frac{\partial u_2}{\partial \nu}(\sigma) - u_2(\sigma)\frac{\partial u_1}{\partial \nu}(\sigma)\right)\mathrm{d}\sigma$$

such that Theorem A.7 (ii) implies

$$\begin{aligned} 0 = &\int_{\partial K_\delta(x^1)} \left(u_1(\sigma)\frac{\partial u_2}{\partial \nu}(\sigma) - u_2(\sigma)\frac{\partial u_1}{\partial \nu}(\sigma)\right)\mathrm{d}\sigma \\ &+ \int_{\partial K_\delta(x^2)} \left(u_1(\sigma)\frac{\partial u_2}{\partial \nu}(\sigma) - u_2(\sigma)\frac{\partial u_1}{\partial \nu}(\sigma)\right)\mathrm{d}\sigma. \end{aligned} \tag{1.56}$$

We are in the same situation now as in (1.23): replacing the off-point 0 by $x^1$, $\gamma_0$ by $u_1 = g(x^1, \cdot)$, and $u$ by $u_2$, we obtain that the first integral on the right-hand side of (1.56) tends to $u_2(x^1)$ for $\delta \to 0$, whereas, by parallel arguments, the second term converges to $-u_1(x^2)$. This finishes the proof of (1.55). □

**Exercise 1.29.** (a) Prove that the (surface) mean value property according to Definition 1.21 is equivalent to the *volume mean value property*

$$u(x^0) = \frac{1}{|K_R(x^0)|} \int_{K_R(x^0)} u(x)\mathrm{d}x \tag{1.57}$$

for any $x^0 \in \Omega$ and any ball $K_R(x^0) \subset \Omega$ given by (1.44).

*Hint:* Use polar coordinates as in (1.52), differentiate and integrate with respect to $r$.

(b) Let $u \in C(\Omega)$ according to Definition A.1 be harmonic in $\Omega$ and let $K$ be a compact subset of $\Omega$ with

$$d = \operatorname{dist}(K, \partial\Omega) = \sup_{x\in K} \inf_{y\in\partial\Omega} |x - y| > 0.$$

Then

$$\sup_{x \in K} \max_{j=1,\dots,n} \left| \frac{\partial u}{\partial x_j}(x) \right| \le \frac{n}{d} \sup_{x \in \Omega} |u(x)|.$$

*Hint:* Apply (a) to the harmonic functions $\dfrac{\partial u}{\partial x_j}$, $j = 1, \dots, n$, in a ball $K_{d-\varepsilon}(x^0)$, $\varepsilon > 0$, $x^0 \in K$, and use Gauß's formula (A.15). Let $\varepsilon \to 0$.

**Exercise 1.30** (Sobolev's mollification method). Let $\omega$ be given for $x \in \mathbb{R}^n$ by

$$\omega(x) = \begin{cases} c\, e^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases} \tag{1.58}$$

where the constant $c > 0$ is chosen such that $\int_{\mathbb{R}^n} \omega(x) \mathrm{d}x = 1$.

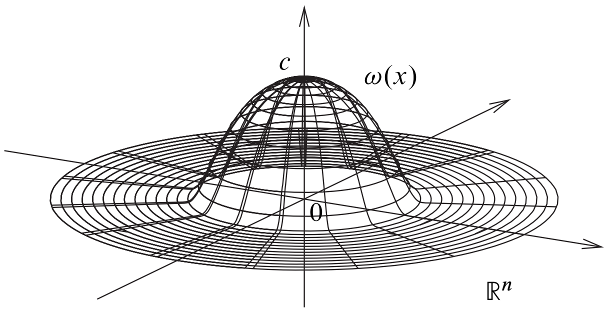



Figure 1.6

For $h > 0$, let $\omega_h(x) = \dfrac{1}{h^n}\, \omega\Big(\dfrac{x}{h}\Big)$, $x \in \mathbb{R}^n$, that is, $\int_{\mathbb{R}^n} \omega_h(x)\, \mathrm{d}x = 1$.

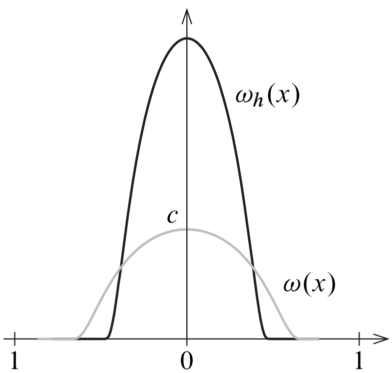



Figure 1.7

Let $u$ be a locally integrable function in $\mathbb{R}^n$. The convolution

$$\begin{aligned} u_h(x) &= (\omega_h * u)(x) \\ &= \int_{\mathbb{R}^n} \omega_h(y) u(x-y) \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \omega_h(x-y) u(y) \mathrm{d}y \end{aligned} \tag{1.59}$$

is called *Sobolev's mollification method*.

(a) Prove that $\omega_h$ are $C^\infty$ functions in $\mathbb{R}^n$ for $h > 0$.

(b) Let $u$ be a locally integrable function in $\mathbb{R}^n$, $h > 0$. Show that $u_h$ is a $C^\infty$ function in $\mathbb{R}^n$.

*Hint:* Either use the mean value theorem for differentiable functions and Lebesgue's bounded convergence theorem to obtain

$$\frac{\partial}{\partial x_j} u_h(x) = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} \omega_h(x-y) u(y) \mathrm{d}y, \tag{1.60}$$

or consult [Tri92a, Sect. 1.3.6].

(c) Prove that any real continuous function in $\mathbb{R}^n$ that satisfies the mean value property according to Definition 1.21 is a $C^\infty$ function.

*Hint:* Show

$$u_h(x) = u(x), \quad h > 0,\ x \in \mathbb{R}^n, \tag{1.61}$$

where $u_h$ is defined by (1.59).

**Exercise 1.31.** Use Exercise 1.30 (c) to replace the assumption $u \in C(\Omega) \cap C^{2,\mathrm{loc}}(\Omega)$ in Corollary 1.27 by $u \in C(\Omega)$.

**Exercise 1.32** (Liouville's theorem). Prove that any bounded harmonic function in $\mathbb{R}^n$ is constant.

*Hint:* Apply the volume mean value property (1.57) to $u(0)$ and $u(x^0)$ and show that $u(x^0) - u(0) \longrightarrow 0$ if $R \to \infty$.

**Exercise 1.33.** Let $\Omega$ be a connected domain in $\mathbb{R}^n$. Then a real function $u \in C^2(\Omega)$ is called *subharmonic* or *superharmonic* in $\Omega$ according as

$$\Delta u(x) \geq 0 \quad \text{or} \quad \Delta u(x) \leq 0 \qquad \text{for } x \in \Omega. \tag{1.62}$$

For convenience we formulate some results for subharmonic functions merely. Their counterparts for superharmonic functions are obvious.

(a) Let $u$ be subharmonic in $\Omega$. Show that for any $x^0 \in \Omega$ and any ball $K_R(x^0) \subset \Omega$ given by (1.44) the mean value properties of harmonic functions (1.43) and (1.57) can be replaced by

$$u(x^0) \leq \frac{1}{|\partial K_R(x^0)|} \int_{\partial K_R(x^0)} u(\sigma)\mathrm{d}\sigma \quad \text{and} \quad u(x^0) \leq \frac{1}{|K_R(x^0)|} \int_{K_R(x^0)} u(x)\mathrm{d}x.$$

(b) Prove that a subharmonic function $u$ in $\Omega$ satisfies that

$$\max_{x \in \overline{\Omega}} u(x) = \max_{y \in \partial\Omega} u(y). \tag{1.63}$$

(c) Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a smooth convex function and $u$ real harmonic in $\Omega$. Then $v = \Phi \circ u$ is subharmonic. Verify that for any real harmonic $u$ in $\Omega$ the function $v = |\nabla u|^2$ is subharmonic, where $\nabla u$ is given as usual,

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right). \tag{1.64}$$

(d) Let $u \in C(\Omega)$ be real harmonic, $v \in C(\Omega)$ subharmonic with $u_{|\partial\Omega} = v_{|\partial\Omega}$. Then $v \le u$ in $\Omega$.

*Hint:* As for parts (a)–(b) adapt the corresponding arguments used for harmonic functions.

**Remark 1.34.** The last assertion (d) explains and justifies the notation *subharmonic*, i.e., a function that is subharmonic in a bounded domain $\Omega$ is dominated by any harmonic function in $\Omega$ having the same boundary values. Similarly for superharmonic functions. For further details see Note 1.7.2.

## 1.5 The Dirichlet problem

We furnish the compact boundary $\partial\Omega$ of a bounded domain $\Omega$ in $\mathbb{R}^n$ with the usual Euclidean metric inherited from $\mathbb{R}^n$. Then $C(\partial\Omega)$ collects all complex-valued continuous functions on $\partial\Omega$. Otherwise we refer for notation again to Sections A.1 and A.2.

**Definition 1.35** (Dirichlet problem for the Laplace equation). Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$ and let $\varphi \in C(\partial\Omega)$. Then one asks for functions $u \in C(\Omega) \cap C^{2,\mathrm{loc}}(\Omega)$ such that

$$\Delta u(x) = 0 \qquad \text{if } x \in \Omega, \tag{1.65}$$

$$u(y) = \varphi(y) \quad \text{if } y \in \partial\Omega. \tag{1.66}$$

**Remark 1.36.** In other words, we look for harmonic functions $u$ according to Definition 1.19 which are continuous on the compact set $\overline{\Omega}$ and take the given boundary values (1.66).

**Theorem 1.37.** *Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$, and $\varphi \in C(\partial\Omega)$. Then the Dirichlet problem according to Definition 1.35 has at most one solution.*

*Proof.* Let $u_1, u_2$ be two solutions of the Dirichlet problem. Then $u = u_1 - u_2$ is a harmonic function in $\Omega$ with $u_{|\partial\Omega} = 0$. If $u$ is real, then Theorem 1.23 (iii) implies $u \equiv 0$ in $\overline{\Omega}$; otherwise $u$ can be decomposed in its real and imaginary part leading by the same arguments as above to $\operatorname{Re} u \equiv 0$, $\operatorname{Im} u \equiv 0$. □

**Exercise 1.38** (Stability). Let $u_1$ and $u_2$ be real solutions of the Dirichlet problem according to Definition 1.35 with respect to boundary data $\varphi_1, \varphi_2 \in C(\partial\Omega)$. Prove that

$$\max_{x\in\overline{\Omega}} |u_1(x) - u_2(x)| \le \max_{y\in\partial\Omega} |\varphi_1(y) - \varphi_2(y)|. \tag{1.67}$$

**Remark 1.39.** Boundary value problems are one of the central objects of the theory of elliptic equations (of second order). They are subject of Chapter 5 in the framework of an $L_2$ theory. In Note 1.7.2 we add a few comments as far as classical methods are concerned. There are only a few cases where the problem (1.65), (1.66) can be treated in an elementary way and where the solution $u$ can be written down explicitly. We restrict ourselves to the case where the underlying domain is a ball. Obviously we may assume that this ball is centred at the origin. The natural candidate for a solution of the Dirichlet problem in a ball is given by (1.36) with $\varphi(\sigma)$ in place of $u(\sigma)$.

**Theorem 1.40** (Poisson's formula). *Let $n \ge 2$, $\Omega = K_R \subset \mathbb{R}^n$ be given by* (1.30), *$R > 0$, and $\varphi \in C(\partial K_R)$. Then the Dirichlet problem according to Definition* 1.35 *has a uniquely determined solution $u$, which is given by*

$$u(x) = \begin{cases} \dfrac{R^2 - |x|^2}{R|\omega_n|} \displaystyle\int_{|\sigma|=R} \dfrac{\varphi(\sigma)}{|\sigma - x|^n}\, d\sigma, & |x| < R, \\ \varphi(x), & |x| = R. \end{cases} \tag{1.68}$$

*Proof. Step* 1. Theorem 1.37 covers the uniqueness; so it remains to prove that $u$ in (1.68) has the required properties. Plainly $u \in C^{2,\mathrm{loc}}(K_R)$ and we wish to show that

$$\Delta_x \left( \frac{R^2 - |x|^2}{|\sigma - x|^n} \right) = 0, \quad x \in K_R,\ |\sigma| = R. \tag{1.69}$$

Let $n \ge 3$ and note that for $|\sigma| = R$,

$$\begin{aligned} \frac{R^2 - |x|^2}{|\sigma - x|^n} &= \frac{|\sigma|^2 - \langle (x-\sigma) + \sigma, (x-\sigma) + \sigma \rangle}{|\sigma - x|^n} \\ &= -\frac{1}{|\sigma - x|^{n-2}} - 2 \sum_{j=1}^n \sigma_j \frac{x_j - \sigma_j}{|\sigma - x|^n}. \end{aligned} \tag{1.70}$$

Since for fixed $\sigma$ both $|\sigma - x|^{-(n-2)}$ and its derivatives,

$$\frac{\partial}{\partial x_j} \frac{1}{|\sigma - x|^{n-2}} = (2-n) \frac{x_j - \sigma_j}{|\sigma - x|^n}$$

are harmonic in $K_R$, we obtain (1.69).

*Step* 2. It remains to prove that $u \in C(K_R)$, which reduces to the question whether for given $\sigma \in \partial K_R$ and $\varepsilon > 0$ one can find a sufficiently small neighbourhood $K_\delta(\sigma) \cap K_R$ of $\sigma$ such that for all $x \in K_\delta(\sigma) \cap K_R$,

$$|u(x) - \varphi(\sigma)| \leq 2\varepsilon. \tag{1.71}$$

We decompose $\partial K_R$ into a neighbourhood $S_1$ of $\sigma$ and $S_2 = \partial K_R \setminus S_1$. Application of (1.41) and (1.68) for $|x| < R$ leads to

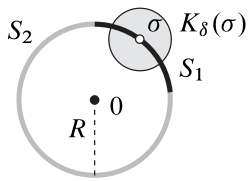



Figure 1.8

$$\begin{aligned} u(x) - \varphi(\sigma) &= \frac{R^2 - |x|^2}{R\,|\omega_n|} \int\limits_{|\tau|=R} \frac{\varphi(\tau) - \varphi(\sigma)}{|\tau - x|^n} \mathrm{d}\tau \\ &= \frac{R^2 - |x|^2}{R\,|\omega_n|} \int\limits_{S_1} \cdots \quad + \frac{R^2 - |x|^2}{R\,|\omega_n|} \int\limits_{S_2} \cdots . \end{aligned}$$

Since $\varphi \in C(\partial K_R)$, we may choose $S_1$ sufficiently small such that

$$\sup_{\tau \in S_1} |\varphi(\tau) - \varphi(\sigma)| \leq \varepsilon.$$

This implies together with another application of (1.41) that

$$\begin{aligned} \frac{R^2 - |x|^2}{R\,|\omega_n|} \left| \int\limits_{S_1} \frac{\varphi(\tau) - \varphi(\sigma)}{|\tau - x|^n} \mathrm{d}\tau \right| &\leq \sup_{\tau \in S_1} |\varphi(\tau) - \varphi(\sigma)| \frac{R^2 - |x|^2}{R\,|\omega_n|} \int\limits_{|\tau|=R} \frac{\mathrm{d}\tau}{|\tau - x|^n} \\ &\leq \varepsilon \end{aligned} \tag{1.72}$$

uniformly for $x \in K_R$. We now choose $\delta > 0$ sufficiently small such that for all $x \in K_R \cap K_\delta(\sigma)$, as indicated in Figure 1.8, $|\tau - x| \geq c_1$ for $\tau \in S_2$, hence

$$\begin{aligned} \frac{R^2 - |x|^2}{R\,|\omega_n|} \left| \int\limits_{S_2} \frac{\varphi(\tau) - \varphi(\sigma)}{|\tau - x|^n} \mathrm{d}\tau \right| &\leq \frac{R^2 - |x|^2}{R\,|\omega_n|} \frac{c_2}{c_1^n} \int\limits_{S_2} \mathrm{d}\tau \\ &\leq c_3\, R^{n-2}\, (R + |x|)(R - |x|) \leq \varepsilon \end{aligned} \tag{1.73}$$

for sufficiently small $\delta > 0$. This gives (1.71). □

**Exercise* 1.41.** Let $\Omega = K_R$, $R > 0$, according to (1.30) and $n = 2$.

(a) Prove (1.69).

(b) Let $C \in \mathbb{R}$ be some constant. What is the unique solution for the Dirichlet problem according to Definition 1.35 in $\Omega$ when $\varphi$ is given by

$$\varphi(R\cos\psi, R\sin\psi) = C\ \sin 4\psi, \quad \psi \in [0, 2\pi)\ ?$$

**Exercise* 1.42.** Let $\Omega = K_R^+$, $R > 0$, that is,

$$K_R^+ = K_R^+(0) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| < R,\ x_n > 0\} \tag{1.74}$$

naturally extending Exercise 1.18 (b).

(a) Let $n = 3$ and $\varphi \in C(\partial K_R^+)$. Solve the Dirichlet problem according to Definition 1.35 for $\Omega = K_R^+$.

*Hint:* Apply Exercise 1.18 (b) and proceed similar to the proof of Theorem 1.40.

(b) Let $n = 2$ and

$$\begin{aligned} \varphi(R\cos\psi, R\sin\psi) &= R^2\cos 2\psi, \quad & 0 \le \psi \le \pi, \\ \varphi(0, x_2) &= -x_2^2, & -R \le x_2 \le R. \end{aligned}$$

What is the unique solution $u = u(x_1, x_2)$ of the Dirichlet problem in $\Omega$ according to Definition 1.35?

## 1.6 The Poisson equation

We refer for notation to Sections A.1, A.2 and the beginning of Section 1.5.

**Definition 1.43** (Dirichlet problem for the Poisson equation). Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$ and $f \in C(\Omega)$, $\varphi \in C(\partial\Omega)$. Then one looks for functions $u \in C(\Omega) \cap C^{2,\text{loc}}(\Omega)$ such that

$$\Delta u(x) = f(x) \quad \text{if } x \in \Omega, \tag{1.75}$$

$$u(y) = \varphi(y) \quad \text{if } y \in \partial\Omega. \tag{1.76}$$

**Remark 1.44.** If $f \equiv 0$, then the above Dirichlet problem for the *Poisson equation* (sometimes denoted as the *Laplace–Poisson equation*) reduces to the Dirichlet problem for the Laplace equation according to Definition 1.35. For given $f$ and $\varphi$ the Dirichlet problem (1.75), (1.76) has at most one solution. This is an immediate consequence of Theorem 1.37. One may try dealing with (1.75), (1.76) in two consecutive steps: first looking for a solution of $\Delta v = f$, and afterwards solving $\Delta w = 0$ with boundary data $w(y) = \varphi(y) - v(y)$. Then $u = v + w$ would give the desired result. However, there are functions $f \in C(\Omega)$ for which $\Delta v = f$ has no solution $v \in C^{2,\text{loc}}(\Omega)$. We return to this question in Note 5.12.11 below.

**Theorem 1.45.** *Let $n \ge 2$ and $f \in C^2(\mathbb{R}^n)$ with $f(x) = 0$ for $|x| > r$ and some $r > 0$. Let $u$ be the* Newtonian potential,

$$u(x) = (\mathcal{N}f)(x) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} f(y)\, \ln|x-y|\, dy, & n = 2, \\ -\dfrac{1}{(n-2)|\omega_n|} \displaystyle\int_{\mathbb{R}^n} \dfrac{f(y)}{|x-y|^{n-2}}\, dy, & n \ge 3, \end{cases} \tag{1.77}$$

*where* $x \in \mathbb{R}^n$. *Then* $u \in C^{2,\mathrm{loc}}(\mathbb{R}^n)$, *and*

$$\Delta u(x) = f(x), \quad x \in \mathbb{R}^n. \tag{1.78}$$

*Proof.* Let $n \geq 3$. It follows from

$$u(x) = -\frac{1}{(n-2)|\omega_n|} \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-2}} \, \mathrm{d}y \tag{1.79}$$

and Lebesgue's bounded convergence theorem that $u$ is continuous in $\mathbb{R}^n$, since $f$ is continuous with compact support in $\mathbb{R}^n$. By the same arguments the assumptions on $f$ imply that $u \in C^2(\mathbb{R}^n)$ and

$$\frac{\partial^2 u}{\partial x_j^2}(x) = -\frac{1}{(n-2)|\omega_n|} \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial x_j^2}(x-y) \, \frac{\mathrm{d}y}{|y|^{n-2}}, \quad x \in \mathbb{R}^n, \tag{1.80}$$

for $j = 1, \ldots, n$, such that

$$\begin{aligned} \Delta u(x) &= -\frac{1}{(n-2)|\omega_n|} \int_{\mathbb{R}^n} \frac{\Delta_x f(x-y)}{|y|^{n-2}} \, \mathrm{d}y \\ &= -\frac{1}{(n-2)|\omega_n|} \int_{\mathbb{R}^n} \frac{\Delta f(y)}{|x-y|^{n-2}} \, \mathrm{d}y, \quad x \in \mathbb{R}^n. \end{aligned}$$

On the other hand, application of Theorem 1.8 for sufficiently large balls $\Omega$ and with $f$ in place of $u$ gives

$$f(x) = -\frac{1}{(n-2)|\omega_n|} \int_{\mathbb{R}^n} \frac{\Delta f(y)}{|y-x|^{n-2}} \, \mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{1.81}$$

□

**Exercise 1.46.** Prove Theorem 1.45 for $n = 2$.

**Remark 1.47.** For $n \geq 3$ one has $u(x) \to 0$ if $|x| \to \infty$ and $u \in C^2(\mathbb{R}^n)$. If $n = 2$, then it may happen that $|u(x)| \to \infty$ if $|x| \to \infty$ and one has only $u \in C^{2,\mathrm{loc}}(\mathbb{R}^2)$. Otherwise the assumptions on $f$ in the above theorem are convenient for us. They assure that $u$ given by (1.77) is a classical solution of (1.78). But one can ensure $u \in C^{2,\mathrm{loc}}(\mathbb{R}^n)$ and (1.78) under weaker and more natural conditions for $f$. We refer to Note 5.12.11. For more general $f$ it is reasonable to ask for distributional solutions $u$ of (1.78). We add a corresponding argument in Note 1.7.3 below.

**Theorem 1.48.** *Let* $n \geq 2$, $\Omega = K_R \subset \mathbb{R}^n$ *be given by* (1.30), $R > 0$, $\varphi \in C(\partial K_R)$, *and* $f \in C^2(\mathbb{R}^n)$ *with* $f(x) = 0$ *for* $|x| > r$ *and some* $r > 0$. *Then the Dirichlet* *problem for the Poisson equation according to Definition 1.43 with $\Omega = K_R$ has a uniquely determined solution u, given by*

$$u(x) = \begin{cases} (\mathcal{N} f)(x) \; + \dfrac{R^2 - |x|^2}{R|\omega_n|} \displaystyle\int\limits_{|\sigma|=R} \dfrac{\varphi(\sigma) - (\mathcal{N} f)(\sigma)}{|\sigma - x|^n} \, \mathrm{d}\sigma, & |x| < R, \\ \varphi(x), & |x| = R, \end{cases} \tag{1.82}$$

*where $x \in \mathbb{R}^n$, and $\mathcal{N} f(x)$ is the Newtonian potential according to (1.77).*

*Proof.* As mentioned in Remark 1.44 the uniqueness is a consequence of Theorem 1.37. By Theorems 1.40 and 1.45 we have

$$\Delta u(x) = \Delta(\mathcal{N} f)(x) + \Delta(u - \mathcal{N} f)(x) = f(x) + 0, \quad |x| < R. \tag{1.83}$$

Furthermore, Theorem 1.40 implies that

$$(\mathcal{N} f)(x) \; + \frac{R^2 - |x|^2}{R|\omega_n|} \int\limits_{|\sigma|=R} \frac{\varphi(\sigma) - (\mathcal{N} f)(\sigma)}{|\sigma - x|^n} \, \mathrm{d}\sigma$$

can be extended continuously from $|x| < R$ to $|x| = R$ with boundary data

$$(\mathcal{N} f)(x) + (\varphi - (\mathcal{N} f))(x) = \varphi(x), \quad |x| = R.$$

This finishes the proof. □

**Exercise* 1.49.** Solve the Dirichlet problem for the Poisson equation where the domain $\Omega \subset \mathbb{R}^2$ is the annulus

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 \; : \; \frac{1}{e^2} < x_1^2 + x_2^2 < 1 \right\},$$

$$\Delta u(x_1, x_2) \equiv 1, \quad (x_1, x_2) \in \Omega,$$

and with boundary data

$$\varphi(x_1, x_2) = \begin{cases} 2 + \dfrac{1}{e^2} & \text{if } x_1^2 + x_2^2 = \dfrac{1}{e^2}, \\ 2 & \text{if } x_1^2 + x_2^2 = 1. \end{cases}$$

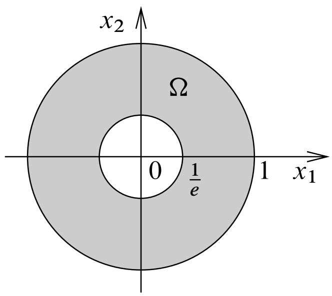



Figure 1.9

*Hint:* Use the above decomposition idea for $u$ twice: first find a (simple) function $u_1$ which satisfies $\Delta u_1(x_1, x_2) \equiv 1$, $(x_1, x_2) \in \Omega$. Then adjust the boundary values by means of a harmonic function $u_2(x_1, x_2)$. Recall, in particular, those radially symmetric functions, harmonic in $\Omega$, which satisfy (1.15) for $n = 2$.

**Remark 1.50.** We add a comment about notation. To call (1.75), (1.76) *Dirichlet problem for the Poisson equation* comes from the long history of this subject. Later on we prefer to denote (1.75), (1.76) as the *inhomogeneous Dirichlet problem for the Laplacian*, whereas the *homogeneous Dirichlet problem for the Laplacian* refers to (1.75), (1.76) with $\varphi = 0$. Both will be studied in Chapter 5 in the framework of an $L_2$ theory for general elliptic differential operators according to Definition 1.1 in bounded $C^\infty$ domains in $\mathbb{R}^n$. In Note 1.7.4 we comment on the assumption $f \in C^2(\mathbb{R}^n)$.

**Exercise 1.51.** Let $u$ be a solution of the Dirichlet problem for the Poisson equation according to Definition 1.43. Then $u$ satisfies the following *a priori estimate*: There exists a constant $c > 0$ depending only on $\Omega$, such that

$$\|u|C(\Omega)\| \le \|\varphi|C(\partial\Omega)\| + c\|f|C(\Omega)\|. \tag{1.84}$$

*Hint:* Assume $\Omega \subset \{x \in \mathbb{R}^n : 0 \le x_1 \le a\}$ for a suitable $a > 0$, and consider

$$h(x) = \|\varphi|C(\partial\Omega)\| + (e^a - e^{x_1})\|f|C(\Omega)\|, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Show that $u - h$ is subharmonic with $(u-h)_{|\partial\Omega} \le 0$. Apply Exercise 1.33 (b) and repeat the argument for $\widetilde{h} = -h$.

**Remark 1.52.** A priori estimates will play an essential rôle in our later investigations in Chapter 5.

## 1.7 Notes

**1.7.1.** Differential equations of second order play a fundamental rôle in many branches of mathematics and physics. The material of Chapter 1 is very classical and the subject of many books and lectures. We followed here the relevant parts of [Tri92a]. More substantial introduction into the classical theory, including detailed studies of boundary value problems in the context of Hölder spaces (which will be shortly mentioned in Exercise 3.21) may be found in [CH53], [CH62], [Mir70], [GT01], [Hel77], [Pet54], [Eva98], [Jac95].

**1.7.2.** The solution of the Dirichlet problem according to Definition 1.35, say, in smooth bounded domains in $\mathbb{R}^n$ is one of the most distinguished problems in the theory of elliptic equations. In Chapter 5 we return to this question in the framework of an $L_2$ theory. As for the classical theory we have so far only for the ball $\Omega = K_R$ a satisfactory solution in Theorem 1.40. For general bounded (smooth) domains there are essentially two classical approaches. The method of single-layer and double-layer potentials reduces (inner, outer) Dirichlet problems and (inner, outer) Neumann problems to Fredholm integral equations on $\partial\Omega$. The theory can be found for $n = 2$ in [Pet54] and for $n \ge 3$ in [Tri92a] with a reference to [Gün67]. The other method goes back to Perron (or Poincaré–Perron according to [Pet54]) and is characterised by the key words *subharmonic* and *superharmonic functions* as briefly mentioned in Exercise 1.33 and Remark 1.34. This may be found in [Pet54], [GT01], [Eva98] and [Jac95]. In particular, the Dirichlet problem according to Definition 1.35 has a unique solution in smooth bounded connected domains.

**1.7.3.** The Newtonian potential (1.77) makes sense on a much larger scale. Its kernel,

$$G(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & n = 2, \\ -\dfrac{1}{(n-2)|\omega_n|} \dfrac{1}{|x|^{n-2}}, & n \geq 3, \end{cases} \tag{1.85}$$

is called the *fundamental solution* of the Laplacian $\Delta$, hence $\Delta G = \delta$, where $\delta$ is the $\delta$-distribution according to Example 2.12 below. If $f \in L_2(\mathbb{R}^n)$ with $f(x) = 0$ if $|x| > r$, then $u(x)$ in (1.77) is locally integrable in $\mathbb{R}^n$ and one has (1.78) in the distributional sense. Details may be found in [Tri92a, Sect. 3.2.3].

**1.7.4.** The assumption $f \in C^2(\mathbb{R}^n)$ in Theorem 1.48 looks slightly incongruous. It can be replaced by the more adequate assumption $f \in C^2(K_R)$ according to Definition A.1. This follows from the extension Theorem 4.1 below and the observation that $u$ is independent of the behaviour of $f$ outside the ball $K_R$ in $\mathbb{R}^n$. But one cannot reduce the smoothness assumptions for $f$ to $f \in C(K_R)$ which would be natural by Definition 1.43. There are counter-examples. We discuss these problems in some detail in Note 5.12.11.

**1.7.5.** Although Theorem 1.23 is very classical, it is a little bit surprising that mostly only the $C^\infty$ smoothness of harmonic functions according to part (ii) is discussed. An explicit proof of the analyticity in the plane $\mathbb{R}^2$ may be found in [Pet54, §30]. As for an $n$-dimensional assertion we refer to [Krz71, §31.8, p. 259/260] and [Eva98, Theorem 2.2.10, p. 31].

# Chapter 2
# Distributions

## 2.1 The spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$

We give a brief, but self-contained introduction to the theory of distributions to an extent as needed later on.

Let $n \in \mathbb{N}$ and $\Omega$ an (arbitrary) domain in $\mathbb{R}^n$. Recall that domain means open set. We use the notation introduced in Appendix A without further explanations. For $f \in C^{\mathrm{loc}}(\Omega)$ we call

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}} \tag{2.1}$$

the *support* of $f$. The closure in (2.1) is taken with respect to $\mathbb{R}^n$. In particular, it may happen that some points $y \in \partial\Omega = \overline{\Omega} \setminus \Omega$ belong to the support of $f$. A function $f \in C^{\mathrm{loc}}(\Omega)$ is said to have *compact support* (with respect to $\Omega$) if

$$\operatorname{supp} f \text{ is bounded (in } \mathbb{R}^n) \text{ and } \operatorname{supp} f \subset \Omega. \tag{2.2}$$

Recall that a set $K \subset \mathbb{R}^n$ is called *compact* in the domain $\Omega$ if $K$ is a closed bounded subset in $\mathbb{R}^n$ and $K \subset \Omega$. In particular, if $f \in C^{\mathrm{loc}}(\Omega)$ has a compact support in $\Omega$, then $f \in C(\Omega)$.

**Remark 2.1.** For continuous functions $f \in C^{\mathrm{loc}}(\Omega)$ the definition (2.1) of a support is not only reasonable, but also in good agreement with the support of a regular distribution $T_f$ generated by $f$ as introduced below. However, if $f$ is only (locally) integrable in $\Omega$, then the right-hand side of (2.1) and the support of an associated regular distribution $T_f$ may be rather different and greater care is necessary. We return to this point in Remark 2.23 below.

**Definition 2.2.** Let $\Omega$ be a domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$, and let $C^\infty(\Omega)$ be as in (A.9). Then

$$\mathcal{D}(\Omega) = \{\varphi \in C^\infty(\Omega) : \operatorname{supp}\varphi \text{ compact in } \Omega\}. \tag{2.3}$$

A sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{D}(\Omega)$ is said to be convergent in $\mathcal{D}(\Omega)$ to $\varphi \in \mathcal{D}(\Omega)$, we shall write $\varphi_j \xrightarrow[\mathcal{D}]{} \varphi$, if there is a compact set $K \subset \Omega$ with

$$\operatorname{supp}\varphi_j \subset K, \quad j \in \mathbb{N}, \tag{2.4}$$

and

$$\mathrm{D}^\alpha \varphi_j \Longrightarrow \mathrm{D}^\alpha \varphi \quad \text{for all } \alpha \in \mathbb{N}_0^n. \tag{2.5}$$

**Remark 2.3.** Recall that (2.5) means *uniform convergence* for all derivatives, that is,

$$\|\varphi_j - \varphi | C^m(\Omega)\| \to 0 \quad \text{if } j \to \infty \tag{2.6}$$

for all $m \in \mathbb{N}_0$, where we used the notation (A.8). Plainly, (2.5) implies

$$\operatorname{supp} \varphi \subset K. \tag{2.7}$$

Sometimes $\mathcal{D}(\Omega)$ is also denoted as $C_0^\infty(\Omega)$ in good agreement with (A.9), and its elements are occasionally called *test functions*.

**Exercise 2.4.** Let $\{\varphi_j\}_{j=1}^\infty$ be a sequence in $\mathcal{D}(\Omega)$ with (2.4) for some compact set $K \subset \Omega$, and for all $m \in \mathbb{N}_0$, $\varepsilon > 0$,

$$\|\varphi_j - \varphi_k | C^m(\Omega)\| \le \varepsilon \quad \text{if } j \ge k \ge k(\varepsilon, m). \tag{2.8}$$

Then there is a function $\varphi \in \mathcal{D}(\Omega)$ with (2.5). Thus any such sequence in $\mathcal{D}(\Omega)$ is convergent in $\mathcal{D}(\Omega)$.

*Hint:* Use Remark A.2. This is the obvious counterpart of the well-known assertion that any Cauchy sequence in a Banach space is a converging sequence. We also refer to Note 2.9.3.

**Definition 2.5.** Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $\mathcal{D}(\Omega)$ be as in Definition 2.2. $\mathcal{D}'(\Omega)$ is the collection of all complex-valued linear continuous functionals $T$ over $\mathcal{D}(\Omega)$, that is,

$$T : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}, \quad T : \varphi \mapsto T(\varphi), \quad \varphi \in \mathcal{D}(\Omega), \tag{2.9}$$

$$T(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 T(\varphi_1) + \lambda_2 T(\varphi_2), \quad \lambda_1, \lambda_2 \in \mathbb{C};\ \varphi_1, \varphi_2 \in \mathcal{D}(\Omega), \tag{2.10}$$

and

$$T(\varphi_j) \to T(\varphi) \quad \text{for } j \to \infty \text{ whenever } \varphi_j \underset{\mathcal{D}}{\longrightarrow} \varphi, \tag{2.11}$$

according to (2.4), (2.5). $T \in \mathcal{D}'(\Omega)$ is called a *distribution*.

**Remark 2.6.** A few historical comments and some references may be found in the Notes 2.9.2, 2.9.3, including a remark that one can look at $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ as the dual pairing of locally convex spaces (just as $X'$ as the dual of a Banach space $X$). In particular,

$$T_1 = T_2 \text{ in } \mathcal{D}'(\Omega) \text{ means } T_1(\varphi) = T_2(\varphi) \text{ for all } \varphi \in \mathcal{D}(\Omega), \tag{2.12}$$

and $\mathcal{D}'(\Omega)$ is converted into a linear space by

$$(\lambda_1 T_1 + \lambda_2 T_2)(\varphi) = \lambda_1 T_1(\varphi) + \lambda_2 T_2(\varphi), \quad \varphi \in \mathcal{D}(\Omega), \tag{2.13}$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$, $T_1, T_2 \in \mathcal{D}'(\Omega)$. For our purpose it is sufficient to furnish $\mathcal{D}'(\Omega)$ with the so-called *simple convergence topology*, that is,

$$T_j \to T \text{ in } \mathcal{D}'(\Omega), \quad T_j \in \mathcal{D}'(\Omega),\ j \in \mathbb{N},\ T \in \mathcal{D}'(\Omega), \tag{2.14}$$

means that

$$T_j(\varphi) \to T(\varphi) \text{ in } \mathbb{C} \quad \text{if } j \to \infty \text{ for any } \varphi \in \mathcal{D}(\Omega). \tag{2.15}$$

If there is no danger of confusion we abbreviate $T(\varphi) = T\varphi$ for $\varphi \in \mathcal{D}(\Omega)$, $T \in \mathcal{D}'(\Omega)$.

## 2.2 Regular distributions, further examples

Distributions are sometimes called *generalised functions*. This notation comes from the observation that complex-valued locally Lebesgue integrable functions $f$ in a domain $\Omega$ in $\mathbb{R}^n$ can be interpreted as so-called *regular distributions* $T_f \in \mathcal{D}'(\Omega)$. We describe the underlying procedure assuming that the reader is familiar with basic measure theory, especially the Lebesgue measure in $\mathbb{R}^n$, and related $L_p$ spaces. But we fix some notation and have a closer look at a few more peculiar properties needed later on.

Again let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. Then $L_p(\Omega)$, $1 \le p < \infty$, is the usual Banach space of all complex-valued Lebesgue measurable functions in $\Omega$ such that

$$\|f \,|L_p(\Omega)\| = \left( \int_\Omega |f(x)|^p \mathrm{d}x \right)^{1/p} < \infty, \tag{2.16}$$

complemented by $L_\infty(\Omega)$, normed by

$$\|f \,|L_\infty(\Omega)\| = \inf\{N : |\{x \in \Omega : |f(x)| > N\}| = 0\}. \tag{2.17}$$

Here $|\Gamma|$ is the Lebesgue measure of a Lebesgue measurable set $\Gamma$ in $\mathbb{R}^n$. Strictly speaking, the elements of $L_p(\Omega)$ are not functions $f$, but their equivalence classes $[f]$ consisting of all Lebesgue measurable functions $g$ that differ from $f$ on a set of measure zero only,

$$[f] = \{g : \ |\{x \in \Omega : f(x) \neq g(x)\}| = 0\}. \tag{2.18}$$

Replacing $f$ in (2.16), (2.17) by any other representative $g \in [f]$ does not change the value. One must have this ambiguity in mind if one wishes to identify (locally integrable) Lebesgue measurable functions with (regular) distributions. Otherwise a detailed discussion of all those questions, including a proof that $L_p(\Omega)$ are Banach spaces, may be found in [Tri92a]. This applies also to the following observations, but we outline proofs to provide a better understanding of the context. Let for $1 \le p \le \infty$,

$$L_p^{\mathrm{loc}}(\Omega) = \{f : f \in L_p(K) \text{ for any bounded domain } K \text{ with } \bar{K} \subset \Omega\}. \tag{2.19}$$

Naturally, $f \in L_p(K)$ means that the restriction $f|_K$ of the Lebesgue measurable function $f$ is contained in $L_p(K)$. Again, $f \in L_p^{\mathrm{loc}}(\Omega)$ must be interpreted as the sloppy, but usual version of $[f] \in L_p^{\mathrm{loc}}(\Omega)$.

**Proposition 2.7.** *Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$.*

(i) *Let $1 \le p < \infty$. Then $\mathcal{D}(\Omega)$ is dense in $L_p(\Omega)$.*

(ii) *Let $f \in L_1^{\mathrm{loc}}(\Omega)$. If*

$$\int_\Omega f(x)\varphi(x)\mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathcal{D}(\Omega), \tag{2.20}$$

*then $[f] = 0$.*

*Proof. Step* 1. We begin with part (i). Any $f \in L_p(\Omega)$ can be approximated by step functions

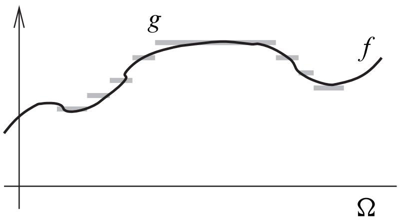



Figure 2.1

$$g = \sum_{j=1}^m a_j \chi_{Q_j}, \quad a_j \in \mathbb{C}, \tag{2.21}$$

where $\chi_{Q_j}$ are the characteristic functions of open cubes $Q_j$ with $\overline{Q_j} \subset \Omega$.

Hence it is sufficient to approximate the characteristic function $\chi_Q$ of a cube $Q$ with $\bar{Q} \subset \Omega$ in $L_p(\Omega)$ by $\mathcal{D}(\Omega)$-functions.

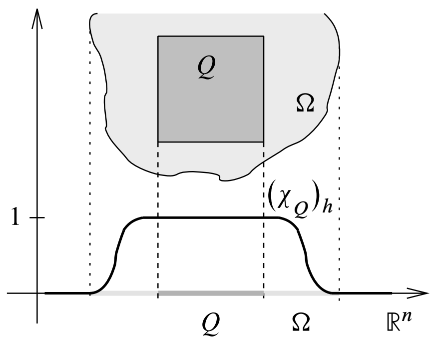



Figure 2.2

Let $h > 0$, and $(\chi_Q)_h$ be the mollified characteristic function according to (1.59). By Exercise 1.30 (or [Tri92a, 1.3.6]) and

$$\operatorname{supp} \omega_h = K_h = \{x \in \mathbb{R}^n : |x| \le h\}$$

in view of (2.1) and (1.58), see also Figure 2.2 aside, one has

$$(\chi_Q)_h \in \mathcal{D}(\Omega) \tag{2.22}$$

for $0 < h \le h_0$, and

$$(\chi_Q)_h \to \chi_Q \text{ in } L_p(\Omega) \quad \text{for } h \to 0. \tag{2.23}$$

This proves (i).

*Step* 2. We deal with (ii). Let $K_1$ and $K_2$ be two bounded domains with $\overline{K_1} \subset K_2 \subseteq \overline{K_2} \subset \Omega$. Let $f \in L_1^{\text{loc}}(\Omega)$ and $f_2 = f\chi_{K_2}$, then $f_2 \in L_1(\mathbb{R}^n)$ (extended by zero outside $\Omega$). Using (1.59) with $\operatorname{supp}\omega_h = \overline{K}_h$, (2.20) implies that

$$\int_{\mathbb{R}^n} f_2(y)\omega_h(x-y)\mathrm{d}y = \int_{\Omega} f(y)\omega_h(x-y)\mathrm{d}y = 0, \quad x \in K_1, \tag{2.24}$$

for $0 < h \le h_0$ and sufficiently small $h_0 > 0$. In view of (1.59), (2.24) can be reformulated as

$$(f_2)_h(x) = 0, \quad x \in K_1,\ 0 < h \le h_0. \tag{2.25}$$

On the other hand, (1.59) also gives

$$(f_2)_h(x) = \int_{\mathbb{R}^n} f_2(x-hy)\omega(y)\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{2.26}$$

such that

$$(f_2)_h(x) - f_2(x) = \int_{\mathbb{R}^n} [f_2(x-hy) - f_2(x)]\omega(y)\mathrm{d}y \tag{2.27}$$

in view of (1.58). We apply a well-known continuity property for $L_1$ norms (see also Exercise 2.8 below) to (2.27) and arrive at

$$\|(f_2)_h - f_2|L_1(\mathbb{R}^n)\| \le \int_{\mathbb{R}^n} \|f_2(\cdot - hy) - f_2(\cdot)|L_1(\mathbb{R}^n)\|\omega(y)\mathrm{d}y \to 0 \tag{2.28}$$

for $h \to 0$. Using (2.25) one obtains

$$\|f_2|L_1(K_1)\| \le \|f_2 - (f_2)_h|L_1(\mathbb{R}^n)\| \to 0 \quad \text{for } h \to 0. \tag{2.29}$$

Since $f_2(x) = f(x)$, $x \in K_1$, it follows finally $[f] = 0$ in any $K_1 \subset \Omega$; thus $[f] = 0$ in $\Omega$. □

**Exercise 2.8.** (a) Let $1 \le p \le \infty$. Prove that

$$\|f_h|L_p(\mathbb{R}^n)\| \le \|f|L_p(\mathbb{R}^n)\|, \quad f \in L_p(\mathbb{R}^n),\ h > 0, \tag{2.30}$$

as a consequence of the triangle inequality for integrals applied to (2.26).

(b) Let $1 \le p < \infty$. Show that for any $f \in L_p(\mathbb{R}^n)$ and any $\varepsilon > 0$ there is a number $\delta(f,\varepsilon) > 0$ such that

$$\|f(\cdot + y) - f(\cdot)|L_p(\mathbb{R}^n)\| \le \varepsilon \quad \text{for all } y,\ |y| \le \delta(f,\varepsilon). \tag{2.31}$$

*Hint:* Use Proposition 2.7 (i).

(c) Show that (2.31) cannot be extended to $p = \infty$ and that $\mathcal{D}(\mathbb{R}^n)$ is not dense in $L_\infty(\mathbb{R}^n)$.

**Exercise 2.9.** A Banach space is called *separable* if there is a countably dense set of elements. Prove that $L_p(\Omega)$ with $1 \le p < \infty$ is separable, unlike $L_\infty(\Omega)$.

*Hint:* Reduce the question to the uncountable set of all characteristic functions of cubes in $\Omega$.

Let $\Omega$ be a domain (i.e., an open set) in $\mathbb{R}^n$ and $f \in L_1^{\mathrm{loc}}(\Omega)$ (that is, $[f] \in L_1^{\mathrm{loc}}(\Omega)$). Then

$$T_f(\varphi) = \int_\Omega f(x)\varphi(x)\mathrm{d}x, \quad \varphi \in \mathcal{D}(\Omega), \tag{2.32}$$

generates a distribution $T_f \in \mathcal{D}'(\Omega)$ according to Definition 2.5. This follows from $f\varphi \in L_1(\Omega)$ which justifies (2.9), (2.10), whereas the continuity (2.11) with (2.4) is a consequence of

$$|T_f(\varphi)| \le \|f|L_1(K)\| \sup_{x\in K} |\varphi(x)|. \tag{2.33}$$

Obviously, $T_f = T_g$ if $g \in [f] \in L_1^{\mathrm{loc}}(\Omega)$. The converse, leading to

$$T_f = T_g \in \mathcal{D}'(\Omega) \quad \text{if, and only if,} \quad [f - g] = 0, \tag{2.34}$$

where $f \in L_1^{\mathrm{loc}}(\Omega)$, $g \in L_1^{\mathrm{loc}}(\Omega)$, follows immediately from Proposition 2.7 (ii).

**Definition 2.10.** Let $\Omega$ be a domain in $\mathbb{R}^n$. Then a distribution $T \in \mathcal{D}'(\Omega)$ is said to be *regular* if there is an $f \in L_1^{\mathrm{loc}}(\Omega)$ such that $T$ can be represented as $T = T_f$ according to (2.32).

**Remark 2.11.** By the above considerations, (2.32) generates a one-to-one correspondence

$$f \in L_1^{\mathrm{loc}}(\Omega) \iff T_f \in \mathcal{D}'(\Omega) \tag{2.35}$$

as indicated in Figure 2.3. But one should always bear in mind that $f$ must be considered as a representative of its equivalence class $[f]$. One avoids this ambiguity if one looks at $f$ as a complex $\sigma$-finite Radon measure in $\Omega$. We comment on this interpretation in Note 2.9.4.

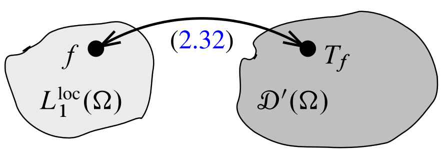



Figure 2.3

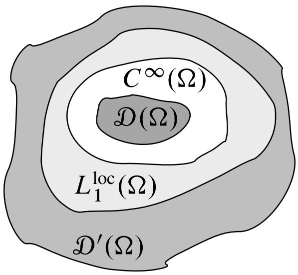



Figure 2.4

But in this book we adopt the usual identification of $f \in L_1^{\mathrm{loc}}(\Omega)$ with $T_f$ according to (2.35) when it comes to distributions, writing $f \in \mathcal{D}'(\Omega)$. This applies also to subspaces of $L_1^{\mathrm{loc}}(\Omega)$, in particular to the inclusions shown in Figure 2.4, and

$$\begin{aligned} \mathcal{D}(\Omega) \subset C^\infty(\Omega) \subset L_p^{\mathrm{loc}}(\Omega) \\ \subset L_1^{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega), \end{aligned} \tag{2.36}$$

with $1 \le p \le \infty$, where the last but one inclusion comes from Hölder's inequality for $L_p$ spaces on bounded domains.

**Example 2.12.** Let $\Omega$ be a domain in $\mathbb{R}^n$ and $a \in \Omega$. Then it follows immediately from Definition 2.5 that $\delta_a$, given by

$$\delta_a(\varphi) = \varphi(a), \quad \varphi \in \mathcal{D}(\Omega), \tag{2.37}$$

is a distribution, $\delta_a \in \mathcal{D}'(\Omega)$. If $a = 0 \in \Omega$, then we put $\delta_0 = \delta$. Both $\delta$ and $\delta_a$ are called $\delta$-*distributions*.

Since $\delta_a(\varphi) = 0$ if $\varphi(a) = 0$, it follows from Proposition 2.7 (ii) applied to $\Omega \setminus \{a\}$ that $\delta_a \in \mathcal{D}'(\Omega)$ cannot be regular according to Definition 2.10. Furthermore, $\delta_a^\gamma$, $a \in \Omega$, $\gamma \in \mathbb{N}_0^n$, with

$$\delta_a^\gamma(\varphi) = (-1)^{|\gamma|} \mathrm{D}^\gamma \varphi(a), \quad \varphi \in \mathcal{D}(\Omega), \tag{2.38}$$

belongs to $\mathcal{D}'(\Omega)$, and also $T_f^\alpha \in \mathcal{D}'(\Omega)$, where

$$T_f^\alpha(\varphi) = (-1)^{|\alpha|} \int_\Omega f(x) \mathrm{D}^\alpha \varphi(x) \mathrm{d}x, \quad \varphi \in \mathcal{D}(\Omega), \tag{2.39}$$

where $\alpha \in \mathbb{N}_0^n$ and $f \in L_1^{\mathrm{loc}}(\Omega)$. The factor $(-1)^{|\alpha|}$ is immaterial, but useful.

**Exercise* 2.13.** A distribution $T \in \mathcal{D}'(\Omega)$ is called *singular* if it is not regular.

(a) Prove that $\delta_a^\gamma$ in (2.38), $a \in \Omega$, $\gamma \in \mathbb{N}_0^n$, is singular.

(b) Show that $T_f^\alpha$ in (2.39), $\alpha \in \mathbb{N}_0^n$, is regular for some non-trivial $f \in L_1^{\mathrm{loc}}(\Omega)$, and singular for other $f \in L_1^{\mathrm{loc}}(\Omega)$.

## 2.3 Derivatives and multiplications with smooth functions

By Remark 2.6 the set $\mathcal{D}'(\Omega)$ of distributions on a domain $\Omega$ in $\mathbb{R}^n$ (according to Definition 2.5) becomes a linear space. Now we additionally equip $\mathcal{D}'(\Omega)$ with two distinguished operations: derivatives, and multiplication with smooth functions.

**Definition 2.14.** Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $\mathcal{D}'(\Omega)$ be as introduced in Definition 2.5 and Remark 2.6.

(i) Let $\alpha \in \mathbb{N}_0^n$ and $T \in \mathcal{D}'(\Omega)$. Then the *derivative* $\mathrm{D}^\alpha T$ is given by

$$(\mathrm{D}^\alpha T)(\varphi) = (-1)^{|\alpha|}\, T(\mathrm{D}^\alpha \varphi), \quad \varphi \in \mathcal{D}(\Omega). \tag{2.40}$$

(ii) Let $g(x) \in C^{\infty,\mathrm{loc}}(\Omega)$ according to (A.7), $T \in \mathcal{D}'(\Omega)$. Then the *multiplication* $gT$ is given by

$$(gT)(\varphi) = T(g\varphi), \quad \varphi \in \mathcal{D}(\Omega). \tag{2.41}$$

**Remark 2.15.** One verifies immediately that for $T \in \mathcal{D}'(\Omega)$, $g \in C^{\infty,\mathrm{loc}}(\Omega)$, and $\alpha \in \mathbb{N}_0^n$, $\mathrm{D}^\alpha T$ and $gT$ are distributions according to Definition 2.5, in particular, since $\varphi_j \underset{\mathcal{D}}{\longrightarrow} \varphi$ implies

$$\mathrm{D}^\alpha \varphi_j \underset{\mathcal{D}}{\longrightarrow} \mathrm{D}^\alpha \varphi, \quad g\varphi_j \underset{\mathcal{D}}{\longrightarrow} g\varphi \tag{2.42}$$

for all $\alpha \in \mathbb{N}_0^n$, $g \in C^{\infty,\mathrm{loc}}(\Omega)$.

If $f \in L_1^{\mathrm{loc}}(\Omega)$ and $g \in C^{\infty,\mathrm{loc}}(\Omega)$, then both $f$ and $gf \in L_1^{\mathrm{loc}}(\Omega)$ can be interpreted as regular distributions according to Definition 2.10 and (2.32). For $\varphi \in \mathcal{D}(\Omega)$ we have

$$T_{gf}(\varphi) = \int_\Omega f(x)g(x)\varphi(x)\mathrm{d}x = T_f(g\varphi) = (gT_f)(\varphi), \tag{2.43}$$

hence $T_{gf} = gT_f$ in $\mathcal{D}'(\Omega)$. In other words, the above definition extends the pointwise multiplication of regular distributions in a consistent way to all distributions in $\mathcal{D}'(\Omega)$. Concerning derivatives, let $f \in C^{k,\mathrm{loc}}(\Omega)$ and let temporarily $\mathrm{D}_\mathrm{c}^\alpha f \in C^{\mathrm{loc}}(\Omega)$, $|\alpha| \le k \in \mathbb{N}$, denote its *classical* derivatives. Then it follows by (2.40) and integration by parts that

$$\begin{aligned}(\mathrm{D}^\alpha T_f)(\varphi) &= (-1)^{|\alpha|}\, T_f(\mathrm{D}_\mathrm{c}^\alpha \varphi)\\ &= (-1)^{|\alpha|}\int_\Omega f(x)(\mathrm{D}_\mathrm{c}^\alpha\varphi)(x)\mathrm{d}x\\ &= \int_\Omega (\mathrm{D}_\mathrm{c}^\alpha f)(x)\varphi(x)\mathrm{d}x = \mathrm{D}_\mathrm{c}^\alpha f(\varphi)\end{aligned} \tag{2.44}$$

for all $\varphi \in \mathcal{D}(\Omega)$. Thus classical derivatives (if they exist) are extended consistently to $\mathcal{D}'(\Omega)$. This observation finally explains the factor $(-1)^{|\alpha|}$ appearing in (2.40). Consequently, in view of the above interpretation, we shall not distinguish between $\mathrm{D}_\mathrm{c}^\alpha$ and $\mathrm{D}^\alpha$ in the sequel.

**Example 2.16.** Let $\delta_a$, $a \in \Omega$, be the $\delta$-distribution according to (2.37), and let $f \in L_1^{\mathrm{loc}}(\Omega)$. Then

$$\mathrm{D}^\gamma \delta_a = \delta_a^\gamma \quad \text{and} \quad \mathrm{D}^\gamma T_f = T_f^\gamma, \quad \gamma \in \mathbb{N}_0^n, \tag{2.45}$$

as defined in (2.38) and (2.39).

**Exercise* 2.17.** Let $\Omega = \mathbb{R}$.

(a) The function

$$\chi(t) = \chi_{[0,\infty)}(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

is called the *Heaviside function*. Prove that $\chi \in L_1^{\mathrm{loc}}(\mathbb{R})$ and $\frac{\mathrm{d}}{\mathrm{d}t}\chi = \delta$ in $\mathcal{D}'(\mathbb{R})$.

(b) Let $g(x) = |x|$, $x \in \mathbb{R}$. Determine $\frac{\mathrm{d}g}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)$ in $\mathcal{D}'(\mathbb{R})$.

**Proposition 2.18.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ and let derivatives and multiplications with $g \in C^{\infty,\mathrm{loc}}(\Omega)$ be explained as in Definition 2.14. Then*

$$\frac{\partial}{\partial x_j}(gT) = \frac{\partial g}{\partial x_j} T + g \frac{\partial T}{\partial x_j}, \quad T \in \mathcal{D}'(\Omega),\ j = 1, \ldots, n, \tag{2.46}$$

*and*

$$\mathrm{D}^{\alpha+\beta} T = \mathrm{D}^\alpha(\mathrm{D}^\beta T) = \mathrm{D}^\beta(\mathrm{D}^\alpha T), \quad T \in \mathcal{D}'(\Omega),\ \alpha, \beta \in \mathbb{N}_0^n. \tag{2.47}$$

**Exercise 2.19.** Prove this proposition by straightforward reasoning or consult [Tri92a, p. 47].

**Remark 2.20.** It is well known that changing the order of classical derivatives in $\mathbb{R}^2$,

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right) = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) \tag{2.48}$$

causes some problems in general – unlike for distributions. They have derivatives of all order which commute arbitrarily without any additional requirements.

## 2.4 Localisations, the spaces $\mathcal{E}'(\Omega)$

Let $\Gamma$ be a compact (that means, bounded and closed) set in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let

$$\operatorname{dist}(x, \Gamma) = \inf\{|x - y| : y \in \Gamma\}, \quad x \in \mathbb{R}^n, \tag{2.49}$$

and

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Gamma) < \varepsilon\}, \quad \varepsilon > 0, \tag{2.50}$$

be an open neighbourhood of $\Gamma$. Then one can construct real non-negative functions $\psi$ with

$$\psi \in \mathcal{D}(\Gamma_\varepsilon) \quad \text{and} \quad \psi(x) = 1,\ x \in \Gamma. \tag{2.51}$$

This can be done by mollification of $\chi = \chi_{\Gamma_{\varepsilon/2}}$ according to (1.59), (2.26),

$$\psi(x) = \chi_h(x) = \int_{\mathbb{R}^n} \omega(y)\chi(x-hy)\mathrm{d}y, \quad x \in \mathbb{R}^n,\ 0 < h < \frac{\varepsilon}{2}\,; \tag{2.52}$$

we also refer to Figure 2.2 as far as this procedure is concerned.

## Resolution of unity

Next we describe the so-called *resolution of unity*. Let the above compact set $\Gamma$ be covered by finitely many open balls $K_j$ of radius $r_j > 0$, $j = 1,\dots,J$.

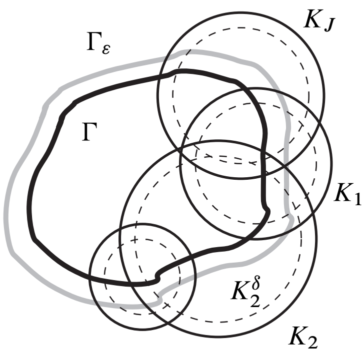



Figure 2.5

Let $K_j^\delta$ be a ball concentric with $K_j$ and of radius $\delta r_j$, where $\delta > 0$. Then we can even refine our assumption

$$\Gamma \subset \bigcup_{j=1}^J K_j$$

by

$$\Gamma_\varepsilon \subset \bigcup_{j=1}^J K_j^\delta \tag{2.53}$$

for suitably chosen $\delta < 1$ and $\varepsilon > 0$, see Figure 2.5 aside. This can be verified by standard reasoning.

In view of our above considerations there are functions $\psi$ with (2.51) and

$$\psi_j \in \mathcal{D}(K_j) \quad \text{with } \psi_j(x) = 1,\ x \in K_j^\delta,\ j = 1,\dots,J. \tag{2.54}$$

We extend $\psi$ outside of $\Gamma_\varepsilon$ and $\psi_j$ outside of $K_j$ by zero. Then

$$\varphi(x) = \sum_{j=1}^J \psi_j(x) \in \mathcal{D}(\mathbb{R}^n) \quad \text{and} \quad \varphi(x) \ge 1,\ x \in \Gamma_\varepsilon. \tag{2.55}$$

Hence

$$\varphi_j(x) = \frac{\psi_j(x)\,\psi(x)}{\varphi(x)} \in \mathcal{D}(K_j \cap \Gamma_\varepsilon), \quad j = 1,\dots,J, \tag{2.56}$$

(extended by zero outside of $K_j \cap \Gamma_\varepsilon$) makes sense and

$$\sum_{j=1}^{J} \varphi_j(x) = 1 \quad \text{if } x \in \Gamma. \tag{2.57}$$

Finally, $\{\varphi_j\}_{j=1}^J$ is the desired *resolution of unity* (subordinate to $\Gamma \subset \bigcup_{j=1}^J K_j$).

Let $\Omega$ be a domain in $\mathbb{R}^n$ and let, say, $f \in L_1^{\mathrm{loc}}(\Omega)$. Assume that

$$\Omega = \bigcup_{j=1}^{\infty} K_j \quad \text{where } K_j \text{ are open balls.} \tag{2.58}$$

Then, plainly, $f$ can be recovered from all its restrictions $f_{|K_j}$. It is remarkable that distributions, though introduced globally according to Definition 2.5, admit a similar *localisation*. If $T \in \mathcal{D}'(\Omega)$, then $T_{|K_j} \in \mathcal{D}'(K_j)$, where

$$(T_{|K_j})(\varphi) = T(\varphi), \quad \varphi \in \mathcal{D}(K_j),\ j \in \mathbb{N}. \tag{2.59}$$

**Theorem 2.21.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $T_1, T_2 \in \mathcal{D}'(\Omega)$ according to Definition 2.5. Let $\{K_j\}_{j=1}^\infty$ be open balls with (2.58). Then $T_1 = T_2$ in $\mathcal{D}'(\Omega)$ if, and only if,*

$$T_{1|K_j} = T_{2|K_j} \text{ in } \mathcal{D}'(K_j), \quad j \in \mathbb{N}. \tag{2.60}$$

*Proof.* Obviously $T_1 = T_2$ in $\mathcal{D}'(\Omega)$ implies (2.60). It remains to prove the converse. Assume that (2.60) is true for $T_1 \in \mathcal{D}'(\Omega)$, $T_2 \in \mathcal{D}'(\Omega)$, let $\varphi \in \mathcal{D}(\Omega)$. Then $\Gamma = \operatorname{supp}\varphi$ is compact (in $\Omega$). It can be covered by finitely many of the balls $K_j$ in (2.58) and we obtain (2.53) for some $J \in \mathbb{N}$. Let $\varphi_j$, $j = 1, \ldots, J$, be as in (2.56), (2.57). Then $\varphi\varphi_j \in \mathcal{D}(K_j)$ such that (2.60) and the linearity of $T_1, T_2$ lead to

$$T_1(\varphi) = T_1\Big(\sum_{j=1}^J \varphi\varphi_j\Big) = \sum_{j=1}^J T_1(\varphi\varphi_j) = \sum_{j=1}^J T_2(\varphi\varphi_j) = T_2(\varphi), \tag{2.61}$$

where we used (2.60). This is just what we wanted to show. □

In (2.1) we said what is meant by the support of a continuous function, complemented in Remark 2.1 by a warning about possible generalisations. Theorem 2.21 paves the way to define the *support of* $T \in \mathcal{D}'(\Omega)$ in such a way that it is consistent with (2.1) when the continuous function $f$ is interpreted as a regular distribution according to Definition 2.10.

If $T(\varphi) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, then $T$ is called the *null distribution*, written as $0 \in \mathcal{D}'(\Omega)$. As before, let

$$K_\delta(x) = \{y \in \mathbb{R}^n : |y - x| < \delta\}, \quad \delta > 0. \tag{2.62}$$

**Definition 2.22** (Support of a distribution). Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $T \in \mathcal{D}'(\Omega)$. Then

$$\operatorname{supp} T = \left\{x \in \bar{\Omega} : T_{|\Omega \cap K_\delta(x)} \neq 0 \text{ for any } \delta > 0\right\} \tag{2.63}$$

is called the *support* of $T$.

**Remark 2.23.** The restriction of $T$ to $\Omega \cap K_\delta(x)$ is defined in analogy to (2.59). We return to Remark 2.1 and interpret $f \in C^{\text{loc}}(\Omega)$ as a regular distribution $T_f$ according to Definition 2.10. Then Proposition 2.7 (ii) implies

$$\operatorname{supp} T_f = \overline{\{x \in \Omega : f(x) \neq 0\}} \tag{2.64}$$

in agreement with (2.1). But for arbitrary $f \in L_1^{\text{loc}}(\Omega)$ the right-hand side of (2.64) and $\operatorname{supp} T_f$ may be different (see also Exercise 2.24 (b) below). For example, let $\Omega = \mathbb{R}$,

$$f(t) = \begin{cases} 1, & t \text{ rational}, \\ 0, & \text{elsewhere}, \end{cases}$$

then $[f] = 0$ and hence $\operatorname{supp} T_f = \emptyset$, whereas

$$\overline{\{t \in \Omega : f(t) \neq 0\}} = \mathbb{R}.$$

*Convention.* We *agree* here that

$$\operatorname{supp} f = \operatorname{supp} T_f \quad \text{whenever } f \in L_1^{\text{loc}}(\Omega) \tag{2.65}$$

and $f$ *is considered as a distribution* (as always in what follows). This does not contradict with our previous notation since we introduced $\operatorname{supp} f$ in (2.1) only for continuous functions where we have (2.64).

**Exercise 2.24.** (a) Let $T \in \mathcal{D}'(\Omega)$. Prove that $K = \Omega \setminus (\Omega \cap \operatorname{supp} T)$ is the largest domain with $K \subset \Omega$ such that $T_{|K} = 0$.

*Hint:* Recall that in this book *domain* means open set. Use the above resolution of unity.

(b) Let $f \in L_1^{\text{loc}}(\Omega)$ and $T_f \in \mathcal{D}'(\Omega)$ the corresponding distribution given by (2.35). Show that a weaker version of (2.64), that is,

$$\operatorname{supp} T_f \subseteq \overline{\{x \in \Omega : f(x) \neq 0\}}$$

is always true.

**Remark 2.25.** Usually it does not matter very much whether one assumes that the underlying domain is connected or not. But in case of Definition 2.22 the situation is different. Even if $\Omega$ is assumed to be connected, $\Omega \cap K_\delta(x)$ need not be connected.

**Exercise 2.26.** Let $a \in \Omega$, $D^\gamma \delta_a$ with $\gamma \in \mathbb{N}_0^n$ be the derivative of $\delta_a$ according to (2.45). Prove that

$$\operatorname{supp} D^\gamma \delta_a = \{a\}, \quad \gamma \in \mathbb{N}_0^n. \tag{2.66}$$

**Definition 2.27.** Let $\Omega$ be a domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$.Then

$$\mathcal{E}'(\Omega) = \{T \in \mathcal{D}'(\Omega) : \operatorname{supp} T \text{ compact in } \Omega\}. \tag{2.67}$$

**Remark 2.28.** One should have in mind that in general $\operatorname{supp} T$ is a subset of $\bar{\Omega}$. The assumption that $\operatorname{supp} T$ is compact in $\Omega$ means that $\Gamma = \operatorname{supp} T \subset \Omega$ and that there are functions $\psi \in \mathcal{D}(\Omega)$ with $\psi(x) = 1$, $x \in \Gamma$, in analogy to (2.51).

**Theorem 2.29.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ and $T \in \mathcal{E}'(\Omega)$. Then there is a number $N \in \mathbb{N}$ and a constant $c > 0$ such that*

$$|T(\varphi)| \le c \sum_{|\alpha| \le N} \sup_{x \in \Omega} |D^\alpha \varphi(x)| \tag{2.68}$$

*for all $\varphi \in \mathcal{D}(\Omega)$.*

*Proof.* We proceed by contradiction. Assume that (2.68) fails and, consequently, for any $c = N = j \in \mathbb{N}$ there is a counter-example $\varphi_j \in \mathcal{D}(\Omega)$ of (2.68). Moreover, since with $\varphi_j$ also $\lambda \varphi_j$ yields such a counter-example for any $\lambda \in \mathbb{C} \setminus \{0\}$, we can find a normalised sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{D}(\Omega)$ such that

$$1 = |T(\varphi_j)| > j \sum_{|\alpha| \le j} \sup_{x \in \Omega} |D^\alpha \varphi_j(x)|, \quad j \in \mathbb{N}. \tag{2.69}$$

Let $\psi \in \mathcal{D}(\Omega)$ with $\psi(x) = 1$ in a neighbourhood of $\operatorname{supp} T$. Then $T(\psi\varphi_j) = T(\varphi_j)$ and hence $|T(\psi\varphi_j)| = 1$. On the other hand, Definitions 2.2, 2.5 and (2.69) imply

$$\psi\varphi_j \xrightarrow[\mathcal{D}]{} 0 \quad \text{and} \quad T(\psi\varphi_j) \to 0 \quad \text{if } j \to \infty. \tag{2.70}$$

But this contradicts $|T(\psi\varphi_j)| = 1$. □

**Remark 2.30.** Let $a \in \Omega$, $N \in \mathbb{N}_0$, $a_\alpha \in \mathbb{C}$, and

$$T = \sum_{|\alpha| \le N} a_\alpha D^\alpha \delta_a \quad \text{with} \sum_{|\alpha| \le N} |a_\alpha| > 0. \tag{2.71}$$

Then one proves immediately that $\operatorname{supp} T = \{a\}$. There is a remarkable converse of this assertion.

**Theorem 2.31.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, and let $a \in \Omega$, $T \in \mathcal{D}'(\Omega)$ with* $\operatorname{supp} T = \{a\}$. *Then*

$$T = \sum_{|\alpha| \le N} a_\alpha D^\alpha \delta_a \tag{2.72}$$

*for some $N \in \mathbb{N}$ and suitable $a_\alpha \in \mathbb{C}$.*

*Proof.* We may assume $a = 0$ and $K_{2\varepsilon} = \{x \in \mathbb{R}^n : |x| < 2\varepsilon\} \subset \Omega$ for sufficiently small $\varepsilon > 0$. We want to apply (2.68). Let $h \in \mathcal{D}(K_{2\varepsilon}) \subset \mathcal{D}(\Omega)$ with $h(x) = 1$, $|x| \le \varepsilon$, and $h_j(x) = h(2^j x)$, $j \in \mathbb{N}$. Let $\varphi \in \mathcal{D}(\Omega)$ and

$$\varphi(x) = \sum_{|\alpha| \le N} \frac{\mathrm{D}^\alpha \varphi(0)}{\alpha!} x^\alpha + r(x) = \sum_{|\alpha| \le N} b_\alpha (\mathrm{D}^\alpha \varphi)(0)\, x^\alpha + r(x) \tag{2.73}$$

its Taylor expansion at $x^0 = 0$, where $r(x)$ is the remainder term with

$$|\mathrm{D}^\gamma r(x)| \le c\, |x|^{N+1-|\gamma|} \quad \text{for } \gamma \in \mathbb{N}_0^n,\ |\gamma| \le N.$$

Thus (2.73) and supp $T = \{0\}$ lead to

$$T(\varphi) = T(h\varphi) = \sum_{|\alpha| \le N} b_\alpha (\mathrm{D}^\alpha \varphi)(0)\, T(x^\alpha h) + T(hr). \tag{2.74}$$

Since

$$|\mathrm{D}^\beta h_j(x) \mathrm{D}^\gamma r(x)| \le c\, 2^{j|\beta|} 2^{-j(N+1-|\gamma|)} \le c'\, 2^{-j} \tag{2.75}$$

if $|\beta| + |\gamma| \le N$, one obtains by (2.68) and supp $T = \{0\}$ that

$$|T(hr)| = |T(h_j r)| \longrightarrow 0 \quad \text{if } j \to \infty. \tag{2.76}$$

Hence the last term in (2.74) disappears and we get

$$T(\varphi) = \sum_{|\alpha| \le N} (-1)^{|\alpha|} b_\alpha\, T(x^\alpha h)(\mathrm{D}^\alpha \delta)(\varphi), \tag{2.77}$$

where we used, in addition, (2.38) and (2.45). This proves (2.72). □

## 2.5 The space $\mathcal{S}(\mathbb{R}^n)$, the Fourier transform

In the special case $\Omega = \mathbb{R}^n$ we have so far the space $\mathcal{D}(\mathbb{R}^n)$ as introduced in Definition 2.2 and its dual space of distributions $\mathcal{D}'(\mathbb{R}^n)$ according to Definition 2.5. The Fourier transform is one of the most powerful instruments in the theory of distributions and, in particular, in the recent theory of function spaces. But for this purpose, $\mathcal{D}(\mathbb{R}^n)$ is too small and, consequently, $\mathcal{D}'(\mathbb{R}^n)$ too large. Asking for something appropriate in between one arrives at the optimally adapted space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$.

**Definition 2.32.** For $n \in \mathbb{N}$ let

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{k,\ell} < \infty \text{ for all } k \in \mathbb{N}_0,\ \ell \in \mathbb{N}_0\}, \tag{2.78}$$

where

$$\|\varphi\|_{k,\ell} = \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{k/2} \sum_{|\alpha| \le \ell} |\mathrm{D}^\alpha \varphi(x)|. \tag{2.79}$$

A sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ is said to converge in $\mathcal{S}(\mathbb{R}^n)$ to $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we shall write $\varphi_j \underset{\mathcal{S}}{\longrightarrow} \varphi$, if

$$\|\varphi_j - \varphi\|_{k,\ell} \longrightarrow 0 \quad \text{for } j \to \infty \text{ and all } k \in \mathbb{N}_0,\ \ell \in \mathbb{N}_0. \tag{2.80}$$

**Remark 2.33.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\ell = 0$ in (2.79), then $|\varphi(x)| \leq c_k(1+|x|^k)^{-1}$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$; similarly for all derivatives $\mathrm{D}^\alpha\varphi(x)$, $\alpha \in \mathbb{N}_0^n$. This explains why $\mathcal{S}(\mathbb{R}^n)$ is usually called *the Schwartz space of all rapidly decreasing infinitely differentiable functions in* $\mathbb{R}^n$ (Schwartz space, for short).

By Definition 2.2,

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n), \quad \text{and} \quad \varphi_j \underset{\mathcal{D}}{\longrightarrow} \varphi \quad \text{implies} \quad \varphi_j \underset{\mathcal{S}}{\longrightarrow} \varphi. \tag{2.81}$$

On the other hand, there are functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ which do not belong to $\mathcal{D}(\mathbb{R}^n)$, the most prominent example might be

$$\varphi(x) = e^{-|x|^2}, \quad x \in \mathbb{R}^n. \tag{2.82}$$

For later use, we introduce the notation

$$\langle\xi\rangle = (1+|\xi|^2)^{1/2}, \quad \xi \in \mathbb{R}^n. \tag{2.83}$$

**Exercise 2.34.** (a) Prove that it is sufficient to restrict (2.78)–(2.80) to

$$\|\varphi\|_\ell = \sup_{x\in\mathbb{R}^n} \langle x\rangle^\ell \sum_{|\alpha|\leq\ell} |\mathrm{D}^\alpha\varphi(x)|, \quad \ell \in \mathbb{N}_0. \tag{2.84}$$

(b) Let $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ be a sequence in $\mathcal{S}(\mathbb{R}^n)$ such that for all $\ell \in \mathbb{N}_0$, and $\varepsilon > 0$,

$$\|\varphi_j - \varphi_k\|_\ell \leq \varepsilon \quad \text{if } j \geq k \geq k(\varepsilon,\ell). \tag{2.85}$$

Prove that there is a (uniquely determined) function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with (2.80). Hence any such sequence in $\mathcal{S}(\mathbb{R}^n)$ is convergent in $\mathcal{S}(\mathbb{R}^n)$.

**Exercise 2.35.** Let for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n)$, the functions $\widetilde{\varrho}\colon \mathcal{S}(\mathbb{R}^n) \to [0,\infty)$ and $\varrho\colon \mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n) \to [0,\infty)$ be defined by

$$\widetilde{\varrho}(\varphi) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|\varphi\|_k}{1+\|\varphi\|_k}, \quad \varrho(\varphi,\psi) = \widetilde{\varrho}(\varphi-\psi). \tag{2.86}$$

Prove that $\varrho$ is a metric and that $\big(\mathcal{S}(\mathbb{R}^n),\varrho\big)$ is a complete metric space with the same topology as $\mathcal{S}(\mathbb{R}^n)$: A sequence converges in $\big(\mathcal{S}(\mathbb{R}^n),\varrho\big)$ if, and only if, it converges in $\mathcal{S}(\mathbb{R}^n)$ according to Definition 2.32.

**Definition 2.36.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\widehat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \varphi(x)\mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{2.87}$$

is called the *Fourier transform* of $\varphi$, and

$$\varphi^{\vee}(\xi) = (\mathcal{F}^{-1}\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi}\, \varphi(x)\mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{2.88}$$

the *inverse Fourier transform* of $\varphi$.

**Remark 2.37.** Recall that $x\xi$ is the scalar product of $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, see (A.5). Since $|\varphi(x)| \le c|x|^{-n-1}$ for $|x| \ge 1$, both (2.87) and (2.88) make sense and

$$\|\widehat{\varphi}\,|L_\infty(\mathbb{R}^n)\| \le c\|\varphi\|_{n+1,0}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.89}$$

and similarly for $\varphi^{\vee}$. As we shall see below $\mathcal{F}\varphi \in \mathcal{S}(\mathbb{R}^n)$ if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and

$$\mathcal{F}^{-1}\mathcal{F}\varphi = \mathcal{F}\mathcal{F}^{-1}\varphi = \varphi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.90}$$

This will justify calling $\mathcal{F}^{-1}$ the *inverse* Fourier transform.

Recall our notation (A.3).

**Theorem 2.38.** (i) *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, $x^\alpha\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\mathrm{D}^\alpha\varphi \in \mathcal{S}(\mathbb{R}^n)$ for $\alpha \in \mathbb{N}_0^n$, and*

$$\mathrm{D}^\alpha(\mathcal{F}\varphi)(\xi) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha\varphi(x))(\xi), \quad \alpha \in \mathbb{N}_0^n,\ \xi \in \mathbb{R}^n, \tag{2.91}$$

*and*

$$\xi^\alpha(\mathcal{F}\varphi)(\xi) = (-i)^{|\alpha|}\mathcal{F}(\mathrm{D}^\alpha\varphi)(\xi), \quad \alpha \in \mathbb{N}_0^n,\ \xi \in \mathbb{R}^n. \tag{2.92}$$

(ii) *Let $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$, and $\varphi_j \underset{\mathcal{S}}{\longrightarrow} \varphi$ according to Definition 2.32. Then*

$$\mathcal{F}\varphi_j \underset{\mathcal{S}}{\longrightarrow} \mathcal{F}\varphi \quad \textit{and} \quad \mathcal{F}^{-1}\varphi_j \underset{\mathcal{S}}{\longrightarrow} \mathcal{F}^{-1}\varphi. \tag{2.93}$$

*Proof. Step* 1. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, then one gets immediately $x^\alpha\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\mathrm{D}^\alpha\varphi \in \mathcal{S}(\mathbb{R}^n)$. Hence the right-hand sides of (2.91) and (2.92) make sense. The mean value theorem and Lebesgue's bounded convergence theorem imply

$$\frac{\partial}{\partial\xi_\ell}(\mathcal{F}\varphi)(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}(-i)\,x_\ell\, e^{-ix\xi}\varphi(x)\mathrm{d}x = (-i)\mathcal{F}(x_\ell\,\varphi(x))(\xi). \tag{2.94}$$

Iteration gives (2.91). As for (2.92) we first remark that

$$\xi_\ell(\mathcal{F}\varphi)(\xi) = (2\pi)^{-n/2}\, i \int_{\mathbb{R}^n} \frac{\partial}{\partial x_\ell}(e^{-ix\xi})\varphi(x)\mathrm{d}x. \tag{2.95}$$

Integration by parts in $x_\ell$-direction for intervals tending to $\mathbb{R}$ leads to

$$\xi_\ell(\mathcal{F}\varphi)(\xi) = (-i)\,(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\,\frac{\partial\varphi}{\partial x_\ell}(x)\mathrm{d}x = (-i)\,\mathcal{F}\Big(\frac{\partial\varphi}{\partial x_\ell}\Big)(\xi) \tag{2.96}$$

and iteration concludes the argument for (2.92).

*Step* 2. By (2.91), (2.92) and (2.89) one obtains

$$\|\mathcal{F}\varphi\|_{k,\ell} \le c\,\|\varphi\|_{\ell+n+1,k}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.97}$$

This proves $\mathcal{F}\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, (2.93) is now an immediate consequence of (2.80) and (2.97). □

**Exercise* 2.39.** What are the counterparts of (2.91), (2.92) for $\mathcal{F}^{-1}$?

**Proposition 2.40.** (i) *Let* $\varepsilon > 0$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^n)$. *Then*

$$\mathcal{F}(\varphi(\varepsilon\cdot))(\xi) = \varepsilon^{-n}\,\mathcal{F}(\varphi)\Big(\frac{\xi}{\varepsilon}\Big), \quad \xi \in \mathbb{R}^n. \tag{2.98}$$

(ii) *Furthermore,*

$$\mathcal{F}(e^{-|x|^2/2})(\xi) = e^{-|\xi|^2/2}, \quad \xi \in \mathbb{R}^n. \tag{2.99}$$

*Proof.* We replace $\varphi(x)$ in (2.87) by $\varphi(\varepsilon x)$ and obtain (2.98) from the dilation $y = \varepsilon x$. In view of the product structure of (2.87) with $\varphi(x) = e^{-|x|^2/2}$ it is sufficient to show (2.99) for $n = 1$. For this purpose we consider

$$h(s) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-t^2/2}e^{-its}\mathrm{d}t, \quad s \in \mathbb{R}, \tag{2.100}$$

and calculate by integration by parts,

$$\begin{aligned} h'(s) &= (2\pi)^{-1/2}\, i \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}e^{-t^2/2}\, e^{-its}\mathrm{d}t \\ &= -s\,(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-t^2/2}e^{-its}\mathrm{d}t = -s\,h(s). \end{aligned} \tag{2.101}$$

Hence

$$h(s) = h(0)e^{-s^2/2} = e^{-s^2/2}, \quad s \in \mathbb{R}, \tag{2.102}$$

since $h(0) = 1$ (the well-known Gauß integral), cf. [Cou37, Chapter X.6.5, p. 496]. By (2.100) this is nothing else than (2.99) for $n = 1$. □

**Remark 2.41.** Due to (2.99), $\varphi(x) = e^{-|x|^2/2}$ is sometimes called an *eigenfunction* of $\mathcal{F}$.

After these preparations we can prove (2.90) now. So far we know by Theorem 2.38 (i) and (2.97) that

$$\mathcal{F}\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{F}^{-1}\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n). \tag{2.103}$$

**Theorem 2.42.** *Let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$. *Then*

$$\varphi = \mathcal{F}^{-1}\mathcal{F}\varphi = \mathcal{F}\mathcal{F}^{-1}\varphi. \tag{2.104}$$

*Furthermore, both* $\mathcal{F}$ *and* $\mathcal{F}^{-1}$ *map* $\mathcal{S}(\mathbb{R}^n)$ *one-to-one onto itself,*

$$\mathcal{F}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) \quad \textit{and} \quad \mathcal{F}^{-1}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n). \tag{2.105}$$

*Proof. Step* 1. We begin with a preparation. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then we get by Fubini's theorem and (2.87) for $x \in \mathbb{R}^n$ that

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathcal{F}\varphi)(\xi)\, e^{ix\xi}\psi(\xi)\mathrm{d}\xi &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\varphi(y)\int_{\mathbb{R}^n} e^{-i(y-x)\xi}\psi(\xi)\mathrm{d}\xi\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\varphi(y)(\mathcal{F}\psi)(y-x)\mathrm{d}y. \end{aligned}\tag{2.106}$$

A change of variables leads to

$$\int_{\mathbb{R}^n} (\mathcal{F}\varphi)(\xi)\, e^{ix\xi}\psi(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^n}\varphi(x+y)(\mathcal{F}\psi)(y)\mathrm{d}y. \tag{2.107}$$

Let $\psi(x) = e^{-\frac{\varepsilon^2|x|^2}{2}}$ for $\varepsilon > 0$, $x \in \mathbb{R}^n$. Then Proposition 2.40 implies that

$$(\mathcal{F}\psi)(y) = \varepsilon^{-n}\,\mathcal{F}(e^{-\frac{|x|^2}{2}})\left(\frac{y}{\varepsilon}\right) = \varepsilon^{-n}\,e^{-\frac{|y|^2}{2\varepsilon^2}}. \tag{2.108}$$

We insert it in (2.107) and the transformation $y = \varepsilon z$ gives

$$\int_{\mathbb{R}^n} (\mathcal{F}\varphi)(\xi)\, e^{ix\xi}\, e^{-\frac{\varepsilon^2|\xi|^2}{2}}\mathrm{d}\xi = \int_{\mathbb{R}^n}\varphi(x+\varepsilon z)\, e^{-\frac{|z|^2}{2}}\mathrm{d}z. \tag{2.109}$$

With $\varepsilon \to 0$ it follows by Lebesgue's bounded convergence theorem that

$$\int_{\mathbb{R}^n} (\mathcal{F}\varphi)(\xi)\, e^{ix\xi}\, \mathrm{d}\xi = \varphi(x) \int_{\mathbb{R}^n} e^{-|z|^2/2} \mathrm{d}z = (2\pi)^{n/2} \varphi(x), \tag{2.110}$$

using again the Gauß integral as in connection with (2.102). In view of (2.88) this proves the first equality in (2.104); similarly for the second equality.

*Step* 2. We apply (2.104) to $\psi = \mathcal{F}^{-1}\varphi$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and obtain $\varphi = \mathcal{F}\psi$. This establishes the first equality in (2.105). Similarly for the second equality. If $\mathcal{F}\varphi_1 = \mathcal{F}\varphi_2$, then one gets by (2.104) that $\varphi_1 = \varphi_2$. Hence $\mathcal{F}$ and, similarly, $\mathcal{F}^{-1}$ are one-to-one mappings of $\mathcal{S}(\mathbb{R}^n)$ onto itself. □

## 2.6 The space $\mathcal{S}'(\mathbb{R}^n)$

We introduced in Definition 2.5 the space $\mathcal{D}'(\Omega)$ as the collection of all linear continuous functionals over $\mathcal{D}(\Omega)$. Now we are doing the same with $\mathcal{S}(\mathbb{R}^n)$ in place of $\mathcal{D}(\Omega)$.

**Definition 2.43.** Let $\mathcal{S}(\mathbb{R}^n)$ be as in Definition 2.32. Then $\mathcal{S}'(\mathbb{R}^n)$ is the collection of all complex-valued linear continuous functionals $T$ over $\mathcal{S}(\mathbb{R}^n)$, that is,

$$T: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}, \quad T: \varphi \mapsto T(\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.111}$$

$$T(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 T(\varphi_1) + \lambda_2 T(\varphi_2), \quad \lambda_1, \lambda_2 \in \mathbb{C};\ \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^n), \tag{2.112}$$

and

$$T(\varphi_j) \longrightarrow T(\varphi) \quad \text{for } j \to \infty \text{ whenever } \varphi_j \underset{\mathcal{S}}{\longrightarrow} \varphi, \tag{2.113}$$

according to (2.79), (2.80).

**Remark 2.44.** We write $T \in \mathcal{S}'(\mathbb{R}^n)$ and call $T$ a *tempered distribution* or *slowly increasing distribution*. This notation will be justified by the examples given below. Some comments may be found in Note 2.9.2.

Similarly to Remark 2.6 with respect to $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, we look at $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, as a dual pairing of locally convex spaces. In particular,

$$T_1 = T_2 \text{ in } \mathcal{S}'(\mathbb{R}^n) \text{ means } T_1(\varphi) = T_2(\varphi) \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.114}$$

and $\mathcal{S}'(\mathbb{R}^n)$ is converted into a linear space by

$$(\lambda_1 T_1 + \lambda_2 T_2)(\varphi) = \lambda_1 T_1(\varphi) + \lambda_2 T_2(\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.115}$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $T_1, T_2 \in \mathcal{S}'(\mathbb{R}^n)$. Again it is sufficient for us to furnish $\mathcal{S}'(\mathbb{R}^n)$ with the *simple convergence topology*, that is,

$$T_j \to T \text{ in } \mathcal{S}'(\mathbb{R}^n), \quad T_j \in \mathcal{S}'(\mathbb{R}^n),\ j \in \mathbb{N},\ T \in \mathcal{S}'(\mathbb{R}^n), \tag{2.116}$$

means that

$$T_j(\varphi) \to T(\varphi) \text{ in } \mathbb{C} \quad \text{if } j \to \infty \text{ for any } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.117}$$

**Remark 2.45.** On $\mathbb{R}^n$ one can compare (with some care) the three types of distributions $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, introduced in Definitions 2.5, 2.27 (with $\Omega = \mathbb{R}^n$) and 2.43, respectively. Appropriately interpreted, one has

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n). \tag{2.118}$$

The second inclusion means that $T \in \mathcal{S}'(\mathbb{R}^n)$ restricted to $\mathcal{D}(\mathbb{R}^n)$ is an element of $\mathcal{D}'(\mathbb{R}^n)$. As for the first inclusion one extends the domain of definition of $T \in \mathcal{E}'(\mathbb{R}^n)$ from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ by

$$T(\varphi) = T(\varphi\psi), \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.119}$$

where $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\psi(x) = 1$ in a neighbourhood of the compact set $\operatorname{supp} T$. One must prove that this definition is independent of $\psi$.

**Exercise 2.46.** Prove (2.118) in the interpretation given above.

*Hint:* Use Exercise 2.24 and Theorem 2.29.

**Example 2.47.** By (2.118) and the given interpretation, (2.66) implies that

$$\mathrm{D}^\gamma \delta_a \in \mathcal{S}'(\mathbb{R}^n), \quad a \in \mathbb{R}^n, \ \gamma \in \mathbb{N}_0^n. \tag{2.120}$$

**Theorem 2.48.** *Let $T$ be a linear form on $\mathcal{S}(\mathbb{R}^n)$ satisfying (2.111) and (2.112). Then $T \in \mathcal{S}'(\mathbb{R}^n)$ if, and only if, there are numbers $c > 0$, $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}_0$, such that*

$$|T(\varphi)| \le c\, \|\varphi\|_{k,\ell} \quad \textit{for all } \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.121}$$

*with $\|\varphi\|_{k,\ell}$ as in (2.79).*

*Proof.* Let $T$ be a linear form on $\mathcal{S}(\mathbb{R}^n)$ with (2.111), (2.112); then (2.121) implies (2.113), i.e., $T \in \mathcal{S}'(\mathbb{R}^n)$. Conversely, let $T \in \mathcal{S}'(\mathbb{R}^n)$. We prove (2.121) by contradiction: Assume that for all $k \in \mathbb{N}$ there exists some $\varphi_k \in \mathcal{S}(\mathbb{R}^n)$ with $|T(\varphi_k)| > k\|\varphi_k\|_{k,k}$. Moreover, for any $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, $\lambda\varphi_k$ satisfies the same inequality, $k \in \mathbb{N}$, such that we can assume

$$1 = |T(\varphi_k)| > k\|\varphi_k\|_{k,k}, \quad k \in \mathbb{N}. \tag{2.122}$$

This implies that $\varphi_k \underset{\mathcal{S}}{\longrightarrow} 0$, and $T(\varphi_k) \to 0$ for $k \to \infty$, since $T \in \mathcal{S}'(\mathbb{R}^n)$. This contradicts (2.122). □

**Remark 2.49.** Recall that by (2.35) with (2.32) one has, appropriately interpreted,

$$L_1^{\text{loc}}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n). \tag{2.123}$$

One may ask for which regular distribution $f \in L_1^{\text{loc}}(\mathbb{R}^n)$,

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.124}$$

generates even a tempered distribution. If this is the case, then it follows by (2.118) and the discussion in Remark 2.11 that one can identify $f$, more precisely, its equivalence class $[f]$, with the tempered distribution generated. Having in mind this ambiguity we write $f \in \mathcal{S}'(\mathbb{R}^n)$ as in (2.35).

**Corollary 2.50.** *Let* $1 \le p \le \infty$. *Then*

$$L_p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \tag{2.125}$$

*in the interpretation* (2.124).

*Proof.* Let $p'$ be given by $\frac{1}{p} + \frac{1}{p'} = 1$. Then (2.125) follows by Hölder's inequality, since for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x \right| = \|f|L_p(\mathbb{R}^n)\| \, \|\varphi|L_{p'}(\mathbb{R}^n)\| \le \|f|L_p(\mathbb{R}^n)\| \, \|\varphi\|_{k,0} \tag{2.126}$$

for some $k \in \mathbb{N}$, $k \ge k(p,n)$. □

**Exercise* 2.51.** Determine $k(p,n)$.

We collect some further examples and counter-examples of distributions in $\mathcal{S}'(\mathbb{R}^n)$, always interpreted as in (2.124).

**Exercise* 2.52.** (a) Let for $m \in \mathbb{N}_0$,

$$p(x) = \sum_{|\alpha| \le m} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{C}, \; \alpha \in \mathbb{N}_0^n, \; x \in \mathbb{R}^n, \tag{2.127}$$

be an arbitrary polynomial, recall notation (A.3). Prove that $p \in \mathcal{S}'(\mathbb{R}^n)$.

(b) Show that $g(x) = e^{|x|^2} \notin \mathcal{S}'(\mathbb{R}^n)$.

(c) Let $1 \le p \le \infty$, $f \in L_p(\mathbb{R}^n)$, $p$ a polynomial (of arbitrary order $m \in \mathbb{N}_0$) according to (2.127). Prove that

$$pf \in \mathcal{S}'(\mathbb{R}^n). \tag{2.128}$$

**Remark 2.53.** The above examples and counter-examples may explain why the distributions $T \in \mathcal{S}'(\mathbb{R}^n)$ are called *tempered* (or *slowly increasing*). Note that Exercise 2.52 (b) implies that one cannot replace $\mathcal{D}'(\mathbb{R}^n)$ in (2.123) by $\mathcal{S}'(\mathbb{R}^n)$.

**Remark 2.54.** Recall that we furnished $\mathcal{S}'(\mathbb{R}^n)$ with the simple convergence topology (2.116), (2.117), see Remark 2.44. If $\{f_j\}_{j=1}^\infty \subset L_p(\mathbb{R}^n)$ is convergent in $L_p(\mathbb{R}^n)$, $1 \le p \le \infty$,

$$f_j \longrightarrow f \text{ in } L_p(\mathbb{R}^n) \quad \text{if } j \to \infty, \tag{2.129}$$

then one gets by (2.126) also

$$f_j \longrightarrow f \text{ in } \mathcal{S}'(\mathbb{R}^n) \quad \text{if } j \to \infty. \tag{2.130}$$

Hence, (2.125) is also a topological embedding.

According to Definition 2.14 and (2.43), (2.44), derivatives and multiplications with smooth functions can be consistently extended from functions to distributions $T \in \mathcal{D}'(\Omega)$. Whereas (2.40) has an immediate counterpart for tempered distributions $T \in \mathcal{S}'(\mathbb{R}^n)$ the multiplication (2.41) with smooth functions requires now some growth restriction at infinity. This follows from the above examples and counter-examples in Exercise 2.52. We restrict ourselves here to

$$p(x) = \begin{cases} \langle x\rangle^\sigma, & \sigma \in \mathbb{R}, \\ \sum_{j=1}^m a_j e^{ih^j x}, & m \in \mathbb{N},\ a_j \in \mathbb{C},\ h^j \in \mathbb{R}^n, \\ \sum_{|\beta|\le m} a_\beta x^\beta, & m \in \mathbb{N},\ \beta \in \mathbb{N}_0^n,\ a_\beta \in \mathbb{C}, \end{cases} \tag{2.131}$$

recall notation (2.83).

**Definition 2.55.** Let $\alpha \in \mathbb{N}_0^n$ and let $p$ be as in (2.131). Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then the derivative $\mathrm{D}^\alpha T$ is given by

$$(\mathrm{D}^\alpha T)(\varphi) = (-1)^{|\alpha|}\, T(\mathrm{D}^\alpha \varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.132}$$

and

$$(pT)(\varphi) = T(p\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.133}$$

**Corollary 2.56.** *Let $\alpha \in \mathbb{N}_0^n$, and $p$ be as in (2.131). Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then*

$$\mathrm{D}^\alpha T \in \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad pT \in \mathcal{S}'(\mathbb{R}^n).$$

*Proof.* This follows immediately from Theorem 2.48. □

**Remark 2.57.** Again by Remark 2.15, Definition 2.55 and Corollary 2.56 are consistent with the above considerations (classical and distributional interpretation as elements of $\mathcal{D}'(\mathbb{R}^n)$).

## 2.7 The Fourier transform in $\mathcal{S}'(\mathbb{R}^n)$

So far we introduced in Definition 2.36 the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ on $\mathcal{S}(\mathbb{R}^n)$. By (2.125) one can consider $\mathcal{S}(\mathbb{R}^n)$ as a subset of $\mathcal{S}'(\mathbb{R}^n)$. In this sense we wish to extend $\mathcal{F}$ and $\mathcal{F}^{-1}$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. Temporarily we reserve $\widehat{\varphi}$ and $\varphi^\vee$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ according to Definition 2.36.

**Definition 2.58.** Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then the Fourier transform $\mathcal{F}T$ and the inverse Fourier transform $\mathcal{F}^{-1}T$ are given by

$$(\mathcal{F}T)(\varphi) = T(\widehat{\varphi}) \quad \text{and} \quad (\mathcal{F}^{-1}T)(\varphi) = T(\varphi^\vee), \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.134}$$

**Remark 2.59.** Theorem 2.38 implies for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ that $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi^\vee \in \mathcal{S}(\mathbb{R}^n)$; hence (2.134) makes sense. Furthermore, one gets by (2.121), (2.97) that

$$|(\mathcal{F}T)(\varphi)| \le c\, \|\varphi\|_{k,\ell}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.135}$$

for some $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}_0$. Then we obtain by Theorem 2.48 that $\mathcal{F}T \in \mathcal{S}'(\mathbb{R}^n)$. Similarly, $\mathcal{F}^{-1}T \in \mathcal{S}'(\mathbb{R}^n)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}\varphi$ be as in (2.134) and $\widehat{\varphi}$ according to (2.87). Let $\psi \in \mathcal{S}(\mathbb{R}^n)$; then Fubini's theorem leads to

$$\begin{aligned}
(\mathcal{F}\varphi)(\psi) = \varphi(\widehat{\psi}) &= \int_{\mathbb{R}^n} \varphi(x)\widehat{\psi}(x)\,\mathrm{d}x \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x) e^{-ixy}\, \psi(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \psi(y)(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ixy}\varphi(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \psi(y)\widehat{\varphi}(y)\,\mathrm{d}y = \widehat{\varphi}(\psi).
\end{aligned} \tag{2.136}$$

Hence $\mathcal{F}\varphi = \widehat{\varphi}$. Similarly, $\mathcal{F}^{-1}\varphi = \varphi^\vee$. In other words, $\mathcal{F}$ and $\mathcal{F}^{-1}$ extend the Fourier transform and its inverse from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, respectively.

**Theorem 2.60.** (i) *Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then*

$$T = \mathcal{F}\mathcal{F}^{-1}T = \mathcal{F}^{-1}\mathcal{F}T. \tag{2.137}$$

*Furthermore, both $\mathcal{F}$ and $\mathcal{F}^{-1}$ map $\mathcal{S}'(\mathbb{R}^n)$ one-to-one onto itself,*

$$\mathcal{F}\mathcal{S}'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n), \quad \mathcal{F}^{-1}\mathcal{S}'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n). \tag{2.138}$$

(ii) *Let* $T \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\alpha \in \mathbb{N}_0^n$. *Then* $x^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$, *and* $\mathrm{D}^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$. *Furthermore,*

$$\mathcal{F}(\mathrm{D}^\alpha T) = i^{|\alpha|}\, x^\alpha (\mathcal{F}T) \quad \textit{and} \quad \mathcal{F}(x^\alpha T) = i^{|\alpha|}\, \mathrm{D}^\alpha(\mathcal{F}T). \tag{2.139}$$

*Proof. Step* 1. By Remark 2.59 one has $\mathcal{F}T \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}^{-1}T \in \mathcal{S}'(\mathbb{R}^n)$ if $T \in \mathcal{S}'(\mathbb{R}^n)$. Then (2.134) and Theorem 2.42 imply for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$(\mathcal{F}\mathcal{F}^{-1}T)(\varphi) = (\mathcal{F}^{-1}T)(\widehat{\varphi}) = T((\widehat{\varphi})^\vee) = T(\varphi). \tag{2.140}$$

This proves (2.137) and also (2.138) by an argument parallel to Step 2 in the proof of Theorem 2.42.

*Step* 2. By Corollary 2.56 we have $x^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{D}^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$ for $T \in \mathcal{S}'(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We obtain by (2.134) and (2.132) that

$$\begin{aligned}\mathcal{F}(\mathrm{D}^\alpha T)(\varphi) &= (\mathrm{D}^\alpha T)(\widehat{\varphi}) = (-1)^{|\alpha|}\, T(\mathrm{D}^\alpha \widehat{\varphi}) \\ &= i^{|\alpha|}\, T(\widehat{x^\alpha \varphi}) = i^{|\alpha|}(\mathcal{F}T)(x^\alpha \varphi) \\ &= i^{|\alpha|}(x^\alpha \mathcal{F}T)(\varphi),\end{aligned} \tag{2.141}$$

where we additionally used (2.91). In a similar way one can prove the second equality in (2.139). □

**Exercise* 2.61.** What is the counterpart of (2.139) for $\mathcal{F}^{-1}$?

We collect further properties and examples of $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}^n)$.

**Exercise 2.62.** (a) Let $\delta = \delta_0$ according to Example 2.12. Prove that

$$\mathcal{F}\Big(\sum_{|\alpha|\le N} a_\alpha \mathrm{D}^\alpha \delta\Big) = (2\pi)^{-n/2} \sum_{|\alpha|\le N} a_\alpha\, i^{|\alpha|} x^\alpha. \tag{2.142}$$

*Hint:* Verify first

$$\mathcal{F}\delta = (2\pi)^{-n/2} \tag{2.143}$$

and use (2.139) afterwards.

(b) Let $h \in \mathbb{R}^n$. Show that the translation

$$(\tau_h \varphi)(x) = \varphi(x+h), \quad x \in \mathbb{R}^n,\ \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.144}$$

can be consistently extended to $\mathcal{S}'(\mathbb{R}^n)$ by

$$(\tau_h T)(\varphi) = T(\tau_{-h}\varphi), \quad T \in \mathcal{S}'(\mathbb{R}^n),\ \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.145}$$

Prove that for $T \in \mathcal{S}'(\mathbb{R}^n)$, $h \in \mathbb{R}^n$,

$$\mathcal{F}(\tau_h T) = e^{ihx}\, \mathcal{F}T, \tag{2.146}$$

and

$$\mathcal{F}(e^{-ihx}T) = \tau_h \mathcal{F} T. \tag{2.147}$$

(c) Let $T \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} T = \{a\}$ for some $a \in \mathbb{R}^n$. Prove that $\mathcal{F}T$ is regular.

*Hint:* Use Theorem 2.31 together with parts (a) and (b) above.

The following simple observation will be of some service for us later on. For $\sigma \in \mathbb{R}$, let $I_\sigma$ be given by

$$I_\sigma f = \mathcal{F}^{-1} \langle \xi \rangle^\sigma \mathcal{F} f, \quad f \in \mathcal{S}'(\mathbb{R}^n). \tag{2.148}$$

**Proposition 2.63.** *Let $\sigma \in \mathbb{R}$ and $I_\sigma$ be given by (2.148). Then $I_\sigma$ maps $\mathcal{S}(\mathbb{R}^n)$ one-to-one onto $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{S}'(\mathbb{R}^n)$ one-to-one onto $\mathcal{S}'(\mathbb{R}^n)$, respectively,*

$$I_\sigma \mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n), \quad I_\sigma \mathcal{S}'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n). \tag{2.149}$$

*Proof.* The multiplication

$$f \mapsto \langle \xi \rangle^\sigma f \quad \text{with } f \in \mathcal{S}(\mathbb{R}^n) \text{ or } f \in \mathcal{S}'(\mathbb{R}^n), \tag{2.150}$$

respectively, maps $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$, and also $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{R}^n)$. This follows from Definitions 2.32, 2.43, Theorem 2.48 and Corollary 2.56. In view of Theorems 2.42 and 2.60 one obtains the desired result. □

**Remark 2.64.** Later on we shall use $I_\sigma$ as *lifts* in the scale of function spaces with fixed integrability and varying smoothness. We refer also to Appendix E.

## 2.8 The Fourier transform in $L_p(\mathbb{R}^n)$

By Corollary 2.50 any $f \in L_p(\mathbb{R}^n)$ with $1 \le p \le \infty$ can be interpreted as a regular distribution according to Definition 2.10 belonging to $\mathcal{S}'(\mathbb{R}^n)$. Hence $\mathcal{F}f \in \mathcal{S}'(\mathbb{R}^n)$. Recall that a linear operator $T \in \mathcal{L}(H)$ in a Hilbert space $H$ is called *unitary* if

$$\|Th|H\| = \|h|H\| \ \text{ for } h \in H, \quad \text{and} \quad TH = H, \tag{2.151}$$

where the latter means that the range of $T$, range$(T)$, coincides with $H$. As for basic notation of operator theory one may consult Section C.1.

**Theorem 2.65.** *Let $n \in \mathbb{N}$.*

(i) *Let $f \in L_p(\mathbb{R}^n)$ with $1 \le p \le 2$, then $\mathcal{F}f \in \mathcal{S}'(\mathbb{R}^n)$ is regular.*

(ii) *If* $f \in L_1(\mathbb{R}^n)$, *then*

$$(\mathcal{F} f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) \mathrm{d}x, \quad \xi \in \mathbb{R}^n. \tag{2.152}$$

*Furthermore,* $\mathcal{F} f$ *is a bounded continuous function on* $\mathbb{R}^n$ *and*

$$\sup_{\xi \in \mathbb{R}^n} |(\mathcal{F} f)(\xi)| \le (2\pi)^{-n/2} \| f | L_1(\mathbb{R}^n)\| \quad \textit{for all} \quad f \in L_1(\mathbb{R}^n). \tag{2.153}$$

(iii) *The restrictions of* $\mathcal{F}$ *and* $\mathcal{F}^{-1}$, *respectively, to* $L_2(\mathbb{R}^n)$ *generate unitary operators in* $L_2(\mathbb{R}^n)$. *Furthermore,*

$$\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \mathrm{id} \quad (\textit{identity in } L_2(\mathbb{R}^n)). \tag{2.154}$$

*Proof. Step* 1. Part (i) follows from parts (ii) and (iii) and the observation that any $f \in L_p(\mathbb{R}^n)$ with $1 \le p \le 2$ can be decomposed as

$$f(x) = f_1(x) + f_2(x), \quad f_1 \in L_1(\mathbb{R}^n), \quad f_2 \in L_2(\mathbb{R}^n), \tag{2.155}$$

where

$$f_1(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.156}$$

*Step* 2. As for part (ii) we first remark that the right-hand side of (2.152), temporarily denoted by $\widehat{f}(\xi)$, makes sense and that we have (2.153) with $\widehat{f}$ in place of $\mathcal{F} f$. The continuity of $\widehat{f}$ in $\mathbb{R}^n$ is again a consequence of Lebesgue's bounded convergence theorem. Finally, $\mathcal{F} f = \widehat{f}$ follows in the same way as in (2.136).

*Step* 3. It remains to show (iii). We apply (2.107) with $x = 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ and obtain the so-called *multiplication formula*,

$$\int_{\mathbb{R}^n} (\mathcal{F}\varphi)(\xi)\, \psi(\xi) \mathrm{d}\xi = \int_{\mathbb{R}^n} \varphi(\eta)(\mathcal{F}\psi)(\eta) \mathrm{d}\eta. \tag{2.157}$$

Let $\varrho \in \mathcal{S}(\mathbb{R}^n)$, then Definition 2.36 gives $\psi(\xi) = \overline{(\mathcal{F}\varrho)}(\xi) = (\mathcal{F}^{-1}\overline{\varrho})(\xi)$ (in obvious notation). We insert it in (2.157) and obtain for the scalar product in $L_2(\mathbb{R}^n)$ that

$$\begin{aligned} \langle \mathcal{F}\varphi, \mathcal{F}\varrho \rangle_{L_2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (\mathcal{F}\varphi)(\xi)\, \overline{(\mathcal{F}\varrho)}(\xi) \mathrm{d}\xi = \int_{\mathbb{R}^n} \varphi(\eta)\overline{\varrho}(\eta) \mathrm{d}\eta \\ &= \langle \varphi, \varrho \rangle_{L_2(\mathbb{R}^n)}. \end{aligned} \tag{2.158}$$

By Proposition 2.7 there is for any $f \in L_2(\mathbb{R}^n)$ a sequence $\{f_j\}_{j=1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$f_j \to f \text{ in } L_2(\mathbb{R}^n) \quad \text{if } j \to \infty. \tag{2.159}$$

It follows by (2.158) that $\{\mathcal{F} f_j\}_{j=1}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ is a Cauchy sequence in $L_2(\mathbb{R}^n)$, hence for some $g \in L_2(\mathbb{R}^n)$,

$$\mathcal{F} f_j \to g \text{ in } L_2(\mathbb{R}^n) \quad \text{if } j \to \infty. \tag{2.160}$$

We use (2.157) with $\varphi = f_j$, $j \in \mathbb{N}$, and obtain for $j \to \infty$ that

$$\int_{\mathbb{R}^n} g(\xi)\psi(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^n} f(\eta)(\mathcal{F}\psi)(\eta)\mathrm{d}\eta = (\mathcal{F} f)(\psi) \tag{2.161}$$

for any $\psi \in \mathcal{S}(\mathbb{R}^n)$. Hence $\mathcal{F} f = g \in L_2(\mathbb{R}^n)$ and by (2.158),

$$\|\mathcal{F} f|L_2(\mathbb{R}^n)\| = \|f|L_2(\mathbb{R}^n)\|, \quad f \in L_2(\mathbb{R}^n), \tag{2.162}$$

that is, $\mathcal{F}$ is isometric on $L_2(\mathbb{R}^n)$. The same argument can be applied to $\mathcal{F}^{-1}$ instead of $\mathcal{F}$. In view of the above approximation procedure, (2.104) can be extended to (2.154). In particular, the range of $\mathcal{F}$ and $\mathcal{F}^{-1}$ is $L_2(\mathbb{R}^n)$. Thus both, $\mathcal{F}$ and $\mathcal{F}^{-1}$, are not only isometric, but unitary. □

**Remark 2.66.** In Note 2.9.5 we add a few comments about Fourier transforms of functions $f \in L_p(\mathbb{R}^n)$. In particular, if $2 < p \leq \infty$, then there are functions $f \in L_p(\mathbb{R}^n)$ such that $\mathcal{F} f$ is not regular. The simplest case is $p = \infty$. By (2.143) the Fourier transform of a constant function $f(x) = c \neq 0$ equals $c'\delta$ with $c' \neq 0$, which is not regular.

**Exercise 2.67.** Let $f \in L_1(\mathbb{R}^n)$.

(a) Prove that

$$(\mathcal{F} f)(\xi) \to 0 \quad \text{for } |\xi| \to \infty. \tag{2.163}$$

*Hint:* Combine (2.103) and (2.153).

(b) Let $g \in L_1(\mathbb{R}^n)$. Show that the *multiplication formula* (2.157) can be extended from $\mathcal{S}(\mathbb{R}^n)$ to $L_1(\mathbb{R}^n)$, i.e.,

$$\int_{\mathbb{R}^n} (\mathcal{F} f)(\xi)\, g(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^n} f(\eta)(\mathcal{F} g)(\eta)\mathrm{d}\eta. \tag{2.164}$$

We add a few standard examples, restricted to the one-dimensional case $\mathbb{R}$, for convenience.

**Exercise* 2.68.** Determine the Fourier transforms $\mathcal{F} f_i$ of the following functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, 4$ :

(a) $f_1(x) = e^{-a|x|}$, $a > 0$;

(b) $f_2(x) = \operatorname{sgn}(x)e^{-|x|} = \begin{cases} e^{-x}, & x > 0, \\ 0, & x = 0, \\ -e^{x}, & x < 0; \end{cases}$

(c) $f_3(x) = \chi_{[-a,a]}(x) = \begin{cases} 1, & -a \le x \le a, \\ 0, & \text{otherwise}, \end{cases}$ where $a > 0$;

(d) $f_4(x) = (1 - |x|)_+ = \begin{cases} 1 - |x|, & |x| \le 1, \\ 0, & \text{otherwise}. \end{cases}$

*Hint:* Verify that $f_i \in L_1(\mathbb{R})$, $i = 1, \ldots, 4$, and use Theorem 2.65 (ii).

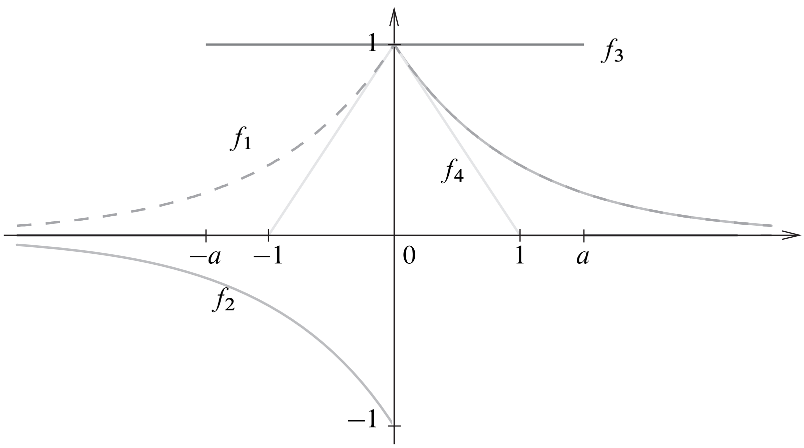



Figure 2.6

**Remark 2.69.** The more complicated $n$-dimensional version of Example 2.68 (a) is given by

$$\mathcal{F}(e^{-a|x|})(\xi) = c\,\frac{a}{(|\xi|^2 + a^2)^{\frac{n+1}{2}}}, \quad \xi \in \mathbb{R}^n, \tag{2.165}$$

where $c$ is a positive constant which is independent of $a > 0$ and $\xi \in \mathbb{R}^n$. It generates the so-called *Cauchy–Poisson semi-group*, see also Exercise 2.70 (c). Details may be found in [Tri78, 2.5.3, pp. 192–196].

We end this section with a short digression to an important feature of the Fourier transform: their interplay with convolutions.

**Exercise* 2.70.** Let $f \in L_1(\mathbb{R}^n)$, $g \in L_1(\mathbb{R}^n)$. Then their *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{2.166}$$

(a) Let $1 \leq p, r \leq \infty$ such that $1 \leq \frac{1}{p} + \frac{1}{r} \leq 2$. Let $f \in L_p(\mathbb{R}^n)$, $g \in L_r(\mathbb{R}^n)$. Then $f * g \in L_q(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$,

$$\|f * g|L_q(\mathbb{R}^n)\| \leq \|g|L_r(\mathbb{R}^n)\| \|f|L_p(\mathbb{R}^n)\|, \tag{2.167}$$

that is, the famous *Young's inequality*, see Theorem D.1. Prove their special cases $p = r = 1$, and $p = r = 2$, corresponding to $q = 1$ and $q = \infty$.

(b) Prove that for $f \in L_1(\mathbb{R}^n)$, $g \in L_1(\mathbb{R}^n)$,

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{n/2} (\mathcal{F} f)(\xi)\, (\mathcal{F} g)(\xi), \quad \xi \in \mathbb{R}^n. \tag{2.168}$$

What is the counterpart for $\mathcal{F}^{-1}(f * g)$?

*Hint:* Recall that we obtained in (2.106) for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ that

$$(2\pi)^{n/2}\, \mathcal{F}^{-1}(\mathcal{F}\varphi \cdot \psi)(x) = (\varphi * \mathcal{F}^{-1}\psi)(x), \tag{2.169}$$

which leads for $f = \varphi$ and $\psi = \mathcal{F} g$ to (2.168) when $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R}^n)$. Adapt the argument to $f \in L_1(\mathbb{R}^n)$, $g \in L_1(\mathbb{R}^n)$. Use Theorem 2.65 (ii).

(c) Let $a > 0$ and consider for $x \in \mathbb{R}$

$$h_a(x) = \frac{a}{\pi(x^2 + a^2)}, \quad g_a(x) = \begin{cases} \dfrac{\sin(ax)}{\pi x}, & x \neq 0, \\ \dfrac{a}{\pi}, & x = 0. \end{cases}$$

Use (the one-dimensional case of) (2.168) to show that for any $b > 0$,

$$h_a * h_b = h_{a+b}, \quad g_a * g_b = g_{\min(a,b)}.$$

*Hint:* Recall Exercise 2.68 (a), (c).

## 2.9 Notes

**2.9.1.** The material in Chapter 2 is rather standard and may be found in many textbooks and monographs. We followed here essentially the relevant parts of [Tri92a] restricting ourselves to the bare minimum needed later on. In [Tri92a] one finds a more elaborated theory of distributions at the same moderate level as here. In this context we refer also to [Hör83] and [Str94] where the latter book contains many exercises.

**2.9.2.** The theory of distributions goes back to S. L. Sobolev (the spaces $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, $W_p^k(\Omega)$ on bounded domains) in the late 1930s as it may be found in [Sob91], and to L. Schwartz (the spaces $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and, in particular, the Fourier analysis of tempered distributions) in [Sch66]. On the one hand, the theory of distributions has a substantial pre-history (especially when taking [Sch66], first edition 1950/51, as a starting point). Some comments and also quotations may be found in [Pie01, Section 4.1.7] and [Går97, Chapter 12], but it is also hidden in [CH53] (first edition 1924) as sequences of smooth functions (approximating distributions). On the other hand, according to [Går97, p. 80],

> ‘*At the time* (1950) *the theory of distributions got a rather lukewarm and sometimes even hostile reception among mathematicians.*’

L. Schwartz’s own description how he discovered distributions may be found in [Sch01, Chapter VI]. But the breakthrough came soon in the 1950s. Nowadays it is accepted as one of the most important developments in mathematics in the second half of the last century influencing significantly not only analysis, but many other branches of mathematics and physics.

**2.9.3.** By Definition 2.32 and the Exercises 2.34, 2.35 one gets that $\mathcal{S}(\mathbb{R}^n)$ is a linear topological (locally convex, metrisable) space and that $\mathcal{S}'(\mathbb{R}^n)$ as introduced in Definition 2.43 and characterised in Theorem 2.48 is its topological dual. The situation for the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ as introduced in the Definitions 2.2, 2.5 is more complicated. But one can furnish $\mathcal{D}(\Omega)$ with a locally convex topology such that converging sequences $\{\varphi_j\}_{j=1}^\infty$ with respect to this topology are just characterised by (2.4), (2.5). The corresponding theory may be found in [Yos80, p. 28, I.8] and [Rud91, Chapter 6] going back to [Sch66, Chapter III]. In particular, the resulting linear topological space is no longer metrisable [Rud91, Remark 6.9] in contrast to $\mathcal{S}(\mathbb{R}^n)$. We adopted in Section 2.1 a more direct approach in good company with many other textbooks and monographs introducing distributions as a tool.

**2.9.4.** The close connection between $\sigma$-finite Borel measures in a domain $\Omega$ in $\mathbb{R}^n$ and distributions belonging to $\mathcal{D}'(\Omega)$ or (in case of $\Omega = \mathbb{R}^n$) to $\mathcal{S}'(\mathbb{R}^n)$ played a decisive rôle in the theory of distributions from the very beginning and had been used for (local) representation of distributions in finite sums of derivatives of measures [Sch66, Chapter III, §7-8]. Theorem 2.31 may serve as a simple example. One may also consult [Rud91, Theorem 6.28, p. 169]. In connection with the identification (2.35) and the discussion in Remark 2.11 the following observation is of some use. Let $M_1^{\mathrm{loc}}(\Omega)$ be the collection of all $\sigma$-finite (locally finite) complex Radon measures on $\Omega$. Then

$$T_\mu : \varphi \longmapsto \int_\Omega \varphi(x)\mu(\mathrm{d}x), \quad \mu \in M_1^{\mathrm{loc}}(\Omega),\ \varphi \in \mathcal{D}(\Omega), \tag{2.170}$$

generates a distribution $T_\mu \in \mathcal{D}'(\Omega)$. Moreover, if $\mu^1 \in M_1^{\mathrm{loc}}(\Omega)$ and $\mu^2 \in M_1^{\mathrm{loc}}(\Omega)$, then

$$T_{\mu^1} = T_{\mu^2} \text{ in } \mathcal{D}'(\Omega) \quad \text{if, and only if,} \quad \mu^1 = \mu^2. \tag{2.171}$$

This follows from the famous Riesz representation theorem according to [Mal95, Theorem 6.6, p. 97]. We refer for a similar discussion and some further details to [Tri06, Section 1.12.2, pp. 80/81]. Hence (2.170) is a one-to-one relation and one may identify $M_1^{\mathrm{loc}}(\Omega)$ with the generated subset of $\mathcal{D}'(\Omega)$. Furthermore, interpreting $f \in L_1^{\mathrm{loc}}(\Omega)$ as $f\mu_L \in M_1^{\mathrm{loc}}(\Omega)$ where $\mu_L$ is the Lebesgue measure in $\mathbb{R}^n$, then one gets

$$L_1^{\mathrm{loc}}(\Omega) \subset M_1^{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega) \tag{2.172}$$

and (2.34), based on Proposition 2.7. In other words, interpreting $L_1^{\mathrm{loc}}(\Omega)$ not as a space of functions (equivalence classes) but as a space of complex measures the ambiguity we discussed in Remark 2.11 disappears.

**2.9.5.** Choosing $f \geq 0$ and $\xi = 0$ in (2.152) it follows that the constant $(2\pi)^{-n/2}$ in (2.153) is sharp. Interpolation of (2.153) and (2.162) gives the famous *Hausdorff–Young inequality*

$$\|\mathcal{F} f \,|L_{p'}(\mathbb{R}^n)\| \leq (2\pi)^{n(\frac{1}{2}-\frac{1}{p})} \|f\,|L_p(\mathbb{R}^n)\|, \quad 1 \leq p \leq 2, \tag{2.173}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, [Tri78, Section 1.18.8]. However, $(2\pi)^{n(\frac{1}{2}-\frac{1}{p})}$ is only the best possible constant when $p = 1$ or when $p = 2$. It turns out that

$$c_{p,n} = \Big((2\pi)^{1-\frac{2}{p}} p^{\frac{1}{p}} (p')^{-\frac{1}{p'}}\Big)^{\frac{n}{2}} \tag{2.174}$$

is the sharp constant in

$$\|\mathcal{F} f \,|L_{p'}(\mathbb{R}^n)\| \leq c_{p,n} \|f\,|L_p(\mathbb{R}^n)\|, \quad 1 < p \leq 2. \tag{2.175}$$

We refer to [LL97, Section 5.7] where the Fourier transform is differently normed. However, if $2 < p \leq \infty$, then there are functions $f \in L_p(\mathbb{R}^n)$ such that $\widehat{f} \in \mathcal{S}'(\mathbb{R}^n)$ is singular, [SW71, 4.13, p. 34]. Hence Theorem 2.65 (i) cannot be extended to $2 < p \leq \infty$. In case of $p = \infty$ this follows also immediately from (2.143).

# Chapter 3

# Sobolev spaces on $\mathbb{R}^n$ and $\mathbb{R}^n_+$

## 3.1 The spaces $W_p^k(\mathbb{R}^n)$

We always interpret $f \in L_p(\mathbb{R}^n)$ with $1 \le p \le \infty$ as a tempered distribution according to Remark 2.49 and Corollary 2.50. In particular, $\mathrm{D}^\alpha f \in \mathcal{S}'(\mathbb{R}^n)$ makes sense for any $\alpha \in \mathbb{N}_0^n$.

**Definition 3.1.** Let $k \in \mathbb{N}$ and $1 \le p < \infty$. Then

$$W_p^k(\mathbb{R}^n) = \{f \in L_p(\mathbb{R}^n) : \mathrm{D}^\alpha f \in L_p(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}_0^n,\ |\alpha| \le k\} \tag{3.1}$$

are the *classical Sobolev spaces*.

**Remark 3.2.** We incorporate notationally $L_p(\mathbb{R}^n) = W_p^0(\mathbb{R}^n)$. Otherwise $W_p^k(\mathbb{R}^n)$ collects all $f \in L_p(\mathbb{R}^n)$ such that the distributional derivatives $\mathrm{D}^\alpha f \in \mathcal{S}'(\mathbb{R}^n)$ with $|\alpha| \le k$ are regular and, in addition, belong to $L_p(\mathbb{R}^n)$. Some references and comments about classical Sobolev spaces on $\mathbb{R}^n$ and on domains may also be found in the Notes 2.9.2, 3.6.1, 3.6.3 and 4.6.1. Strictly speaking, the elements of $W_p^k(\mathbb{R}^n)$ are equivalence classes $[f]$. But by (2.40), (2.124) one has $\mathrm{D}^\alpha f = \mathrm{D}^\alpha g$ in $\mathcal{S}'(\mathbb{R}^n)$ for $g \in [f]$ and all $\alpha \in \mathbb{N}_0^n$. If, in addition, $\mathrm{D}^\alpha f \in L_p(\mathbb{R}^n)$, then $\mathrm{D}^\alpha g \in [\mathrm{D}^\alpha f]$. As usual, we do not care about this ambiguity which we discussed in detail in (2.32)–(2.34), Definition 2.10 and Remark 2.11.

**Theorem 3.3.** *Let $1 \le p < \infty$ and $k \in \mathbb{N}_0$. Then $W_p^k(\mathbb{R}^n)$ furnished with the norm*

$$\|f\,|W_p^k(\mathbb{R}^n)\| = \Big(\sum_{|\alpha|\le k} \|\mathrm{D}^\alpha f\,|L_p(\mathbb{R}^n)\|^p\Big)^{1/p} \tag{3.2}$$

*becomes a Banach space and*

$$\mathcal{S}(\mathbb{R}^n) \subset W_p^k(\mathbb{R}^n) \subset L_p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n). \tag{3.3}$$

*Furthermore, $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are dense in $W_p^k(\mathbb{R}^n)$.*

*Proof. Step* 1. By (2.115), (2.132) we have for any $\alpha \in \mathbb{N}_0^n$,

$$\mathrm{D}^\alpha(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \mathrm{D}^\alpha f_1 + \lambda_2 \mathrm{D}^\alpha f_2, \quad f_1, f_2 \in \mathcal{S}'(\mathbb{R}^n);\ \lambda_1, \lambda_2 \in \mathbb{C}. \tag{3.4}$$

In particular, $W_p^k(\mathbb{R}^n)$ is a linear space, a subspace of $\mathcal{S}'(\mathbb{R}^n)$. Since both $L_p(\mathbb{R}^n)$ and the related (finite-dimensional) sequence space $\ell_p$ are normed spaces, one gets

$$\|f_1 + f_2|W_p^k(\mathbb{R}^n)\| \le \|f_1|W_p^k(\mathbb{R}^n)\| + \|f_2|W_p^k(\mathbb{R}^n)\|,\ f_1, f_2 \in W_p^k(\mathbb{R}^n), \tag{3.5}$$

and

$$\|\lambda f|W_p^k(\mathbb{R}^n)\| = |\lambda|\|f|W_p^k(\mathbb{R}^n)\|, \quad f \in W_p^k(\mathbb{R}^n),\ \lambda \in \mathbb{C}. \tag{3.6}$$

If $\|f|W_p^k(\mathbb{R}^n)\| = 0$, then, in particular, $\|f|L_p(\mathbb{R}^n)\| = 0$, hence $[f] = 0$, that is, $f = 0$ in $\mathcal{S}'(\mathbb{R}^n)$. This implies that (3.2) is a norm.

*Step* 2. Let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence in $W_p^k(\mathbb{R}^n)$. Then $\{D^\alpha f_j\}_{j=1}^\infty$ are Cauchy sequences in $L_p(\mathbb{R}^n)$ for $|\alpha| \le k$. Hence there are $f^\alpha \in L_p(\mathbb{R}^n)$ with

$$D^\alpha f_j \to f^\alpha \text{ in } L_p(\mathbb{R}^n),\ |\alpha| \le k, \quad \text{and} \quad f^0 = f. \tag{3.7}$$

It follows from (2.40),

$$\int_{\mathbb{R}^n} D^\alpha f_j(x)\varphi(x)\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_j(x)(D^\alpha\varphi)(x)\mathrm{d}x, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{3.8}$$

and Hölder's inequality applied to $D^\alpha f_j - f^\alpha$, $f_j - f \in L_p(\mathbb{R}^n)$ and $\varphi$, $D^\alpha\varphi \in L_{p'}(\mathbb{R}^n)$ that

$$\int_{\mathbb{R}^n} f^\alpha(x)\varphi(x)\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)(D^\alpha\varphi)(x)\mathrm{d}x, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{3.9}$$

Then $f^\alpha = D^\alpha f$, $|\alpha| \le k$, and $f \in W_p^k(\mathbb{R}^n)$ with

$$f_j \to f \text{ in } W_p^k(\mathbb{R}^n) \quad \text{for } j \to \infty. \tag{3.10}$$

Consequently, $W_p^k(\mathbb{R}^n)$ is a Banach space. Furthermore, (3.3) is obvious by Corollary 2.50.

*Step* 3. Since $\mathcal{S}(\mathbb{R}^n) \subset W_p^k(\mathbb{R}^n)$, it remains to prove that $\mathcal{D}(\mathbb{R}^n)$ is dense. Let $f \in W_p^k(\mathbb{R}^n)$. By the mollification according to Exercise 1.30 with the functions $\omega_h \in \mathcal{D}(\mathbb{R}^n)$, $h > 0$, especially (1.59), (1.60), one obtains

$$\begin{aligned}(D^\alpha f_h)(x) &= \int_{\mathbb{R}^n} D_x^\alpha \omega_h(x-y) f(y)\mathrm{d}y \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} D_y^\alpha \omega_h(x-y) f(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \omega_h(x-y)(D^\alpha f)(y)\mathrm{d}y = (D^\alpha f)_h(x).\end{aligned} \tag{3.11}$$

The same argument as in (2.28) with (2.30), (2.31) implies

$$\|D^\alpha f - D^\alpha f_h|L_p(\mathbb{R}^n)\| \to 0 \quad \text{if } h \to 0,\ |\alpha| \le k. \tag{3.12}$$

Hence $f_h \in C^\infty(\mathbb{R}^n) \cap W_p^k(\mathbb{R}^n)$ and

$$f_h \to f \text{ in } W_p^k(\mathbb{R}^n) \quad \text{for } h \to 0. \tag{3.13}$$

In particular, it is sufficient to approximate functions $g \in C^\infty(\mathbb{R}^n) \cap W_p^k(\mathbb{R}^n)$ in $W_p^k(\mathbb{R}^n)$ by functions belonging to $\mathcal{D}(\mathbb{R}^n)$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi(x) = 1$ if $|x| \le 1$. Then

$$\mathcal{D}(\mathbb{R}^n) \ni \varphi(2^{-j}x)g(x) \to g(x) \text{ in } W_p^k(\mathbb{R}^n) \quad \text{if } j \to \infty \tag{3.14}$$

by straightforward calculation. □

**Exercise 3.4.** Let $k \in \mathbb{N}_0$ and $1 \le p < \infty$. Prove the following *homogeneity estimates* for Sobolev spaces: There are positive constants $c, c'$ and $C, C'$ such that

$$c\ R^{-\frac{n}{p}} \| f | L_p(\mathbb{R}^n)\| \le \| f(R\cdot) | W_p^k(\mathbb{R}^n)\| \le c' R^{-\frac{n}{p}} \| f | W_p^k(\mathbb{R}^n)\| \tag{3.15}$$

for all $f \in W_p^k(\mathbb{R}^n)$ and all $0 < R \le 1$, and

$$C\ R^{-\frac{n}{p}} \| f | W_p^k(\mathbb{R}^n)\| \le \| f(R\cdot) | W_p^k(\mathbb{R}^n)\| \le C' R^{k-\frac{n}{p}} \| f | W_p^k(\mathbb{R}^n)\| \tag{3.16}$$

for all $f \in W_p^k(\mathbb{R}^n)$ and all $R > 1$.

**Remark 3.5.** Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a Banach space $B$ are called *equivalent*, denoted by $\|\cdot\|_1 \sim \|\cdot\|_2$, if there are two numbers $0 < c_1 \le c_2 < \infty$ such that

$$c_1 \|b\|_1 \le \|b\|_2 \le c_2 \|b\|_1 \quad \text{for all } b \in B. \tag{3.17}$$

We do not distinguish between equivalent norms in the sequel and may switch from one norm to an equivalent one if appropriate. For example,

$$\| f | W_p^k(\mathbb{R}^n)\|^* = \sum_{|\alpha| \le k} \| \mathrm{D}^\alpha f | L_p(\mathbb{R}^n)\|, \quad f \in W_p^k(\mathbb{R}^n), \tag{3.18}$$

is equivalent to (3.2).

**Exercise 3.6.** (a) Let $n \in \mathbb{N}$, $n \ge 2$, and $\varkappa > 0$. Consider the radial functions

$$h_\varkappa(x) = \varphi(x) |\log|x||^\varkappa, \quad x \in \mathbb{R}^n,$$

where $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is some cut-off function, say, with

$$\varphi(x) = 1,\ |x| \le \frac{1}{4} \quad \text{and} \quad \varphi(x) = 0,\ |x| \ge \frac{1}{2}, \tag{3.19}$$

see Figure 3.1. Show that $h_\varkappa \in W_n^1(\mathbb{R}^n)$ if $\varkappa < 1 - \frac{1}{n}$.

(b) Let $n \in \mathbb{N}$, $n \ge 2$, and

$$g(x) = \varphi(x) \log|\log|x||, \quad x \in \mathbb{R}^n,$$

with $\varphi \in \mathcal{D}(\mathbb{R}^n)$ according to (3.19). Prove that $g \in W_n^1(\mathbb{R}^n)$.

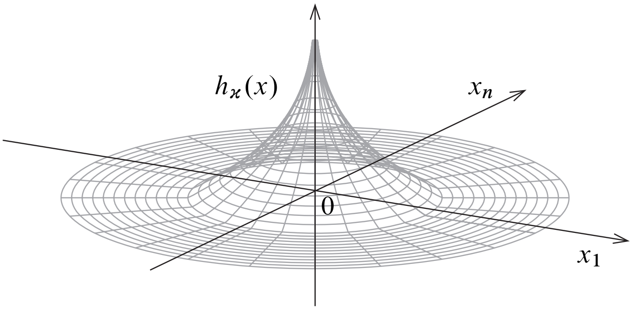



Figure 3.1

**Remark 3.7.** We shall return to these examples in connection with so-called *Sobolev embeddings* in Section 3.3 below. We refer, in particular, to Theorem 3.32 and Exercise 3.33.

## 3.2 The spaces $H^s(\mathbb{R}^n)$

For $p = 2$ and $k \in \mathbb{N}_0$ the classical Sobolev spaces $W_2^k(\mathbb{R}^n)$ according to Definition 3.1, Theorem 3.3, equipped with the scalar product

$$\langle f, g \rangle_{W_2^k(\mathbb{R}^n)} = \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} \mathrm{D}^\alpha f(x) \overline{\mathrm{D}^\alpha g(x)} \mathrm{d}x \tag{3.20}$$

become Hilbert spaces. We wish to characterise the spaces $W_2^k(\mathbb{R}^n)$ in terms of the Fourier transform generalising Theorem 2.65 (iii). For this purpose we need weighted $L_2$ spaces.

**Definition 3.8.** Let $n \in \mathbb{N}$ and let $w$ be a continuous positive function in $\mathbb{R}^n$. Then

$$L_2(\mathbb{R}^n, w) = \{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : wf \in L_2(\mathbb{R}^n) \}. \tag{3.21}$$

**Remark 3.9.** Quite obviously, $L_2(\mathbb{R}^n, w)$ becomes a Hilbert space when furnished with the scalar product

$$\langle f, g \rangle_{L_2(\mathbb{R}^n, w)} = \int_{\mathbb{R}^n} w(x) f(x) \overline{w(x) g(x)} \mathrm{d}x = \langle wf, wg \rangle_{L_2(\mathbb{R}^n)}. \tag{3.22}$$

Furthermore, $f \mapsto wf$ maps $L_2(\mathbb{R}^n, w)$ unitarily onto $L_2(\mathbb{R}^n)$. Of special interest are the weights

$$w_s(x) = \langle x \rangle^s = (1 + |x|^2)^{s/2}, \quad s \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{3.23}$$

recall (2.83).

**Proposition 3.10.** *Let $L_2(\mathbb{R}^n, w_s)$ be given by (3.21), (3.23) with $s \in \mathbb{R}$. Then $L_2(\mathbb{R}^n, w_s)$ together with the scalar product (3.22) with $w = w_s$ is a Hilbert space. Furthermore,*

$$\mathcal{S}(\mathbb{R}^n) \subset L_2(\mathbb{R}^n, w_s) \subset \mathcal{S}'(\mathbb{R}^n) \tag{3.24}$$

*in the interpretation of Definition 2.55 and Corollary 2.56. Both $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are dense in $L_2(\mathbb{R}^n, w_s)$.*

*Proof.* As mentioned above, $L_2(\mathbb{R}^n, w_s)$ is a Hilbert space. The left-hand side of (3.24) follows from Definition 2.32 whereas the right-hand side is covered by Corollary 2.56 (and $L_2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$). If $\varphi(2^{-j}\cdot)$ are the same cut-off functions as in connection with (3.14), then $\varphi(2^{-j}\cdot)f$ approximate any given $f \in L_2(\mathbb{R}^n, w_s)$. By Proposition 2.7 the compactly supported function $\varphi(2^{-j}\cdot)f$ can be approximated in $L_2(\mathbb{R}^n)$, and hence in $L_2(\mathbb{R}^n, w_s)$, by functions belonging to $\mathcal{D}(\mathbb{R}^n)$. □

According to Definition 2.58 and Theorem 2.60 the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$, respectively, are defined on $\mathcal{S}'(\mathbb{R}^n)$. In view of (3.3) and (3.24) one can restrict $\mathcal{F}$ and $\mathcal{F}^{-1}$ to $W_2^k(\mathbb{R}^n)$ and to $L_2(\mathbb{R}^n, w_s)$ (denoting these restrictions by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively, again).

**Theorem 3.11.** *Let $k \in \mathbb{N}_0$. The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ generate unitary maps of $W_2^k(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n, w_k)$, and of $L_2(\mathbb{R}^n, w_k)$ onto $W_2^k(\mathbb{R}^n)$,*

$$\mathcal{F} W_2^k(\mathbb{R}^n) = \mathcal{F}^{-1} W_2^k(\mathbb{R}^n) = L_2(\mathbb{R}^n, w_k). \tag{3.25}$$

*Proof.* Let $f \in W_2^k(\mathbb{R}^n)$. Equations (3.2), (2.139), and Theorem 2.65 (iii) imply

$$\begin{aligned} \|f|W_2^k(\mathbb{R}^n)\|^2 &= \sum_{|\alpha|\le k} \|\mathrm{D}^\alpha f|L_2(\mathbb{R}^n)\|^2 \\ &= \sum_{|\alpha|\le k} \|\mathcal{F}(\mathrm{D}^\alpha f)|L_2(\mathbb{R}^n)\|^2 \\ &= \int_{\mathbb{R}^n} \Big(\sum_{|\alpha|\le k} |x^\alpha|^2\Big)|\mathcal{F} f(x)|^2 \mathrm{d}x. \end{aligned} \tag{3.26}$$

Since $\sum_{|\alpha|\le k} |x^\alpha|^2 \sim w_k^2(x)$, one obtains with respect to this equivalent norm, denoted by $\|\cdot\|_*$, that

$$\|f|W_2^k(\mathbb{R}^n)\| = \|\mathcal{F} f|L_2(\mathbb{R}^n, w_k)\|_*. \tag{3.27}$$

Hence $\mathcal{F}$ is an isometric map from $W_2^k(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n, w_k)$. Conversely, let $g \in L_2(\mathbb{R}^n, w_k)$ and $f = \mathcal{F}^{-1} g$. By (2.139) and Theorem 2.65 (iii) one has

$$\mathrm{D}^\alpha f = i^{|\alpha|}\mathcal{F}^{-1}(x^\alpha g) \in L_2(\mathbb{R}^n), \quad |\alpha| \le k. \tag{3.28}$$

This proves $f \in W_2^k(\mathbb{R}^n)$. Hence $\mathcal{F}$ in (3.27) maps $W_2^k(\mathbb{R}^n)$ unitarily onto $L_2(\mathbb{R}^n, w_k)$. Similarly one proceeds for the other cases. □

**Remark 3.12.** One can rewrite (3.25) as

$$W_2^k(\mathbb{R}^n) = \mathcal{F} L_2(\mathbb{R}^n, w_k) = \mathcal{F}^{-1} L_2(\mathbb{R}^n, w_k). \tag{3.29}$$

In other words, one could define $W_2^k(\mathbb{R}^n)$ as the Fourier image of $L_2(\mathbb{R}^n, w_k)$. But for such a procedure one does not need that $k \in \mathbb{N}_0$. We have (3.24) for any $s \in \mathbb{R}$. The resulting spaces $W_2^s(\mathbb{R}^n)$, especially if $s = \frac{1}{2} + k$, where $k \in \mathbb{N}_0$, will be of great service for us later in connection with boundary value problems. Furthermore, one may ask whether one can replace $W_2^k(\mathbb{R}^n)$ in Theorem 3.11 by $W_p^k(\mathbb{R}^n)$ with $1 \leq p < \infty$. But if $p \neq 2$, then the situation is more complicated. We return to this point in the Notes 3.6.1, 3.6.2.

**Definition 3.13.** Let $s \in \mathbb{R}$ and $w_s$ as in (3.23). Then

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : w_s \mathcal{F} f \in L_2(\mathbb{R}^n)\}. \tag{3.30}$$

**Remark 3.14.** It follows by Theorem 3.11 and (3.27) that

$$H^k(\mathbb{R}^n) = W_2^k(\mathbb{R}^n), \quad k \in \mathbb{N}_0. \tag{3.31}$$

One can replace $\mathcal{F}$ in (3.30) by $\mathcal{F}^{-1}$. In any case the spaces $H^s(\mathbb{R}^n)$ extend naturally the classical Sobolev spaces $W_2^k(\mathbb{R}^n)$ according to Definition 3.1 from $k \in \mathbb{N}_0$ to $s \in \mathbb{R}$. As for a corresponding extension $H_p^s(\mathbb{R}^n)$ of $W_p^k(\mathbb{R}^n)$ from $k \in \mathbb{N}_0$ to $s \in \mathbb{R}$ and with $1 < p < \infty$ we refer to Note 3.6.1. It is usual nowadays to call $H_p^s(\mathbb{R}^n)$ *Sobolev spaces* for all $s \in \mathbb{R}$, $1 < p < \infty$.

**Exercise 3.15.** Let $k \in \mathbb{N}$. Prove that

$$\|f|L_2(\mathbb{R}^n)\| + \sum_{j=1}^n \left\| \frac{\partial^k f}{\partial x_j^k} |L_2(\mathbb{R}^n)\right\| \tag{3.32}$$

is an equivalent norm in $W_2^k(\mathbb{R}^n)$.

*Hint:* Use (3.31).

We start with some elementary examples for $s \geq 0$; there are further ones related to $s < 0$ in Exercise 3.18 below.

**Exercise* 3.16.** Let $s \geq 0$.

(a) Show that $e^{-|x|} \in H^s(\mathbb{R})$ if, and only if, $s < \frac{3}{2}$.

*Hint:* Use Exercise 2.68 (a).

(b) Let $a > 0$. Prove that $\chi_{[-a,a]} \in H^s(\mathbb{R})$ if, and only if, $s < \frac{1}{2}$.

*Hint:* Use Exercise 2.68 (c).

(c) Let $a > 0$ and $A = [-a,a]^n = \{x \in \mathbb{R}^n : |x_j| \le a,\ j = 1,\dots,n\}$. Prove that $\chi_A \in H^s(\mathbb{R}^n)$ if, and only if, $s < \frac{1}{2}$.
*Hint:* Use (b) and the special product structure of $A$ and $\chi_A$.

(d) Let $a > 0$, $r \in \mathbb{N}$, and $f$ the $r$-fold convolution of $\chi_{[-a,a]}$,

$$f(x) = (\underbrace{\chi_{[-a,a]} * \cdots * \chi_{[-a,a]}}_{r\text{-times}})(x), \quad x \in \mathbb{R}.$$

Prove that $f \in H^s(\mathbb{R})$ if, and only if, $s < r - \frac{1}{2}$, $r \in \mathbb{N}$.
*Hint:* Use Exercises 2.68 (c) and 2.70 (b).

**Proposition 3.17.** *Let $s \in \mathbb{R}$. The spaces $H^s(\mathbb{R}^n)$, furnished with the scalar product*

$$\langle f, g\rangle_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} w_s(x)\mathcal{F}f(x)\,\overline{w_s(x)\mathcal{F}g(x)}\,\mathrm{d}x, \tag{3.33}$$

*are Hilbert spaces. Furthermore,*

$$\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \tag{3.34}$$

*and $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.*

*Proof.* By definition,

$$f \mapsto w_s\mathcal{F}f : H^s(\mathbb{R}^n) \to L_2(\mathbb{R}^n), \tag{3.35}$$

generates an isometric map into $L_2(\mathbb{R}^n)$. Choosing $f = \mathcal{F}^{-1}(w_{-s}g) \in \mathcal{S}'(\mathbb{R}^n)$ for a given $g \in L_2(\mathbb{R}^n)$ it follows that (3.35) is a unitary map onto $L_2(\mathbb{R}^n)$. Hence $H^s(\mathbb{R}^n)$ is a Hilbert space. Furthermore, by the same arguments as in the proof of Proposition 2.63 it follows that (3.35) maps also $\mathcal{S}(\mathbb{R}^n)$ onto itself and $\mathcal{S}'(\mathbb{R}^n)$ onto itself. Then both (3.34) and the density of $\mathcal{S}(\mathbb{R}^n)$ first in $L_2(\mathbb{R}^n)$, and subsequently in $H^s(\mathbb{R}^n)$ follow from Theorem 3.3. □

**Exercise* 3.18.** (a) Prove that

$$\delta \in H^s(\mathbb{R}^n) \quad \text{if, and only if,} \quad s < -\frac{n}{2}. \tag{3.36}$$

*Hint:* Use (2.143).

(b) Prove that for given $s < 0$ there are singular distributions $f \in H^s(\mathbb{R})$.
*Hint:* Apply (2.147) to

$$f(x) = \varphi(x)\sum_{k=0}^{\infty} 2^{k\sigma} e^{i2^k x}, \quad x \in \mathbb{R},\ 0 < \sigma < |s|, \tag{3.37}$$

with $\varphi \in \mathcal{D}(\mathbb{R})$ and $\varphi(x) = 1$ if $|x| \le \pi$.

(c) Construct a singular distribution which belongs to all $H^s(\mathbb{R})$ with $s < 0$ simultaneously.

*Hint:* What about $\sigma = 0$ in (3.37)? Use Exercise 2.67 (a).

(d) Let $n \in \mathbb{N}$. Prove that for given $s < 0$ there are singular distributions $f \in H^s(\mathbb{R}^n)$.

*Hint:* Multiply (3.37) with a suitable function belonging to $\mathcal{D}(\mathbb{R}^{n-1})$.

By (3.30) one has

$$H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n) \quad \text{if } -\infty < s_2 \le s_1 < \infty, \tag{3.38}$$

and, in particular,

$$H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n) \subset L_2(\mathbb{R}^n) \quad \text{if } 0 \le s_2 \le s_1 < \infty. \tag{3.39}$$

Hence the gaps between the smoothness parameters $k \in \mathbb{N}_0$ of $W_2^k(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ are filled by the continuous smoothness parameter $s$.

However, it would be highly desirable to have descriptions of $H^s(\mathbb{R}^n)$, $s > 0$, parallel to $H^k(\mathbb{R}^n) = W_2^k(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, given in Definition 3.1 at our disposal, avoiding, in particular, the Fourier transform. This is not only of interest for its own sake, but essential when switching from $\mathbb{R}^n$ to (bounded) domains, subject to Section 4.

It is well known that fractional smoothness, say, for continuous functions, can be expressed in terms of differences resulting in Hölder spaces. Let

$$(\Delta_h f)(x) = f(x+h) - f(x), \quad h \in \mathbb{R}^n,\ x \in \mathbb{R}^n. \tag{3.40}$$

**Exercise 3.19.** Let $\Delta_h^1 = \Delta_h$, and define for $m \in \mathbb{N}$ the iterated differences by

$$(\Delta_h^{m+1} f)(x) = \Delta_h^1(\Delta_h^m f)(x), \quad x \in \mathbb{R}^n,\ h \in \mathbb{R}^n. \tag{3.41}$$

(a) Prove that

$$(\Delta_h^m f)(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(x+jh), \quad x \in \mathbb{R}^n,\ h \in \mathbb{R}^n,$$

where $\binom{m}{j} = \frac{m!}{(m-j)!j!}$ are the usual binomial coefficients.

(b) Show that for all $m \in \mathbb{N}$, $h \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $\lambda > 0$,

$$(\Delta_h^m[f(\lambda\cdot)])(x) = (\Delta_{\lambda h}^m f)(\lambda x).$$

(c) Prove that

$$2(\Delta_h f)(x) = (\Delta_{2h} f)(x) - (\Delta_h^2 f)(x) \tag{3.42}$$

and its iteration

$$2^m(\Delta_h^m f)(x) = (\Delta_{2h}^m f)(x) + \Delta_h^{m+1}\Big(\sum_{l=0}^{m-1} a_{m,l} f(\cdot + lh)\Big)(x) \tag{3.43}$$

for $m \in \mathbb{N}$, where the real coefficients $a_{m,l}$ are independent of $f$ and $h$.

*Hint:* One may also consult [Tri83, p. 99].

Let $C^k(\mathbb{R}^n)$ be the spaces according to Definition A.1 where $k \in \mathbb{N}_0$. Let $s = k + \sigma$ with $k \in \mathbb{N}_0$ and $0 < \sigma < 1$. Then the fractional extension of (A.8) for $\Omega = \mathbb{R}^n$ are the Hölder spaces normed by

$$\begin{aligned} \|f|C^s(\mathbb{R}^n)\| &= \|f|C^k(\mathbb{R}^n)\| + \sum_{|\alpha|=k} \sup_{x\neq y} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|}{|x-y|^\sigma} \\ &= \|f|C^k(\mathbb{R}^n)\| + \sum_{|\alpha|=k} \sup_{0\neq h\in\mathbb{R}^n} \sup_{x\in\mathbb{R}^n} \frac{|(\Delta_h \mathrm{D}^\alpha f)(x)|}{|h|^\sigma} \\ &\sim \|f|C^k(\mathbb{R}^n)\| + \sum_{|\alpha|=k} \sup_{0<|h|<1} \sup_{x\in\mathbb{R}^n} \frac{|(\Delta_h \mathrm{D}^\alpha f)(x)|}{|h|^\sigma}, \end{aligned} \tag{3.44}$$

the latter being an equivalent norm.

**Exercise* 3.20.** (a) Prove that $C^s(\mathbb{R}^n)$, $s > 0$, $s \notin \mathbb{N}$, is a Banach space.

(b) Let $\mathrm{Lip}(\mathbb{R}^n)$ be normed by

$$\|f|\mathrm{Lip}(\mathbb{R}^n)\| = \|f|C(\mathbb{R}^n)\| + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|}. \tag{3.45}$$

Prove that $\mathrm{Lip}(\mathbb{R}^n)$ is a Banach space and that

$$C^1(\mathbb{R}^n) \subsetneq \mathrm{Lip}(\mathbb{R}^n). \tag{3.46}$$

**Exercise 3.21.** Let $0 < s = k + \sigma$ with $k \in \mathbb{N}_0$, $0 < \sigma < 1$. Prove that for any $m \in \mathbb{N}$ and $\Delta_h^m$ given by (3.41),

$$\|f|C^k(\mathbb{R}^n)\| + \sum_{|\alpha|=k} \sup_{0\neq h\in\mathbb{R}^n} \sup_{x\in\mathbb{R}^n} \frac{|(\Delta_h^m \mathrm{D}^\alpha f)(x)|}{|h|^\sigma} \tag{3.47}$$

is an equivalent norm in $C^s(\mathbb{R}^n)$.

*Hint:* Use Exercise 3.19 (c).

Some further information will be given in the Notes 3.6.1, 3.6.5.

We ask now for an $L_2$ counterpart of (3.44). Let again $s = k + \sigma$ with $k \in \mathbb{N}_0$, and $0 < \sigma < 1$, and let $W_2^k(\mathbb{R}^n)$ be normed according to (3.2) with $p = 2$. Then the appropriate replacement of the two $L_\infty$ norms in (3.44) is given by

$$\begin{aligned}
&\|f\,|W_2^s(\mathbb{R}^n)\| \\
&= \Big(\|f\,|W_2^k(\mathbb{R}^n)\|^2 + \sum_{|\alpha|=k} \iint\limits_{\mathbb{R}^{2n}} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/2} \\
&= \Big(\|f\,|W_2^k(\mathbb{R}^n)\|^2 + \sum_{|\alpha|=k} \int\limits_{\mathbb{R}^n} |h|^{-2\sigma}\|\Delta_h \mathrm{D}^\alpha f\,|L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n}\Big)^{1/2} \\
&\sim \Big(\|f\,|W_2^k(\mathbb{R}^n)\|^2 + \sum_{|\alpha|=k} \int\limits_{|h|\le 1} |h|^{-2\sigma}\|\Delta_h \mathrm{D}^\alpha f\,|L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n}\Big)^{1/2}, \qquad (3.48)
\end{aligned}$$

where the latter is an equivalent norm due to

$$\|\Delta_h g|L_2(\mathbb{R}^n)\| \le 2\|g|L_2(\mathbb{R}^n)\| \quad \text{for any } h \in \mathbb{R}^n, \qquad (3.49)$$

and, hence,

$$\int\limits_{|h|>1} |h|^{-2\sigma}\|\Delta_h \mathrm{D}^\alpha f\,|L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n} \le c\|\mathrm{D}^\alpha f\,|L_2(\mathbb{R}^n)\|^2. \qquad (3.50)$$

This shows, in addition, that both $n$ and $\sigma > 0$ in the exponent of the denominator in (3.48) are needed to compensate the integration (compared with $\sigma > 0$ in (3.44)). Furthermore,

$$\|\varphi|W_2^s(\mathbb{R}^n)\| < \infty \quad \text{if } \varphi \in \mathcal{S}(\mathbb{R}^n) \qquad (3.51)$$

as a consequence of

$$|(\Delta_h\varphi)(x)| \le c_\gamma |h| \langle x\rangle^{-\gamma}, \quad |h| < 1,\ x \in \mathbb{R}^n, \qquad (3.52)$$

for all $\gamma > 0$ and appropriate $c_\gamma > 0$, recall notation (2.83). Here $\sigma < 1$ is essential.

**Definition 3.22.** Let $s = k + \sigma$ with $k \in \mathbb{N}_0$, $0 < \sigma < 1$. Then

$$W_2^s(\mathbb{R}^n) = \{f \in L_2(\mathbb{R}^n) : \|f\,|W_2^s(\mathbb{R}^n)\| < \infty\} \qquad (3.53)$$

with $\|\cdot|W_2^s(\mathbb{R}^n)\|$ as in (3.48).

**Remark 3.23.** The norm $\|\cdot|W_2^s(\mathbb{R}^n)\|$ is related to the scalar product

$$\langle f, g\rangle_{W_2^s(\mathbb{R}^n)} = \langle f, g\rangle_{W_2^k(\mathbb{R}^n)} + \sum_{|\alpha|=k} \int\limits_{|h|\le 1} \frac{\langle \Delta_h \mathrm{D}^\alpha f, \Delta_h \mathrm{D}^\alpha g\rangle_{L_2(\mathbb{R}^n)}}{|h|^{2\sigma}} \frac{\mathrm{d}h}{|h|^n}, \qquad (3.54)$$

where we used (3.20).

Next we wish to extend (3.31) to all $s > 0$.

**Theorem 3.24.** *Let* $0 < s = k + \sigma$ *with* $k \in \mathbb{N}_0$ *and* $0 < \sigma < 1$. *Let* $H^s(\mathbb{R}^n)$ *and* $W_2^s(\mathbb{R}^n)$ *be the spaces according to Definitions* 3.13 *and* 3.22, *respectively. Then*

$$H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n) \tag{3.55}$$

(*equivalent norms*). *Both* $\mathcal{D}(\mathbb{R}^n)$ *and* $\mathcal{S}(\mathbb{R}^n)$ *are dense in* $W_2^s(\mathbb{R}^n)$.

*Proof. Step* 1. Obviously $W_2^s(\mathbb{R}^n)$ is a linear space with respect to the norm generated by the scalar product (3.54). If $f \in W_2^s(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $f\varphi \in W_2^s(\mathbb{R}^n)$. This follows from (3.48), together with the old trick of calculus,

$$\begin{aligned}(\varphi f)(x+h) - (\varphi f)(x) \\ = f(x)(\varphi(x+h) - \varphi(x)) + \varphi(x+h)(f(x+h) - f(x)),\end{aligned} \tag{3.56}$$

and

$$|\varphi(x+h) - \varphi(x)| \le |h| \sum_{l=1}^{n} \sup_{x\in\mathbb{R}^n} \left|\frac{\partial\varphi}{\partial x_l}(x)\right|, \quad |h| \le 1,\ x \in \mathbb{R}^n. \tag{3.57}$$

If we apply this observation to $\varphi_j f$ with $\varphi_j(x) = \varphi(2^{-j}x)$, $j \in \mathbb{N}$, where $\varphi$ is a cut-off function, that is, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi(x) = 1$ if $|x| \le 1$, then one obtains by (3.56), (3.57) with $\varphi_j$ in place of $\varphi$ uniform estimates. This yields an integrable upper bound for the corresponding integrals in (3.48) with $\varphi_j f$ in place of $f$. Thus $\varphi_j f$ tends to $f$ in $W_2^s(\mathbb{R}^n)$ due to $(\varphi_j f)(x) \to f(x)$ in $\mathbb{R}^n$ and Lebesgue's bounded convergence theorem.

We wish to prove that $\mathcal{D}(\mathbb{R}^n)$ – and hence, by (3.51) also $\mathcal{S}(\mathbb{R}^n)$ – is dense in $W_2^s(\mathbb{R}^n)$. By the above consideration it is sufficient to approximate compactly supported functions $f \in W_2^s(\mathbb{R}^n)$. Let

$$f_t(x) = \int_{\mathbb{R}^n} \omega(y) f(x - ty)\,\mathrm{d}y, \quad x \in \mathbb{R}^n,\ 0 < t \le 1, \tag{3.58}$$

be the mollification of a compactly supported function $f \in W_2^s(\mathbb{R}^n)$ according to (2.26), based on (1.58), (1.59). The triangle inequality for integrals (and the translation-invariance of $\|\cdot|L_2(\mathbb{R}^n)\|$) imply

$$\|f_t|W_2^s(\mathbb{R}^n)\| \le \|f|W_2^s(\mathbb{R}^n)\|. \tag{3.59}$$

Similarly one finds for given $\varepsilon > 0$ and $\alpha$ with $|\alpha| \le k$ a number $\delta > 0$ such that

$$\begin{aligned}&\Big(\int_{|h|\le\delta} |h|^{-2\sigma} \|\Delta_h \mathrm{D}^\alpha f_t|L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n}\Big)^{1/2} \\ &\qquad \le \Big(\int_{|h|\le\delta} |h|^{-2\sigma} \|\Delta_h \mathrm{D}^\alpha f|L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n}\Big)^{1/2} \le \varepsilon\end{aligned} \tag{3.60}$$

uniformly in $t$, $0 < t \le 1$. This leads to

$$f_t \to f \text{ in } W_2^s(\mathbb{R}^n) \quad \text{if } t \to 0, \tag{3.61}$$

in view of (3.60) and Step 3 of the proof of Theorem 3.3. Since $f_t$ is a compactly supported $C^\infty$ function, where the latter is covered by Exercise 1.30, it follows that $\mathcal{D}(\mathbb{R}^n)$ is dense in $W_2^s(\mathbb{R}^n)$.

*Step* 2. By Proposition 3.17 and the above considerations $\mathcal{S}(\mathbb{R}^n)$ is dense both in $H^s(\mathbb{R}^n)$ and $W_2^s(\mathbb{R}^n)$. Then (3.55) will follow from the equivalence

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |(\mathcal{F} f)(\xi)|^2 \mathrm{d}\xi \sim \|f|W_2^s(\mathbb{R}^n)\|^2, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{3.62}$$

which we are going to show now. Assume first $0 < s = \sigma < 1$. By Theorem 2.65 (iii) the Fourier transform $\mathcal{F}$ is a unitary operator in $L_2(\mathbb{R}^n)$. Hence,

$$\begin{aligned} &\int_{\mathbb{R}^n} |h|^{-2\sigma} \|f(\cdot + h) - f(\cdot)|L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n} \\ &\qquad = \int_{\mathbb{R}^n} |h|^{-2\sigma} \|\mathcal{F}(f(\cdot + h) - f(\cdot))|L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n}. \end{aligned} \tag{3.63}$$

We insert

$$\begin{aligned} \mathcal{F}(f(\cdot + h) - f(\cdot))(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi x} (f(x+h) - f(x)) \mathrm{d}x \\ &= (e^{i\xi h} - 1)(\mathcal{F} f)(\xi) \end{aligned} \tag{3.64}$$

in (3.63) and obtain that

$$\begin{aligned} &\int_{\mathbb{R}^n} |h|^{-2\sigma} \|\Delta_h f|L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n} \\ &\qquad = \int_{\mathbb{R}^n} |\mathcal{F} f(\xi)|^2 \int_{\mathbb{R}^n} |e^{i\xi h} - 1|^2 |h|^{-2\sigma} \frac{\mathrm{d}h}{|h|^n} \, \mathrm{d}\xi \\ &\qquad = \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\mathcal{F} f(\xi)|^2 \int_{\mathbb{R}^n} |e^{i\frac{\xi}{|\xi|}h} - 1|^2 |h|^{-2\sigma} \frac{\mathrm{d}h}{|h|^n} \, \mathrm{d}\xi \\ &\qquad = c \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\mathcal{F} f(\xi)|^2 \mathrm{d}\xi \end{aligned} \tag{3.65}$$

for some $c > 0$, where we used that the converging integral over $h$ in the last but one line is independent of $\xi \in \mathbb{R}^n$, $\xi \neq 0$. This proves (3.62) in case of $0 < s = \sigma < 1$.

If $s = k + \sigma$ where $k \in \mathbb{N}$ and $0 < \sigma < 1$, then one applies (3.65) to $D^\alpha f$, $|\alpha| = k$, instead of $f$. By the same arguments as in (3.26) and

$$\sum_{|\alpha|=k} |\xi^\alpha|^2 \sim |\xi|^{2k}$$

one obtains the desired result as far as the respective second terms in (3.48) are concerned. Together with Theorem 3.11 and (3.26) this gives (3.62). □

**Corollary 3.25.** *Let $W_2^s(\mathbb{R}^n)$ with $s \geq 0$ be the spaces as introduced in Definition 3.1 if $s = k \in \mathbb{N}_0$ and in Definition 3.22 if $s \notin \mathbb{N}_0$, respectively. Let $H^s(\mathbb{R}^n)$ be the spaces according to Definition 3.13. Then*

$$H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n), \quad s \geq 0, \tag{3.66}$$

*(equivalent norms). Both $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are dense in $W_2^s(\mathbb{R}^n)$.*

*Proof.* This is an immediate consequence of (3.31) and the Theorems 3.3 and 3.24. □

**Remark 3.26.** In Note 3.6.1 we describe some extensions of $H^s(\mathbb{R}^n)$ and $W_2^s(\mathbb{R}^n)$ from $p = 2$ to $1 < p < \infty$. But then the situation is more complicated.

**Exercise 3.27.** Let $s = k + \sigma$ with $k \in \mathbb{N}_0$ and $0 < \sigma < 1$. Let $\Delta_h^m$, $m \in \mathbb{N}$, $h \in \mathbb{R}^n$, be the iterated differences according to (3.41). Prove that

$$\|f \,|W_2^k(\mathbb{R}^n)\| + \sum_{|\beta|\leq k} \Big( \int_{\mathbb{R}^n} |h|^{-2\sigma} \|\Delta_h^m D^\beta f \,|L_2(\mathbb{R}^n)\|^2 \,\frac{\mathrm{d}h}{|h|^n} \Big)^{1/2} \tag{3.67}$$

and

$$\|f \,|L_2(\mathbb{R}^n)\| + \sum_{j=1}^n \Big( \int_{\mathbb{R}^n} |h|^{-2\sigma} \Big\|\Delta_h^m \frac{\partial^k f}{\partial x_j^k} \Big|L_2(\mathbb{R}^n)\Big\|^2 \,\frac{\mathrm{d}h}{|h|^n} \Big)^{1/2} \tag{3.68}$$

are equivalent norms in $W_2^s(\mathbb{R}^n)$ where the integral over $\mathbb{R}^n$ can be replaced by an integral over $\{h \in \mathbb{R}^n : |h| \leq 1\}$.

*Hint:* Use (3.42), (3.43) first to reduce the question to $m = 1$. Afterwards, study the terms with $|\beta| < k$, modify (3.65) and rely on (3.32).

The following observation will be of some use for us later on.

**Proposition 3.28.** *Let $\Delta_h^m$, $m \in \mathbb{N}$, $h \in \mathbb{R}^n$, be the iterated differences according to (3.41). Let for $s \in \mathbb{R}$ and $f \in H^s(\mathbb{R}^n)$,*

$$\sup_{0<|h|<1} |h|^{-m} \|\Delta_h^m f \,|H^s(\mathbb{R}^n)\| < \infty. \tag{3.69}$$

*Then* $f \in H^{s+m}(\mathbb{R}^n)$ *and for some* $c > 0$,

$$\| f \,| H^{s+m}(\mathbb{R}^n)\| \le c \| f \,| H^s(\mathbb{R}^n)\| + c \sup_{0<|h|<1} |h|^{-m} \| \Delta_h^m f \,| H^s(\mathbb{R}^n)\|. \tag{3.70}$$

*Proof.* Iteration of (3.64) yields that

$$\mathcal{F}(\Delta_h^m f)(\xi) = (e^{i\xi h} - 1)^m \mathcal{F} f(\xi), \quad \xi \in \mathbb{R}^n. \tag{3.71}$$

Then it follows from (3.69) and Definition 3.13 that

$$\int_{\mathbb{R}^n} |h|^{-2m} |1 - e^{i\xi h}|^{2m} \langle \xi \rangle^{2s} |(\mathcal{F} f)(\xi)|^2 \mathrm{d}\xi \le C < \infty \tag{3.72}$$

uniformly in $h$, $0 < |h| < 1$. Choosing $h = (h_1, 0, \ldots, 0)$ leads to

$$\int_{\mathbb{R}^n} |\xi_1|^{2m} \langle \xi \rangle^{2s} |(\mathcal{F} f)(\xi)|^2 \mathrm{d}\xi \le C \tag{3.73}$$

in view of Fatou's lemma according to [Mal95, I.7.7, p. 38] (or [Tri86, 14.2.5, p. 125]). Similarly for the other directions. This proves (3.70). □

**Exercise* 3.29.** (a) Let $m \in \mathbb{N}$ and $0 < s < m$. Prove that

$$\| f \,| L_2(\mathbb{R}^n)\| + \left( \int_{\mathbb{R}^n} |h|^{-2s} \| \Delta_h^m f \,| L_2(\mathbb{R}^n)\|^2 \frac{\mathrm{d}h}{|h|^n} \right)^{1/2} \tag{3.74}$$

is an equivalent norm in $H^s(\mathbb{R}^n)$ where the integral over $\mathbb{R}^n$ can be replaced by an integral over $\{h \in \mathbb{R}^n : |h| \le 1\}$.

*Hint:* The case $0 < s < 1$ is covered by (3.68) with $k = 0$. For $m > s \ge 1$ use (3.71) and modify (3.65). See also Note 3.6.1.

(b) Let $\gamma > 0$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be some smooth cut-off function, say, as in connection with (3.14) with $\operatorname{supp} \varphi \subset K_2$. Prove that

$$f_\gamma(x) = |x|^\gamma \varphi(x), \quad x \in \mathbb{R}^n,$$

belongs to $H^s(\mathbb{R}^n)$ if $0 < s < \gamma + \frac{n}{2}$.

*Hint:* Apply (3.74) with $m \in \mathbb{N}$, $m > \gamma + \frac{n}{2}$.

(c) Extend the homogeneity estimates as formulated in Exercise 3.4 for spaces $W_2^k(\mathbb{R}^n)$ $(p = 2)$ to spaces $H^s(\mathbb{R}^n)$, $s \ge 0$.

*Hint:* Use (3.74) together with Exercise 3.19 (b).

## 3.3 Embeddings

Embedding theorems play a central rôle in the theory of function spaces. We concentrate here on the Sobolev spaces $H^s(\mathbb{R}^n)$ as introduced in Definition 3.13. According to Corollary 3.25 they coincide with the spaces $W_2^s(\mathbb{R}^n)$ for $s \geq 0$.

**Theorem 3.30.** *Let*

$$-\infty < s_1 < s < s_2 < \infty \quad \textit{and} \quad s = (1-\theta)s_1 + \theta s_2 \tag{3.75}$$

*with* $0 < \theta < 1$. *Then there is a positive constant $c$ such that for any $\varepsilon > 0$ and any $f \in H^{s_2}(\mathbb{R}^n)$,*

$$\begin{aligned} \|f\,|H^s(\mathbb{R}^n)\| &\leq \|f\,|H^{s_1}(\mathbb{R}^n)\|^{1-\theta}\,\|f\,|H^{s_2}(\mathbb{R}^n)\|^{\theta} \\ &\leq \varepsilon\|f\,|H^{s_2}(\mathbb{R}^n)\| + c\,\varepsilon^{-\frac{\theta}{1-\theta}}\,\|f\,|H^{s_1}(\mathbb{R}^n)\|. \end{aligned} \tag{3.76}$$

*Proof.* Equation (3.30) with $g = \mathcal{F} f$ and Hölder's inequality imply

$$\begin{aligned} \|f\,|H^s(\mathbb{R}^n)\| &= \left(\int_{\mathbb{R}^n} (\langle\xi\rangle^{2s_1}|g(\xi)|^2)^{1-\theta}(\langle\xi\rangle^{2s_2}|g(\xi)|^2)^{\theta}\mathrm{d}\xi\right)^{1/2} \\ &\leq (\varepsilon^{-\frac{1}{1-\theta}}\|f\,|H^{s_1}(\mathbb{R}^n)\|)^{1-\theta}(\varepsilon^{\frac{1}{\theta}}\|f\,|H^{s_2}(\mathbb{R}^n)\|)^{\theta} &(3.77)\\ &\leq c'\,\varepsilon^{-\frac{1}{1-\theta}}\,\|f\,|H^{s_1}(\mathbb{R}^n)\| + c'\,\varepsilon^{\frac{1}{\theta}}\|f\,|H^{s_2}(\mathbb{R}^n)\|, &(3.78) \end{aligned}$$

where we used the well-known inequality $ab \leq \frac{a^r}{r} + \frac{b^{r'}}{r'}$ for $a,b > 0$ and $r \in (1,\infty)$, $\frac{1}{r} + \frac{1}{r'} = 1$ in the last line. Now (3.77) with $\varepsilon = 1$ and (3.78) with $\varepsilon' = c'\varepsilon^{1/\theta}$ prove (3.76). ☐

**Corollary 3.31.** *Let $W_2^s(\mathbb{R}^n)$ with $s \geq 0$ be the spaces as introduced in Definitions 3.1 and 3.22 for $s = k \in \mathbb{N}_0$ and $s \notin \mathbb{N}_0$, respectively. Let $t$ be such that $0 \leq s < t < \infty$. Then there are constants $c > 0$ and $c' > 0$ such that for all $\varepsilon > 0$ and all $f \in W_2^t(\mathbb{R}^n)$,*

$$\begin{aligned} \|f\,|W_2^s(\mathbb{R}^n)\| &\leq c\|f\,|L_2(\mathbb{R}^n)\|^{\frac{t-s}{t}}\,\|f\,|W_2^t(\mathbb{R}^n)\|^{\frac{s}{t}} \\ &\leq \varepsilon\|f\,|W_2^t(\mathbb{R}^n)\| + c'\,\varepsilon^{-\frac{s}{t-s}}\,\|f\,|L_2(\mathbb{R}^n)\|. \end{aligned} \tag{3.79}$$

*Proof.* This follows from Corollary 3.25 and Theorem 3.30 with $s_1 = 0$ and $\theta = \frac{s}{t}$. ☐

Next we are interested in the so-called *Sobolev embedding*

$$\mathrm{id}\colon W_2^s(\mathbb{R}^n) \hookrightarrow C^{\ell}(\mathbb{R}^n), \quad \ell \in \mathbb{N}_0, \tag{3.80}$$

where $C^\ell(\mathbb{R}^n)$ are the spaces introduced in Definition A.1 with $\Omega = \mathbb{R}^n$. Here id is the identity interpreted as a linear and bounded map between the spaces indicated, hence

$$\|f|C^\ell(\mathbb{R}^n)\| \le c\,\|f|W_2^s(\mathbb{R}^n)\|, \quad f \in W_2^s(\mathbb{R}^n). \tag{3.81}$$

As for the use of '$\hookrightarrow$' one may consult Appendix C.1.

Strictly speaking, $W_2^s(\mathbb{R}^n)$ consists of equivalence classes $[f]$ whereas the elements of $C^\ell(\mathbb{R}^n)$ are functions. Then (3.80) must be understood in the sense that one finds in any equivalence class $[f] \in W_2^s(\mathbb{R}^n)$ a representative $f \in C^\ell(\mathbb{R}^n)$ (being unique if it exists) with (3.80). If one proves (3.81) first for smooth functions (being their own unique representatives), then one obtains in the limit just what one wants. We do not stress this point in the sequel.

**Theorem 3.32** (Sobolev embedding). *Let $C^\ell(\mathbb{R}^n)$, $\ell \in \mathbb{N}_0$, be the spaces introduced in Definition A.1 and let $W_2^s(\mathbb{R}^n)$ be the Sobolev spaces according to Definitions 3.1 and 3.22, respectively, with $s > \ell + \frac{n}{2}$. Then the embedding*

$$\mathrm{id}\colon W_2^s(\mathbb{R}^n) \hookrightarrow C^\ell(\mathbb{R}^n) \tag{3.82}$$

*exists in the sense explained above.*

*Proof.* We know by Corollary 3.25 that $\mathcal{S}(\mathbb{R}^n)$ is dense in $W_2^s(\mathbb{R}^n)$. In view of the above explanations it is thus sufficient to prove that there is a number $c > 0$ such that

$$|\mathrm{D}^\alpha\varphi(x)| \le c\|\varphi|W_2^s(\mathbb{R}^n)\|, \quad |\alpha| \le \ell,\ x \in \mathbb{R}^n, \tag{3.83}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Equations (2.104), (2.91) and Hölder's inequality imply that

$$\begin{aligned}
|\mathrm{D}^\alpha\varphi(x)| &= |\mathrm{D}^\alpha(\mathcal{F}^{-1}\mathcal{F}\varphi)(x)| = |\mathcal{F}^{-1}(\xi^\alpha\mathcal{F}\varphi(\xi))(x)| \\
&= c\left|\int_{\mathbb{R}^n} e^{ix\xi}\xi^\alpha(\mathcal{F}\varphi)(\xi)\mathrm{d}\xi\right| \\
&\le c'\int_{\mathbb{R}^n}\langle\xi\rangle^s|\mathcal{F}\varphi(\xi)|\langle\xi\rangle^{\ell-s}\mathrm{d}\xi \\
&\le c'\left(\int_{\mathbb{R}^n}\langle\xi\rangle^{2s}|\mathcal{F}\varphi(\xi)|^2\mathrm{d}\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}\langle\xi\rangle^{-2(s-\ell)}\mathrm{d}\xi\right)^{1/2}.
\end{aligned} \tag{3.84}$$

The last integral converges due to $2(s-\ell) > n$, and the first term in (3.84) represents an equivalent norm in $W_2^s(\mathbb{R}^n)$ according to Corollary 3.25 and $w_s(\xi) = \langle\xi\rangle^s$. □

**Exercise* 3.33.** Show that (3.82) fails for $s = \ell + \frac{n}{2}$ in general.

*Hint:* Review the examples considered in Exercise 3.6.

**Exercise 3.34.** Let $C^r(\mathbb{R}^n)$ with $r = k + \sigma$, $k \in \mathbb{N}_0$, $0 < \sigma < 1$, be the Hölder spaces normed by (3.44). Prove that Theorem 3.32 can be extended to $C^t(\mathbb{R}^n)$,

$$\mathrm{id}\colon W_2^s(\mathbb{R}^n) \hookrightarrow C^t(\mathbb{R}^n) \quad \text{if } s - \frac{n}{2} > t \geq 0. \tag{3.85}$$

*Hint:* Combine (3.84) with the arguments from (3.64), (3.65).

**Remark 3.35.** By the above proof (and the extension indicated in Exercise 3.34) it follows that

$$\mathrm{id}\colon W_2^s(\mathbb{R}^n) \hookrightarrow \mathring{C}^t(\mathbb{R}^n) \quad \text{if } s - \frac{n}{2} > t \geq 0, \tag{3.86}$$

where the latter spaces stand for the completion of $\mathcal{S}(\mathbb{R}^n)$ in $C^t(\mathbb{R}^n)$. In particular, one obtains for all $f \in W_2^s(\mathbb{R}^n)$ and all $\alpha$ with $0 \leq |\alpha| < s - \frac{n}{2}$ uniformly (with respect to $|x|$) that

$$(\mathrm{D}^\alpha f)(x) \to 0 \quad \text{if } |x| \to \infty. \tag{3.87}$$

**Exercise* 3.36.** (a) Prove that $\mathring{C}^t(\mathbb{R}^n)$, $t \geq 0$, is also the completion of $\mathcal{D}(\mathbb{R}^n)$ in $C^t(\mathbb{R}^n)$.

(b) Show that $\mathring{C}^t(\mathbb{R}^n)$ is a closed proper subspace of $C^t(\mathbb{R}^n)$, i.e.,

$$\mathring{C}^t(\mathbb{R}^n) \subsetneq C^t(\mathbb{R}^n).$$

*Hint:* What about $f \equiv 1$?

## 3.4 Extensions

For arbitrary domains (i.e., open sets) $\Omega$ in $\mathbb{R}^n$ we introduced in Definition A.1 the Banach spaces $C^\ell(\Omega)$ where $\ell \in \mathbb{N}_0$. Now we are interested in Sobolev spaces on $\Omega$ considered as subspaces of $L_p(\Omega)$, where the latter has the same meaning as at the beginning of Section 2.2, always interpreted as distributions on $\Omega$.

**Definition 3.37.** Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. Let $W_p^s(\mathbb{R}^n)$ be either the classical Sobolev spaces according to Definition 3.1 with $1 \leq p < \infty$ and $s \in \mathbb{N}_0$, or the Sobolev spaces as introduced in Definition 3.22 with $p = 2$ and $s > 0$, $s \notin \mathbb{N}$. Then

$$W_p^s(\Omega) = \{f \in L_p(\Omega) : \text{there exists } g \in W_p^s(\mathbb{R}^n) \text{ with } g\big|_\Omega = f\}. \tag{3.88}$$

**Remark 3.38.** Here $g\big|_\Omega$ denotes the restriction of $g$ to $\Omega$ considered as an element of $\mathcal{D}'(\Omega)$, hence $g\big|_\Omega = f$ means

$$g(\varphi) = f(\varphi) \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{3.89}$$

One can replace $f \in L_p(\Omega)$ in (3.88) by $f \in \mathcal{D}'(\Omega)$ and extend this definition to the spaces $H^s$ with $s < 0$ according to Definition 3.13.

**Proposition 3.39.** *Let* $\Omega$ *be an (arbitrary) domain in* $\mathbb{R}^n$ *and let either* $1 \le p < \infty$, $s \in \mathbb{N}_0$, *or* $p = 2$, $s > 0$. *Then* $W_p^s(\Omega)$, *furnished with the norm*

$$\|f|W_p^s(\Omega)\| = \inf\{\|g|W_p^s(\mathbb{R}^n)\| : g \in W_p^s(\mathbb{R}^n),\ g\big|_\Omega = f\}, \tag{3.90}$$

*becomes a Banach space (and a Hilbert space if* $p = 2$*), and*

$$\mathcal{D}(\Omega) \subset W_p^s(\Omega) \subset L_p(\Omega) \subset \mathcal{D}'(\Omega). \tag{3.91}$$

*Furthermore, the restriction* $\mathcal{D}(\mathbb{R}^n)\big|_\Omega$ *of* $\mathcal{D}(\mathbb{R}^n)$ *to* $\Omega$ *and the corresponding restriction* $\mathcal{S}(\Omega) = \mathcal{S}(\mathbb{R}^n)\big|_\Omega$ *are dense in* $W_p^s(\Omega)$.

*Proof.* Let $\Omega^c = \mathbb{R}^n \setminus \Omega$ and

$$\widetilde{W}_p^s(\Omega^c) = \{h \in W_p^s(\mathbb{R}^n) : \operatorname{supp} h \subset \Omega^c\}. \tag{3.92}$$

Since $\Omega^c$ is a closed set it follows from

$$\{h_k\}_{k=1}^\infty \subset \widetilde{W}_p^s(\Omega^c) \quad \text{and} \quad h_k \to h \ \text{ in } W_p^s(\mathbb{R}^n) \tag{3.93}$$

that $h \in \widetilde{W}_p^s(\Omega^c)$ as a consequence of Definition 2.22 and, say, (2.11). Hence $\widetilde{W}_p^s(\Omega^c)$ is a closed subspace of $W_p^s(\mathbb{R}^n)$ and

$$W_p^s(\Omega) \approx W_p^s(\mathbb{R}^n)\big/ \widetilde{W}_p^s(\Omega^c) \tag{3.94}$$

is isomorphic to the indicated factor space which, in turn, is a Banach space (and a Hilbert space if $p = 2$). As for the latter assertions one may consult [Yos80, I.11, pp. 59/60] (or [Rud91, pp. 30–32] where factor spaces are called quotient spaces). Finally, (3.91) is just the interpretation of $W_p^s(\Omega)$ as a space of distributions (2.36). The density of $\mathcal{D}(\mathbb{R}^n)\big|_\Omega$ and $\mathcal{S}(\Omega)$ in $W_p^s(\Omega)$ follows from the corresponding assertions in Theorems 3.3 and 3.24. □

**Remark 3.40.** Note that (3.91) is the counterpart of (3.3). We return to Sobolev spaces on smooth bounded domains later on in Chapter 4. At this moment we are only interested in $\Omega = \mathbb{R}^n_+$ where

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \ x_n > 0\}, \tag{3.95}$$

occasionally written as $x = (x', x_n) \in \mathbb{R}^n_+$ with $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n > 0$. We shall deal with the *extension problem* first, asking for linear and bounded (common) extension operators $\mathrm{ext}^L$, $L \in \mathbb{N}$, such that

$$\mathrm{ext}^L : \begin{cases} C^l(\mathbb{R}^n_+) \hookrightarrow C^l(\mathbb{R}^n), & l = 0, \dots, L, \\ W_p^l(\mathbb{R}^n_+) \hookrightarrow W_p^l(\mathbb{R}^n), & l = 0, \dots, L,\ 1 \le p < \infty, \\ W_2^s(\mathbb{R}^n_+) \hookrightarrow W_2^s(\mathbb{R}^n), & 0 < s < L, \end{cases} \tag{3.96}$$

with

$$\operatorname{ext}^L f\,|_{\mathbb{R}^n_+} = f. \tag{3.97}$$

This is the usual somewhat sloppy notation which means that $\operatorname{ext}^L$ is defined on the union of all these spaces such that its restriction to an admitted specific space has the properties (3.96), (3.97).

Recall that we said in Remark 3.5 what is meant by equivalent norms.

**Theorem 3.41.** (i) *For any $L \in \mathbb{N}$ there are extension operators* $\operatorname{ext}^L$ *according to* (3.96), (3.97).

(ii) *Let $1 \le p < \infty$ and $l \in \mathbb{N}_0$. Then*

$$\big\| f \,|W^l_p(\mathbb{R}^n_+)\big\|_* = \Big( \sum_{|\alpha| \le l} \big\| \mathrm{D}^\alpha f \,|L_p(\mathbb{R}^n_+)\big\|^p \Big)^{1/p} \sim \big\| f \,|W^l_p(\mathbb{R}^n_+)\big\| \tag{3.98}$$

*is an equivalent norm on $W^l_p(\mathbb{R}^n_+)$.*

(iii) *Let $s = l + \sigma$ with $l \in \mathbb{N}_0$ and $0 < \sigma < 1$. Then*

$$\begin{aligned} &\| f \,|W^s_2(\mathbb{R}^n_+)\|_* \\ &= \Big( \sum_{|\alpha| \le l} \|\mathrm{D}^\alpha f \,|L_2(\mathbb{R}^n_+)\|^2 + \sum_{|\alpha| = l} \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y \Big)^{1/2} \\ &\sim \| f \,|W^s_2(\mathbb{R}^n_+)\| \end{aligned} \tag{3.99}$$

*is an equivalent norm on $W^s_2(\mathbb{R}^n_+)$.*

*Proof. Step* 1. Let $\mathbb{Z}^n$ be as in Section A.1 and let $\{\psi_m : m \in \mathbb{Z}^n\}$ be a related resolution of unity in $\mathbb{R}^n$ with respect to suitable congruent balls $K_m$ centred at $m \in \mathbb{Z}^n$ in adaption of (2.53)–(2.57) where we may assume $\psi_m(x) = \psi(x - m)$, $m \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$. Hence

$$\psi_m \in \mathcal{D}(K_m), \quad 0 \le \psi_m \le 1, \quad \sum_{m \in \mathbb{Z}^n} \psi_m(x) = 1 \quad \text{if } x \in \mathbb{R}^n, \tag{3.100}$$

see the left-hand side of Figure 3.2 below.

Consequently,

$$\big\| f \,|C^l(\mathbb{R}^n_+)\big\| \sim \sup_{m \in \mathbb{Z}^n} \big\| \psi_m f \,|C^l(\mathbb{R}^n_+)\big\|, \quad f \in C^l(\mathbb{R}^n_+). \tag{3.101}$$

Furthermore one gets by Proposition 2.18 and iterates that

$$\big\| f \,|W^l_p(\mathbb{R}^n)\big\| \sim \Big( \sum_{m \in \mathbb{Z}^n} \big\| \psi_m f \,|W^l_p(\mathbb{R}^n)\big\|^p \Big)^{1/p}, \quad f \in W^l_p(\mathbb{R}^n). \tag{3.102}$$

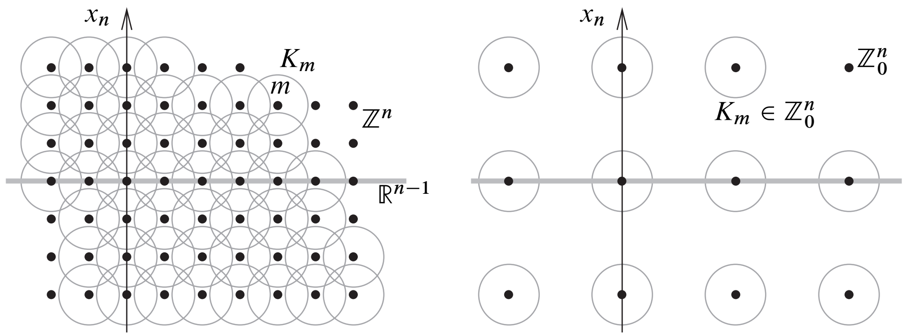



Figure 3.2

We decompose the lattice $\mathbb{Z}^n$ into finitely many sub-lattices

$$\mathbb{Z}^n = \bigcup_{j=0}^{J} \mathbb{Z}^n_j, \quad \mathbb{Z}^n_0 = \{x \in \mathbb{R}^n : x = Mm,\ m \in \mathbb{Z}^n\}, \\ \mathbb{Z}^n_j = m^{(j)} + \mathbb{Z}^n_0, \quad j = 1, \ldots, J, \tag{3.103}$$

where $M \in \mathbb{N}$ and $m^{(j)} \in \mathbb{Z}^n$ with $j = 1, \ldots, J$, are suitably chosen, see the right-hand side of Figure 3.2. In particular, balls $K_m$ belonging to the same sub-lattice $\mathbb{Z}^n_j$ have pairwise distance of, say, at least 1. Then one obtains a corresponding decomposition for the resolution of unity (3.100),

$$\psi^{(j)}(x) = \sum_{m \in \mathbb{Z}^n_j} \psi_m(x), \quad \sum_{j=0}^{J} \psi^{(j)}(x) = 1, \quad x \in \mathbb{R}^n, \tag{3.104}$$

and as in (3.102),

$$\| f \,|W^l_p(\mathbb{R}^n)\| \sim \sum_{j=0}^{J} \| \psi^{(j)} f \,|W^l_p(\mathbb{R}^n)\|, \quad f \in W^l_p(\mathbb{R}^n). \tag{3.105}$$

This localises the extension problem and leads, in particular, to

$$\| f \,|W^l_p(\mathbb{R}^n_+)\| \sim \Big( \sum_{m \in \mathbb{Z}^n} \| \psi_m f \,|W^l_p(\mathbb{R}^n_+)\|^p \Big)^{1/p}, \quad f \in W^l_p(\mathbb{R}^n_+). \tag{3.106}$$

*Step* 2. By Step 1 it is sufficient to extend functions $f$ in $\mathbb{R}^n_+$ with

$$\operatorname{supp} f \subset \{y \in \mathbb{R}^n : |y| < 1,\ y_n \geq 0\} \tag{3.107}$$

to $\mathbb{R}^n$. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_{L+1} < -1$ and

$$\big(\mathrm{ext}^L f\big)(x) = \begin{cases} f(x), & x_n \geq 0, \\ \sum_{k=1}^{L+1} a_k f(x', \lambda_k x_n), & x_n < 0, \end{cases} \tag{3.108}$$

for $x = (x', x_n) \in \mathbb{R}^n$. If $x_n < 0$, then $\lambda_k x_n > 0$; hence (3.108) makes sense, see Figure 3.3 aside.

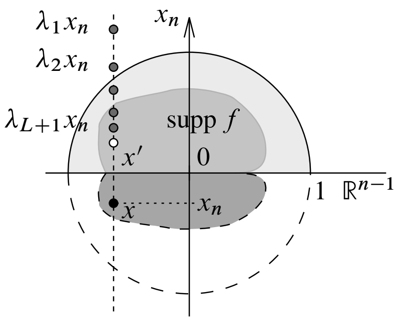



Figure 3.3

Let $f \in C^l(\mathbb{R}^n_+)$. We want to choose the coefficients $a_k, k = 1, \ldots, L+1$, in such a way that $\mathrm{ext}^L f \in C^l(\mathbb{R}^n)$. For that reason one has to care for the derivatives in the $x_n$-direction at $x_n = 0$. Since

$$\lim_{x_n \downarrow 0} \frac{\partial^r}{\partial x_n^r}\big(\mathrm{ext}^L f\big)(x) = \lim_{x_n \downarrow 0} \frac{\partial^r f}{\partial x_n^r}(x) = \frac{\partial^r f}{\partial x_n^r}(x', 0), \quad r = 0, \ldots, L, \tag{3.109}$$

and

$$\begin{aligned} \lim_{x_n \uparrow 0} \frac{\partial^r}{\partial x_n^r}\big(\mathrm{ext}^L f\big)(x) &= \lim_{x_n \uparrow 0} \sum_{k=1}^{L+1} a_k \lambda_k^r \frac{\partial^r f}{\partial y_n^r}(x', \lambda_k x_n) \\ &= \frac{\partial^r f}{\partial x_n^r}(x', 0) \sum_{k=1}^{L+1} a_k \lambda_k^r, \quad r = 0, \ldots, L, \end{aligned} \tag{3.110}$$

the coefficients have to satisfy

$$\sum_{k=1}^{L+1} a_k \lambda_k^r = 1, \quad r = 0, \ldots, L. \tag{3.111}$$

We can always choose the $a_k$'s appropriately since Vandermonde's determinant,

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_{L+1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^L & \lambda_2^L & \ldots & \lambda_{L+1}^L \end{vmatrix} = \prod_{k>\ell} (\lambda_k - \lambda_\ell) \tag{3.112}$$

is different from zero. In particular, $\operatorname{ext}^L f \in C^l(\mathbb{R}^n)$,

$$\big\| \operatorname{ext}^L f | C^l(\mathbb{R}^n) \big\| \le c \big\| f | C^l(\mathbb{R}^n_+) \big\| \tag{3.113}$$

for some $c > 0$ and all $f \in C^l(\mathbb{R}^n_+)$ with (3.107). In addition, one has

$$\operatorname{supp} \operatorname{ext}^L f \subset \{ y \in \mathbb{R}^n : |y| < 1 \}. \tag{3.114}$$

*Step* 3. Let $1 \le p < \infty$ and $l \in \mathbb{N}$ with $l \le L$. It follows from Proposition 3.39 that the restriction $\mathcal{D}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ to $\mathbb{R}^n_+$ is dense in $W^l_p(\mathbb{R}^n_+)$. But for smooth functions one gets by Step 2 that

$$\big\| \operatorname{ext}^L f | W^l_p(\mathbb{R}^n) \big\| \le c \big\| f | W^l_p(\mathbb{R}^n_+) \big\|_* \tag{3.115}$$

for some $c > 0$ and $l \in \mathbb{N}$ with $l \le L$ where we used (3.98). The rest is now a matter of completion having in mind the discussion before Theorem 3.32. This proves part (i) for $W^l_p$ and also (ii).

*Step* 4. Let $0 < s = k + \sigma < L$ with $k \in \mathbb{N}_0$ and $0 < \sigma < 1$. First we claim that the counterpart of (3.102) is given by

$$\begin{aligned} \big\| f | W^s_2(\mathbb{R}^n) \big\|^2 &\sim \sum_{m \in \mathbb{Z}^n} \big\| \psi_m f | W^s_2(\mathbb{R}^n) \big\|^2 \\ &\sim \sum_{m \in \mathbb{Z}^n} \Big( \big\| \psi_m f | W^k_2(\mathbb{R}^n) \big\|^2 \\ &\qquad + \sum_{|\alpha| = k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big| \mathrm{D}^\alpha (\psi_m f)(x) - \mathrm{D}^\alpha (\psi_m f)(y) \big|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y \Big) \end{aligned} \tag{3.116}$$

which can be reduced to the question whether

$$\begin{aligned} &\int_{|x-m| \le c} \int_{|x-y| \le 1} \frac{|(\psi_m g)(x) - (\psi_m g)(y)|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad \le c' \int_{|x-m| \le c} \int_{|x-y| \le 1} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y + c' \int_{|x-m| \le c} |g(x)|^2 \, \mathrm{d}x \end{aligned} \tag{3.117}$$

for some $c > 0$, $c' > 0$. However, the old trick of calculus

$$(\psi_m g)(x) - (\psi_m g)(y) = \psi_m(y)(g(x) - g(y)) + g(x)(\psi_m(x) - \psi_m(y)) \tag{3.118}$$

and $|\psi_m(x) - \psi_m(y)|^2 \le c|x-y|^2$ prove (3.117). By Proposition 3.39 the restrictions $\mathcal{D}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ and $\mathcal{S}(\mathbb{R}^n_+) = \mathcal{S}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ are dense in $W^s_2(\mathbb{R}^n_+)$. Now we are in the same position as in Step 3 asking for the counterpart of (3.115),

$$\big\| \operatorname{ext}^L f | W^s_2(\mathbb{R}^n) \big\| \le c \| f | W^s_2(\mathbb{R}^n_+) \|_* \tag{3.119}$$

for compactly supported $C^\infty$ functions as in Figure 3.3. Compared with the preceding steps it remains to prove that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\mathrm{D}^\alpha(\mathrm{ext}^L f)(x) - \mathrm{D}^\alpha(\mathrm{ext}^L f)(y)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y \le c\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.120}$$

Let $x = (x', x_n)$ and $y = (y', y_n)$. The integration over $\mathbb{R}^n \times \mathbb{R}^n$ on the left-hand side of (3.120) can be decomposed into

$$\{(x,y) \in \mathbb{R}^{2n} : x_n y_n \ge 0\} \quad \text{and} \quad \{(x,y) \in \mathbb{R}^{2n} : x_n y_n < 0\}. \tag{3.121}$$

The corresponding parts with $x_n y_n \ge 0$ can be estimated from above by the right-hand side of (3.120). As for $x_n > 0$, $y_n < 0$ (and, similarly, for $x_n < 0$, $y_n > 0$) one has to show that

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_-} \frac{|1\cdot \mathrm{D}^\alpha f(x', x_n) - \sum_{k=1}^{L+1} a_k \lambda_k^r (\mathrm{D}^\alpha f)(y', \lambda_k y_n)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y \le c\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.122}$$

where $r = |\alpha|$ with (3.111) and $\mathbb{R}^n_- = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : y_n < 0\}$. Replacing the factor 1 in front of $\mathrm{D}^\alpha f(x', x_n)$ by the left-hand side of (3.111) this question can be reduced to

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_-} \frac{|g(x', x_n) - g(y', \lambda y_n)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y \le c\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y \tag{3.123}$$

with $\lambda < -1$. If $0 < \lambda y_n \le x_n$, then $0 \le x_n - \lambda y_n \le x_n - y_n$. If $\lambda y_n > x_n$, then one obtains

$$|x_n - \lambda y_n| \le |\lambda||y_n| \le |\lambda||x_n - y_n|, \tag{3.124}$$

see also Figure 3.4 aside. Hence if one replaces $|x-y|^2$ on the left-hand side of (3.123) by

$$|x' - y'|^2 + (x_n - \lambda y_n)^2 \le |x-y|^2,$$

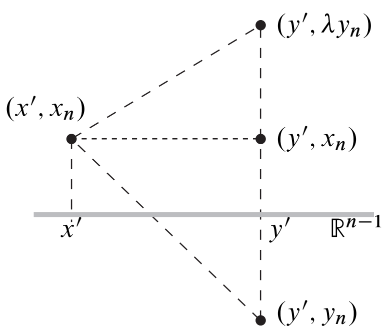


Figure 3.4

one obtains an estimate from above which results in (3.123). This proves part (i) of the theorem and also part (iii). □

**Exercise 3.42.** Prove the equality in (3.112).

*Hint:* Develop Vandermonde's determinant (of order $L+1$) by its last column, determine the zeros of the corresponding polynomial and reduce by this method the problem to the corresponding one of order $L$. Iterate the process.

**Exercise 3.43.** Justify the counterpart of (3.106) for $W_2^s(\mathbb{R}^n)$, $s>0$, $s\notin\mathbb{N}$.

*Hint:* Rely on (3.48) and the arguments in (3.117).

We give a simple application of Theorem 3.41 (i) which will be of some use for us later on.

**Corollary 3.44.** *Let $W_2^s(\mathbb{R}^n_+)$ and $W_2^t(\mathbb{R}^n_+)$ be the same spaces as in Theorem 3.41 (i) with $0\le s<t<\infty$. Then there are constants $c>0$ and $c'>0$ such that for all $\varepsilon>0$ and all $f\in W_2^t(\mathbb{R}^n_+)$,*

$$\begin{aligned}\|f\,|W_2^s(\mathbb{R}^n_+)\| &\le c\|f\,|L_2(\mathbb{R}^n_+)\|^{\frac{t-s}{t}}\,\|f\,|W_2^t(\mathbb{R}^n_+)\|^{\frac{s}{t}}\\ &\le \varepsilon\|f\,|W_2^t(\mathbb{R}^n_+)\|+c'\,\varepsilon^{-\frac{s}{t-s}}\|f\,|L_2(\mathbb{R}^n_+)\|.\end{aligned}\tag{3.125}$$

*Proof.* Let $\mathrm{ext}^L$ be a common extension operator for the spaces involved. Then (3.125) follows from Corollary 3.31 and

$$\begin{aligned}\|f\,|W_2^s(\mathbb{R}^n_+)\| &\le c\big\|\mathrm{ext}^L f\,|L_2(\mathbb{R}^n)\big\|^{\frac{t-s}{t}}\,\big\|\mathrm{ext}^L f\,|W_2^t(\mathbb{R}^n)\big\|^{\frac{s}{t}}\\ &\le c'\big\|f\,|L_2(\mathbb{R}^n_+)\big\|^{\frac{t-s}{t}}\,\big\|f\,|W_2^t(\mathbb{R}^n_+)\big\|^{\frac{s}{t}}.\end{aligned}\tag{3.126}$$

□

## 3.5 Traces

Let $C^l(\mathbb{R}^n)$ and $C^l(\mathbb{R}^n_+)$ with $l\in\mathbb{N}_0$ be the spaces as introduced in Definition A.1, where $\mathbb{R}^n_+$ is given by (3.95). Obviously, any $f\in C^l(\mathbb{R}^n)$ or $f\in C^l(\mathbb{R}^n_+)$ has pointwise trace

$$(\mathrm{tr}_\Gamma f)(x)=f(x',0)\tag{3.127}$$

on

$$\Gamma=\{x=(x',x_n)\in\mathbb{R}^n: x_n=0\}.\tag{3.128}$$

If $C^l(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n)$ and $C^l(\mathbb{R}^n_+)\subset\mathcal{D}'(\mathbb{R}^n_+)$ are considered as distributions, then (3.127) means that the trace is taken for the distinguished (and unique) representative of the corresponding equivalence class $[f]$. But this point of view can be extended to the Sobolev spaces considered so far, $W_p^k(\mathbb{R}^n)$, $W_p^k(\Omega)$ according to Definitions 3.1, 3.37, and $H^s(\mathbb{R}^n)$, $W_2^s(\mathbb{R}^n)$ according to Definitions 3.13, 3.22

and Theorem 3.24. In other words, one may ask whether there are distinguished representatives of elements (equivalence classes) of some of these spaces for which (3.127), (3.128) make sense. In this context one may also consult Note 4.6.3. As a first, but not very typical example may serve the Sobolev embedding in Theorem 3.32, Exercise 3.34, where the distinguished representatives of $f \in W_2^s(\mathbb{R}^n)$ according to (3.81), (3.82), (3.85) have (pointwise) traces according to (3.127), (3.128). It is usual to adopt an easier and slightly different approach (resulting in the same traces as indicated).

Let $A(\mathbb{R}^n)$ or $A(\mathbb{R}^n_+)$ be one of the above spaces, for example $W_p^1(\mathbb{R}^n_+)$, and let $\mathcal{S}(\mathbb{R}^n_+) = \mathcal{S}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ be again the restriction of $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{R}^n_+$. Then one asks whether there is a constant $c > 0$ such that

$$\|\varphi|L_p(\Gamma)\| \le c\|\varphi|A(\mathbb{R}^n_+)\| \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n_+), \tag{3.129}$$

and an obvious counterpart for $A(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Assuming that $\mathcal{S}(\mathbb{R}^n_+)$ is dense in $A(\mathbb{R}^n_+)$ one approximates $f \in A(\mathbb{R}^n_+)$ by $\varphi_j \in \mathcal{S}(\mathbb{R}^n_+)$ where $j \in \mathbb{N}$. If one has (3.129), then $\{\varphi_j(x',0)\}_{j=1}^\infty$ is a Cauchy sequence in $L_p(\Gamma)$. Its limit element is called the *trace* of $f \in A(\mathbb{R}^n_+)$ and denoted by $\operatorname{tr}_\Gamma f \in L_p(\Gamma)$. Completion implies

$$\|\operatorname{tr}_\Gamma f|L_p(\Gamma)\| \le c\|f|A(\mathbb{R}^n_+)\|, \quad f \in A(\mathbb{R}^n_+), \tag{3.130}$$

and

$$\operatorname{tr}_\Gamma \colon A(\mathbb{R}^n_+) \hookrightarrow L_p(\Gamma) \tag{3.131}$$

is a linear and bounded operator. By (3.129) the resulting trace $\operatorname{tr}_\Gamma f \in L_p(\Gamma)$ is independent of the approximating sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{S}(\mathbb{R}^n_+)$. Later on we deal in detail with traces of spaces $W_p^k(\Omega)$ and $H^s(\Omega) = W_2^s(\Omega)$ on the boundary $\partial\Omega$ of bounded $C^\infty$ domains in $\mathbb{R}^n$. At this moment we are more interested in the above description of the trace problem which will now be exemplified by having a closer look at the spaces $W_p^1(\mathbb{R}^n_+)$ and $W_p^1(\mathbb{R}^n)$.

**Theorem 3.45.** *Let $n \ge 2$, $1 \le p < \infty$, and let $W_p^1(\mathbb{R}^n)$ and $W_p^1(\mathbb{R}^n_+)$ be the spaces according to the Definitions 3.1 and 3.37 with $\Omega = \mathbb{R}^n_+$, respectively. Let $\Gamma$ be as in (3.128). Then the trace operators,*

$$\operatorname{tr}_\Gamma \colon W_p^1(\mathbb{R}^n) \hookrightarrow L_p(\Gamma), \tag{3.132}$$

*and*

$$\operatorname{tr}_\Gamma^+ \colon W_p^1(\mathbb{R}^n_+) \hookrightarrow L_p(\Gamma), \tag{3.133}$$

*exist and one has for some $c > 0$,*

$$\|\operatorname{tr}_\Gamma f|L_p(\Gamma)\| \le c\|f|W_p^1(\mathbb{R}^n)\|, \quad f \in W_p^1(\mathbb{R}^n), \tag{3.134}$$

*and*

$$\| \operatorname{tr}_\Gamma^+ f | L_p(\Gamma) \| \le c \| f | W_p^1(\mathbb{R}_+^n) \|, \quad f \in W_p^1(\mathbb{R}_+^n). \tag{3.135}$$

*Furthermore,*

$$\operatorname{tr}_\Gamma f = \operatorname{tr}_\Gamma^+ g \quad \textit{for } f \in W_p^1(\mathbb{R}^n) \textit{ and } g = f\,|_{\mathbb{R}_+^n}. \tag{3.136}$$

*Proof. Step* 1. Let $f \in C^1(\mathbb{R}_+^n)$ be real and assume

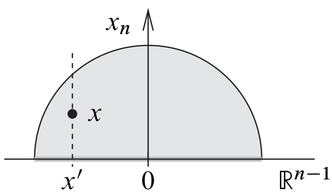



Figure 3.5

$$\operatorname{supp} f \subset \{x \in \overline{\mathbb{R}_+^n} : |x| < 1\}. \tag{3.137}$$

For fixed $x' \in \mathbb{R}^{n-1}$, $|x'| < 1$, we may choose $\tau = \tau(x') \in [0, 1]$ such that

$$\int_0^1 f(x', x_n)\mathrm{d}x_n = f(x', \tau). \tag{3.138}$$

Then one obtains by Hölder's inequality

$$\begin{aligned} |f(x',0)|^p &= \left| f(x',\tau) - \int_0^\tau \frac{\partial f}{\partial x_n}(x', x_n)\, \mathrm{d}x_n \right|^p \\ &\le c \int_0^1 \left( |f(x', x_n)|^p + \left| \frac{\partial f}{\partial x_n}(x', x_n) \right|^p \right) \mathrm{d}x_n. \end{aligned} \tag{3.139}$$

Integration over $x' \in \mathbb{R}^{n-1}$ gives (3.135) where one may use (3.98). This inequality can be extended to complex-valued functions $f \in C^1(\mathbb{R}_+^n)$ with (3.137). Recall that $\mathcal{S}(\mathbb{R}_+^n)$ is dense in $W_p^1(\mathbb{R}_+^n)$ according to Proposition 3.39. Then the above approximation procedure with (3.130) is a consequence of (3.129), and the decomposition argument (3.106) together with its obvious counterpart for $L_p(\Gamma)$ proves (3.135).

*Step* 2. The above arguments cover also (3.134) and one obtains as a by-product (3.136). □

**Remark 3.46.** Obviously one can replace $W_p^1(\mathbb{R}^n)$ in (3.134), (3.135) by $W_p^l(\mathbb{R}^n)$ with $l \in \mathbb{N}$ and, if $p = 2$, by $W_2^s(\mathbb{R}^n)$ with $s \ge 1$. We refer later on in Section 4.5 to traces of $W_2^s(\Omega)$ spaces on the boundary $\partial\Omega$. Then it comes out that (3.134), (3.135) with $p = 2$ remains valid even for $s > \frac{1}{2}$.

## 3.6 Notes

**3.6.1.** It is not the aim of Chapters 3 and 4 to develop the theory of Sobolev spaces (or even more general spaces) for their own sake. We restrict ourselves to those topics which are needed later on, that is, we concentrate preferably on the Hilbert spaces $W_2^s$ on $\mathbb{R}^n$ and $\mathbb{R}^n_+$ with $s > 0$ (Chapter 3) and on bounded $C^\infty$ domains $\Omega$ and their boundaries $\Gamma = \partial\Omega$ (Chapter 4). On the other hand, if no additional effort is needed or if it is simply more natural, then we adopt(ed) a wider point of view. This applies to the spaces $H^s(\mathbb{R}^n)$ which we introduced in Definition 3.13 for all $s \in \mathbb{R}$, though we are mainly interested in the case $s \geq 0$ where they coincide with the spaces $W_2^s(\mathbb{R}^n)$ according to Definition 3.22, Theorem 3.24 and Corollary 3.25. Similarly, the classical Sobolev spaces $W_p^k(\mathbb{R}^n)$ have been introduced in (3.1) for all $p$, $1 \leq p < \infty$ (and $k \in \mathbb{N}_0$) though the Fourier-analytical characterisation of interest, (3.31), is restricted to $p = 2$. One may ask to which extent these assertions and also the descriptions in terms of differences according to (3.48) and (3.74) have counterparts if $L_2$ is replaced by $L_p$. It is not the subject of this book and will not be needed later on, but it seems reasonable to add a few relevant comments. One may also consult Appendix E for further information.

First we extend $H^s(\mathbb{R}^n)$ in (3.30) by

$$H_p^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(w_s \mathcal{F} f) \in L_p(\mathbb{R}^n)\} \tag{3.140}$$

from $p = 2$, $s \in \mathbb{R}$, to $1 < p < \infty$, $s \in \mathbb{R}$, where $w_s$ is given by (3.23). (Recall that in case of $p = 2$ one has (2.162) both for $\mathcal{F}$ and $\mathcal{F}^{-1}$.) Nowadays it is quite usual to call $H_p^s(\mathbb{R}^n)$, naturally normed by

$$\| f \,|\, H_p^s(\mathbb{R}^n)\| = \|\mathcal{F}^{-1}(w_s \mathcal{F} f) \,|\, L_p(\mathbb{R}^n)\|, \tag{3.141}$$

*Sobolev spaces*, where the notation *classical Sobolev spaces* is reserved for the spaces $W_p^k(\mathbb{R}^n)$ as introduced in Definition 3.1. The crucial assertion (3.31) can be generalised to

$$H_p^k(\mathbb{R}^n) = W_p^k(\mathbb{R}^n), \quad 1 < p < \infty,\ k \in \mathbb{N}_0. \tag{3.142}$$

In contrast to (3.31) which is an easy consequence of the observation that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are unitary operators in $L_2(\mathbb{R}^n)$, (3.142) relies on the *Michlin–Hörmander Fourier multiplier theorem* in $L_p(\mathbb{R}^n)$, $1 < p < \infty$, which is much deeper. It says that there is a constant $c > 0$ such that for all $m \in C^l(\mathbb{R}^n)$ according to Definition A.1 with $l \in \mathbb{N}$, $l > \frac{n}{2}$ (one may choose the smallest $l$ with this property, $l = [\frac{n}{2}] + 1$) and all $f \in L_p(\mathbb{R}^n)$,

$$\|\mathcal{F}^{-1}(m \mathcal{F} f) \,|\, L_p(\mathbb{R}^n)\| \leq c \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq l} |x|^{\alpha} |\mathrm{D}^\alpha m(x)| \, \| f \,|\, L_p(\mathbb{R}^n)\|. \tag{3.143}$$

A proof of (a vector-valued version of) (3.143), comments and references may be found in [Tri78, Section 2.2.4]. Taking (3.143) for granted the justification of (3.142) is not complicated.

What about $L_p$ counterparts of (3.48), (3.74) (and of (3.50), (3.47)) and also of Theorem 3.24? It turns out that the situation is now more complicated and it seems to be reasonable to say what is meant by the *classical Besov spaces* $B^s_{p,q}(\mathbb{R}^n)$, where $s > 0$, $1 \le p \le \infty$, $1 \le q \le \infty$. Let $\Delta^m_h$ be the differences according to (3.41) and let for $0 < s < m \in \mathbb{N}$,

$$\|f\,|B^s_{p,q}(\mathbb{R}^n)\|_m = \|f\,|L_p(\mathbb{R}^n)\| + \Bigg( \int\limits_{|h|\le 1} |h|^{-sq} \|\Delta^m_h f\,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}h}{|h|^n} \Bigg)^{1/q} \tag{3.144}$$

when $q < \infty$, modified by

$$\|f\,|B^s_{p,\infty}(\mathbb{R}^n)\|_m = \|f\,|L_p(\mathbb{R}^n)\| + \sup_{|h|\le 1} |h|^{-s} \|\Delta^m_h f\,|L_p(\mathbb{R}^n)\|. \tag{3.145}$$

In particular, with $\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$, $s > 0$,

$$\|f\,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \|f\,|C(\mathbb{R}^n)\| + \sup_{|h|\le 1} \sup_{x\in\mathbb{R}^n} |h|^{-s} |\Delta^m_h f(x)|. \tag{3.146}$$

Then

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in L_p(\mathbb{R}^n) : \|f\,|B^s_{p,q}(\mathbb{R}^n)\|_m < \infty\}. \tag{3.147}$$

These spaces are independent of $m$ (in the sense of equivalence of norms) in generalisation of the Exercises 3.21, 3.27, 3.29, but with the same hints. Similarly as in (3.48), (3.67), (3.68) one can replace some differences in (3.144) by derivatives. In particular,

$$B^s_{p,p}(\mathbb{R}^n) \quad \text{with } 1 < p < \infty,\ s = k + \sigma,\ k \in \mathbb{N}_0,\ 0 < \sigma < 1, \tag{3.148}$$

can be equivalently normed by

$$\begin{aligned} &\big\| f\,|B^s_{p,p}(\mathbb{R}^n)\big\|_* \\ &= \|f\,|W^k_p(\mathbb{R}^n)\| + \sum_{|\alpha|=k} \Bigg( \iint\limits_{\mathbb{R}^{2n}} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y \Bigg)^{1/p}. \end{aligned} \tag{3.149}$$

In generalisation of Theorem 3.24 it seems to be natural to ask whether the above Sobolev spaces $H^s_p(\mathbb{R}^n)$ and the *special Besov spaces* $B^s_{p,p}(\mathbb{R}^n)$ coincide. But this is not the case whenever $p \ne 2$. More precisely, for given $s > 0$ and $1 < p < \infty$, then

$$B^s_{p,p}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n) \quad \text{if, and only if,} \quad p = 2. \tag{3.150}$$

It should be mentioned that the spaces $B^s_{p,p}(\mathbb{R}^n)$ with (3.148) had been introduced in the late 1950s, denoted as $W^s_p(\mathbb{R}^n)$ and called *Slobodeckij spaces*. But the notation $W^s_p(\mathbb{R}^n)$ is slightly dangerous as it seems to suggest that $W^s_p(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n)$ naturally fill the gaps between the classical Sobolev spaces $W^k_p(\mathbb{R}^n)$, $k \in \mathbb{N}_0$. However, this is not the case if $p \neq 2$, also for structural reasons we are going to discuss now. The natural extension of *classical* Sobolev spaces $W^s_p(\mathbb{R}^n)$ with $s = k \in \mathbb{N}_0$ to arbitrary $s \in \mathbb{R}$ are the Sobolev spaces $H^s_p(\mathbb{R}^n)$ in good agreement with (3.142).

**3.6.2.** Two (complex) Banach spaces $B_1$ and $B_2$ are called *isomorphic*, written as $B_1 \approx B_2$, if there is a linear and bounded one-to-one map $T$ of $B_1$ *onto* $B_2$ such that

$$\|Tb|B_2\| \sim \|b|B_1\|, \quad b \in B_1 \tag{3.151}$$

(equivalent norms). Then the inverse $T^{-1}$ maps $B_2$ isomorphically *onto* $B_1$. Let, as usual, $\ell_p$, $1 \le p \le \infty$, be the Banach space of all sequences $b = \{b_j\}_{j=1}^{\infty}$ with $b_j \in \mathbb{C}$, $j \in \mathbb{N}$, such that

$$\|b|\ell_p\| = \Big(\sum_{j=1}^{\infty} |b_j|^p\Big)^{1/p} < \infty \tag{3.152}$$

if $p < \infty$, modified by

$$\|b|\ell_\infty\| = \sup_{j \in \mathbb{N}} |b_j| < \infty. \tag{3.153}$$

All Hilbert spaces in the Chapters 3 and 4 are complex and separable and, hence, isomorphic to $\ell_2$. This applies especially to the spaces $H^s(\mathbb{R}^n)$ in Definition 3.13, Proposition 3.17 and hence to $W^s_2(\mathbb{R}^n)$ in Corollary 3.25, but also to their restrictions to bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$, and to $\Gamma = \partial\Omega$ as considered in Chapter 4 below. If $p \neq 2$, then the situation is different.

Let $I = (0,1) = \{t \in \mathbb{R} : 0 < t < 1\}$ be the open unit interval in $\mathbb{R}$. We collect some isomorphic relations between $L_p$ spaces and $\ell_p$ spaces going back essentially to the famous book by S. Banach [Ban32, Chapter 12]:

$(L1)$ Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$ and let $1 < p < \infty$. Then

$$L_p(\Omega) \approx L_p(I).$$

$(LL)$ Let $1 < p_0 < \infty$, $1 < p_1 < \infty$. Then

$$L_{p_0}(I) \approx L_{p_1}(I) \quad \text{if, and only if,} \quad p_0 = p_1.$$

$(\ell\ell)$ Let $1 \le p_0 \le \infty$, $1 \le p_1 \le \infty$. Then

$$\ell_{p_0} \approx \ell_{p_1} \quad \text{if, and only if,} \quad p_0 = p_1.$$

$(L\ell)$ Let $1 < p_0 < \infty$, $1 < p_1 < \infty$. Then

$$L_{p_0}(I) \approx \ell_{p_1} \quad \text{if, and only if,} \quad p_0 = p_1 = 2.$$

We followed essentially [Tri78, Section 2.11.1] where one finds the necessary references, especially to the original literature. A more recent account on problems of this type has been given in [AK06]. The isomorphic structure of $H^s_p(\mathbb{R}^n)$ in (3.140), of $B^s_{p,p}(\mathbb{R}^n)$ in (3.148), and of $\mathcal{C}^s(\mathbb{R}^n)$ normed by (3.146) is the following:

$(H)$ Let $1 < p < \infty$ and $s \in \mathbb{R}$. Then

$$H^s_p(\mathbb{R}^n) \approx L_p(I).$$

$(B)$ Let $1 < p < \infty$ and $s > 0$. Then

$$B^s_{p,p}(\mathbb{R}^n) \approx \ell_p.$$

$(\mathcal{C})$ Let $s > 0$. Then

$$\mathcal{C}^s(\mathbb{R}^n) \approx \ell_\infty.$$

We refer to [Tri78, Section 2.11.2, pp. 237–240, and p. 343], and, more recently, to [Tri06, Section 3.1.4, p. 157] where one also finds further isomorphic assertions, comments, and references to the (original) literature. In particular, in view of $(L\ell)$, $(H)$ and $(B)$, the spaces $H^s_p(\mathbb{R}^n)$ and $B^s_{p,p}(\mathbb{R}^n)$ belong to different isomorphic classes (unless $p = 2$) which sheds some new light on (3.150).

**3.6.3.** The spaces on $\mathbb{R}^n$ and $\mathbb{R}^n_+$ considered in this Chapter 3 and also their restrictions to domains $\Omega$ and to the related boundaries $\partial\Omega$, treated in Chapter 4, including $H^s_p$, $B^s_{p,q}$, $\mathcal{C}^s$ mentioned above, are special cases of the two scales of spaces

$$B^s_{p,q}, \; F^s_{p,q}, \quad \text{where } 0 < p \le \infty, \; 0 < q \le \infty, \; s \in \mathbb{R}, \tag{3.154}$$

($p < \infty$ for $F$-spaces) which are subject of several books and many papers, especially of [Tri78], [Tri83], [Tri92b], [Tri06]. We refer, in particular, to [Tri92b, Chapter 1] and to [Tri06, Chapter 1] which are historically oriented surveys from the roots up to our time. There one finds many references and discussions, including the classical spaces treated in the present book. This will not be repeated here, but we wish to mention the outstanding Russian contributions subject to [BIN75], [Maz85], [Nik77], [Sob91]. Here we are mainly interested in Sobolev spaces. There are several books devoted especially to diverse aspects of Sobolev spaces on $\mathbb{R}^n$ and in domains. We refer, in particular, to [AF03], [Bur98], [Maz85], [Zie89]. In Appendix E we recall briefly the formal definitions of the spaces in (3.154), complemented by a few properties mentioned in diverse Notes in this book. We list some special cases covering especially the spaces mentioned in Notes 3.6.1, 3.6.2.

**3.6.4.** In the Sections 3.4, 3.5 we dealt with extensions and traces in connection with $\mathbb{R}^n_+$. Both topics deserve to be commented. But we return in Chapter 4 to these outstanding problems in connection with bounded $C^\infty$ domains and shift some discussion and also references to the Notes in Section 4.6.

**3.6.5.** We wish to discuss a peculiar point in connection with the 'sinister' rôle played by the differences $\Delta_h^m$ according to (3.41) in (3.44), (3.47), (3.67), (3.68), (3.74), but also in (3.144), (3.146). One may have the impression that the more handsome first differences are sufficient as in (3.44) and (3.149). This is also largely correct as long as $0 < s \notin \mathbb{N}$. But if $s \in \mathbb{N}$, then one needs at least second differences $\Delta_h^2$. This has been observed 1854 in connection with the above spaces $\mathcal{C}^1(\mathbb{R}^n) = B^1_{\infty,\infty}(\mathbb{R}^n)$ (different from $C^1(\mathbb{R}^n)$ being a genuine subset of $\mathcal{C}^1(\mathbb{R}^n)$) more than 150 years ago by B. Riemann in his *Habilitationsschrift* (German, meaning *habilitation thesis*) [Rie54] and has again been treated by A. Zygmund 1945 in [Zyg45]. More details may be found in [Tri01, Section 14.5, pp. 225/226]. A log-term is coming in which is rather typical for the recent theory of envelopes of spaces of this type which may be found in [Har07] and the references given there.

# Chapter 4
# Sobolev spaces on domains

## 4.1 Basic definitions

For arbitrary domains $\Omega$ in $\mathbb{R}^n$ we introduced in Definition A.1 the spaces $C^\ell(\Omega)$ where $\ell \in \mathbb{N}_0$ or $\ell = \infty$, in Definition 3.37 the spaces

$$W_p^k(\Omega), \quad 1 \le p < \infty,\ k \in \mathbb{N}_0, \tag{4.1}$$

and

$$W_2^s(\Omega), \quad s \ge 0. \tag{4.2}$$

Recall that domain means open set. Plainly, $W_2^s(\Omega)$ in (4.2) coincides with $W_p^k(\Omega)$ in (4.1) when $s = k \in \mathbb{N}_0$, $p = 2$. Proposition 3.39 implies that $W_2^s(\Omega)$, $s \ge 0$, are Hilbert spaces. We wish to extend Theorem 3.41 from $\mathbb{R}^n_+$ to domains. But this is not possible for arbitrary domains. One needs some specifications. We rely on bounded $C^\ell$ domains in $\mathbb{R}^n$ and, especially, on bounded $C^\infty$ domains in $\mathbb{R}^n$ according to Definition A.3. Recall that by definition $C^\ell$ domains and $C^\infty$ domains are, in particular, *connected open sets*. We complement Definition A.3, Remark A.4 and Figure A.1 as follows.

Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let the balls $K_j$ with $j = 1, \dots, J$, form a suitable covering as in (A.11). Then there are diffeomorphic $C^\infty$ maps (curvilinear coordinates)

$$y = \psi^{(j)}(x)\colon K_j \Longleftrightarrow V_j = \psi^{(j)}(K_j), \quad j = 1, \dots, J, \tag{4.3}$$

such that

$$\psi^{(j)}(K_j \cap \Omega) \subset \mathbb{R}^n_+, \quad \psi^{(j)}(K_j \cap \partial\Omega) \subset \mathbb{R}^{n-1}, \tag{4.4}$$

as indicated in Figure 4.1 below.

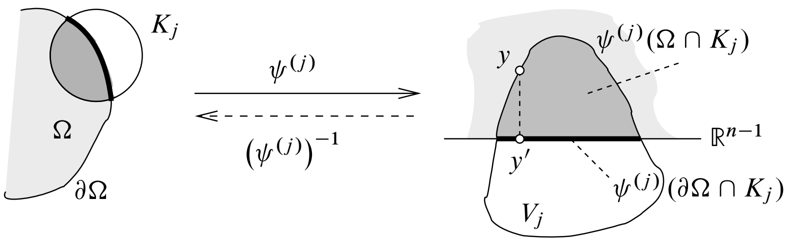



Figure 4.1

One may assume that $V_j$ is simply connected and that the upper boundary of $\psi^{(j)}(K_j \cap \Omega)$ can be described by $y_n = \tau^{(j)}(y')$, where $\tau^{(j)}$ is a $C^\infty$ function. In the canonical situation as sketched in Figure A.1 together with (A.12) one may choose $y' = x'$ and $y_n = x_n - h(x')$ locally. For our later purpose the fibre-preserving specification of (4.3) as indicated in Figure 4.2 is of some use:

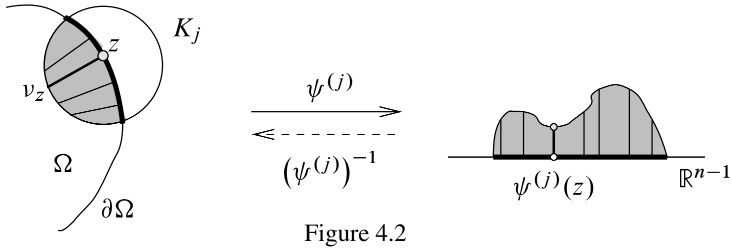



Figure 4.2

Here the inner normal directions $\nu_z$ with $z \in \partial\Omega \cap K_j$ are mapped in normal $y_n$-directions with the foot-points $\psi^{(j)}(z)$.

## 4.2 Extensions and intrinsic norms

We are looking for the counterpart of Theorem 3.41 with bounded $C^\infty$ domains $\Omega$ in place of $\mathbb{R}^n_+$. One can rely largely on the techniques developed so far.

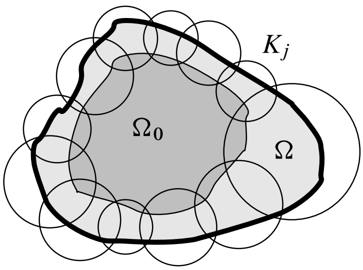



Figure 4.3

Let $K_j$ with $j = 1, \ldots, J$ be the same balls as in Definition A.3 and in the preceding Section 4.1. Let $\Omega_0$ be an inner domain with $\overline{\Omega_0} \subset \Omega$ as indicated in Figure 4.3 aside; hence

$$\partial\Omega \subset \bigcup_{j=1}^J K_j \quad \text{and} \quad \Omega \subset \Omega_0 \cup \Big(\bigcup_{j=1}^J K_j\Big). \tag{4.5}$$

Let $\{\varphi_j\}_{j=0}^J$ be a related resolution of unity of $\overline{\Omega}$ according to Section 2.4, hence $\varphi_j$ are non-negative functions with

$$\varphi_0 \in \mathcal{D}(\Omega_0), \quad \varphi_j \in \mathcal{D}(K_j),\ j = 1, \ldots, J, \tag{4.6}$$

and

$$\sum_{j=0}^J \varphi_j(x) = 1 \quad \text{if } x \in \overline{\Omega}. \tag{4.7}$$

Let $\psi^{(j)}$ be the diffeomorphic maps (4.3). As in (3.96), (3.97) we ask for (common) extension operators $\mathrm{ext}^L_\Omega$, where $L \in \mathbb{N}$, such that

$$\mathrm{ext}^L_\Omega : \begin{cases} C^\ell(\Omega) \hookrightarrow C^\ell(\mathbb{R}^n), & \ell = 0, \ldots, L, \\ W^\ell_p(\Omega) \hookrightarrow W^\ell_p(\mathbb{R}^n), & \ell = 0, \ldots, L,\ 1 \le p < \infty, \\ W^s_2(\Omega) \hookrightarrow W^s_2(\mathbb{R}^n), & 0 < s < L, \end{cases} \tag{4.8}$$

with

$$\mathrm{ext}^L_\Omega f_{|\Omega} = f. \tag{4.9}$$

The understanding of (4.8), (4.9) is the same as the corresponding one for (3.97) with $\Omega$ in place of $\mathbb{R}^n_+$.

**Theorem 4.1.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $C^\ell(\Omega)$, $W^\ell_p(\Omega)$, $W^s_2(\Omega)$ be the spaces recalled in Section 4.1.*

(i) *For any $L \in \mathbb{N}$ there are extension operators $\mathrm{ext}^L_\Omega$ according to (4.8), (4.9).*

(ii) *Let $1 \le p < \infty$ and $\ell \in \mathbb{N}_0$. Then*

$$\|f\,|W^\ell_p(\Omega)\|_* = \Big(\sum_{|\alpha|\le\ell} \|\mathrm{D}^\alpha f\,|L_p(\Omega)\|^p\Big)^{1/p} \sim \|f\,|W^\ell_p(\Omega)\| \tag{4.10}$$

*is an equivalent norm in $W^\ell_p(\Omega)$.*

(iii) *Let $0 < s = \sigma + \ell$ with $\ell \in \mathbb{N}_0$ and $0 < \sigma < 1$. Then*

$$\begin{aligned} &\big\|f\,|W^s_2(\Omega)\big\|_* \\ &= \Big(\sum_{|\alpha|\le\ell} \|\mathrm{D}^\alpha f\,|L_2(\Omega)\|^2 + \sum_{|\alpha|=\ell} \iint\limits_{\Omega\times\Omega} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/2} \\ &\sim \|f\,|W^s_2(\Omega)\| \end{aligned} \tag{4.11}$$

*is an equivalent norm in $W^s_2(\Omega)$.*

*Proof.* Let $\{\varphi_j\}^J_{j=0}$ be the resolution of unity according to (4.5)–(4.7) and let $\{\psi^{(j)}\}^J_{j=1}$ be the diffeomorphic maps as described in (4.3) and Figure 4.1. We are essentially in the same position as in the proof of Theorem 3.41. It follows from Proposition 3.39 that it is sufficient to extend smooth functions $f$ from $\Omega$ to $\mathbb{R}^n$ and to control the norms of the corresponding spaces $C^\ell$, $W^\ell_p$ and $W^s_2$. Again we decompose $f$ according to (4.6), (4.7), hence

$$f(x) = \varphi_0(x) f(x) + \sum_{j=1}^J \varphi_j(x) f(x), \quad x \in \Omega, \tag{4.12}$$

where the term $\varphi_0 f$ can be extended outside of $\Omega$ by zero.

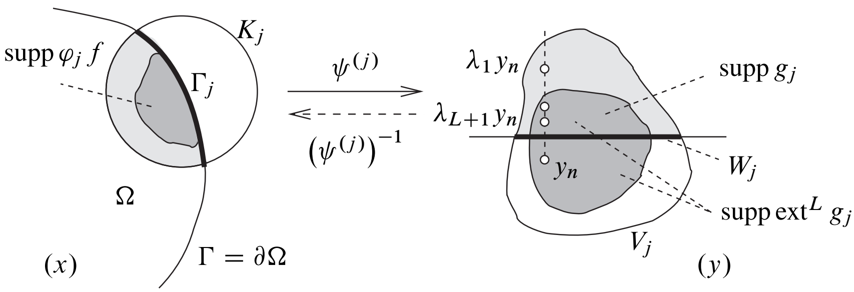



Figure 4.4

The functions $\varphi_j f$ have supports as indicated in Figure 4.4. Using the $C^\infty$ diffeomorphisms $\psi^{(j)}$ according to (4.3), (4.4) and Figure 4.1 one obtains functions

$$g_j(y) = (\varphi_j f) \circ (\psi^{(j)})^{-1}(y), \quad j = 1, \ldots, J, \tag{4.13}$$

to which Theorem 3.41 can be applied. The procedure as indicated in Figure 3.3 and according to Step 2 of the proof of Theorem 3.41 results in functions

$$\mathrm{ext}^L g_j \quad \text{with } \mathrm{supp}(\mathrm{ext}^L g_j) \subset V_j = \psi^{(j)}(K_j). \tag{4.14}$$

Returning to the $x$-variables one gets functions

$$h_j(x) = (\mathrm{ext}^L g_j) \circ \psi^{(j)}(x), \quad \mathrm{supp}\, h_j \subset K_j,\ h_j|_\Omega = \varphi_j f \tag{4.15}$$

with the desired properties, assuming, in addition, that we put $h_j(x) = 0$ if $x \in \mathbb{R}^n \setminus K_j$. Then

$$\mathrm{ext}^L_\Omega f = \varphi_0 f + \sum_{j=1}^J h_j \tag{4.16}$$

is the extension operator we are looking for. Since everything is reduced to the $\mathbb{R}^n_+$-case one obtains (4.10) from (3.98) by standard arguments which are also the subject of Exercise 4.3 below. Transformation of (3.99) gives (4.11), but with $\sum_{|\alpha|\le\ell}$ also in the terms with respect to the integration over $\Omega \times \Omega$. However, if $\ell \in \mathbb{N}$, then Corollary 3.44 implies that

$$\| f \,|W_2^{\ell-1+\sigma}(\Omega)\| \le c \| f \,|W_2^\ell(\Omega)\|. \tag{4.17}$$

This shows that the disturbing terms with $|\alpha| < \ell$ in the integration over $\Omega \times \Omega$ can be incorporated in the first sum in (4.11). □

**Exercise 4.2.** Let $\Omega = K_{1/2}(0) \subset \mathbb{R}^n$, $k \in \mathbb{N}_0$, $1 \le p < \infty$.

(a) Let $\alpha \in \mathbb{R}$ and $g_\alpha(x) = |x|^\alpha$, $x \in \mathbb{R}^n$. Show that

$$g_\alpha \in W_p^k(\Omega) \quad \text{if, and only if,} \quad \begin{cases} \text{either} & \alpha = 2r,\ r \in \mathbb{N}_0, \\ \text{or} & \alpha > k - \frac{n}{p}. \end{cases}$$

*Hint:* Show that for $\alpha \neq 2r$ with $r \in \mathbb{N}_0$,

$$|\Delta^m g_\alpha(x)| \sim |x|^{\alpha - 2m}, \quad \text{and} \quad \sum_{j=1}^n \left| \frac{\partial}{\partial x_j} \Delta^m g_\alpha(x) \right| \sim |x|^{\alpha-2m-1}, \quad m \in \mathbb{N},$$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, as usual. Use (1.12)–(1.14). Compare it with Exercise 3.29 (b).

(b) Let $\Theta = (-1,1) \times (-1,1) = \{(x_1,x_2) \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\} \subset \mathbb{R}^2$, and

$$u(x_1,x_2) = \begin{cases} 1 - |x_1| & \text{if } |x_2| < |x_1|, \\ 1 - |x_2| & \text{if } |x_1| < |x_2|, \end{cases}$$

as indicated in Figure 4.5 below. Show that $u \in W_p^1(\Theta)$ for $1 \le p < \infty$.
*Hint:* Use (2.44) and Exercise 2.17 (b) extended to $\mathbb{R}^2$.

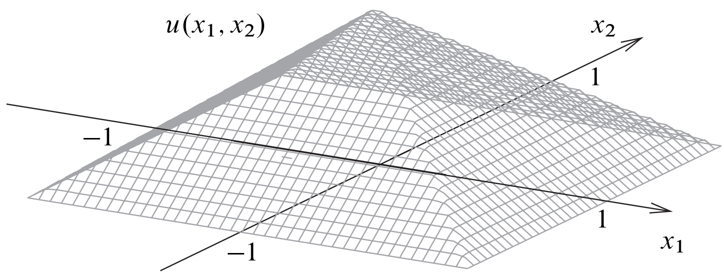



Figure 4.5

**Exercise 4.3.** A continuous one-to-one map of $\mathbb{R}^n$ onto itself,

$$\begin{aligned} y &= \psi(x) = (\psi_1(x), \ldots, \psi_n(x)), \\ x &= \psi^{-1}(y) = (\psi_1^{-1}(y), \ldots, \psi_n^{-1}(y)), \end{aligned} \tag{4.18}$$

is called a *diffeomorphism* if all components $\psi_j(x)$ and $\psi_j^{-1}(y)$ are real $C^\infty$ functions on $\mathbb{R}^n$ and for $j = 1, \ldots, n$,

$$\sup_{x \in \mathbb{R}^n} \left( |\mathrm{D}^\alpha \psi_j(x)| + |\mathrm{D}^\alpha \psi_j^{-1}(x)| \right) < \infty \quad \text{for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| > 0. \tag{4.19}$$

Extend consistently the composition

$$f \mapsto f \circ \psi \tag{4.20}$$

from functions to distributions $f \in \mathcal{S}'(\mathbb{R}^n)$. Let $A(\mathbb{R}^n)$ be one of the spaces $C^\ell(\mathbb{R}^n)$, $W_p^\ell(\mathbb{R}^n)$, $W_2^s(\mathbb{R}^n)$ as in (4.8). Prove that (4.20) is a linear and isomorphic map of $A(\mathbb{R}^n)$ onto $A(\mathbb{R}^n)$. Show that (4.20) maps also $\mathcal{S}(\mathbb{R}^n)$ onto itself, and $\mathcal{S}'(\mathbb{R}^n)$ onto itself.

*Hint:* Concerning the extension of the composition $f \circ \psi$ to $\mathcal{S}'(\mathbb{R}^n)$ the Jacobian should appear. In case of $W_p^\ell(\mathbb{R}^n)$ and $W_2^s(\mathbb{R}^n)$ one may use the density of $\mathcal{S}(\mathbb{R}^n)$ in these spaces.

**Exercise 4.4.** Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let

$$\omega = \psi(\Omega) = \{y \in \mathbb{R}^n : \text{there exists an } x \in \Omega \text{ with } y = \psi(x)\}, \tag{4.21}$$

where $\psi$ is the above diffeomorphism. Let $A(\Omega)$ be one of the spaces $C^\ell(\Omega)$, $W_p^\ell(\Omega)$, $W_2^s(\Omega)$ as in (4.8). Prove that (4.20) is a linear and isomorphic map of $A(\omega)$ onto $A(\Omega)$.

**Remark 4.5.** In modification of fibre-preserving maps as indicated in Figure 4.2 the following version of the extension procedure will be of some use. If $\varepsilon > 0$ is sufficiently small, then the strip

$$S_\varepsilon = \{x \in \mathbb{R}^n : \text{there exists a } y \in \partial\Omega \text{ with } |x-y| < \varepsilon\}, \tag{4.22}$$

in Figure 4.6 below around the boundary $\partial\Omega$ of the above bounded $C^\infty$ domain $\Omega$ can be furnished at least *locally* with curvilinear $C^\infty$ coordinates

$$\sigma = (\sigma', \sigma_n) \quad \text{where } \sigma' = (\sigma_1, \dots, \sigma_{n-1}) \in \partial\Omega,\ |\sigma_n| < \varepsilon, \tag{4.23}$$

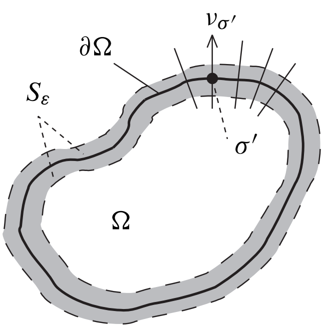


Figure 4.6

and $\sigma_n$ measures the distance to $\partial\Omega$ along the $C^\infty$ normal vector field $\nu_{\sigma'}$ (or any other non-trivial, non-tangential $C^\infty$ vector field on $\partial\Omega$). Also the resolution of unity in (4.6) can be adapted assuming that $\varphi_j \in \mathcal{D}(K_j)$, $j = 1, \dots, J$, is given by

$$\begin{aligned} \varphi_j(\sigma) &= \varphi_j'(\sigma')\varphi_{j,n}(\sigma_n), \\ \varphi_j' &\in \mathcal{D}(K_j \cap \partial\Omega), \end{aligned} \tag{4.24}$$

and $\varphi_{j,n} \in \mathcal{D}((-\varepsilon, \varepsilon))$ with $\varphi_{j,n}(\sigma_n) = 1$ when $|\sigma_n| < \varepsilon/2$.

Afterwards one can repeat the above extension procedure as indicated in Figure 4.4 with the (local) curvilinear $C^\infty$ coordinates $(\sigma', \sigma_n)$ in place of $(y', y_n)$.

Recall that $\mathcal{D}(\mathbb{R}^n)|_\Omega$ and $\mathcal{S}(\Omega) = \mathcal{S}(\mathbb{R}^n)|_\Omega$ are the restrictions of $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ to $\Omega$, respectively, as in Proposition 3.39.

**Corollary 4.6.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $L \in \mathbb{N}$. Let $C^\ell(\Omega)$, $W_p^\ell(\Omega)$ and $W_2^s(\Omega)$ be the same spaces as in (4.8) and Theorem 4.1. Then*

$$\mathcal{D}(\mathbb{R}^n)|_\Omega = \mathcal{S}(\Omega) \quad \text{and} \quad C^L(\Omega) \tag{4.25}$$

*are dense in these spaces.*

*Proof.* By Proposition 3.39 both $\mathcal{D}(\mathbb{R}^n)|_\Omega$ and $\mathcal{S}(\Omega)$ are dense in $W_p^\ell(\Omega)$ and $W_2^s(\Omega)$. Since $\Omega$ is bounded these two sets coincide. By Theorem 4.1 any $f \in C^\ell(\Omega)$ is the restriction of

$$g = \mathrm{ext}_\Omega^L f \in C^\ell(\mathbb{R}^n) \quad \text{with } \operatorname{supp} g \text{ compact.} \tag{4.26}$$

By the same mollification argument as in the proof of Proposition 2.7 one can approximate $g$ in $C^\ell(\mathbb{R}^n)$ by functions belonging to $\mathcal{D}(\mathbb{R}^n)$. In particular, $\mathcal{S}(\Omega)$ is dense in $C^\ell(\Omega)$ and, as a consequence,

$$C^L(\Omega) \subset W_p^\ell(\Omega), \quad C^L(\Omega) \subset W_2^s(\Omega), \tag{4.27}$$

which completes the proof. □

**Exercise* 4.7.** Let $\ell \in \mathbb{N}$, $1 \le p < \infty$. Prove that for bounded $C^1$ domains $\Omega$ in $\mathbb{R}^n$ the counterpart of Theorem 3.3 (for $\Omega = \mathbb{R}^n$) is not true, that is, $\mathcal{D}(\Omega)$ is not dense in $W_p^\ell(\Omega)$.

*Hint:* First reduce the situation to $W_1^1(\Omega)$ as a consequence of Hölder's inequality. Choose $u \equiv 1$ on $\Omega$ and show that there is some positive constant $c$ (depending on $\Omega$ only) such that

$$\|u - \varphi | W_1^1(\Omega)\| \ge c \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

**Exercise* 4.8.** (a) Let $\Omega = (0,1)$ be the unit interval in $\mathbb{R}$. Prove *Poincaré's inequality* in $W_p^1(\Omega)$, $1 \le p < \infty$, i.e.,

$$\|u|L_p(\Omega)\| \le \|u'|L_p(\Omega)\| \quad \text{for all } u \in W_p^1(\Omega) \text{ with } \int_\Omega u(x)\mathrm{d}x = 0. \tag{4.28}$$

*Hint:* Integrate $u(x) - u(y) = \int_y^x u'(z)\mathrm{d}z$ over $y$ and apply Hölder's inequality.

(b) Show that (4.28) does not hold for $\Omega = \mathbb{R}$.

*Hint:* Construct odd functions $u_k$, $k \in \mathbb{N}$, with $\|u_k|L_p(\mathbb{R})\| = 1$ and $\|u_k'|L_p(\mathbb{R})\| \to 0$ for $k \to \infty$.

## 4.3 Odd and even extensions

Our later considerations of elliptic equations in bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$ rely at least partly on subspaces and traces of $W_2^s(\Omega)$ on $\partial\Omega$. This section may be considered as a preparation, but also as a continuation of the preceding section.

The boundary $\Gamma = \partial\Omega$ of a bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$ will be furnished in the usual naïve way with a surface measure $\mathrm{d}\sigma$ as used in Section A.3 in connection with integral formulas and in Section 1. The corresponding complex-valued Lebesgue spaces $L_p(\Gamma)$, where $1 \le p < \infty$, are normed by

$$\|g|L_p(\Gamma)\| = \Big( \int_\Gamma |g(\gamma)|^p \, \mathrm{d}\sigma(\gamma) \Big)^{1/p}. \tag{4.29}$$

Let $A(\Omega)$ be one of the spaces covered by Corollary 4.6. Then one can look for traces of $f \in A(\Omega)$ on $\Gamma$ by the same type of reasoning as in Section 3.5. Since $\mathcal{S}(\Omega)$ is dense in $A(\Omega)$ one asks first whether there is a constant $c > 0$ such that

$$\|\varphi|L_p(\Gamma)\| \le c\|\varphi|A(\Omega)\| \quad \text{for all } \varphi \in \mathcal{S}(\Omega). \tag{4.30}$$

If this is the case, then one defines $\mathrm{tr}_\Gamma\, f \in L_p(\Gamma)$ for $f \in A(\Omega)$ by completion and obtains

$$\|\,\mathrm{tr}_\Gamma\, f|L_p(\Gamma)\| \le c\|f|A(\Omega)\|, \quad f \in A(\Omega), \tag{4.31}$$

for the linear and bounded trace operator

$$\mathrm{tr}_\Gamma\colon A(\Omega) \hookrightarrow L_p(\Gamma). \tag{4.32}$$

All this must be done in the understanding as presented in Section 3.5. Parallel to our discussion in Section 3.5 in connection with Theorem 3.45 one finds that $A(\Omega)$ can be replaced by $A(\mathbb{R}^n)$ in (4.32) with the same outcome (as a consequence of the density assertions in Corollary 4.6). In Section 4.5 we shall deal with traces of $W_2^s(\Omega)$ in detail. Here we discuss some consequences of Section 3.5.

Let $W_p^\ell(\Omega)$ with $\ell \in \mathbb{N}$ and $1 \le p < \infty$ be the classical Sobolev spaces as introduced in Definition 3.37 where $\Omega$ is again a bounded $C^\infty$ domain in $\mathbb{R}^n$. Combining the decomposition technique of Section 4.2 with Theorem 3.45 it follows that the spaces $W_p^\ell(\Omega)$ with $\ell \in \mathbb{N}$ have traces on $\Gamma = \partial\Omega$,

$$\mathrm{tr}_\Gamma\colon W_p^\ell(\Omega) \hookrightarrow L_p(\Gamma), \quad \ell \in \mathbb{N},\ 1 \le p < \infty, \tag{4.33}$$

and

$$\|\,\mathrm{tr}_\Gamma\, f|L_p(\Gamma)\| \le c\|f|W_p^\ell(\Omega)\|, \quad f \in W_p^\ell(\Omega). \tag{4.34}$$

Furthermore, by the discussion in Remark 4.5,

$$\frac{\partial f}{\partial \nu} \in W_p^{\ell-1}(S_\varepsilon \cap \Omega) \quad \text{with } f \in W_p^\ell(\Omega),\ \ell \in \mathbb{N}, \tag{4.35}$$

makes sense. Taking the trace, (4.33), (4.34) imply that

$$\operatorname{tr}_\Gamma \frac{\partial}{\partial \nu} : W_p^\ell(\Omega) \hookrightarrow L_p(\Gamma), \quad \ell \in \mathbb{N},\ \ell \geq 2,\ 1 \leq p < \infty, \tag{4.36}$$

and

$$\Big\| \operatorname{tr}_\Gamma \frac{\partial f}{\partial \nu} \,\Big| L_p(\Gamma) \Big\| \leq c \| f | W_p^\ell(\Omega) \|, \quad f \in W_p^\ell(\Omega), \tag{4.37}$$

are well-defined traces. In particular, the following definitions make sense.

**Definition 4.9.** Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $1 \leq p < \infty$.

(i) Then

$$W_{p,0}^2(\Omega) = \{ f \in W_p^2(\Omega) \,:\, \operatorname{tr}_\Gamma f = 0 \}, \tag{4.38}$$

and $w_{p,0}^2(\Omega)$ is the completion of

$$\{ f \in C^2(\Omega) \,:\, \operatorname{tr}_\Gamma f = 0 \} \tag{4.39}$$

in $W_p^2(\Omega)$.

(ii) Then

$$W_p^{2,0}(\Omega) = \Big\{ f \in W_p^2(\Omega) \,:\, \operatorname{tr}_\Gamma \frac{\partial f}{\partial \nu} = 0 \Big\}, \tag{4.40}$$

and $w_p^{2,0}(\Omega)$ is the completion of

$$\Big\{ f \in C^2(\Omega) \,:\, \operatorname{tr}_\Gamma \frac{\partial f}{\partial \nu} = 0 \Big\} \tag{4.41}$$

in $W_p^2(\Omega)$.

**Remark 4.10.** In view of the above discussion and Corollary 4.6 the definitions are reasonable. Furthermore, $W_{p,0}^2(\Omega)$, $W_p^{2,0}(\Omega)$ as well as $w_{p,0}^2(\Omega)$, $w_p^{2,0}(\Omega)$ are closed subspaces of $W_p^2(\Omega)$. Corollary 4.6 and the discussion about traces imply

$$w_{p,0}^2(\Omega) \subset W_{p,0}^2(\Omega) \quad \text{and} \quad w_p^{2,0}(\Omega) \subset W_p^{2,0}(\Omega). \tag{4.42}$$

One can prove that

$$w_{p,0}^2(\Omega) = W_{p,0}^2(\Omega) \quad \text{and} \quad w_p^{2,0}(\Omega) = W_p^{2,0}(\Omega) \quad \text{if } 1 < p < \infty. \tag{4.43}$$

But this will not be done here in general. However, in case of $p = 2$ we return to (4.43) in Proposition 4.32 and Remark 4.33 below. We postpone this point for the moment and deal with the $w$-spaces subject to a special extension procedure.

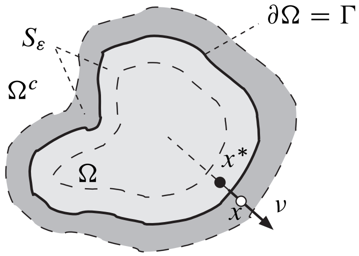



Figure 4.7

We rely on Remark 4.5 and Figure 4.6 with the outer normals $\nu = \nu_{\sigma'}$. Let again $\Omega^c = \mathbb{R}^n \setminus \Omega$. There is a one-to-one relation between $x \in S_\varepsilon \cap \Omega^c$ and its mirror point $x^* \in S_\varepsilon \cap \overline{\Omega}$,

$$x^* = x + \lambda \nu, \quad \operatorname{dist}(x, \Gamma) = \operatorname{dist}(x^*, \Gamma) \geq 0, \tag{4.44}$$

as shown in Figure 4.7 aside.

Let $\chi \in \mathcal{D}(\Omega \cup S_\varepsilon)$ be a cut-off function with $\chi(y) = 1$ if $y \in \Omega \cup S_{\varepsilon/2}$. Then the odd and the even extensionof $f \in C^2(\Omega)$, given by

$$\mathsf{O}\text{-}\operatorname{ext} f(x) = \begin{cases} \chi(x) f(x), & x \in \Omega, \\ -\chi(x) f(x^*), & x \in S_\varepsilon \cap \Omega^c, \end{cases} \tag{4.45}$$

and

$$\mathsf{E}\text{-}\operatorname{ext} f(x) = \begin{cases} \chi(x) f(x), & x \in \Omega, \\ \chi(x) f(x^*), & x \in S_\varepsilon \cap \Omega^c, \end{cases} \tag{4.46}$$

respectively, and extended by zero outside of $\Omega \cup S_\varepsilon$, are well-defined. Of course, $\mathsf{O}\text{-}\operatorname{ext} f$ may be discontinuous at $\Gamma$ and $\mathsf{E}\text{-}\operatorname{ext} f$ may have discontinuous first derivatives at $\Gamma$. But restricted to the spaces introduced in Definition 4.9 one obtains the following assertion.

**Theorem 4.11.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $1 \leq p < \infty$. Then*

$$\mathsf{O}\text{-}\operatorname{ext}\colon w^2_{p,0}(\Omega) \hookrightarrow W^2_p(\mathbb{R}^n) \tag{4.47}$$

*and*

$$\mathsf{E}\text{-}\operatorname{ext}\colon w^{2,0}_p(\Omega) \hookrightarrow W^2_p(\mathbb{R}^n) \tag{4.48}$$

*are linear and bounded extension operators for the spaces indicated.*

*Proof.* The above discussions, especially in connection with Remark 4.5, imply that it is sufficient to deal with the same standard situation as in the proof of Theorem 3.41, Figure 3.3 and (3.108). But this is essentially a one-dimensional affair. Hence, let $f \in C^2(\mathbb{R}_+)$ according to Definition A.1 with $f(x) = 0$ if $x > 1$ and $\mathbb{R}_+ = (0, \infty)$ according to (3.95). Then $f$ and its first and second derivatives $f'$ and $f''$ are continuous in $\overline{\mathbb{R}_+} = [0, \infty)$.

For convenience, we deal with the odd extension only, i.e., we consider (4.47). Let $g = \mathsf{O}\text{-}\operatorname{ext} f$ be the odd extension of $f$ according to (4.45), which in our case may be simplified by

$$g(x) = \mathsf{O}\text{-}\operatorname{ext} f(x) = \begin{cases} f(x), & x > 0, \\ -f(-x), & x \leq 0. \end{cases} \tag{4.49}$$

Let

$$\widetilde{g}\,'(x) = \begin{cases} f'(x), & x > 0, \\ f'(-x), & x \le 0, \end{cases} \tag{4.50}$$

and

$$\widetilde{g}\,''(x) = \begin{cases} f''(x), & x > 0, \\ -f''(-x), & x \le 0, \end{cases} \tag{4.51}$$

be the pointwise derivatives. The distributional derivatives are denoted by $g'$ and $g''$. As usual, $\delta$ is the $\delta$-distribution with respect to the origin. Then

$$g' = 2f(0)\delta + \widetilde{g}\,'. \tag{4.52}$$

This can be seen as follows. Let $\varphi \in \mathcal{S}(\mathbb{R})$, then integration by parts yields

$$\begin{aligned} g'(\varphi) = -g(\varphi') &= -\int_0^\infty f(x)\varphi'(x)\mathrm{d}x + \int_{-\infty}^0 f(-x)\varphi'(x)\mathrm{d}x \\ &= 2f(0)\varphi(0) + \int_0^\infty f'(x)\varphi(x)\mathrm{d}x + \int_{-\infty}^0 f'(-x)\varphi(x)\mathrm{d}x \\ &= (2f(0)\delta + \widetilde{g}\,')(\varphi). \end{aligned} \tag{4.53}$$

Since $f(0) = 0$ in our case we get $g' = \widetilde{g}\,'$. As for $g''$ we are led in a similar way to $g'' = \widetilde{g}\,''$ since $\widetilde{g}\,'$ is continuous at the origin. In particular, $g \in W_p^2(\mathbb{R})$ and

$$\|g|W_p^2(\mathbb{R})\| \le c\|f|W_p^2(\mathbb{R}_+)\|. \tag{4.54}$$

This proves (4.47). The argument concerning (4.48) is similar and left to the reader. □

**Exercise 4.12.** Prove (4.48).

*Hint:* Modify the above proof appropriately, using this time $f'(0) = 0$.

## 4.4 Periodic representations and compact embeddings

Let $n \in \mathbb{N}$, and

$$\mathbb{Q}^n = (-\pi, \pi)^n = \{x \in \mathbb{R}^n : -\pi < x_j < \pi,\ j = 1, \ldots, n\}, \tag{4.55}$$

as in (B.1), and

$$K = K_1(0) = \{x \in \mathbb{R}^n : |x| < 1\} \tag{4.56}$$

be the unit ball in $\mathbb{R}^n$, see (1.30). According to Theorem B.1 any $f \in L_2(\mathbb{Q}^n)$ can be represented as

$$f(x) = \sum_{m \in \mathbb{Z}^n} a_m h_m(x), \quad x \in \mathbb{Q}^n, \tag{4.57}$$

where

$$h_m(x) = (2\pi)^{-n/2} \, e^{imx}, \quad m \in \mathbb{Z}^n, \; x \in \mathbb{Q}^n, \tag{4.58}$$

is an orthonormal basis in the complex Hilbert space $L_2(\mathbb{Q}^n)$. In particular,

$$\|f \,|L_2(\mathbb{Q}^n)\|^2 = \sum_{m \in \mathbb{Z}^n} |a_m|^2 \quad \text{where } a_m = (2\pi)^{-n/2} \int_{\mathbb{Q}^n} f(x) e^{-imx} \mathrm{d}x, \tag{4.59}$$

are the related Fourier coefficients. With the interpretation of $\mathbb{Q}^n$ as the $n$-torus $\mathbb{T}^n$ one can develop a theory of periodic Sobolev spaces $H^s(\mathbb{T}^n)$, $s \in \mathbb{R}$, and $W_2^s(\mathbb{T}^n)$, $s \geq 0$, which is largely parallel to the theory of spaces $H^s(\mathbb{R}^n)$ and $W_2^s(\mathbb{R}^n)$, respectively, in Section 3.2. This will not be done here. A few comments may be found in Note 4.6.5. We use expansions of type (4.57) for elements $f$ belonging to $W_2^s(\mathbb{Q}^n)$, $s \geq 0$, according to Definition 3.37 with compact supports in $\mathbb{Q}^n$ as a vehicle for a more detailed study of $W_2^s(\Omega)$ in bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$.

**Theorem 4.13.** *Let* $\mathbb{Q}^n$ *and* $K$ *be given by* (4.55) *and* (4.56). *Let* $s \geq 0$. *Then*

$$f \in L_2(\mathbb{Q}^n), \quad \operatorname{supp} f \subset \bar{K}, \tag{4.60}$$

*belongs to* $W_2^s(\mathbb{Q}^n)$ *if, and only if, it can be represented by* (4.57)–(4.60) *such that*

$$\|f \,|W_2^s(\mathbb{Q}^n)\|_+ = \Big( \sum_{m \in \mathbb{Z}^n} (1+|m|^2)^s |a_m|^2 \Big)^{1/2} < \infty. \tag{4.61}$$

*Furthermore,*

$$\|f \,|W_2^s(\mathbb{Q}^n)\|_+ \sim \|f \,|W_2^s(\mathbb{Q}^n)\| \tag{4.62}$$

(*equivalent norms*).

*Proof.* It is sufficient to prove (4.62) for $f \in D(\mathbb{Q}^n)$ with $\operatorname{supp} f \subset K$. The rest is a matter of approximation or mollification as in the proof of Corollary 4.6. Let $f \in D(\mathbb{Q}^n)$ be expanded by (4.57). Then

$$\mathrm{D}^\alpha f(x) = \sum_{m \in \mathbb{Z}^n} i^{|\alpha|} m^\alpha a_m h_m(x), \quad x \in \mathbb{Q}^n, \tag{4.63}$$

where $\alpha \in \mathbb{N}_0^n$ and $m^\alpha = m_1^{\alpha_1} \cdots m_n^{\alpha_n}$ as in (A.3). For $s = k \in \mathbb{N}_0$ one obtains by Theorem 4.1 with $\mathbb{Q}^n$ in place of $\Omega$ that

$$\begin{aligned} \|f\,|W_2^k(\mathbb{Q}^n)\|^2 &= \sum_{|\alpha|\le k} \|\mathrm{D}^\alpha f\,|L_2(\mathbb{Q}^n)\|^2 \\ &= \sum_{|\alpha|\le k} \sum_{m\in\mathbb{Z}^n} |m^\alpha|^2 |a_m|^2 \\ &\sim \|f\,|W_2^k(\mathbb{Q}^n)\|_+^2 \end{aligned} \tag{4.64}$$

in view of supp $f \subset K$. If $0 < s < 1$, then it follows again by supp $f \subset K$ and Theorem 4.1 that

$$\begin{aligned} &\big\|f\,|W_2^s(\mathbb{Q}^n)\big\|^2 \\ &\quad = \|f\,|L_2(\mathbb{Q}^n)\|^2 + \int_{\mathbb{Q}^n}\int_{\mathbb{Q}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad \sim \|f\,|L_2(\mathbb{Q}^n)\|^2 + \int_{|h|\le 1} |h|^{-2s} \int_{\mathbb{Q}^n} |f(x+h)-f(x)|^2\,\mathrm{d}x\,\frac{\mathrm{d}h}{|h|^n}. \end{aligned} \tag{4.65}$$

The following estimate of the second integral in (4.65) is more or less the discrete version of (3.65). Consequently we can proceed similarly for its evaluation, that is,

$$\begin{aligned} f(x+h) - f(x) &= (2\pi)^{-n/2} \sum_{m\in\mathbb{Z}^n} a_m (e^{imh}-1) e^{imx} \\ &= \sum_{m\in\mathbb{Z}^n} a_m (e^{imh}-1) h_m(x) \end{aligned} \tag{4.66}$$

implies

$$\begin{aligned} &\int_{|h|\le 1} |h|^{-2s} \int_{\mathbb{Q}^n} |f(x+h)-f(x)|^2\,\mathrm{d}x\,\frac{\mathrm{d}h}{|h|^n} \\ &\qquad = \sum_{|m|>0} |a_m|^2 \int_{|h|\le 1} |h|^{-2s} |1-e^{imh}|^2\,\frac{\mathrm{d}h}{|h|^n} \\ &\qquad = \sum_{|m|>0} |a_m|^2\,|m|^{2s} \int_{|h|\le |m|} |h|^{-2s} |1-e^{i\frac{m}{|m|}h}|^2\,\frac{\mathrm{d}h}{|h|^n} \end{aligned} \tag{4.67}$$

where we replaced $h$ by $h/|m|$. However, the last integral can be estimated uniformly in $m \in \mathbb{Z}^n \setminus \{0\}$ from above and below by positive constants. Then (4.65) and (4.59) prove (4.62). □

In Corollary 3.31 and Theorem 3.32 we dealt with embeddings and $\varepsilon$-inequalities. This can be carried over immediately to spaces on domains including an $\varepsilon$-inequality for (3.82) using

$$W_2^t(\mathbb{R}^n) \hookrightarrow W_2^s(\mathbb{R}^n) \hookrightarrow C^\ell(\mathbb{R}^n) \tag{4.68}$$

if

$$t > s > \ell + \frac{n}{2} \tag{4.69}$$

and Corollary 3.31. But we prefer here a direct approach based on Theorem 4.13 since we need later on some technicalities of the arguments given.

**Exercise 4.14.** Derive an $\varepsilon$-version of (3.82).

*Hint:* Use (4.68), (4.69).

We introduce temporarily

$$\widetilde{W}_2^s(\bar{K}) = \{f \in W_2^s(\mathbb{Q}^n) : \operatorname{supp} f \subset \bar{K}\} \tag{4.70}$$

as a closed subspace of $W_2^s(\mathbb{Q}^n)$, where one can replace $W_2^s(\mathbb{Q}^n)$ by $W_2^s(\mathbb{R}^n)$. Here we assume $s \geq 0$ and $\mathbb{Q}^n$, $K$ as in (4.55), (4.56). Similarly, let

$$\widetilde{C}^\ell(\bar{K}) = \{f \in C^\ell(\mathbb{Q}^n) : \operatorname{supp} f \subset \bar{K}\}, \quad \ell \in \mathbb{N}_0, \tag{4.71}$$

with $\widetilde{C}(\bar{K}) = \widetilde{C}^0(\bar{K})$. This notation is consistent with (3.92).

**Proposition 4.15.** *Let $\mathbb{Q}^n$ and $K$ be as in (4.55), (4.56). Let $\widetilde{W}_2^s(\bar{K})$ with $s \geq 0$ and $\widetilde{C}^\ell(\bar{K})$ with $\ell \in \mathbb{N}_0$ be as above based on $W_2^s(\mathbb{Q}^n)$ and $C^\ell(\mathbb{Q}^n)$ according to Definitions 3.37 and A.1.*

(i) *Let $0 \leq t < s < \infty$. Then the embedding*

$$\mathrm{id}\colon \widetilde{W}_2^s(\bar{K}) \hookrightarrow W_2^t(\mathbb{Q}^n) \tag{4.72}$$

*is compact. Furthermore, there is a constant $c > 0$ such that for all $\varepsilon > 0$ and all $f \in \widetilde{W}_2^s(\bar{K})$,*

$$\|f \,|\widetilde{W}_2^t(\bar{K})\| \leq \varepsilon \|f \,|\widetilde{W}_2^s(\bar{K})\| + c\, \varepsilon^{-\frac{t}{s-t}} \|f \,|L_2(\mathbb{Q}^n)\|. \tag{4.73}$$

(ii) *Let $s > \ell + \frac{n}{2}$. Then the embedding*

$$\mathrm{id}\colon \widetilde{W}_2^s(\bar{K}) \hookrightarrow C^\ell(\mathbb{Q}^n) \tag{4.74}$$

*is compact. Furthermore, for any $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that for all $f \in \widetilde{W}_2^s(\bar{K})$,*

$$\|f \,|\widetilde{C}^\ell(\bar{K})\| \leq \varepsilon \|f \,|\widetilde{W}_2^s(\bar{K})\| + c_\varepsilon \|f \,|L_2(\mathbb{Q}^n)\|. \tag{4.75}$$

*Proof. Step* 1. Let $f \in \widetilde{W}_2^s(\bar{K})$ be given by (4.57) with (4.61), (4.62). We obtain for $0 \le t < s$ and $M \in \mathbb{N}$ that

$$\|f|W_2^t(\mathbb{Q}^n)\|^2 \le c\, M^{2t} \sum_{|m|\le M} |a_m|^2 + c\, M^{-2(s-t)} \sum_{|m|>M} |m|^{2s}\, |a_m|^2, \quad (4.76)$$

which reads for $\varepsilon = M^{-(s-t)}$ as (4.73). Assume now $s > t > \ell + \frac{n}{2}$, then

$$\begin{aligned} \|f|C^\ell(\mathbb{Q}^n)\| &\le c \sum_{m\in\mathbb{Z}^n} (1+|m|)^\ell |a_m| \\ &\le c\Big(\sum_{m\in\mathbb{Z}^n} (1+|m|)^{2t}|a_m|^2\Big)^{1/2}\Big(\sum_{m\in\mathbb{Z}^n} (1+|m|)^{-2(t-\ell)}\Big)^{1/2} \end{aligned} \quad (4.77)$$

by Hölder's inequality. Since $2(t-\ell) > n$, the last factor converges,

$$\sum_{m\in\mathbb{Z}^n} (1+|m|)^{-2(t-\ell)} \sim \int_{\mathbb{R}^n} (1+|x|)^{-2(t-\ell)} \mathrm{d}x < \infty. \quad (4.78)$$

Combining (4.77) with (4.76) or (4.73), respectively, leads to (4.75).

*Step* 2. We first prove the compactness of the identity (4.72) for $t = 0$ and split

$$\mathrm{id}\colon \widetilde{W}_2^s(\bar{K}) \hookrightarrow L_2(\mathbb{Q}^n), \quad s > 0, \quad (4.79)$$

into $\mathrm{id} = \mathrm{id}_M + \mathrm{id}^M$, where $M \in \mathbb{N}$, and

$$\mathrm{id}_M f = \sum_{|m|\le M} a_m h_m, \quad \mathrm{id}^M f = \sum_{|m|>M} a_m h_m, \quad (4.80)$$

assuming that $f \in \widetilde{W}_2^s(\bar{K})$ is given by (4.57) with the Fourier coefficients $a_m$ as in (4.59). Then it follows by (4.59) for the finite-rank operator $\mathrm{id}_M$ mapping $\widetilde{W}_2^s(\bar{K})$ into $L_2(\mathbb{Q}^n)$ that $\|\mathrm{id}_M\| \le c$ independently of $M$. In particular, $\mathrm{id}_M$ is compact. As in (4.76) with $t = 0$ one gets

$$\|\mathrm{id} - \mathrm{id}_M\| = \|\mathrm{id}^M\| \le c\, M^{-2s} \longrightarrow 0 \quad \text{for } M \to \infty. \quad (4.81)$$

Thus id is compact. This covers (i) for $t = 0$. Next we prove that id in (4.72) is also compact if $0 < t < s$. Since the image of the unit ball $U_s$ in $\widetilde{W}_2^s(\bar{K})$ is precompact in $L_2(\mathbb{Q}^n)$ we find for any $\delta > 0$ a finite $\delta$-net

$$\{f_\ell\}_{\ell=1}^L, \quad L = L(\delta), \quad \|f_\ell|\widetilde{W}_2^s(\bar{K})\| \le 1 \quad \text{for } \ell = 1,\dots,L, \quad (4.82)$$

and

$$\min_{\ell=1,\dots,L} \|f - f_\ell|L_2(\mathbb{Q}^n)\| \le \delta, \quad f \in U_s. \quad (4.83)$$

Consequently, (4.73) implies

$$\min_{\ell=1,\dots,L} \|f - f_\ell | \widetilde{W}_2^t(\overline{K})\| \le c\varepsilon + c\,\varepsilon^{-\frac{t}{s-t}}\delta \le c'\,\varepsilon \tag{4.84}$$

with $\delta = \varepsilon^{\frac{t}{s-t}+1}$. It follows that (4.72) is compact. Using (4.75) one obtains in the same way that id in (4.74) is compact, too. □

**Exercise 4.16.** Prove that one may choose $c_\varepsilon = c\varepsilon^{-\varkappa}$ in (4.75) with

$$\varkappa = \frac{\ell + \frac{n}{2}}{s - \ell - \frac{n}{2}}$$

and some $c > 0$ independent of $\varepsilon > 0$.

**Theorem 4.17.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3. Let $C^\ell(\Omega)$, $\ell \in \mathbb{N}_0$, be the spaces as introduced in Definition A.1 and let $W_2^s(\Omega)$, $s \ge 0$, be the Sobolev spaces as in (4.2).*

(i) *Let $0 \le t < s < \infty$. Then the embedding*

$$\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow W_2^t(\Omega) \tag{4.85}$$

*is compact. Furthermore, there is a constant $c > 0$ such that for all $\varepsilon > 0$ and all $f \in W_2^s(\Omega)$,*

$$\|f|W_2^t(\Omega)\| \le \varepsilon \|f|W_2^s(\Omega)\| + c\,\varepsilon^{-\frac{t}{s-t}}\|f|L_2(\Omega)\|. \tag{4.86}$$

(ii) *Let $\ell \in \mathbb{N}_0$ and $s > \ell + \frac{n}{2}$. Then the embedding*

$$\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow C^\ell(\Omega) \tag{4.87}$$

*is compact. Furthermore, for any $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that for all $f \in W_2^s(\Omega)$,*

$$\|f|C^\ell(\Omega)\| \le \varepsilon \|f|W_2^s(\Omega)\| + c_\varepsilon \|f|L_2(\Omega)\|. \tag{4.88}$$

*Proof.* We may assume that

$$\Omega \subset \left\{ x \in \mathbb{R}^n : |x| < \frac{1}{2} \right\}. \tag{4.89}$$

Let $K$ be the unit ball according to (4.56) and let $\chi \in \mathcal{D}(K)$ with $\chi(x) = 1$ when $|x| < \frac{1}{2}$. If one multiplies the extension operator $\mathrm{ext}_\Omega^L$ according to (4.8) and

Theorem 4.1 with $\chi$, then one obtains again an extension operator. In other words, one may assume that there is a common extension operator

$$\mathrm{ext}^L_\Omega : \begin{cases} C^\ell(\Omega) \hookrightarrow \widetilde{C}^\ell(\bar{K}), & \ell = 0, \dots, L, \\ W_2^s(\Omega) \hookrightarrow \widetilde{W}_2^s(\bar{K}), & 0 < s < L, \end{cases} \tag{4.90}$$

using the notation (4.70), (4.71). Denoting temporarily id in (4.72), (4.74) by $\widetilde{\mathrm{id}}$, then the embedding in (4.85), (4.87) can be decomposed into

$$\mathrm{id} = \mathrm{re} \circ \widetilde{\mathrm{id}} \circ \mathrm{ext}^L_\Omega, \tag{4.91}$$

where re is the restriction operator. Then all assertions of the theorem follow from (4.91) and the corresponding assertions of Proposition 4.15. □

**Exercise 4.18.** Give a direct proof of (4.86), (4.88) based on (4.8).

*Hint:* Use Corollary 3.31 and the $\varepsilon$-version of (3.82) subject to Exercise 4.14.

**Exercise* 4.19.** The assumption that $\Omega$ is bounded is essential for the compactness of the embedding (4.85) (unlike its continuity). Take, for instance, $\Omega = \mathbb{R}^n$, $t = \ell \in \mathbb{N}_0$, $s = k \in \mathbb{N}$, $\ell < k$. Prove that there exists a set $\Phi \subset W_p^k(\mathbb{R}^n)$ which is bounded but not precompact in $W_p^\ell(\mathbb{R}^n)$.

*Hint:* Choose $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subset K_{1/2}(0)$, $\varphi \not\equiv 0$, and consider the set $\Phi = \{\varphi(\cdot - m)\}_{m \in \mathbb{Z}^n}$.

## 4.5 Traces

As in Section 4.3 we furnish the boundary $\Gamma = \partial\Omega$ of a bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$, where $n \geq 2$, with a surface measure $d\sigma$. There we introduced the spaces $L_p(\Gamma)$, $1 \leq p < \infty$, and explained our understanding of traces as limits of pointwise traces of smooth functions (which are dense in the spaces considered). This will not be repeated here. So far we have (4.34) for $\mathrm{tr}_\Gamma$ in (4.33) and (4.37) for $\mathrm{tr}_\Gamma \frac{\partial}{\partial \nu}$ in (4.36). We are now interested in the precise trace spaces of $W_2^s(\Omega)$ where the latter have the meaning as in (4.2). This requires the introduction of Sobolev spaces on $\Gamma$. We rely on the resolution of unity according to (4.6), (4.7) and the local diffeomorphisms $\psi^{(j)}$ mapping $\Gamma_j = \Gamma \cap K_j$ onto $W_j = \psi^{(j)}(\Gamma_j)$ as indicated in Figure 4.4. Let $g_j(y)$ be as in (4.13). Restricted to $y = (y', 0) \in W_j$,

$$g_j(y') = (\varphi_j f) \circ (\psi^{(j)})^{-1}(y'), \quad j = 1, \dots, J, \quad f \in L_2(\Gamma), \tag{4.92}$$

makes sense. This results in functions $g_j \in L_2(W_j)$ with compact supports in the $(n-1)$-dimensional $C^\infty$ domain in $W_j$. (Strictly speaking, $y' \in W_j$ in (4.92) must be interpreted as $(y', 0) \in W_j$, but we do not distinguish notationally between $g_j$ and $(\psi^{(j)})^{-1}$ as functions of $(y', 0)$ and of $y'$.)

**Definition 4.20.** Let $n \geq 2$, and let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ with $\Gamma = \partial\Omega$, and $\varphi_j$, $\psi^{(j)}$, $W_j$ be as above. Assume $s > 0$. Then we introduce

$$W_2^s(\Gamma) = \{f \in L_2(\Gamma) : g_j \in W_2^s(W_j),\ j = 1, \dots, J\}, \tag{4.93}$$

equipped with

$$\|f|W_2^s(\Gamma)\| = \Big(\sum_{j=1}^{J} \|g_j|W_2^s(W_j)\|^2\Big)^{1/2}, \tag{4.94}$$

where $g_j$ is given by (4.92).

**Remark 4.21.** We furnish $W_2^s(W_j)$ with the intrinsic $(n-1)$-dimensional norms (and related scalar products) $\|\cdot|W_2^s(W_j)\|_*$ according to Theorem 4.1. A few further comments may be found in Note 4.6.4.

**Proposition 4.22.** *Let $\Gamma = \partial\Omega$ be the boundary of a bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$ with $n \geq 2$.*

(i) *Let $s > 0$. Then the spaces $W_2^s(\Gamma)$ according to Definition 4.20 are Hilbert spaces. They are independent of admissible resolutions of unity $\{\varphi_j\}_j$ and local diffeomorphisms $\{\psi^{(j)}\}_j$.*

(ii) *If $0 < s_1 < s_2 < \infty$, then*

$$W_2^{s_2}(\Gamma) \hookrightarrow W_2^{s_1}(\Gamma) \hookrightarrow L_2(\Gamma) \tag{4.95}$$

*are compact embeddings.*

(iii) *If $0 < s < 1$, then*

$$\|f|W_2^s(\Gamma)\|_* = \Big(\|f|L_2(\Gamma)\|^2 + \int_\Gamma\int_\Gamma \frac{|f(\gamma) - f(\zeta)|^2}{|\gamma - \zeta|^{n-1+2s}}\, \mathrm{d}\sigma(\gamma)\, \mathrm{d}\sigma(\zeta)\Big)^{1/2} \tag{4.96}$$

*is an equivalent norm on $W_2^s(\Gamma)$.*

(iv) *If $s = \ell \in \mathbb{N}$, then*

$$W_2^\ell(\Gamma) = \{f \in L_2(\Gamma) : \mathrm{D}_\mathrm{t}^\alpha f \in L_2(\Gamma),\ |\alpha| \leq \ell\}, \tag{4.97}$$

*where $\mathrm{D}_\mathrm{t}^\alpha$ are tangential derivatives (in local curvilinear coordinates on $\Gamma$).*

(v) *If $s = \ell + \sigma$ with $\ell \in \mathbb{N}_0$ and $0 < \sigma < 1$, then*

$$W_2^s(\Gamma) = \{f \in W_2^\ell(\Gamma) : \mathrm{D}_\mathrm{t}^\alpha f \in W_2^\sigma(\Gamma) \text{ for } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = \ell\}. \tag{4.98}$$

*Proof.* In view of Definition 4.20 all assertions can be carried over from $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$ to boundaries $\Gamma = \partial\Omega$ using Theorem 4.1 and, as far as the compactness in (4.95) is concerned, from Theorem 4.17. □

**Remark 4.23.** The assumption $n \geq 2$ in Proposition 4.22 is natural. However, in what follows it is reasonable to incorporate also $n = 1$ where the formulations given must be interpreted appropriately: If $n = 1$, then according to Definition A.3 (iv), $\Omega$ is a bounded interval $I = \Omega$ and its boundary $\Gamma = \partial\Omega$ consists of the endpoints of $I$, say, $a$ and $b$ with $a < b$. By Theorem 3.32,

$$f(a), f(b) \text{ make sense if } f \in W_2^s(I), \quad s > \frac{1}{2}, \tag{4.99}$$

and

$$f'(a), f'(b) \text{ are well-defined if } f \in W_2^s(I), \quad s > \frac{3}{2}. \tag{4.100}$$

This is the correct interpretation of $\mathrm{tr}_\Gamma$ and $\mathrm{tr}_\Gamma \frac{\partial}{\partial\nu}$ if $n = 1$ in what follows.

**Theorem 4.24.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $\Gamma = \partial\Omega$ be its boundary.*

(i) *Let $s > \frac{1}{2}$. Then $\mathrm{tr}_\Gamma$ (in the explanations given above) is a linear and bounded map of $W_2^s(\Omega)$ onto $W_2^{s-\frac{1}{2}}(\Gamma)$,*

$$\mathrm{tr}_\Gamma \colon W_2^s(\Omega) \hookrightarrow W_2^{s-\frac{1}{2}}(\Gamma), \quad \mathrm{tr}_\Gamma\, W_2^s(\Omega) = W_2^{s-\frac{1}{2}}(\Gamma). \tag{4.101}$$

(ii) *Let $s > \frac{3}{2}$. Then $\mathrm{tr}_\Gamma \frac{\partial}{\partial\nu}$ (in the explanations given above) is a linear and bounded map of $W_2^s(\Omega)$ onto $W_2^{s-\frac{3}{2}}(\Gamma)$,*

$$\mathrm{tr}_\Gamma \frac{\partial}{\partial\nu} \colon W_2^s(\Omega) \hookrightarrow W_2^{s-\frac{3}{2}}(\Gamma), \quad \mathrm{tr}_\Gamma \frac{\partial}{\partial\nu} W_2^s(\Omega) = W_2^{s-\frac{3}{2}}(\Gamma). \tag{4.102}$$

*Proof. Step* 1. In view of Remark 4.23 we may assume $n \geq 2$. By the above localisations one can reduce these problems to the functions $g_j$ in (4.13) and their traces in (4.92). Furthermore, the extended functions $\mathrm{ext}_\Omega^L\, g_j$ according to Theorem 4.1 and $g_j$ have the same traces. We discussed this point in some detail in connection with Theorem 3.45. This applies not only to $\mathrm{tr}_\Gamma$ but also to $\mathrm{tr}_\Gamma \frac{\partial}{\partial\nu}$ having in mind the discussion in Remark 4.5 about fibre-preserving maps. Hence we can restrict our attention to the model case considered in Theorem 4.13 with

$$\mathrm{tr}_\Gamma \colon f(x) \longmapsto f(x', 0), \quad x \in \mathbb{Q}^n = (-\pi, \pi)^n, \tag{4.103}$$

and

$$\mathrm{tr}_\Gamma \frac{\partial}{\partial\nu} \colon f(x) \longmapsto \frac{\partial f}{\partial x_n}(x', 0), \quad x \in \mathbb{Q}^n. \tag{4.104}$$

In other words, we may assume that

$$f \in W_2^s(\mathbb{Q}^n), \quad \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \le 1\} \tag{4.105}$$

is represented by

$$f(x) = (2\pi)^{-n/2} \sum_{m \in \mathbb{Z}^n} a_m e^{imx}, \quad a_m = (2\pi)^{-n/2} \int_{\mathbb{Q}^n} f(x) e^{-imx} \mathrm{d}x, \tag{4.106}$$

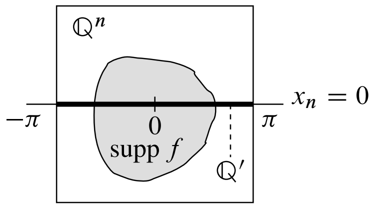



Figure 4.8

relying on the equivalent norm in (4.61). Let, as indicated in Figure 4.8 aside,

$$\begin{aligned} \mathbb{Q}^n &= (-\pi, \pi)^n \qquad \text{and} \\ \mathbb{Q}' &= (-\pi, \pi)^{n-1} \\ &= \{x = (x', x_n) \in \mathbb{Q}^n : x_n = 0\}. \end{aligned} \tag{4.107}$$

*Step* 2. We prove the first assertion in (4.101). Let $f$ be given by (4.106) with

$$f(x', 0) = (2\pi)^{-n/2} \sum_{m' \in \mathbb{Z}^{n-1}} b_{m'} e^{im'x'} \quad \text{where } b_{m'} = \sum_{m_n = -\infty}^{\infty} a_{(m', m_n)}. \tag{4.108}$$

Let $\varkappa$ be chosen such that $2s > \varkappa > 1$. For $m' \neq 0$ one obtains by Hölder's inequality that

$$\begin{aligned} |b_{m'}| &\le \sum_{|m_n| \le |m'|} |a_{(m',m_n)}| + \sum_{|m_n| > |m'|} |a_{(m',m_n)}| |m_n|^{\frac{\varkappa}{2}} |m_n|^{-\frac{\varkappa}{2}} \\ &\le c |m'|^{\frac{1}{2}} \Big( \sum_{|m_n| \le |m'|} |a_{(m',m_n)}|^2 \Big)^{\frac{1}{2}} \\ &\quad + c |m'|^{\frac{1-\varkappa}{2}} \Big( \sum_{|m_n| > |m'|} |a_{(m',m_n)}|^2 |m_n|^{\varkappa} \Big)^{\frac{1}{2}}. \end{aligned} \tag{4.109}$$

Since $|m'| \le |m|$ and $|m_n| \le |m|$ for $m = (m', m_n)$, we can further estimate

$$\begin{aligned} |m'|^{2s-1} |b_{m'}|^2 &\le c |m|^{2s} \sum_{|m_n| \le |m'|} |a_{(m',m_n)}|^2 + c |m|^{\varkappa + 2s - \varkappa} \sum_{|m_n| > |m'|} |a_{(m',m_n)}|^2 \\ &\le c' |m|^{2s} \sum_{m_n = -\infty}^{\infty} |a_m|^2, \end{aligned} \tag{4.110}$$

where we used, in addition, that $0 < \varkappa < 2s$. Hence,

$$\sum_{m' \in \mathbb{Z}^{n-1}} (1 + |m'|^2)^{s-\frac{1}{2}} |b_{m'}|^2 \le c \sum_{m \in \mathbb{Z}^n} (1 + |m|^2)^s |a_m|^2 \tag{4.111}$$

and it follows from Theorem 4.13 applied to $\mathbb{Q}^n$ and $\mathbb{Q}'$ that

$$\| f(x', 0) \, | \, W_2^{s-\frac{1}{2}}(\mathbb{Q}')\| \le c \| f \, | W_2^s(\mathbb{Q}^n)\| \tag{4.112}$$

for $f$ with (4.105). By Step 1 we get the first assertion in (4.101) (the map *into*).

*Step* 3. We prove that $\mathrm{tr}_\Gamma$ in (4.101) is a map *onto*, which again can be reduced to the above model situation. Let

$$g(x') \in W_2^{s-\frac{1}{2}}(\mathbb{Q}') \quad \text{with } \operatorname{supp} g \subset \{x' \in (-\pi, \pi)^{n-1} : |x'| \le 1\} \tag{4.113}$$

be represented by

$$g(x') = \sum_{m' \in \mathbb{Z}^{n-1}} b_{m'} e^{im'x'}, \quad b_{m'} = (2\pi)^{-\frac{n-1}{2}} \int_{\mathbb{Q}'} g(x') e^{-im'x'} \mathrm{d}x', \tag{4.114}$$

where $\mathbb{Q}'$ is interpreted as in (4.107). By Theorem 4.13 we have

$$\|g | W_2^{s-\frac{1}{2}}(\mathbb{Q}')\|^2 \sim \sum_{m' \in \mathbb{Z}^{n-1}} |b_{m'}|^2 (1 + |m'|^2)^{s-\frac{1}{2}}. \tag{4.115}$$

Let $\{a_m\}_{m \in \mathbb{Z}^n}$ be such that

$$\begin{aligned} G(x', x_n) &= \sum_{m \in \mathbb{Z}^n} a_m e^{imx} \\ &= b_0 + \sum_{0 \ne m' \in \mathbb{Z}^{n-1}} \frac{b_{m'}}{|m'|} e^{im'x'} \sum_{m_n = |m'|}^{2|m'|-1} e^{im_n x_n}, \end{aligned} \tag{4.116}$$

in particular, with $a_m = 0$ for the remaining terms. Firstly we observe that

$$G(x', 0) = g(x'), \quad x' \in \mathbb{Q}'. \tag{4.117}$$

Secondly, (4.116) implies

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} |a_m|^2 (1 + |m|^2)^s &= |b_0|^2 + \sum_{0 \ne m' \in \mathbb{Z}^{n-1}} \frac{|b_{m'}|^2}{|m'|^2} \sum_{m_n = |m'|}^{2|m'|-1} (1 + |m|^2)^s \\ &\sim \sum_{m' \in \mathbb{Z}^{n-1}} |b_{m'}|^2 (1 + |m'|^2)^{s-\frac{1}{2}}. \end{aligned} \tag{4.118}$$

Theorem 4.13 suggests

$$G(x) \in W_2^s(\mathbb{Q}^n), \tag{4.119}$$

but this is not immediately covered since $G$ need not to have a compact support in $\mathbb{Q}^n$ (which would be sufficient to apply Theorem 4.13). We return to this point in Remark 4.26 below and take temporarily (4.119) for granted. Then

$$f(x) = \chi(x) G(x) \in W_2^s(\mathbb{Q}^n) \quad \text{with } f(x', 0) = g(x'), \tag{4.120}$$

where $\chi \in \mathcal{D}(\mathbb{Q}^n)$ with $\chi(x) = 1$ if $|x| \le 2$. But this is just the extension of $g \in W_2^{s-\frac{1}{2}}(\mathbb{Q}')$ to $f \in W_2^s(\mathbb{Q}^n)$ we are looking for. By the above considerations it follows that $\mathrm{tr}_\Gamma$ in (4.101) maps $W_2^s(\Omega)$ onto $W_2^{s-\frac{1}{2}}(\Gamma)$.

*Step* 4. We prove part (ii) which can be reduced to the model situation as described in Step 1, in particular, in (4.104). If $f$ is given by (4.105) now with $s > \frac{3}{2}$, then

$$\frac{\partial f}{\partial x_n} \in W_2^{s-1}(\mathbb{Q}^n), \quad \operatorname{supp} \frac{\partial f}{\partial x_n} \subset \{x \in \mathbb{R}^n : |x| \le 1\}. \tag{4.121}$$

Application of part (i) gives the first assertion in (4.102), hence $\mathrm{tr}_\Gamma \frac{\partial}{\partial \nu}$ is a map from $W_2^s(\Omega)$ *into* $W_2^{s-\frac{3}{2}}(\Gamma)$. It remains to verify that this map is also *onto*. Let

$$g(x') \in W_2^{s-\frac{3}{2}}(\mathbb{Q}'), \quad \operatorname{supp} g \subset \{x' \in \mathbb{R}^{n-1} : |x'| \le 1\} \tag{4.122}$$

be represented by (4.114) with

$$\left\| g | W_2^{s-\frac{3}{2}}(\mathbb{Q}') \right\|^2 \sim \sum_{m' \in \mathbb{Z}^{n-1}} |b_{m'}|^2 (1 + |m'|^2)^{s-\frac{3}{2}} \tag{4.123}$$

as the counterpart of (4.115). The substitute of $G$ in (4.116) is given by

$$\begin{aligned} G(x', x_n) &= \sum_{m \in \mathbb{Z}^n} a_m e^{imx} \\ &= b_0 + \sum_{0 \ne m' \in \mathbb{Z}^{n-1}} \frac{b_{m'}}{i\,|m'|} e^{im'x'} \sum_{m_n = |m'|}^{2|m'|-1} \frac{e^{im_n x_n}}{m_n}, \end{aligned} \tag{4.124}$$

with $a_m = 0$ for the remaining terms. Then

$$\frac{\partial G}{\partial x_n}(x', 0) = g(x') \quad \text{if} \quad x' \in \mathbb{Q}', \tag{4.125}$$

and

$$\sum_{m\in\mathbb{Z}^n} |a_m|^2(1+|m|^2)^s = |b_0|^2 + \sum_{0\neq m'\in\mathbb{Z}^{n-1}} \frac{|b_{m'}|^2}{|m'|^2} \sum_{m_n=|m'|}^{2|m'|-1} \frac{(1+|m|^2)^s}{m_n^2}$$
$$\sim \sum_{m'\in\mathbb{Z}^{n-1}} |b_{m'}|^2(1+|m'|^2)^{s-\frac{3}{2}} \qquad (4.126)$$

are the counterparts of (4.117), (4.118). The rest is now the same as in Step 3. This concludes the proof of (4.102). □

**Exercise 4.25.** Review Theorem 4.24 and its proof in case of $n = 1$.

*Hint:* Rely on (4.99), (4.100).

**Remark 4.26.** We justify (4.119). For this purpose we extend $G(x)$ periodically to neighbouring cubes of the same size and multiply the outcome with suitable cut-off functions. It follows by the same arguments as in the proof of Theorem 4.13 that these functions belong to $W_2^s(\mathbb{R}^n)$, hence $G \in W_2^s(\mathbb{Q}^n)$ by restriction. Moreover,

$$\|G\,|W_2^s(\mathbb{Q}^n)\| \le c\Big(\sum_{m\in\mathbb{Z}^n} (1+|m|^2)^s|a_m|^2\Big)^{1/2}, \qquad (4.127)$$

where we used (3.90). On the other hand, the reverse estimate follows from the arguments in the proof of Theorem 4.13. In other words, one obtains not only (4.119) (which would be sufficient for our purpose), but also the norm-equivalence (4.61), (4.62) for these periodic functions belonging to $W_2^s(\mathbb{Q}^n)$.

**Exercise 4.27.** Let $k \in \mathbb{N}$ and $s > k + \frac{1}{2}$. Prove by the same method as explicated for Theorem 4.24 that

$$\operatorname{tr}_\Gamma \frac{\partial^k}{\partial\nu^k} : W_2^s(\Omega) \hookrightarrow W_2^{s-k-\frac{1}{2}}(\Gamma), \quad \operatorname{tr}_\Gamma \frac{\partial^k}{\partial\nu^k} W_2^s(\Omega) = W_2^{s-k-\frac{1}{2}}(\Gamma), \quad (4.128)$$

and that

$$\operatorname{tr}_\Gamma \mathrm{D}^\alpha : W_2^s(\Omega) \hookrightarrow W_2^{s-k-\frac{1}{2}}(\Gamma) \quad \text{if } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le k. \qquad (4.129)$$

*Hint:* Reduce (4.129) to (4.128).

**Remark 4.28.** Again let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and $\Gamma = \partial\Omega$ its boundary. Let $\mu = \mu_\gamma$ with $\gamma \in \Gamma$ be a non-tangential $C^\infty$ vector field on $\Gamma$ which means that the components of $\mu_\gamma = (\mu_\gamma^1, \dots, \mu_\gamma^n)$ are $C^\infty$ functions on $\Gamma$ and that $\mu_\gamma \nu_\gamma > 0$ for the related scalar product of $\mu_\gamma$ and the outer normal $\nu_\gamma$. Then one obtains by the same arguments as above that (4.102) can be generalised by

$$\operatorname{tr}_\Gamma \frac{\partial}{\partial\mu} : W_2^s(\Omega) \hookrightarrow W_2^{s-\frac{3}{2}}(\Gamma), \quad \operatorname{tr}_\Gamma \frac{\partial}{\partial\mu} W_2^s(\Omega) = W_2^{s-\frac{3}{2}}(\Gamma), \qquad (4.130)$$

where $s > \frac{3}{2}$.

We proved a little bit more than stated. Both the extension of $g(x')$ in (4.114) to $G(x)$ and to $f(x)$ in (4.116), (4.120) and its counterpart (4.124) are linear in $g$ and apply simultaneously to all admitted spaces. Clipping together these model cases according to Step 1 of the proof of Theorem 4.24 one gets a universal extension operator

$$\mathrm{ext}_\Gamma : W_2^{s-\frac{1}{2}}(\Gamma) \hookrightarrow W_2^s(\Omega), \quad s > \frac{1}{2}, \tag{4.131}$$

such that

$$\mathrm{tr}_\Gamma \circ \mathrm{ext}_\Gamma = \mathrm{id} \quad (\text{identity in } W_2^{s-\frac{1}{2}}(\Gamma)). \tag{4.132}$$

*Universal* means that $\mathrm{ext}_\Gamma$ is defined on $\bigcup_{\sigma > \frac{1}{2}} W_2^{\sigma-\frac{1}{2}}(\Gamma)$ so that its restriction to a specific space has the properties (4.131), (4.132). We formulate the outcome.

**Corollary 4.29.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $\Gamma = \partial\Omega$ be its boundary.*

(i) *Let $s > \frac{1}{2}$ and let* $\mathrm{tr}_\Gamma$ *be the trace operator according to Theorem* 4.24 (i). *Then there is a universal extension operator* $\mathrm{ext}_\Gamma$ *with* (4.131), (4.132).

(ii) *Let $s > \frac{3}{2}$ and let $\mu$ be a non-tangential $C^\infty$ vector field on $\Gamma$ according to Remark* 4.28 (*with the field $\nu$ of outer normals as a distinguished example*). *Then there is a universal extension operator* $\mathrm{ext}_{\Gamma,\mu}$ *with*

$$\mathrm{ext}_{\Gamma,\mu} : W_2^{s-\frac{3}{2}}(\Gamma) \hookrightarrow W_2^s(\mathbb{R}^n), \quad s > \frac{3}{2}, \tag{4.133}$$

*such that*

$$\mathrm{tr}_\Gamma \frac{\partial}{\partial\mu} \circ \mathrm{ext}_{\Gamma,\mu} = \mathrm{id} \quad (\textit{identity in } W_2^{s-\frac{3}{2}}(\Gamma)). \tag{4.134}$$

*Proof.* All assertions are covered by the proof of Theorem 4.24 and the above comments. □

**Definition 4.30.** Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $\Gamma = \partial\Omega$ be its boundary.

(i) Let $s > \frac{1}{2}$. Then

$$W_{2,0}^s(\Omega) = \{f \in W_2^s(\Omega) : \mathrm{tr}_\Gamma f = 0\}. \tag{4.135}$$

(ii) Let $s > \frac{3}{2}$ and let $\mu$ be a non-tangential $C^\infty$ vector field on $\Gamma$ according to Remark 4.28 (with the field $\nu$ of outer normals as a distinguished example). Then

$$W_2^{s,\mu}(\Omega) = \Big\{ f \in W_2^s(\Omega) \ : \ \mathrm{tr}_\Gamma \frac{\partial f}{\partial\mu} = 0 \Big\}. \tag{4.136}$$

**Remark 4.31.** This complements Definition 4.9 for $p = 2$. By Theorem 4.24 and Remark 4.28 the above definition makes sense and both $W^s_{2,0}(\Omega)$ and $W^{s,\mu}_2(\Omega)$ are closed genuine subspaces of $W^s_2(\Omega)$. The related orthogonal complements are denoted by $W^s_{2,0}(\Omega)^\perp$ and $W^{s,\mu}_2(\Omega)^\perp$.

**Proposition 4.32.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $\Gamma = \partial\Omega$ be its boundary.*

(i) *Let $s > \frac{1}{2}$. Then*

$$\{f \in C^\infty(\Omega) \ : \ \mathrm{tr}_\Gamma f = 0\} \tag{4.137}$$

*is dense in $W^s_{2,0}(\Omega)$. Furthermore,*

$$W^s_2(\Omega) = W^s_{2,0}(\Omega) \oplus W^s_{2,0}(\Omega)^\perp \tag{4.138}$$

*and*

$$\mathrm{tr}_\Gamma \colon W^s_{2,0}(\Omega)^\perp \rightleftarrows W_2^{s-\frac{1}{2}}(\Gamma) \tag{4.139}$$

*is an isomorphic map of $W^s_{2,0}(\Omega)^\perp$* onto $W_2^{s-\frac{1}{2}}(\Gamma)$.

(ii) *Let $s > \frac{3}{2}$ and let $\mu$ be as in Definition 4.30* (ii). *Then*

$$\left\{f \in C^\infty(\Omega) \ : \ \mathrm{tr}_\Gamma \frac{\partial f}{\partial \mu} = 0\right\} \tag{4.140}$$

*is dense in $W^{s,\mu}_2(\Omega)$. Furthermore,*

$$W^s_2(\Omega) = W^{s,\mu}_2(\Omega) \oplus W^{s,\mu}_2(\Omega)^\perp \tag{4.141}$$

*and*

$$\mathrm{tr}_\Gamma \frac{\partial}{\partial \mu} \colon W^{s,\mu}_2(\Omega)^\perp \rightleftarrows W_2^{s-\frac{3}{2}}(\Gamma) \tag{4.142}$$

*is an isomorphic map of $W^{s,\mu}_2(\Omega)^\perp$* onto $W_2^{s-\frac{3}{2}}(\Gamma)$.

*Proof.* Both (4.138), (4.141) are obvious by definition. Furthermore, (4.139), (4.142) follow from Hilbert space theory and (4.101), (4.130). Corollary 4.6 implies that (4.137) is a subset of $W^s_{2,0}(\Omega)$ and that (4.140) is a subset of $W^{s,\mu}_2(\Omega)$.

It remains to prove the density assertions. Let $f \in W^s_{2,0}(\Omega)$. In view of Corollary 4.6 one finds for any $\varepsilon > 0$ a function $g_\varepsilon \in \mathcal{S}(\Omega)$ with

$$\|f - g_\varepsilon | W^s_2(\Omega)\| \le \varepsilon \quad \text{and} \quad \|\,\mathrm{tr}_\Gamma\, g_\varepsilon | W_2^{s-\frac{1}{2}}(\Gamma)\| \le \varepsilon. \tag{4.143}$$

Then we conclude by (4.131) that

$$h_\varepsilon = \mathrm{ext}_\Gamma \circ \mathrm{tr}_\Gamma\, g_\varepsilon \in W^s_2(\Omega) \quad \text{and} \quad \|h_\varepsilon | W^s_2(\Omega)\| \le c\varepsilon \tag{4.144}$$

for some $c > 0$ independent of $\varepsilon$. By Corollary 4.29 the extension operator is universal. Thus $h_\varepsilon \in C^\infty(\Omega)$ since $\mathrm{tr}_\Gamma\, g_\varepsilon \in C^\infty(\Gamma)$ using Theorem 3.32. With $f_\varepsilon = g_\varepsilon - h_\varepsilon$ this leads to

$$\| f - f_\varepsilon | W_2^s(\Omega)\| \le c'\varepsilon \quad \text{and} \quad f_\varepsilon \in C^\infty(\Omega), \quad \mathrm{tr}_\Gamma\, f_\varepsilon = 0. \tag{4.145}$$

This proves that (4.137) is dense in $W^s_{2,0}(\Omega)$. Similarly one can show that (4.140) is dense in $W_2^{s,\mu}(\Omega)$. □

**Remark 4.33.** Note that, in particular, this covers (4.43) when $p = 2$.

## 4.6 Notes

**4.6.1.** The *Extension Theorems* 3.41 (for spaces on $\mathbb{R}^n_+$ to $\mathbb{R}^n$) and 4.1 (for spaces on domains to spaces on $\mathbb{R}^n$) are cornerstones of the theory of function spaces not only for the special cases treated in this book, but also for the more general spaces briefly mentioned in the Notes 3.6.1, 3.6.3; see also Appendix E. They have a long history. The procedure described in Step 2 of the proof of Theorem 3.41 and in Figures 3.3, 4.4, 4.7, is called the *reflection method* for obvious reasons. The first step in this direction was taken in 1929 by L. Lichtenstein in [Lic29] extending $C^1$ functions in domains in $\mathbb{R}^3$ beyond the boundary using *'dachziegelartige Überdeckungen'* (German, meaning *tiling*) on $\partial\Omega$, hence (4.5) and Figure 4.3 (referring to some needs in hydrodynamics as an excuse for publishing such elementary stuff). The extension of this method as described in Step 2 of the proof of Theorem 3.41 for $C^\ell$ spaces (not Sobolev spaces as occasionally suggested, which are unknown at this time, at least in the West) is due to M. R. Hestenes [Hes41]. This method was extended to Sobolev spaces in smooth domains in [Bab53], [Nik53], [Nik56] in the 1950s and to the classical Besov spaces as briefly described in (3.144) (restricted to domains) in [Bes62], [Bes67a], [Bes67b] in the 1960s. (The last is O. V. Besov's own report of the main results of his doctoral dissertation, *doktorskaja*, the second Russian doctor degree.) This method has been extended step by step and could be applied finally to all spaces briefly mentioned in (3.154). We refer to [Tri92b, Section 4.5.5] and the literature quoted there. It should be remarked that there is a second method based on integral representations and (weakly) singular integrals, especially well adapted for the extension of Sobolev spaces from bounded Lipschitz domains to $\mathbb{R}^n$. It goes back to [Cal61] and [Smi61] around 1960. But this will not be used in this book. A more detailed account on these methods and further references may also be found in [Tri78, Section 4].

**4.6.2.** According to Definition 3.37 we introduced the spaces $W_p^s(\Omega)$ on domains $\Omega$ in $\mathbb{R}^n$ by restriction of $W_p^s(\mathbb{R}^n)$ to $\Omega$. But at least in case of classical Sobolev spaces $W_p^k(\Omega)$ with $1 \le p < \infty$, $k \in \mathbb{N}$, one can also introduce corresponding spaces intrinsically, hence

$$L_p^k(\Omega) = \{f \in L_p(\Omega) : \mathrm{D}^\alpha f \in L_p(\Omega),\ \alpha \in \mathbb{N}_0^n,\ |\alpha| \le k\}, \tag{4.146}$$

where $\mathrm{D}^\alpha f \in \mathcal{D}'(\Omega)$ are the distributional derivatives similarly as in Definition 3.1 and Remark 3.2. Obviously, $L_p^k(\Omega)$, normed by

$$\|f\,|L_p^k(\Omega)\| = \Big(\sum_{|\alpha|\le k} \|\mathrm{D}^\alpha f\,|L_p(\Omega)\|^p\Big)^{1/p}, \tag{4.147}$$

is a Banach space. One may ask under which conditions the spaces

$$W_p^k(\Omega) \quad \text{and} \quad L_p^k(\Omega), \qquad 1 \le p < \infty,\ k \in \mathbb{N}, \tag{4.148}$$

coincide. According to [Ste70, Theorem 5, p. 181] for bounded Lipschitz domains $\Omega$ in $\mathbb{R}^n$ there exists a universal extension operator

$$\mathrm{ext} : L_p^k(\Omega) \hookrightarrow W_p^k(\mathbb{R}^n), \qquad 1 \le p < \infty,\ k \in \mathbb{N}. \tag{4.149}$$

In particular, one has $W_p^k(\Omega) = L_p^k(\Omega)$ in this case. But for more general domains the situation might be different. This problem has been studied in great detail. We refer, in particular, to [Maz85, §1.5]. On the one hand, there are easy examples of (cusp) domains in which the spaces $L_p^k(\Omega)$ and $W_p^k(\Omega)$ differ. On the other hand, one has for huge classes of bounded domains in $\mathbb{R}^n$ with irregular fractal boundaries that $L_p^k(\Omega) = W_p^k(\Omega)$, including the *snowflake domain* in $\mathbb{R}^2$ illustrated in Figure 4.9 below. We refer to [Jon81], [Maz85].

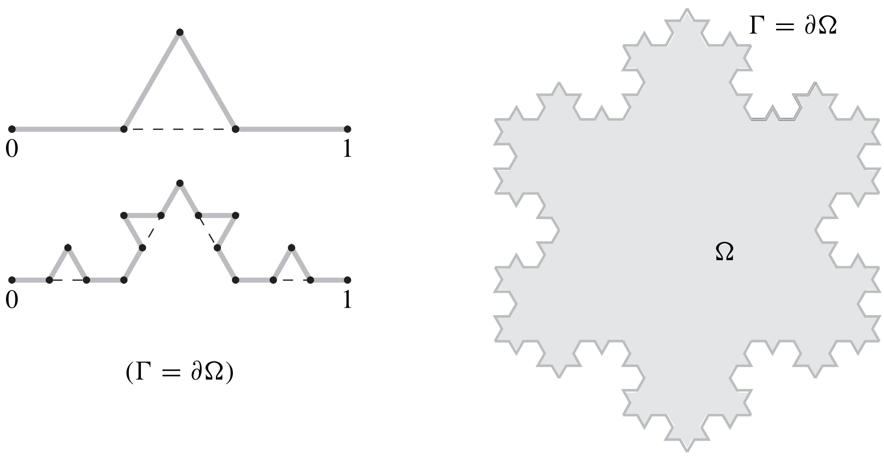



Figure 4.9

**4.6.3.** We described our understanding of traces at the beginning of Section 3.5: First one asks for inequalities of type (3.129) for smooth functions, having pointwise traces and defines afterwards the trace operator $\mathrm{tr}_\Gamma$ according to (3.130), (3.131) by completion avoiding the ambiguity of selecting distinguished representatives within the equivalence classes both in the source space and the target space. The same point of view was adopted in the Sections 4.3, 4.5 in connection with Definition 4.9 and Theorem 4.24. Moreover, the interpretation of the embeddings (3.81), (3.82) requires to select the uniquely determined continuous representative $f$ of the equivalence class $[f] \in W_2^s(\mathbb{R}^n)$, $s > \frac{n}{2}$. If $s \le \frac{n}{2}$, then such a continuous distinguished representative does not exist in general. (It is well known that there are elements of $W_2^s(\mathbb{R}^n)$ with $0 \le s \le \frac{n}{2}$ which are essentially unbounded. One may consult [Tri01, Theorem 11.4] and the literature given there in the framework of the more general spaces briefly mentioned in (3.154).) Nevertheless in any equivalence class $[f] \in W_2^s(\mathbb{R}^n)$ with $s > \frac{1}{2}$, or, more generally,

$$[f] \in H_p^s(\mathbb{R}^n), \quad 1 < p < \infty,\ s > \frac{1}{p}, \tag{4.150}$$

according to (3.140), there is a uniquely determined distinguished representative $f$ for which traces on $\Gamma$ according to (3.128)–(3.131) or $\Gamma = \partial\Omega$ as in the Sections 4.3, 4.5 make sense more directly. We give a brief description following [Tri01, pp. 260/261] where one finds the necessary detailed references, especially to [AH96].

First we recall that a point $x \in \mathbb{R}^n$ is called a *Lebesgue point* for $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ according to (2.19) if

$$f(x) = \lim_{r\to 0} \frac{1}{|K_r(x)|} \int_{K_r(x)} f(y)\,\mathrm{d}y, \tag{4.151}$$

where $K_r(x)$ stands for a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ with radius $r$, $0 < r < 1$, see (1.30). It is one of the outstanding observations of real analysis that one gets (4.151) with exception of a set $\Gamma$ having Lebesgue measure $|\Gamma| = 0$, [Ste70, p. 5]. If $[f] \in H_p^s(\mathbb{R}^n)$ with $s > \frac{n}{p}$, then one has (4.151) for all $x \in \mathbb{R}^n$ and the uniquely determined continuous representative $f \in [f]$. For the general case $[f] \in H_p^s(\mathbb{R}^n)$, with $0 < s \le \frac{n}{p}$, one needs the $(s,p)$-capacity of compact sets $K$ given by

$$\mathsf{C}_{s,p}(K) = \inf\{\|\varphi | H_p^s(\mathbb{R}^n)\| : \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ real},\ \varphi \ge 1 \text{ on } K\}, \tag{4.152}$$

where this definition can be extended to arbitrary sets $K \subset \mathbb{R}^n$. It turns out that in each equivalence class $[f] \in H_p^s(\mathbb{R}^n)$ there is a uniquely determined representative $f \in [f]$ for which (4.151) is true with exception of a set $K$ with capacity $\mathsf{C}_{s,p}(K) = 0$. Furthermore, if $\Gamma$ is either given by (3.128) or $\Gamma = \partial\Omega$, that is, the boundary of a bounded $C^\infty$ domain, and if $K$ is a set in $\mathbb{R}^n$ with $\mathsf{C}_{s,p}(K) = 0$ where $1 < p < \infty$, $s > \frac{1}{p}$, then $K \cap \Gamma$ has $(n-1)$-dimensional Lebesgue measure (surface measure) zero. Hence (4.151) makes sense on $\Gamma$ pointwise with exception of a subset of $(n-1)$-dimensional Lebesgue measure zero. It coincides with $\mathrm{tr}_\Gamma f$ introduced via a limiting procedure. In other words, these uniquely determined distinguished representatives pave the way for a direct definition of traces. We refer to [Tri01, pp. 260/261] for details. This is reminiscent of the famous final slogan in G. Orwell's novel *'Animal farm'*, [Orw46, p. 114], which reads, adapted to our situation, as

*All representatives of an equivalence class are equal,*
*but some representatives are more*
*equal than others.*

**4.6.4.** In Definition 4.20 we introduced the spaces $W_2^s(\Gamma)$ on the boundary $\Gamma = \partial\Omega$ of a bounded $C^\infty$ domain $\Omega$ via finitely many local charts characterised by (4.92). The description given in Proposition 4.22 is satisfactory but not totally intrinsic. To get intrinsic norms one can convert $\Gamma$ into a compact Riemannian manifold, characterised by the same local charts. Afterwards one can replace $|\gamma - \zeta|$ in (4.96) by the Riemannian distance and $\mathrm{D}_{\mathrm{t}}^\alpha$ in (4.97), (4.98) by covariant derivatives. One can do the same with the more general spaces considered in Note 3.6.1 where the $\Gamma$-counterpart $B^s_{p,p}(\Gamma)$ of $B^s_{p,p}(\mathbb{R}^n)$ in (3.148), (3.149) is of special interest. This can be extended to all spaces mentioned in (3.154) and to complete (non-compact) Riemannian $C^\infty$ manifolds (of bounded geometry and of positive injectivity radius). We refer to [Tri92b, Chapter 7]. But this is not the subject of this book. The only point which we wish to mention here is the extension of the characterisation of the trace of $W_2^s(\Omega)$ in (4.101) from $p = 2$ to $1 < p < \infty$, which is also related to (4.150). Let $H_p^s(\Omega)$ be the restriction of $H_p^s(\mathbb{R}^n)$ according to (3.140) to the bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$ as in Definition 3.37 where now $1 < p < \infty, s > \frac{1}{p}$. Let $B^s_{p,p}(\Gamma)$ be as indicated above. Then the extension of (4.101) from $p = 2$ to $1 < p < \infty$ is given by

$$\mathrm{tr}_\Gamma : H_p^s(\Omega) \hookrightarrow B^{s-\frac{1}{p}}_{p,p}(\Gamma), \quad \mathrm{tr}_\Gamma\, H_p^s(\Omega) = B^{s-\frac{1}{p}}_{p,p}(\Gamma). \tag{4.153}$$

We refer to the books mentioned after (3.154) where the above special case may be found in [Tri78, Section 4.7.1]. In Note 3.6.2 we discussed the isomorphic structure of $H_p^s(\mathbb{R}^n)$ and $B^s_{p,p}(\mathbb{R}^n)$. There are counterparts for the two spaces in (4.153),

$$H_p^s(\Omega) \approx L_p(I) \quad \text{and} \quad B^{s-\frac{1}{p}}_{p,p}(\Gamma) \approx \ell_p. \tag{4.154}$$

According to $(L\ell)$ in Note 3.6.2 the space $H_p^s(\Omega)$ and its exact trace space $B^{s-\frac{1}{p}}_{p,p}(\Gamma)$ belong to different isomorphic classes if $1 < p < \infty$, $p \neq 2$.

**4.6.5.** Let $\mathbb{Q}^n = \mathbb{T}^n$ be as in (4.55) and let $\mathcal{D}(\mathbb{T}^n)$ be the restriction of

$$\{f \in C^\infty(\mathbb{R}^n) : f(x) = f(y) \text{ if } x - y \in 2\pi\mathbb{Z}^n\} \tag{4.155}$$

to $\mathbb{T}^n$, the space of $C^\infty$ functions on the torus $\mathbb{T}^n$. It is the periodic substitute of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ in Definition 2.32. The counterpart of the space $\mathcal{S}'(\mathbb{R}^n)$ according to Definition 2.43 is now the space $\mathcal{D}'(\mathbb{T}^n)$ of periodic distributions. The rôle of the Fourier transform in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ is taken over by the Fourier coefficients,

$$f \in \mathcal{D}'(\mathbb{T}^n) \longmapsto \{a_m\}_{m\in\mathbb{Z}^n}, \quad a_m = (2\pi)^{-n/2} f(e^{imx}). \tag{4.156}$$

On this basis one can develop a theory of the periodic counterparts $H^s(\mathbb{T}^n)$, $W_2^s(\mathbb{T}^n)$ of the spaces $H^s(\mathbb{R}^n)$, $W_2^s(\mathbb{R}^n)$ according to the Definitions 3.13, 3.22 but also of other spaces mentioned above, including the periodic counterpart of the spaces in (3.154). This may be found in [ST87, Chapter 3]. We relied in Section 4.4 on periodic expansions, but avoided to refer directly to results from the theory of periodic spaces to keep the presentation self-contained. This caused occasionally some extra work, for example in connection with the spaces in (4.70), (4.71) and Proposition 4.15.

**4.6.6.** The theory of spaces $B^s_{p,q}$, $F^s_{p,q}$ with $s, p, q$ as in (3.154) on $\mathbb{R}^n$ and in domains and their use for the study of pseudo-differential operators relies on some key problems,

- extensions,
- traces,
- pointwise multipliers,
- diffeomorphisms,

and, in case of spaces on domains,

- intrinsic characterisations.

The full satisfactory solutions of these key problems needed years, even almost two decades, from the early 1970s up to the early 1990s and is the subject of [Tri92b], including diverse applications, in particular, to (elliptic) pseudo-differential equations. In the above Chapters 3, 4 we dealt with the same problems, having applications to boundary value problems for elliptic differential operators of second order in mind, the subject of the following chapters, but now restricted mainly to $W_2^s$ in $\mathbb{R}^n$ and on domains. Then the task is significantly easier and we tried to find direct arguments as simple as possible. But there remains a hard core which cannot be circumvented and which lies in the nature of the subject. We try to continue in this way in what follows true to Einstein's advice,

*Present your subject as simply as possible, but not simpler.*

# Chapter 5
# Elliptic operators in $L_2$

## 5.1 Boundary value problems

In Section 1.1 we outlined the plan of the book. Chapter 1 dealt with some classical assertions for the Laplace–Poisson equation. For the homogeneous and inhomogeneous Dirichlet problem according to the Definitions 1.35, 1.43 we merely got in case of balls some (more or less) satisfactory assertions in the Theorems 1.40, 1.48. On the one hand, the Chapters 2–4 are self-contained introductions to the theory of distributions and Sobolev spaces. On the other hand, they prepare the study of boundary value problems for elliptic equations of second order as outlined in Section 1.1. We stick at the same moderate level as in the preceding chapters avoiding any additional complications. In particular, we deal mostly (but not exclusively) with (homogeneous and inhomogeneous) boundary value problems in an $L_2$ setting.

First we recall some definitions adapted to what follows. As for basic notation we refer to Appendix A. In particular, $\mathrm{D}^\alpha f$ indicates derivatives as introduced in (A.1), (A.2). Domain in $\mathbb{R}^n$ means simply open set. Moreover, according to Definition A.3 a bounded domain $\Omega$ in $\mathbb{R}^n$ is called a *bounded $C^\ell$ domain* or *bounded $C^\infty$ domain* if it is connected and if its boundary $\partial\Omega$ has the smoothness properties described there. We use the notation $C(\Omega)$ as in Definition A.1 as the collection of all complex-valued bounded functions which are continuous on the closure $\bar{\Omega}$ of the domain $\Omega$. Next we recall and adapt Definition 1.1.

**Definition 5.1.** Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3. Let

$$\{a_{jk}\}_{j,k=1}^n \subset C(\Omega), \quad \{a_l\}_{l=1}^n \subset C(\Omega), \quad a \in C(\Omega) \tag{5.1}$$

with

$$a_{jk}(x) = a_{kj}(x) \in \mathbb{R}, \quad x \in \bar{\Omega}, \; j,k = 1,\ldots,n. \tag{5.2}$$

Then the differential expression $A$,

$$(Au)(x) = -\sum_{j,k=1}^n a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{l=1}^n a_l(x)\frac{\partial u}{\partial x_l}(x) + a(x)u(x), \tag{5.3}$$

of second order is called *elliptic* if there is a constant $E > 0$ (*ellipticity constant*) such that for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$ the *ellipticity condition*

$$\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \;\geq\; E|\xi|^2 \tag{5.4}$$

is satisfied.

**Remark 5.2.** Recall that according to Example 1.3 the most distinguished example of an elliptic differential expression of second order is the *Laplacian*

$$A = -\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \tag{5.5}$$

where one may choose $E = 1$ in (5.4). Otherwise we refer for some discussion to Section 1.1. In particular, Remark 1.4 implies that

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j \overline{\xi_k} \geq E|\xi|^2, \quad \xi \in \mathbb{C}^n. \tag{5.6}$$

Furthermore, (1.9) and (1.10) indicate what is meant by boundary value problems. Now we give some more precise definitions adapted to the $L_2$ theory we have in mind.

For bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$ we have the equivalent norms for the Sobolev spaces $W_2^s(\Omega)$, $s > 0$, as described in Theorem 4.1. In particular,

$$\|f \,|W_2^2(\Omega)\| \sim \Big( \sum_{|\alpha|\leq 2} \|\mathrm{D}^\alpha f \,| L_2(\Omega)\|^2 \Big)^{1/2}, \quad f \in W_2^2(\Omega). \tag{5.7}$$

In the Sections 4.3 and 4.5 we dealt in detail with traces

$$\mathrm{tr}_\Gamma\, u \quad \text{and} \quad \mathrm{tr}_\Gamma \frac{\partial}{\partial \mu} u \quad \text{on } \Gamma = \partial\Omega \text{ for } u \in W_2^2(\Omega), \tag{5.8}$$

now restricted to $s = 2$. Here $\mu = \mu_\gamma$ with $\gamma \in \Gamma$ is a non-tangential $C^\infty$ vector field on $\Gamma$ as introduced in Remark 4.28 with the usual $C^\infty$ vector field of outer normals $\nu = \nu_\gamma$, $\gamma \in \Gamma$, as a distinguished case. In terms of the Sobolev spaces $W_2^s(\Gamma)$ at the boundary $\Gamma = \partial\Omega$ according to Definition 4.20 we obtained in Theorem 4.24 complemented by (4.130) that

$$\mathrm{tr}_\Gamma : W_2^2(\Omega) \hookrightarrow W_2^{3/2}(\Gamma), \quad \mathrm{tr}_\Gamma\, W_2^2(\Omega) = W_2^{3/2}(\Gamma), \tag{5.9}$$

and

$$\mathrm{tr}_\Gamma \frac{\partial}{\partial \mu} : W_2^2(\Omega) \hookrightarrow W_2^{1/2}(\Gamma), \quad \mathrm{tr}_\Gamma \frac{\partial}{\partial \mu} W_2^2(\Omega) = W_2^{1/2}(\Gamma). \tag{5.10}$$

Of interest for us are now the special cases

$$W_{2,0}^2(\Omega) = \{ f \in W_2^2(\Omega) : \mathrm{tr}_\Gamma\, f = 0 \} \tag{5.11}$$

and

$$W_2^{2,\mu}(\Omega) = \Big\{ f \in W_2^2(\Omega) : \mathrm{tr}_\Gamma \frac{\partial f}{\partial \mu} = 0 \Big\} \tag{5.12}$$

of the spaces introduced in Definition 4.30. They are closed subspaces of $W_2^2(\Omega)$ and one has the orthogonal decompositions according to Proposition 4.32 which (in slight abuse of notation) can be written as

$$W_2^2(\Omega) = W_{2,0}^2(\Omega) \oplus W_2^{3/2}(\Gamma) \tag{5.13}$$

and

$$W_2^2(\Omega) = W_2^{2,\mu}(\Omega) \oplus W_2^{1/2}(\Gamma), \tag{5.14}$$

including the density assertion with respect to (4.137), (4.140). As for $L_p$ counterparts one may consult Definition 4.9 and Remarks 4.10, 4.33.

**Definition 5.3.** Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ as introduced in Definition A.3 with the boundary $\Gamma = \partial\Omega$ and let $A$ be an elliptic differential expression of second order according to Definition 5.1.

(i) Let $f \in L_2(\Omega)$ and $g \in W_2^{3/2}(\Gamma)$. The *inhomogeneous Dirichlet problem* asks for functions $u \in W_2^2(\Omega)$ with

$$Au = f \quad \text{in } \Omega \quad \text{and} \quad \mathrm{tr}_\Gamma\, u = g \quad \text{on } \Gamma. \tag{5.15}$$

The *homogeneous Dirichlet problem* asks for functions $u$ with

$$Au = f \quad \text{in } \Omega \quad \text{and} \quad u \in W_{2,0}^2(\Omega). \tag{5.16}$$

(ii) Let $\mu$ be a non-tangential $C^\infty$ vector field on $\Gamma$ according to Remark 4.28. Let $f \in L_2(\Omega)$ and $g \in W_2^{1/2}(\Gamma)$. The *inhomogeneous Neumann problem* asks for functions $u \in W_2^2(\Omega)$ with

$$Au = f \quad \text{in } \Omega \quad \text{and} \quad \mathrm{tr}_\Gamma \frac{\partial u}{\partial \mu} = g \quad \text{on } \Gamma. \tag{5.17}$$

The *homogeneous Neumann problem* asks for functions $u$ with

$$Au = f \quad \text{in } \Omega \quad \text{and} \quad u \in W_2^{2,\mu}(\Omega). \tag{5.18}$$

**Remark 5.4.** Since all coefficients in (5.3) are bounded (5.7) implies that

$$Au = f \in L_2(\Omega) \quad \text{if } u \in W_2^2(\Omega). \tag{5.19}$$

Together with (5.9), (5.10) it follows that the above *boundary value problems* make sense. Of course, the homogeneous problems are simply the corresponding inhomogeneous problems with vanishing boundary data. If $A = -\Delta$ is the Laplacian according to (5.5) and if $\mu = \nu$ is the $C^\infty$ vector field of the outer normals on $\Gamma$, then (1.9), (1.10) give first descriptions of the above boundary value problems. For $A = -\Delta$ it is natural to choose the vector field $\mu = \nu$ of outer normals for the

Neumann problems. For more general $A$ this is no longer the case and one may ask for a distinguished substitute. This is the so-called *co-normal* on $\Gamma$,

$$\nu^A = (\nu_j^A)_{j=1}^n, \quad \nu_j^A = \sum_{k=1}^n a_{jk}(\gamma)\nu_k(\gamma), \ \gamma \in \Gamma, \tag{5.20}$$

where $\nu = (\nu_k(\gamma))_{k=1}^n$ is the outer normal on $\Gamma$. Recall that the coefficients $a_{jk}$ are continuous on $\overline{\Omega}$ and, hence, on $\Gamma$. However, we deal here mainly with the Dirichlet problem and look at the Neumann problem only if no substantial additional efforts are needed. This means that in case of the Neumann problem we restrict ourselves to the Laplacian $A = -\Delta$ and the outer normals $\mu = \nu$ on $\Gamma$ in (5.17), (5.18). But we comment on the more general cases in Note 5.12.1.

**Exercise 5.5.** Prove that $\nu^A$ according to (5.20) generates a continuous non-tangential vector field on $\Gamma$.

*Hint:* Show that $\langle \nu^A, \nu \rangle > 0$ for the scalar product of $\nu^A$ and $\nu$.

**Exercise 5.6.** Justify for the co-normal $\nu^A$ according to (5.20) with constant coefficients $a_{jk} = a_{kj}$ the generalisation

$$\begin{aligned}&\int_\Omega \sum_{j,k=1}^n a_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(x) g(x) \mathrm{d}x \\ &\quad = -\int_\Omega \sum_{j,k=1}^n a_{jk} \frac{\partial f}{\partial x_j}(x) \, \frac{\partial g}{\partial x_k}(x) \mathrm{d}x + \int_\Gamma g(\gamma) \frac{\partial f}{\partial \nu^A}(\gamma) \mathrm{d}\sigma(\gamma)\end{aligned} \tag{5.21}$$

of the Green's formula (A.16). (This makes clear that for given $\{a_{jk}\}_{j,k=1}^n$ the co-normal $\nu^A$ is a distinguished vector field on $\Gamma$.)

## 5.2 Outline of the programme, and some basic ideas

First we discuss what follows on a somewhat heuristical and provisional level. So far we have for the inhomogeneous Dirichlet problem and the Laplacian $A = -\Delta$, given by (5.5), Definition 1.43, the existence and uniqueness Theorem 1.48, and the discussion in Remark 1.50 hinting at the present chapter. Now we deal with boundary value problems of this type in the framework of an $L_2$ theory in arbitrary bounded $C^\infty$ domains in $\mathbb{R}^n$ and for general elliptic equations according to Definition 5.3. The precise assumptions for the given data $f$ and $g$ in, say, Definition 5.3 (i) suggest that one gets also precise answers for possible solutions $u$ in (5.15), their existence, uniqueness and smoothness with the ideal outcome

$$\|u|W_2^2(\Omega)\| \sim \|f|L_2(\Omega)\| + \|g|W_2^{3/2}(\Gamma)\|. \tag{5.22}$$

The proof of Theorem 1.48 advises, also in the framework of an $L_2$ theory, the reduction of the inhomogeneous Dirichlet problem (5.15) to the homogeneous one in (5.16) with the optimal outcome

$$\|u|W_2^2(\Omega)\| \sim \|f|L_2(\Omega)\|, \quad u \in W_{2,0}^2(\Omega). \tag{5.23}$$

It is usual and one of the most fruitful developments since more than fifty years to incorporate (homogeneous) boundary value problems into the operator theory in Hilbert spaces (or, more general, Banach spaces) resulting in our case in the (unbounded closed) *elliptic operator* $A$,

$$Au = -\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{l=1}^{n} a_l \frac{\partial u}{\partial x_l} + au, \quad \mathrm{dom}(A) = W_{2,0}^2(\Omega), \tag{5.24}$$

in $L_2(\Omega)$, where $\mathrm{dom}(A)$ is its *domain of definition*. In other words, we interpret $A$ either as a continuous operator from $W_2^2(\Omega)$ or $W_{2,0}^2(\Omega)$ into $L_2(\Omega)$ according to (5.19), or within $L_2(\Omega)$ as an unbounded operator described by (5.24). The adopted point of view will be clear from the context. However, if there is any danger of confusion, the spaces involved will be indicated. In particular, (5.24) reduces the (homogeneous) Dirichlet problem (5.16) to the study of the mapping properties of the unbounded operator $A$. However, the suggested uniqueness according to Theorem 1.48, tacitly underlying also (5.22), (5.23), cannot be expected in general. On the contrary, under the influence of the needs of quantum mechanics in the late 1920s, 1930s and (as far as differential operators are concerned) in the 1950s it came out that it is reasonable to deal not with isolated operators $A$, but with the scale

$$A - \lambda\,\mathrm{id} \quad \text{where } \lambda \in \mathbb{C} \text{ and } \mathrm{id}\, u = u,\ u \in \mathrm{dom}(A), \tag{5.25}$$

is the identity. In this context the non-trivial *null spaces* or *kernels*

$$\ker(A - \lambda\,\mathrm{id}) = \{u \in \mathrm{dom}(A) : Au = \lambda u\} \tag{5.26}$$

(having dimension of at least 1) are of peculiar interest. Then $\lambda$ is called *eigenvalue* of $A$ and

$$Au = \lambda u, \quad u \in \mathrm{dom}(A),\ u \neq 0, \tag{5.27}$$

are the related *eigenfunctions*, spanning $\ker(A - \lambda\,\mathrm{id})$. The *spectral theory* for $A$, in particular, the distribution of its eigenvalues, will be considered later in detail in Chapter 7. But some decisive preparations will be made in this chapter. This may explain that we do not deal exclusively with $A$ but also with its translates in (5.25). As mentioned before, (5.22), (5.23) cannot be expected if 0 is an eigenvalue of $A$ according to (5.24). The adequate replacement of (5.23) is given by

$$\|u|W_2^2(\Omega)\| \sim \|Au|L_2(\Omega)\| + \|u|L_2(\Omega)\|, \quad u \in W_{2,0}^2(\Omega), \tag{5.28}$$

or, explaining the equivalence $\sim$, there are constants $0 < c_1 \le c_2$ such that for all $u \in W^2_{2,0}(\Omega)$,

$$c_1 \, \|u|W^2_2(\Omega)\| \le \|Au|L_2(\Omega)\| + \|u|L_2(\Omega)\| \le c_2 \, \|u|W^2_2(\Omega)\|. \tag{5.29}$$

Usually equivalences of this type are called *a priori estimates*. Assuming that the homogeneous Dirichlet problem (5.16) is solved, then one gets (5.28) with little effort from the operator theory in Hilbert spaces. But usually one follows just the opposite way of reasoning proving first (i.e., *a priori*) (5.28) and using these substantial assertions afterwards to deal with the homogeneous (and then with the inhomogeneous) Dirichlet problem. One may summarise what follows in the next three chapters as follows:

- In this Chapter 5 we concentrate first on the indicated a priori estimates, preferably for the Dirichlet problem, but also for the Neumann problem (if no additional effort is needed). Afterwards we deal with the boundary value problems according to Definition 5.3. This will be complemented by some assertions about degenerate elliptic equations and a related $L_p$ theory.

- In Chapter 7 we have a closer look at the spectral theory of operators of type (5.24) including assertions about the distribution of eigenvalues.

- It is expected that the reader has some basic knowledge of abstract functional analysis, and, in particular, of the theory of unbounded closed operators in Hilbert spaces. But we collect what we need (with references) in Appendix C. Some more specific assertions, especially about approximation numbers and entropy numbers, respectively, and their relation to eigenvalues will be the subject of Chapter 6 preparing, in particular, Chapter 7.

## 5.3 A priori estimates

Let $\{a_v : v \in V\}$ and $\{b_v : v \in V\}$ be two sets of non-negative numbers indexed by $v \in V$. If there are two numbers $0 < c_1 \le c_2 < \infty$ such that

$$c_1 a_v \le b_v \le c_2 a_v \quad \text{for all } v \in V, \text{ then we write } \; a_v \sim b_v, \; v \in V, \tag{5.30}$$

and call it an *equivalence*; (5.28) with the explanation (5.29) and $V = W^2_{2,0}(\Omega)$ may serve as an example. According to the programme outlined in Section 5.2 we deal first with the *a priori estimate* (5.28). Recall that bounded $C^\infty$ domains in $\mathbb{R}^n$ as introduced in Definition A.3 are connected. The spaces $W^2_2(\Omega)$ and $W^2_{2,0}(\Omega)$ have the same meaning as in Theorem 4.1 and (5.11), always assumed to be normed by (5.7).

**Theorem 5.7.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$ and let $A$ be an elliptic differential expression according to Definition 5.1. Then*

$$\|Au|L_2(\Omega)\| + \|u|L_2(\Omega)\| \sim \|u|W_2^2(\Omega)\|, \quad u \in W_{2,0}^2(\Omega). \tag{5.31}$$

*Proof. Step* 1. By (5.1), (5.7) we have for some $c > 0$,

$$\|Au|L_2(\Omega)\| + \|u|L_2(\Omega)\| \le c\|u|W_2^2(\Omega)\|, \quad u \in W_{2,0}^2(\Omega). \tag{5.32}$$

*Step* 2. As for the converse it is sufficient to prove that there is a constant $c > 0$ such that

$$c\|u|W_2^2(\Omega)\| \le \|Au|L_2(\Omega)\| + \|u|L_2(\Omega)\|, \quad u \in C^\infty(\Omega),\ \mathrm{tr}_\Gamma\, u = 0. \tag{5.33}$$

This follows from Proposition 4.32 (i) with $s = 2$ and a standard completion argument. In particular, we may assume in the sequel that $u$ has classical derivatives.

We prove (5.33) by reducing it in several steps to standard situations. First we assume that we had already shown

$$c\|\varphi_k u|W_2^2(\Omega)\| \le \|A(\varphi_k u)|L_2(\Omega)\| + \|\varphi_k u|L_2(\Omega)\| \tag{5.34}$$

for $u \in C^\infty(\Omega)$, $\mathrm{tr}_\Gamma\, u = 0$, where $\{\varphi_k\}_{k=0}^J$ is the resolution of unity according to (4.5)–(4.7) and Figure 4.3. Of course, this implies $\varphi_k u \in C^\infty(\Omega)$ with $\mathrm{tr}_\Gamma(\varphi_k u) = 0$ for $k = 1, \dots, J$. Then one obtains by (5.3) and Theorem 4.17 that

$$\begin{aligned} \|u|W_2^2(\Omega)\| &\le c\|Au|L_2(\Omega)\| + c\|u|W_2^1(\Omega)\| \\ &\le c\|Au|L_2(\Omega)\| + \varepsilon\|u|W_2^2(\Omega)\| + c_\varepsilon\|u|L_2(\Omega)\|, \end{aligned} \tag{5.35}$$

where $\varepsilon > 0$ is at our disposal. This proves (5.33). Hence the proof of (5.33) is a local matter where the boundary terms in (5.34) are of interest, i.e., $k \ge 1$. One gets the term with $\varphi_0$ as an easy by-product.

*Step* 3. By Step 2 it is sufficient to prove (5.33) for a local standard situation as considered in connection with the Figures A.1 and 4.1.

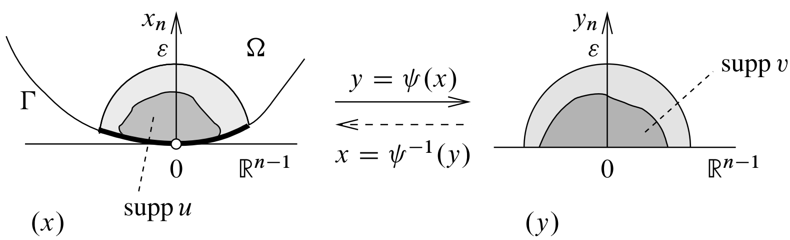



Figure 5.1

We may assume that $0 \in \Gamma = \partial\Omega$ and that $\Gamma$ near the origin is given by

$$x_n = \tau(x'), \quad x' = (x_1, \dots, x_{n-1}), \; |x'| \le \varepsilon, \tag{5.36}$$

where $\tau$ is a $C^\infty$ function in $\mathbb{R}^{n-1}$ with

$$\tau(0) = 0 \quad \text{and} \quad \frac{\partial \tau}{\partial x_k}(0) = 0 \text{ for } k = 1, \dots, n-1. \tag{5.37}$$

Then the indicated locally diffeomorphic map $\psi$ given by

$$y_k = x_k \text{ for } k = 1, \dots, n-1 \quad \text{and} \quad y_n = x_n - \psi(x'), \tag{5.38}$$

flattens $\Omega_\varepsilon = \Omega \cap \{x \in \mathbb{R}^n : |x| < \varepsilon\}$ and $\Gamma_\varepsilon = \Gamma \cap \{x \in \mathbb{R}^n : |x| < \varepsilon\}$ such that

$$\begin{aligned} \psi(\Omega_\varepsilon) &\subset \mathbb{R}^n_+ \cap \{y \in \mathbb{R}^n : |y| < \varepsilon\} \quad \text{and} \\ \psi(\Gamma_\varepsilon) &\subset \mathbb{R}^{n-1} \cap \{y' \in \mathbb{R}^{n-1} : |y'| < \varepsilon\}. \end{aligned} \tag{5.39}$$

If $u \in C^\infty(\Omega)$ with $\operatorname{supp} u \subset \overline{\Omega_\varepsilon}$ and $\operatorname{tr}_\Gamma u = 0$, then

$$v(y) = (u \circ \psi^{-1})(y), \quad y \in \overline{\mathbb{R}^n_+} \text{ and } |y| \le \varepsilon, \tag{5.40}$$

has corresponding properties. In particular, $v(y', 0) = 0$ if $|y'| \le \varepsilon$. Transforming $Au$ given by (5.3) according to (5.38), (5.40), leads to

$$\begin{aligned} (Au)(x) &= (\tilde{A}v)(y) \\ &= -\sum_{j,k=1}^n \tilde{a}_{jk}(y) \frac{\partial^2 v}{\partial y_j \partial y_k}(y) + \sum_{l=1}^n \tilde{a}_l(y) \frac{\partial v}{\partial y_l}(y) + \tilde{a}(y) v(y) \end{aligned} \tag{5.41}$$

with

$$\tilde{a}_{jk}(y) = a_{jk}(\psi^{-1}(y)) + \varepsilon_{jk}(y), \quad |\varepsilon_{jk}(y)| \le \delta, \tag{5.42}$$

where $\delta > 0$ is at our disposal (choosing the above $\varepsilon > 0$ sufficiently small). In particular, one obtains a transformed ellipticity condition (5.4),

$$\sum_{j,k=1}^n \tilde{a}_{jk}(0) \xi_j \xi_k \ge \frac{E}{2} |\xi|^2, \quad \xi \in \mathbb{R}^n, \tag{5.43}$$

at the origin. Furthermore, since $\tilde{a}_{jk}(y)$ are continuous at the origin, we have

$$\tilde{a}_{jk}(y) = \tilde{a}_{jk}(0) + b_{jk}(y) \quad \text{with } |b_{jk}(y)| \le \delta, \; |y| \le \varepsilon, \tag{5.44}$$

where $\delta$ is at our disposal (at the expense of $\varepsilon$). Inserting (5.44) in (5.41) results in the main term now with $\tilde{a}_{jk}(0)$ in place of $a_{jk}(y)$ and perturbations of second order terms with small coefficients. Altogether one arrives at the following model case: Let

$$(Au)(x) = -\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k}(x), \quad x \in \mathbb{R}^n, \tag{5.45}$$

with constant coefficients $a_{jk}$ satisfying the ellipticity condition (5.4). Then we wish to prove that there is a constant $c > 0$ such that

$$c\, \|u|W_2^2(\mathbb{R}^n_+)\| \le \|Au|L_2(\mathbb{R}^n_+)\| + \|u|L_2(\mathbb{R}^n_+)\| \tag{5.46}$$

for

$$u \in C^\infty(\mathbb{R}^n_+), \quad \operatorname{supp} u \subset \overline{\mathbb{R}^n_+} \cap \{x \in \mathbb{R}^n : |x| < \varepsilon\}, \quad u(x',0) = 0 \tag{5.47}$$

for small $\varepsilon > 0$. Afterwards the above reductions and re-transformations prove (5.3) in the same way as in Step 2.

*Step* 4. Next we wish to reduce the desired estimate (5.46) with (5.47) on $\mathbb{R}^n_+$ to a corresponding estimate on $\mathbb{R}^n$ using the odd extension procedure according to (4.45) and Theorem 4.11, hence

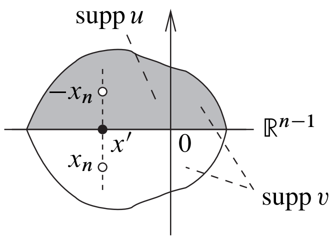



Figure 5.2

$$v(x) = \text{O-ext}\, u(x) = \begin{cases} u(x) & \text{if } x_n \ge 0, \\ -u(x', -x_n) & \text{if } x_n < 0. \end{cases} \tag{5.48}$$

By Theorem 4.11 one has $v \in W_2^2(\mathbb{R}^n)$ and $\operatorname{supp} v \subset \{x \in \mathbb{R}^n : |x| < \varepsilon\}$. If one replaces $u(x)$, $x \in \mathbb{R}^n_+$, in (5.45) by $v(x)$, $x \in \mathbb{R}^n$, according to (5.48), then the differential expression is preserved with the exception of the terms with $j = n$, $1 \le k \le n-1$, which change sign. We remove this unpleasant effect applying first an orthogonal rotation $\boldsymbol{H} = (h_{ml})^n_{m,l=1}$ in $\mathbb{R}^n$ and afterwards dilations with respect to (new) axes of coordinates,

$$y_j = \sum_{m=1}^{n} h_{jm} x_m, \quad z_j = \frac{y_j}{d_j}, \quad j = 1, \ldots, n, \tag{5.49}$$

with $d_j > 0$. Analytic geometry tells us that $\boldsymbol{H}$ can be chosen such that

$$-\sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k}(x) = -\sum_{l=1}^{n} d_l \frac{\partial^2 \widetilde{u}}{\partial y_l^2}(y) = -\Delta u^*(z), \tag{5.50}$$

where $u(x) = \widetilde{u}(y) = u^*(z)$ are the transformed functions according to (5.49). As for the corresponding quadratic forms one has

$$\sum_{j,k=1}^{n} a_{jk} \xi_j \xi_k = \sum_{l=1}^{n} d_l \eta_l^2 \ge E|\eta|^2 = E|\xi|^2, \tag{5.51}$$

and for $l = 1, \dots, n$,

$$E \le d_l \le dM \quad \text{with } M = \max_{j,k=1,\dots,n} |a_{jk}|, \tag{5.52}$$

where $d \ge 1$ is independent of $E$ and $M$ (this will be of some use for us later on). If $u$ is given by (5.47), then $u^*(z) = u(x)$ has similar properties with $u^*(z) = 0$ on the transformed upper hyper-plane $\{z \in \mathbb{R}^n : x_n(z) = 0\}$. We arrive finally at the Laplacian in a half-space. Of course, we may assume that this half-space is $\mathbb{R}^n_+$ and we wish to prove that

$$c\, \|u|W_2^2(\mathbb{R}^n_+)\| \le \|\Delta u|L_2(\mathbb{R}^n_+)\| + \|u|L_2(\mathbb{R}^n_+)\| \tag{5.53}$$

for $u$ with (5.47). Hence (5.46), (5.47) can be reduced to (5.53), (5.47). Now we apply the odd extension (5.48) described above which reduces (5.53) to

$$c\, \|v|W_2^2(\mathbb{R}^n)\| \le \|\Delta v|L_2(\mathbb{R}^n)\| + \|v|L_2(\mathbb{R}^n)\| \tag{5.54}$$

for $v \in W_2^2(\mathbb{R}^n)$.

*Step* 5. We observe that (5.54) is essentially covered by Theorem 3.11. In particular, (3.26) implies that

$$\begin{aligned}
\|v|W_2^2(\mathbb{R}^n)\|^2 &= \int_{\mathbb{R}^n} \sum_{|\alpha|\le 2} |\mathcal{F}(\mathrm{D}^\alpha v)(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^n} \Big( \sum_{|\alpha|\le 2} |\xi^\alpha|^2 \Big) |\mathcal{F} v(\xi)|^2 \,\mathrm{d}\xi \\
&\le c \int_{\mathbb{R}^n} |\xi|^4 |\mathcal{F} v(\xi)|^2 \,\mathrm{d}\xi + c \int_{\mathbb{R}^n} |\mathcal{F} v(\xi)|^2 \,\mathrm{d}\xi \\
&= c\|\Delta v|L_2(\mathbb{R}^n)\|^2 + c\|v|L_2(\mathbb{R}^n)\|^2.
\end{aligned} \tag{5.55}$$

This completes the proof of the theorem. □

**Exercise* 5.8.** Justify (5.51). How does $\eta$ depend on $\xi$?

**Corollary 5.9** (Gårding's inequality). *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$. Let $A$ be an elliptic differential expression according to Definition 5.1 where, in addition, the functions $a_{jk}(x)$ are Lipschitz continuous, hence*

$$|a_{jk}(x)| \le M, \quad |a_l(x)| \le M, \quad |a(x)| \le M \quad \text{for } x \in \bar{\Omega} \tag{5.56}$$

*and all admitted $j, k, l$, and*

$$|a_{jk}(x) - a_{jk}(y)| \le M|x-y|, \quad x \in \bar{\Omega},\ y \in \bar{\Omega}, \tag{5.57}$$

*for $j, k = 1, \dots, n$, and some $M > 0$. Then*

$$\|Au|L_2(\Omega)\| \ge c_1\, E\|u|W_2^2(\Omega)\| - c_2\|u|L_2(\Omega)\|, \quad u \in \overset{\circ}{W}{}^2_{2,0}(\Omega), \tag{5.58}$$

*with positive constants $c_1, c_2$ depending only on $M$ and $M/E$.*

*Proof.* One can follow the above proof of Theorem 5.7. The additional assumption (5.57) is helpful in connection with (5.44). The quotient $M/E$ influences the estimates transforming $y$ in $z$ in (5.50), (5.52). □

**Remark 5.10.** One may ask for a counterpart of Theorem 5.7 with respect to homogeneous Neumann problems according to Definition 5.3 (ii). This can be done but requires some extra efforts which we wish to avoid. Some information will be given in Note 5.12.1 below. Otherwise we use the same notation as in Definition 5.3 and Remark 5.4. In particular, $\nu$ is the $C^\infty$ vector field of outer normals on $\Gamma$. Recall that $A = -\Delta$ is the Laplacian (5.5). Furthermore, $W_2^2(\Omega)$ and $W_2^{2,\nu}(\Omega)$ have the same meaning as in Theorem 4.1, Definition 4.30 and (5.12) with $\mu = \nu$ always assumed to be normed by (5.7).

**Theorem 5.11.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$. Then*

$$\|\Delta u|L_2(\Omega)\| + \|u|L_2(\Omega)\| \sim \|u|W_2^2(\Omega)\|, \quad u \in W_2^{2,\nu}(\Omega). \tag{5.59}$$

*Proof.* We follow the proof of Theorem 5.7 indicating the necessary modifications. Step 1 remains unchanged. The counterpart of (5.33) is given by

$$c\|u|W_2^2(\Omega)\| \le \|\Delta u|L_2(\Omega)\| + \|u|L_2(\Omega)\|, \quad u \in C^\infty(\Omega),\ \mathrm{tr}_\Gamma \frac{\partial u}{\partial \nu} = 0, \tag{5.60}$$

where the restriction to smooth functions is justified by Proposition 4.32 (ii). We now base the localisation described in Step 2 on the special resolution of unity according to Remark 4.5, in particular, (4.24). Then $\mathrm{tr}_\Gamma \frac{\partial u}{\partial \nu} = 0$ is preserved. Similarly one modifies (5.38) in Step 3 by curvilinear coordinates with $y_n$ pointing in the normal direction as indicated in Figure 4.6. Then one arrives at the counterparts of (5.45)–(5.47), hence

$$c\,\|u|W_2^2(\mathbb{R}^n_+)\| \le \|\Delta u|L_2(\mathbb{R}^n_+)\| + \|u|L_2(\mathbb{R}^n_+)\| \tag{5.61}$$

for

$$u \in C^\infty(\mathbb{R}^n_+), \quad \mathrm{supp}\, u \subset \overline{\mathbb{R}^n_+} \cap \{x \in \mathbb{R}^n : |x| < \varepsilon\}, \quad \frac{\partial u}{\partial x_n}(x', 0) = 0. \tag{5.62}$$

Instead of the odd extension (5.48) we use now the even extension,

$$v(x) = \mathsf{E}\text{-}\mathrm{ext}\, u(x) = \begin{cases} u(x) & \text{if } x_n \ge 0, \\ u(x', -x_n) & \text{if } x_n < 0. \end{cases} \tag{5.63}$$

By Theorem 4.11 (with $w_2^{2,0} = W_2^{2,\nu}(\mathbb{R}^n_+)$) we have $v \in W_2^2(\mathbb{R}^n)$. There is no need now for the rotation and dilations as in Step 4 which would not preserve $\frac{\partial v}{\partial x_n}(x', 0) = 0$ in general. We get immediately (5.54) and its proof in (5.55). □

**Remark 5.12.** There is a full counterpart of Theorem 5.7 with $u \in W_2^{2,\mu}(\Omega)$ in place of $W_{2,0}^2(\Omega)$ where again $\mu$ is an arbitrary non-tangential $C^\infty$ vector field on $\Gamma = \partial\Omega$. Following the arguments in the proof of Theorem 5.7, then the rotation and dilations in Step 3 transform $\mu$ into another non-tangential vector field $\widetilde{\mu}$ now on $\Gamma = \mathbb{R}^{n-1}$. This would require an assertion of type (4.48) with, say, $W_2^{2,\widetilde{\mu}}(\mathbb{R}^n_+)$ in place of $w_2^{2,0}(\Omega)$. This can be done, but it is not covered by our arguments. In addition, application of such an extension to $\Delta$ in $\mathbb{R}^n_+$ does not produce in general $\Delta$ in $\mathbb{R}^n$ which we used stepping from (5.53) to (5.54).

**Exercise 5.13.** Let $\{a_j\}_{j=1}^n \subset C(\Omega)$ and $a \in C(\Omega)$. Prove Theorem 5.11 for the perturbed Laplacian

$$Au = -\Delta u + \sum_{l=1}^{n} a_l(x)\, \frac{\partial u}{\partial x_l} + a(x)u \tag{5.64}$$

in place of $\Delta$.

*Hint:* Use the arguments in Step 2 of the proof of Theorem 5.7.

As outlined in Section 5.2 the spectral theory of elliptic operators $A$ of type (5.24) requires to deal with the scale (5.25). This will be done in detail below. We prepare these considerations by the following assertion. Let $C^1(\Omega)$ be the spaces as introduced in Definition A.1.

**Corollary 5.14.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$.*

(i) *Let $A$ be an elliptic differential expression according to Definition 5.1 where, in addition, $a_{jk} \in C^1(\Omega)$. Let $E$ be the ellipticity constant and let*

$$\|a_{jk}|C^1(\Omega)\| \le M, \quad \|a_l|C(\Omega)\| \le M, \quad \|a|C(\Omega)\| \le M, \tag{5.65}$$

*for all admitted $k, j, l$ and some $M > 0$. Then there are positive constants $\lambda_0$, $c_1, c_2$ depending only on $E$, $M$ (and $\Omega$) such that*

$$\|(A + \lambda\, \mathrm{id})u|L_2(\Omega)\| \ge c_1\|u|W_2^2(\Omega)\| + c_2\,\lambda\, \|u|L_2(\Omega)\| \tag{5.66}$$

*for all $u \in W_{2,0}^2(\Omega)$ and all $\lambda \in \mathbb{R}$ with $\lambda \ge \lambda_0$.*

(ii) *Again let $\nu$ be the $C^\infty$ vector field of the outer normals on $\Gamma = \partial\Omega$. There are positive constants $\lambda_0$, $c_1$, $c_2$ such that*

$$\|(-\Delta + \lambda\, \mathrm{id})u|L_2(\Omega)\| \ge \; c_1\, \|u|W_2^2(\Omega)\| + c_2\,\lambda\, \|u|L_2(\Omega)\| \tag{5.67}$$

*for all $u \in W_2^{2,\nu}(\Omega)$ and all $\lambda \in \mathbb{R}$ with $\lambda \ge \lambda_0$.*

*Proof.* *Step* 1. We prove (i). As before it follows from Proposition 4.32 (i) that it is sufficient to deal with $C^\infty$ functions $u \in W^2_{2,0}(\Omega)$. Let

$$A_0 u = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right), \quad u \in C^\infty(\Omega),\ \mathrm{tr}_\Gamma\, u = 0, \tag{5.68}$$

which makes sense since we assumed $a_{jk} \in C^1(\Omega)$. Integration by parts implies for the scalar product $\langle A_0 u, u\rangle$ in $L_2(\Omega)$,

$$\begin{aligned}\langle A_0 u, u\rangle &= -\int_\Omega \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right)\overline{u}(x)\,\mathrm{d}x\\ &= \int_\Omega \sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial u}{\partial x_k}(x)\frac{\partial \overline{u}}{\partial x_j}(x)\,\mathrm{d}x \ \geq\ 0, \end{aligned}\tag{5.69}$$

where the last follows from (5.6). Furthermore,

$$Au = A_0 u + \sum_{l=1}^{n} b_l \frac{\partial u}{\partial x_l} + bu = A_0 u + A_1 u, \tag{5.70}$$

where $b_l$ and $b = a$ can be similarly estimated as in (5.65). Then one obtains for $\lambda > 0$,

$$\begin{aligned}&\|(A + \lambda\,\mathrm{id})u|L_2(\Omega)\|^2\\ &\quad= \langle Au + \lambda u, Au + \lambda u\rangle\\ &\quad= \|Au|L_2(\Omega)\|^2 + 2\lambda\langle A_0 u, u\rangle + 2\lambda\,\mathrm{Re}\langle A_1 u, u\rangle + \lambda^2\|u|L_2(\Omega)\|^2\\ &\quad\geq \|Au|L_2(\Omega)\|^2 + \lambda^2\|u|L_2(\Omega)\|^2 + 2\lambda\,\mathrm{Re}\langle A_1 u, u\rangle. \end{aligned}\tag{5.71}$$

Using Theorem 4.17 one can estimate the last term from above by

$$\begin{aligned}2\lambda|\langle A_1 u, u\rangle| &\leq c\ \|u|W_2^1(\Omega)\|\lambda\|u|L_2(\Omega)\|\\ &\leq \frac{\lambda^2}{2}\|u|L_2(\Omega)\|^2 + \varepsilon\|u|W_2^2(\Omega)\|^2 + c_\varepsilon\|u|L_2(\Omega)\|^2. \end{aligned}\tag{5.72}$$

We insert (5.72) in (5.71) (estimate from below), use (5.58) and choose $\varepsilon$ (independently of $\lambda$) sufficiently small. Hence,

$$\|(A + \lambda\,\mathrm{id})u|L_2(\Omega)\|^2 \geq\ c_1\|u|W_2^2(\Omega)\|^2 + \left(\frac{\lambda^2}{2} - c\right)\|u|L_2(\Omega)\|^2 \tag{5.73}$$

for some $c > 0$ independent of $\lambda > 0$. Choosing $\lambda \geq \lambda_0$ and $\lambda_0$ sufficiently large results in (5.66).

*Step* 2. The proof of (ii) is the same using now Theorem 5.11. □

**Exercise 5.15.** Let $\Omega$ and $A$ be as in Corollary 5.14 (i) and let $\nu^A$ be the co-normal according to (5.20). Prove the generalisation of (5.67),

$$\|(A+\lambda\,\mathrm{id})u|L_2(\Omega)\| \geq c_1\|u|W_2^2(\Omega)\| + c_2\lambda\|u|L_2(\Omega)\|, \quad u \in W_2^{2,\nu^A}(\Omega), \tag{5.74}$$

for some $c_1 > 0$, $c_2 > 0$, $\lambda_0 > 0$ and all $\lambda \geq \lambda_0$.

*Hint:* Take the generalisation of (5.59),

$$\|Au|L_2(\Omega)\| + \|u|L_2(\Omega)\| \sim \|u|W_2^2(\Omega)\|, \quad u \in W_2^{2,\mu}(\Omega), \tag{5.75}$$

for non-tangential $C^\infty$ vector fields $\mu$ on $\Gamma = \partial\Omega$ for granted and rely on Exercise 5.6.

## 5.4 Some properties of Sobolev spaces on $\mathbb{R}^n_+$

In the preceding Section 5.3 we always assumed that $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3, where we also explained what is meant by bounded $C^\ell$ domains, $\ell \in \mathbb{N}$. By the arguments given it is quite clear that it would be sufficient for the assertions in Section 5.3 to assume that $\Omega$ is a bounded $C^2$ domain (or $C^3$ domain in case of Neumann problems), not to speak about some minor technicalities. But this is not so interesting and will not be needed in the sequel. On the other hand, we reduced assertions for bounded $C^\infty$ domains via localisations and diffeomorphic maps to $\mathbb{R}^n$ and $\mathbb{R}^n_+$. This technique will also be of some use in what follows. For this purpose we first fix the $\mathbb{R}^n_+$ counterparts of the above key assertions complemented afterwards by some density and smoothness properties playing a crucial rôle in the sequel.

Let $C(\mathbb{R}^n_+)$ and $C^1(\mathbb{R}^n_+)$ be the spaces as introduced in Definition A.1 where again

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x = (x', x_n),\ x' \in \mathbb{R}^{n-1},\ x_n > 0\}, \tag{5.76}$$

for $n \in \mathbb{N}$. The spaces $W_2^\ell(\mathbb{R}^n_+)$ have the same meaning as in Theorem 3.41 and are assumed to be normed by (3.98). Traces must be understood as in (3.127), (3.128) with the obvious counterparts of Definition 4.30,

$$W_{2,0}^s(\mathbb{R}^n_+) = \{f \in W_2^s(\mathbb{R}^n_+) : \mathrm{tr}_\Gamma\, f = 0\}, \tag{5.77}$$

if $s > \frac{1}{2}$ and

$$W_2^{s,\nu}(\mathbb{R}^n_+) = \left\{ f \in W_2^s(\mathbb{R}^n_+) : \mathrm{tr}_\Gamma\, \frac{\partial f}{\partial \nu} = 0 \right\}, \tag{5.78}$$

if $s > \frac{3}{2}$, where we now assume that $\nu$ is the outer normal, hence

$$(\mathrm{tr}_\Gamma\, f)(x) = f(x', 0), \quad \left(\mathrm{tr}_\Gamma\, \frac{\partial f}{\partial \nu}\right)(x) = -\frac{\partial f}{\partial x_n}(x', 0). \tag{5.79}$$

We strengthen the $\mathbb{R}^n_+$ counterpart of Definition 5.1 by (5.65) with $\mathbb{R}^n_+$ in place of $\Omega$.

**Theorem 5.16.** (i) *Let $A$ be a second order elliptic differential expression in* $\mathbb{R}^n_+$,

$$(Au)(x) = -\sum_{j,k=1}^{n} a_{jk}(x)\,\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{l=1}^{n} a_l(x)\frac{\partial u}{\partial x_l}(x) + a(x)u(x), \quad (5.80)$$

*with*

$$a_{jk}(x) = a_{kj}(x) \in \mathbb{R}, \quad x \in \overline{\mathbb{R}^n_+},\ 1 \le j,k \le n, \quad (5.81)$$

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \ge E|\xi|^2, \quad x \in \overline{\mathbb{R}^n_+},\ \xi \in \mathbb{R}^n, \quad (5.82)$$

*for some $E > 0$ (ellipticity constant), and*

$$\|a_{jk}|C^1(\mathbb{R}^n_+)\| \le M, \quad \|a_l|C(\mathbb{R}^n_+)\| \le M, \quad \|a|C(\mathbb{R}^n_+)\| \le M \quad (5.83)$$

*for all admitted $j,k,l$ and some $M > 0$. Then*

$$\|Au|L_2(\mathbb{R}^n_+)\| + \|u|L_2(\mathbb{R}^n_+)\| \sim \|u|W^2_2(\mathbb{R}^n_+)\|, \quad u \in W^2_{2,0}(\mathbb{R}^n_+). \quad (5.84)$$

*Furthermore, there are positive constants $\lambda_0$, $c_1$ and $c_2$ depending only on $E$ and $M$ such that*

$$\|(A + \lambda\,\mathrm{id})u|L_2(\mathbb{R}^n_+)\| \ge c_1\|u|W^2_2(\mathbb{R}^n_+)\| + c_2\,\lambda\|u|L_2(\mathbb{R}^n_+)\| \quad (5.85)$$

*for all $u \in W^2_{2,0}(\mathbb{R}^n_+)$ and all $\lambda \in \mathbb{R}$ with $\lambda \ge \lambda_0$.*

(ii) *Let $A = -\Delta$ be the Laplacian according to* (5.5) *and let $\nu$ be the outer normal as in* (5.79)*. Then*

$$\|\Delta u|L_2(\mathbb{R}^n_+)\| + \|u|L_2(\mathbb{R}^n_+)\| \sim \|u|W^2_2(\mathbb{R}^n_+)\|, \quad u \in W^{2,\nu}_2(\mathbb{R}^n_+). \quad (5.86)$$

*Furthermore, there are positive constants $\lambda_0$, $c_1$ and $c_2$ depending only on $E$ and $M$ such that*

$$\|(-\Delta + \lambda\,\mathrm{id})u|L_2(\mathbb{R}^n_+)\| \ge c_1\|u|W^2_2(\mathbb{R}^n_+)\| + c_2\,\lambda\|u|L_2(\mathbb{R}^n_+)\| \quad (5.87)$$

*for all $u \in W^{2,\nu}_2(\mathbb{R}^n_+)$ and all $\lambda \in \mathbb{R}$ with $\lambda \ge \lambda_0$.*

*Proof.* As for part (i) one can follow the proof of Theorem 5.7 with a reference to Corollary 3.44 instead of Theorem 4.17 in connection with the counterpart of (5.35). Then one obtains (5.84) and also a counterpart of Corollary 5.9. Similarly one gets (5.85) as a modification of (5.66). Furthermore, part (ii) is the counterpart of Theorem 5.11 and Corollary 5.14 (ii). □

The theory of elliptic operators in $\mathbb{R}^n_+$ is in some aspects (but not all) parallel to the corresponding theory in bounded $C^\infty$ domains. But some technical instruments are more transparent in $\mathbb{R}^n_+$. This is the main reason for having a closer look at $\mathbb{R}^n_+$ in preparation of what follows. Recall (5.77), (5.78) and that $\mathcal{S}(\mathbb{R}^n_+) = \mathcal{S}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ as in Proposition 3.39.

**Proposition 5.17.** *Let* $f \in L_2(\mathbb{R}^n_+)$.

(i) *Let* $u \in W^1_{2,0}(\mathbb{R}^n_+)$ *and*

$$\int_{\mathbb{R}^n_+} \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)\frac{\partial \varphi}{\partial x_j}(x)\mathrm{d}x = \int_{\mathbb{R}^n_+} f(x)\varphi(x)\mathrm{d}x \tag{5.88}$$

*for all* $\varphi \in \mathcal{S}(\mathbb{R}^n_+)$ *with* $\operatorname{tr}_\Gamma \varphi = 0$. *Then* $u \in W^2_{2,0}(\mathbb{R}^n_+)$.

(ii) *Let* $u \in W^1_2(\mathbb{R}^n_+)$ *and*

$$\int_{\mathbb{R}^n_+} \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)\frac{\partial \varphi}{\partial x_j}(x)\mathrm{d}x = \int_{\mathbb{R}^n_+} f(x)\varphi(x)\mathrm{d}x \tag{5.89}$$

*for all* $\varphi \in \mathcal{S}(\mathbb{R}^n_+)$ *with* $\operatorname{tr}_\Gamma \frac{\partial \varphi}{\partial \nu} = 0$. *Then* $u \in W^{2,\nu}_2(\mathbb{R}^n_+)$.

*Proof. Step* 1. Let $x = (x', x_n) \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$. We insert

$$\varphi(x', x_n) = \psi(x', x_n) - \psi(x', -x_n) \quad \text{with } \psi \in \mathcal{D}(\mathbb{R}^n) \tag{5.90}$$

in (5.88). The arguments in the proof of Theorem 4.11 and Remark 4.33 imply for the odd extensions

$$\begin{aligned} U(x) &= \mathsf{O}\text{-}\operatorname{ext} u(x) = \begin{cases} u(x', x_n) & \text{if } x_n \ge 0,\\ -u(x', -x_n) & \text{if } x_n < 0, \end{cases} \\ F(x) &= \mathsf{O}\text{-}\operatorname{ext} f(x) = \begin{cases} f(x', x_n) & \text{if } x_n \ge 0,\\ -f(x', -x_n) & \text{if } x_n < 0, \end{cases} \end{aligned} \tag{5.91}$$

that $U \in W^1_2(\mathbb{R}^n)$, $F \in L_2(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} \sum_{j=1}^n \frac{\partial U}{\partial x_j}(x)\frac{\partial \psi}{\partial x_j}(x)\mathrm{d}x = \int_{\mathbb{R}^n} F(x)\psi(x)\mathrm{d}x, \quad \psi \in \mathcal{D}(\mathbb{R}^n). \tag{5.92}$$

For the justification of the transformation of the left-hand side of (5.88) into the left-hand side of (5.92) one may approximate $u$ by smooth functions using the $\mathbb{R}^n_+$ counterpart of Proposition 4.32 (i). However, one gets by (5.92) and (3.30), (3.31) that

$$U - \Delta U = F + U = G \in L_2(\mathbb{R}^n), \tag{5.93}$$

$$(1 + |\xi|^2)\widehat{U}(\xi) = \widehat{G}(\xi) \in L_2(\mathbb{R}^n), \tag{5.94}$$

and hence $U \in W_2^2(\mathbb{R}^n)$. This proves $u \in W_{2,0}^2(\mathbb{R}^n_+)$.

*Step* 2. We insert

$$\varphi(x', x_n) = \psi(x', x_n) + \psi(x', -x_n) \quad \text{with } \psi \in \mathcal{D}(\mathbb{R}^n) \tag{5.95}$$

in (5.89). Again by the arguments in the proof of Theorem 4.11 this leads for the even extensions

$$\begin{aligned} U(x) &= \text{E-ext}\, u(x) = \begin{cases} u(x', x_n) & \text{if } x_n \geq 0, \\ u(x', -x_n) & \text{if } x_n < 0, \end{cases} \\ F(x) &= \text{E-ext}\, f(x) = \begin{cases} f(x', x_n) & \text{if } x_n \geq 0, \\ f(x', -x_n) & \text{if } x_n < 0, \end{cases} \end{aligned} \tag{5.96}$$

to $U \in W_2^1(\mathbb{R}^n)$, $F \in L_2(\mathbb{R}^n)$, and (5.92). By Proposition 3.39 the set $\mathcal{S}(\mathbb{R}^n_+)$ is dense in $W_2^1(\mathbb{R}^n_+)$. This justifies by approximation the transformation of the left-hand side of (5.89) into the left-hand side of (5.92). Otherwise we get by the same arguments as in Step 1 that $U \in W_2^2(\mathbb{R}^n)$ and, hence, $u \in W_2^2(\mathbb{R}^n_+)$. Furthermore one obtains

$$\text{tr}_\Gamma \frac{\partial U}{\partial \nu} = \frac{\partial u}{\partial x_n}(x', 0) = -\frac{\partial u}{\partial x_n}(x', 0) = 0. \tag{5.97}$$

Hence $u \in W_2^{2,\nu}(\mathbb{R}^n_+)$. □

**Remark 5.18.** We wish to discuss the effect that the identity (5.89) for $u \in W_2^1(\mathbb{R}^n_+)$ does not only improve the smoothness properties, $u \in W_2^2(\mathbb{R}^n_+)$, but even ensures that $\frac{\partial u}{\partial x_n}(x', 0) = 0$ in the interpretation of (5.79) (or (5.97)). Let $\mathcal{D}(\overline{\mathbb{R}^n_+})$ be the restriction of $\mathcal{D}(\mathbb{R}^n)$ to $\mathbb{R}^n_+$ (denoted previously for arbitrary domains $\Omega$ by $\mathcal{D}(\mathbb{R}^n)_{|\Omega}$) and let

$$\mathcal{D}(\overline{\mathbb{R}^n_+})^\nu = \Big\{ f \in \mathcal{D}(\overline{\mathbb{R}^n_+}) : \frac{\partial f}{\partial x_n}(x', 0) = 0 \Big\}. \tag{5.98}$$

**Proposition 5.19.** (i) *The set $\mathcal{D}(\overline{\mathbb{R}^n_+})^\nu$ is dense in $W_2^1(\mathbb{R}^n_+)$.*

(ii) *The set $\mathcal{D}(\mathbb{R}^n_+)$ is dense both in $W_{2,0}^1(\mathbb{R}^n_+)$ (also denoted by $\mathring{W}_2^1(\mathbb{R}^n_+)$) and in*

$$\mathring{W}_2^2(\mathbb{R}^n_+) = \Big\{ f \in W_2^2(\mathbb{R}^n_+) : \text{tr}_\Gamma f = \text{tr}_\Gamma \frac{\partial f}{\partial \nu} = 0 \Big\}. \tag{5.99}$$

*Proof. Step* 1. We prove (i). According to Proposition 3.39 it is sufficient to approximate $f \in \mathcal{D}(\overline{\mathbb{R}^n_+})$ by functions belonging to (5.98). For $f \in \mathcal{D}(\overline{\mathbb{R}^n_+})$ and $\varepsilon > 0$ let

$$f^\varepsilon \in \mathcal{D}(\overline{\mathbb{R}^n_+}), \quad \text{supp}\, f^\varepsilon \subset \{x \in \overline{\mathbb{R}^n_+} : 0 \leq x_n \leq 2\varepsilon\} \tag{5.100}$$

and

$$f^\varepsilon(x', x_n) = f(x', 0)x_n \quad \text{if } 0 \le x_n \le \varepsilon \text{ and } x' \in \mathbb{R}^{n-1}. \tag{5.101}$$

There are functions of this type with

$$\|f^\varepsilon | W_2^1(\mathbb{R}^n_+)\| \le c\,\varepsilon \quad \text{for some } c > 0 \text{ and all } \varepsilon \text{ with } 0 < \varepsilon \le 1. \tag{5.102}$$

Then $f_\varepsilon = f - f^\varepsilon \in \mathcal{D}(\overline{\mathbb{R}^n_+})^\nu$ approximates $f$ in $W_2^1(\mathbb{R}^n_+)$.

*Step* 2. We prove (ii) for the spaces $\mathring{W}_2^2(\mathbb{R}^n_+)$. First remark that $f \in \mathring{W}_2^2(\mathbb{R}^n_+)$ can be approximated by functions

$$\varphi(x')\psi(x_n) f \in \mathring{W}_2^2(\mathbb{R}^n_+) \tag{5.103}$$

where $\varphi \in \mathcal{D}(\mathbb{R}^{n-1})$, $\psi \in \mathcal{D}(\mathbb{R})$ with

$$\varphi(x') = 1 \text{ if } |x'| \le c \quad \text{and} \quad \psi(x_n) = 1 \text{ if } |x_n| \le c. \tag{5.104}$$

Hence we may assume that $f \in \mathring{W}_2^2(\mathbb{R}^n_+)$ has compact support in $\overline{\mathbb{R}^n_+}$. Let $f_j \in \mathcal{D}(\overline{\mathbb{R}^n_+})$ be an approximating sequence of $f$ in $W_2^2(\mathbb{R}^n_+)$ if $j \to \infty$. Then it follows by Theorem 4.24 and its proof that for $0 \le |\alpha| \le 1$,

$$(\mathrm{D}^\alpha f_j)(x', 0) \to \mathrm{tr}_\Gamma\, \mathrm{D}^\alpha f = 0 \quad \text{in } W_2^{\frac{3}{2}-|\alpha|}(\mathbb{R}^{n-1}) \text{ if } j \to \infty. \tag{5.105}$$

This is immediately covered by Theorem 4.24 if $\alpha = 0$ or if $\mathrm{D}^\alpha = -\frac{\partial}{\partial \nu}$ is the normal derivative. As for tangential derivatives $\mathrm{D}^\alpha = \frac{\partial}{\partial x_k}$, $k = 1, \ldots, n-1$, we have

$$\mathrm{tr}_\Gamma \frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k} \mathrm{tr}_\Gamma, \quad k = 1, \ldots, n-1. \tag{5.106}$$

This is obvious if applied to smooth functions and it follows in general from our definition of traces as limits of traces of smooth functions as indicated several times, for example in (4.30)–(4.32) (extended to target spaces of type $W_2^s(\Gamma)$). Next we use (5.105) for the approximating sequence $f_j \to f \in \mathring{W}_2^2(\mathbb{R}^n_+)$ in $W_2^2(\mathbb{R}^n_+)$ to prove that for $|\alpha| \le 2$,

$$\int_{\mathbb{R}^n_+} (\mathrm{D}^\alpha f)(x)\varphi(x)\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n_+} f(x)(\mathrm{D}^\alpha \varphi)(x)\mathrm{d}x, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{5.107}$$

We use integration by parts for $f_j \in \mathcal{D}(\overline{\mathbb{R}^n_+})$ and get

$$\int_{\mathbb{R}^n_+} (\mathrm{D}^\alpha f_j)(x)\varphi(x)\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n_+} f_j(x)(\mathrm{D}^\alpha \varphi)(x)\mathrm{d}x + \int_{\mathbb{R}^{n-1}} \cdots, \tag{5.108}$$

with at most first order boundary terms for $f_j$. Then (5.107) follows from (5.108), (5.105) and $j \to \infty$. We extend $f$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ by zero. Then one obtains by (5.107) that

$$\operatorname{ext} f(x) = \begin{cases} f(x) & \text{if } x_n > 0, \\ 0 & \text{if } x_n < 0, \end{cases} \tag{5.109}$$

is a linear and bounded extension operator from $\mathring{W}^2_2(\mathbb{R}^n_+)$ into $W^2_2(\mathbb{R}^n)$. If $F \in W^2_2(\mathbb{R}^n)$ and

$$F_h(x) = F(x+h), \text{ then } F_h \to F \text{ in } W^2_2(\mathbb{R}^n) \text{ for } h \to 0. \tag{5.110}$$

This is clear for smooth functions, say, $F \in \mathcal{D}(\mathbb{R}^n)$, and follows for arbitrary $F \in W^2_2(\mathbb{R}^n)$ by approximation. We apply this observation to $F = \operatorname{ext} f$ in (5.109) and $h = (0, h_n)$ with $h_n < 0$ and $h_n \to 0$. This proves that $F$ in $W^2_2(\mathbb{R}^n)$ and, as a consequence, $f \in \mathring{W}^2_2(\mathbb{R}^n_+)$ in $\mathbb{R}^n_+$ can be approximated by $f_h$ having a compact support in $\mathbb{R}^n_+$. The rest is now a matter of mollification as detailed in (3.58), (3.61). Hence $\mathcal{D}(\mathbb{R}^n_+)$ is dense in $\mathring{W}^2_2(\mathbb{R}^n_+)$. But the above arguments also show that $\mathcal{D}(\mathbb{R}^n_+)$ is dense in $\mathring{W}^1_2(\mathbb{R}^n_+)$. □

**Exercise* 5.20.** Construct explicitly functions $f^\varepsilon$ with (5.100)–(5.102).

**Exercise 5.21.** Let $k \in \mathbb{N}$. Prove that $\mathcal{D}(\mathbb{R}^n_+)$ is dense in

$$\mathring{W}^k_2(\mathbb{R}^n_+) = \left\{ f \in W^k_2(\mathbb{R}^n_+) : \operatorname{tr}_\Gamma \frac{\partial^l f}{\partial \nu^l} = 0 \text{ for } l = 0, \ldots, k-1 \right\}. \tag{5.111}$$

*Hint:* Use Exercise 4.27. One may also consult Note 5.12.2 for a more general result.

## 5.5 The Laplacian

We are mainly interested in boundary value problems for second order elliptic differential equations in bounded $C^\infty$ domains in $\mathbb{R}^n$ as described in Definition 5.3. The first candidate is the Laplacian,

$$Au = -\Delta u = -\sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2}. \tag{5.112}$$

Usually we reduced questions for Sobolev spaces and elliptic equations in bounded $C^\infty$ domains to corresponding problems on the half-space $\mathbb{R}^n_+$. The same will be done in what follows. This may justify dealing with boundary value problems for the Laplacian first, both in $\mathbb{R}^n_+$ and in bounded $C^\infty$ domains. We rely now on the theory of self-adjoint operators in Hilbert spaces $H = L_2(\Omega)$. We refer to Appendix C where we collected what we need now. We use the notation introduced there.

The *Dirichlet Laplacian* in the Hilbert spaces $H = L_2(\mathbb{R}^n_+)$ is defined by

$$A_{\mathsf{D}} u = -\Delta u \quad \text{with } \operatorname{dom}(A_{\mathsf{D}}) = \mathcal{D}(\mathbb{R}^n_+), \tag{5.113}$$

and the *Neumann Laplacian* by

$$A_{\mathsf{N}} u = -\Delta u \quad \text{with } \operatorname{dom}(A_{\mathsf{N}}) = \mathcal{D}(\overline{\mathbb{R}^n_+})^{\nu}, \tag{5.114}$$

where $\mathcal{D}(\overline{\mathbb{R}^n_+})^{\nu}$ is given by (5.98). Integration by parts implies that

$$\langle A_{\mathsf{D}} u, v\rangle = \int_{\mathbb{R}^n_+} (-\Delta u)(x)\overline{v(x)}\mathrm{d}x = \int_{\mathbb{R}^n_+} \sum_{k=1}^n \frac{\partial u}{\partial x_k}(x)\frac{\partial \overline{v}}{\partial x_k}(x)\mathrm{d}x \tag{5.115}$$

for $u \in \operatorname{dom}(A_{\mathsf{D}})$, $v \in \operatorname{dom}(A_{\mathsf{D}})$, and similarly,

$$\langle A_{\mathsf{N}} u, v\rangle = \int_{\mathbb{R}^n_+} \sum_{k=1}^n \frac{\partial u}{\partial x_k}(x)\frac{\partial \overline{v}}{\partial x_k}(x)\mathrm{d}x \tag{5.116}$$

for $u \in \operatorname{dom}(A_{\mathsf{N}})$, $v \in \operatorname{dom}(A_{\mathsf{N}})$. One may consult Section A.3. In particular,

$$\langle A_{\mathsf{D}} u, u\rangle \geq 0,\ u \in \operatorname{dom}(A_{\mathsf{D}}) \quad \text{and} \quad \langle A_{\mathsf{N}} u, u\rangle \geq 0,\ u \in \operatorname{dom}(A_{\mathsf{N}}), \tag{5.117}$$

and hence both $A_{\mathsf{D}}$ and $A_{\mathsf{N}}$ are symmetric positive operators in $L_2(\mathbb{R}^n)$ according to Definition C.9. If $\varepsilon > 0$, then both $A_{\mathsf{D}} + \varepsilon\,\mathrm{id}$ and $A_{\mathsf{N}} + \varepsilon\,\mathrm{id}$ are positive-definite. We choose $\varepsilon = 1$ and abbreviate for convenience,

$$A^{\mathsf{D}} = A_{\mathsf{D}} + \mathrm{id} \quad \text{and} \quad A^{\mathsf{N}} = A_{\mathsf{N}} + \mathrm{id}. \tag{5.118}$$

Let $A^{\mathsf{D}}_F$ and $A^{\mathsf{N}}_F$ be the respective self-adjoint Friedrichs extensions according to Theorem C.13 with spectra $\sigma(A^{\mathsf{D}}_F) \subset [1,\infty)$ and $\sigma(A^{\mathsf{N}}_F) \subset [1,\infty)$. In particular,

$$A^{\mathsf{D}}_F : \operatorname{dom}(A^{\mathsf{D}}_F) \rightleftarrows L_2(\mathbb{R}^n_+), \quad A^{\mathsf{N}}_F : \operatorname{dom}(A^{\mathsf{N}}_F) \rightleftarrows L_2(\mathbb{R}^n_+) \tag{5.119}$$

are one-to-one mappings, the corresponding inverse operators exist and belong to $\mathcal{L}(L_2(\mathbb{R}^n_+))$. Let $W^1_{2,0}(\mathbb{R}^n_+) = \mathring{W}^1_2(\mathbb{R}^n_+)$, $W^2_{2,0}(\mathbb{R}^n_+)$, and $W^{2,\nu}_2(\mathbb{R}^n_+)$ be as in (5.77), (5.78) and Proposition 5.19.

**Theorem 5.22.** *Let $A^{\mathsf{D}}$ and $A^{\mathsf{N}}$ be the above operators in the Hilbert space $L_2(\mathbb{R}^n_+)$. Then one has*

$$H_{A^{\mathsf{D}}_F} = \mathring{W}^1_2(\mathbb{R}^n_+) \quad \textit{and} \quad H_{A^{\mathsf{N}}_F} = W^1_2(\mathbb{R}^n_+) \tag{5.120}$$

*for the corresponding energy spaces, and*

$$A_F^{\mathsf{D}} u = -\Delta u + u, \quad \operatorname{dom}(A_F^{\mathsf{D}}) = W_{2,0}^2(\mathbb{R}_+^n), \tag{5.121}$$

*and*

$$A_F^{\mathsf{N}} u = -\Delta u + u, \quad \operatorname{dom}(A_F^{\mathsf{N}}) = W_2^{2,\nu}(\mathbb{R}_+^n). \tag{5.122}$$

*Proof. Step* 1. The density assertions in Proposition 5.19 and (C.36) imply (5.120).

*Step* 2. In view of the $\mathbb{R}_+^n$ counterpart of Proposition 4.32 (i) any $u \in W_{2,0}^2(\mathbb{R}_+^n)$ can be approximated in $W_2^2(\mathbb{R}_+^n)$ by functions belonging to

$$\{f \in \mathcal{D}(\overline{\mathbb{R}_+^n}) : f(x', 0) = 0\}. \tag{5.123}$$

Then integration by parts and approximation imply

$$\begin{aligned} \big\langle(-\Delta + \mathrm{id})u, v\big\rangle &= \int\limits_{\mathbb{R}_+^n} \Big(\sum_{k=1}^n \frac{\partial u}{\partial x_k}\frac{\partial \overline{v}}{\partial x_k}(x) + u(x)\overline{v(x)}\Big)\mathrm{d}x \\ &= \big\langle u, (-\Delta + \mathrm{id})v\big\rangle \end{aligned} \tag{5.124}$$

for $u \in W_{2,0}^2(\mathbb{R}_+^n)$ and $v \in W_{2,0}^2(\mathbb{R}_+^n)$. In particular, Theorem C.13 leads to

$$W_{2,0}^2(\mathbb{R}_+^n) \subset \operatorname{dom}\big((A^{\mathsf{D}})^*\big) \cap H_{A^{\mathsf{D}}} = \operatorname{dom}(A_F^{\mathsf{D}}). \tag{5.125}$$

Let $u \in \operatorname{dom}(A_F^{\mathsf{D}})$. Then $u \in W_{2,0}^1(\mathbb{R}_+^n)$ and $A_F^{\mathsf{D}} u = g \in L_2(\mathbb{R}_+^n)$. In view of Remark C.14 one obtains for any $\varphi \in \mathcal{S}(\mathbb{R}_+^n)$, $\operatorname{tr}_\Gamma \varphi = 0$, and with $f = g - u \in L_2(\mathbb{R}_+^n)$,

$$\int\limits_{\mathbb{R}_+^n} \sum_{k=1}^n \frac{\partial u}{\partial x_k}(x)\frac{\partial \varphi}{\partial x_k}(x)\mathrm{d}x = \big\langle A_F^{\mathsf{D}} u - u, \overline{\varphi}\big\rangle = \int\limits_{\mathbb{R}_+^n} f(x)\varphi(x)\mathrm{d}x. \tag{5.126}$$

Now one obtains by Proposition 5.17 (i) that $u \in W_{2,0}^2(\mathbb{R}_+^n)$. This is the converse of (5.125). Hence we have (5.121).

*Step* 3. The proof of (5.122) follows the same line of arguments using the $\mathbb{R}_+^n$ counterpart of Proposition 4.32 (ii). It follows that $\mathcal{D}(\overline{\mathbb{R}_+^n})^\nu$ according to (5.98) is dense in $W_2^{2,\nu}(\mathbb{R}_+^n)$ and one gets a counterpart of (5.124) resulting in

$$W_2^{2,\nu}(\mathbb{R}_+^n) \subset \operatorname{dom}\big((A^{\mathsf{N}})^*\big) \cap H_{A^{\mathsf{N}}} = \operatorname{dom}(A_F^{\mathsf{N}}). \tag{5.127}$$

We have (5.126) with $u \in \operatorname{dom}(A_F^{\mathsf{N}}) \subset W_2^1(\mathbb{R}_+^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}_+^n)$, $\operatorname{tr}_\Gamma \frac{\partial \varphi}{\partial \nu} = 0$. Then one obtains by Proposition 5.17 (ii) that $u \in W_2^{2,\nu}(\mathbb{R}_+^n)$ which completes the proof. □

**Remark 5.23.** Although $H_{A^{\mathsf{D}}} = H_{A_F^{\mathsf{D}}} = \mathring{W}_2^1(\mathbb{R}^n_+)$ is a genuine subspace of $H_{A^{\mathsf{N}}} = H_{A_F^{\mathsf{N}}} = W_2^1(\mathbb{R}^n_+)$ a corresponding assertion for $\mathrm{dom}(A_F^{\mathsf{D}})$ and $\mathrm{dom}(A_F^{\mathsf{N}})$ cannot be valid. We refer to Note 5.12.3, too.

**Remark 5.24.** As remarked above one has $\sigma(A_F^{\mathsf{D}}) \subset [1,\infty)$ and $\sigma(A_F^{\mathsf{N}}) \subset [1,\infty)$ for the spectra of $A_F^{\mathsf{D}}$ and $A_F^{\mathsf{N}}$. It turns out that

$$\sigma(A_F^{\mathsf{D}}) = \sigma(A_F^{\mathsf{N}}) = [1,\infty) \tag{5.128}$$

and

$$\sigma_{\mathsf{p}}(A_F^{\mathsf{D}}) = \sigma_{\mathsf{p}}(A_F^{\mathsf{N}}) = \emptyset, \tag{5.129}$$

where $\sigma_{\mathsf{p}}(A_F^{\mathsf{D}})$ and $\sigma_{\mathsf{p}}(A_F^{\mathsf{N}})$ are the corresponding point spectra (collection of eigenvalues, see Section C.1, (C.9)). Hence neither $A_F^{\mathsf{D}}$ nor $A_F^{\mathsf{N}}$ possesses eigenvalues.

**Exercise 5.25.** Justify (5.129).

*Hint:* Prove first that $A$, given by

$$Au = (-\Delta + \mathrm{id})u, \quad \mathrm{dom}(A) = W_2^2(\mathbb{R}^n), \tag{5.130}$$

is a positive-definite self-adjoint operator in $L_2(\mathbb{R}^n)$ and that $\sigma_{\mathsf{p}}(A) = \emptyset$. Use the Fourier transform. Reduce (5.129) to this case relying on the above technique of odd and even extensions. We refer also to Note 5.12.4.

Next we deal with the counterparts of (5.113), (5.114) and Theorem 5.22 in bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$ as introduced in Definition A.3. First we need the $\Omega$-versions of the Propositions 5.17 and 5.19. The spaces $W_{2,0}^1(\Omega)$, $W_{2,0}^2(\Omega)$ and $W_2^{2,\nu}(\Omega)$ with the $C^\infty$ vector field $\nu$ of outer normals have the same meaning as in Definition 4.30. We use $C^\infty(\Omega)$ as in (A.9).

**Proposition 5.26.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $f \in L_2(\Omega)$.*

(i) *Let $u \in W_{2,0}^1(\Omega)$ and*

$$\int_\Omega \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x) \frac{\partial \varphi}{\partial x_j}(x)\mathrm{d}x = \int_\Omega f(x)\varphi(x)\mathrm{d}x \tag{5.131}$$

*for all $\varphi \in C^\infty(\Omega)$ with $\mathrm{tr}_\Gamma\, \varphi = 0$. Then $u \in W_{2,0}^2(\Omega)$.*

(ii) *Let $u \in W_2^1(\Omega)$ and*

$$\int_\Omega \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x) \frac{\partial \varphi}{\partial x_j}(x)\mathrm{d}x = \int_\Omega f(x)\varphi(x)\mathrm{d}x \tag{5.132}$$

*for all $\varphi \in C^\infty(\Omega)$ with $\mathrm{tr}_\Gamma\, \dfrac{\partial \varphi}{\partial \nu} = 0$. Then $u \in W_2^{2,\nu}(\Omega)$.*

*Proof.* As indicated in Figure 5.3 below we furnish a neighbourhood of $\gamma_0 \in \Gamma = \partial\Omega$ with orthogonal curvilinear coordinates

$$y_j = h_j(x), \quad j = 1, \dots, n, \tag{5.133}$$

such that $y_n$ points in the direction of the outer normal $\nu$, hence $y_n = 0$ refers locally to $\Gamma = \partial\Omega$ and the level sets $y_n = c$ are parallel to $\Gamma$, where $|c|$ is small.

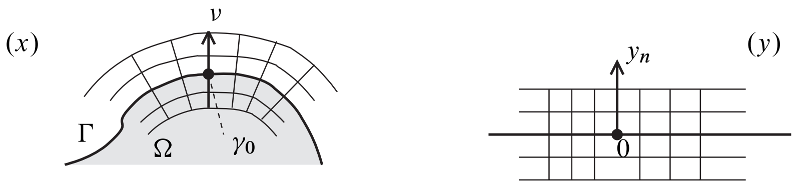



Figure 5.3

Let

$$\boldsymbol{J} = \left( \left( \frac{\partial h_j}{\partial x_k} \right)_{j,k=1}^{n} \right) \quad \text{and} \quad v(y) = u(x),\ \psi(y) = \varphi(x),\ g(y) = f(x), \tag{5.134}$$

where $u$, $\varphi$ and $f$ are as in the proposition. Let $\boldsymbol{J}^*$ be the adjoint matrix of the Jacobian $\boldsymbol{J}$. The integrands on the left-hand sides of (5.131), (5.132) can be written as the scalar product $\langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle$ of the related gradients. In [Tri92a, Sections 6.3.2, 6.3.3, pp. 373–378] we discussed in detail the impact of orthogonal curvilinear coordinates on gradients and the Laplacian. We may assume additionally that $\boldsymbol{J}^* \boldsymbol{J}$ is the unit matrix and that $\det \boldsymbol{J} = 1$ for the Jacobian determinant. Then

$$\langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle = \langle \boldsymbol{J}^* \boldsymbol{J} \operatorname{grad} v, \operatorname{grad} \psi \rangle = \langle \operatorname{grad} v, \operatorname{grad} \psi \rangle \tag{5.135}$$

and also

$$\Delta u(x) = \Delta v(y), \quad \operatorname{tr}_\Gamma \frac{\partial \varphi}{\partial \nu} = \frac{\partial \psi}{\partial y_n}(y', 0) = 0. \tag{5.136}$$

(We shall not use directly that the Laplacian is preserved, but it illuminates what happens.) Restricting (5.131), (5.132) to a neighbourhood of a point $\gamma_0 \in \Gamma$ which corresponds to $y = 0$ in the curvilinear coordinates, now interpreted as Cartesian coordinates, then one obtains

$$\int_{\mathbb{R}^n_+} \sum_{j=1}^{n} \frac{\partial v}{\partial y_j}(y) \frac{\partial \psi}{\partial y_j}(y) \mathrm{d}y = \int_{\mathbb{R}^n_+} g(y) \psi(y) \mathrm{d}y \tag{5.137}$$

for $\psi \in \mathcal{S}(\mathbb{R}^n_+)$ with $\psi(y) = 0$ if $|y| > 1$, and either $\psi(y', 0) = 0$ or $\frac{\partial \psi}{\partial y_n}(y', 0) = 0$, respectively. It follows from Proposition 5.17 and its proof with $U$ and $F$ replaced by $V$ and $G$, respectively, that

$$-\Delta V = G \in L_2(K) \quad \text{with, say, } K = \{ y \in \mathbb{R}^n : |y| < 1 \}, \tag{5.138}$$

where $V \in W_2^1(K)$ as a counterpart of (5.93). Let $\chi \in \mathcal{D}(K)$, then

$$-\Delta(\chi V) \in L_2(\mathbb{R}^n), \quad \text{and, hence, } \chi V \in W_2^2(\mathbb{R}^n), \tag{5.139}$$

in the same way as in (5.94). This proves $v \in W_2^2$ near $\Gamma$ and $u \in W_2^2(\Omega)$. Furthermore, $u \in W_2^{2,\nu}(\Omega)$ in case of part (ii) of the proposition. □

**Remark 5.27.** The question arises whether one can always find orthogonal curvilinear coordinates (5.133) with the desired properties.

If $n = 2$, then this can be done even globally in a neighbourhood of $\Gamma$. But otherwise it is a matter of trajectories of the vector field of normals on a sequence of, say, $(n-1)$-dimensional $C^\infty$ surfaces $F(x', \lambda) = 0$ for a parameter $\lambda$ as indicated in Figure 5.4. The generated trajectories are orthogonal to the surfaces $F(x', \lambda) = 0$ in $\mathbb{R}^n$. Afterwards one repeats this procedure on a fixed surface $F(x', \lambda) = 0$, and so on.

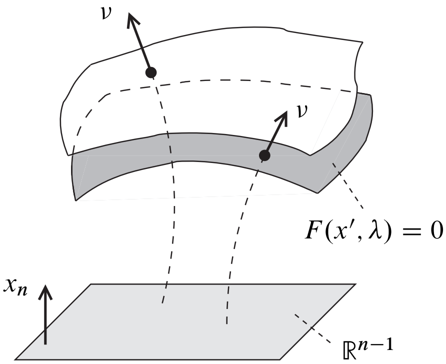



Figure 5.4

Next we deal with the $\Omega$ counterpart of Proposition 5.19 and a crucial inequality. Recall that bounded $C^\infty$ domains in $\mathbb{R}^n$ according to Definition A.3 are connected. Furthermore $C^\infty(\Omega)$ was introduced in (A.9).

**Proposition 5.28.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$. Let $\nu$ be the $C^\infty$ vector field of outer normals.*

(i) *The set*

$$C^\infty(\Omega)^\nu = \left\{ f \in C^\infty(\Omega) : \mathrm{tr}_\Gamma \frac{\partial f}{\partial \nu} = 0 \right\} \tag{5.140}$$

*is dense in $W_2^1(\Omega)$.*

(ii) *The set $\mathcal{D}(\Omega)$ is dense both in $W_{2,0}^1(\Omega)$ (also denoted as $\mathring{W}_2^1(\Omega)$) and in*

$$\mathring{W}_2^2(\Omega) = \left\{ f \in W_2^2(\Omega) : \mathrm{tr}_\Gamma f = \mathrm{tr}_\Gamma \frac{\partial f}{\partial \nu} = 0 \right\}. \tag{5.141}$$

(iii) (Friedrichs's inequality) *There is a number $c > 0$ such that*

$$\| f \,|L_2(\Omega)\| \le c \Big( \int_\Omega \sum_{k=1}^n \Big| \frac{\partial f}{\partial x_k}(x) \Big|^2 \mathrm{d}x \Big)^{1/2} \quad \textit{for } f \in \mathring{W}_2^1(\Omega). \tag{5.142}$$

*Proof. Step* 1. As for the proof of the parts (i) and (ii) one can follow the proof of Proposition 5.19. By Proposition 3.39 it is sufficient to approximate $f \in C^\infty(\bar{\Omega})$ in $W_2^1(\Omega)$ by functions belonging to (5.140). But this can be done in obvious modification of (5.101), (5.102). This proves (i). Concerning part (ii) one can follow Step 2 of the proof of Proposition 5.19.

*Step* 2. We prove part (iii) by contradiction and assume that there is no such $c$ with (5.142). Then there exists a sequence of functions $\{f_j\}_{j=1}^\infty \subset \mathring{W}_2^1(\Omega)$ such that

$$1 = \| f_j \,|L_2(\Omega)\| > j \Big( \int_\Omega \sum_{k=1}^n \Big| \frac{\partial f_j}{\partial x_k}(x) \Big|^2 \mathrm{d}x \Big)^{1/2}. \tag{5.143}$$

In particular, the sequence $\{f_j\}_{j=1}^\infty$ is bounded in $W_2^1(\Omega)$ and hence Theorem 4.17 implies that it is precompact in $L_2(\Omega)$. We may assume that

$$f_j \to f \text{ in } L_2(\Omega) \quad \text{with } \| f \,|L_2(\Omega)\| = 1. \tag{5.144}$$

By (5.143) the sequence $\{f_j\}_{j=1}^\infty$ converges also in $\mathring{W}_2^1(\Omega)$ (to the same $f$) and one obtains

$$\frac{\partial f}{\partial x_k}(x) = 0, \quad k = 1, \dots, n, \; f \in \mathring{W}_2^1(\Omega). \tag{5.145}$$

In particular, $\Delta f = 0$ in $\Omega$. If $\varphi \in \mathcal{D}(\Omega)$, then

$$\Delta(\varphi f) \in L_2(\Omega) \quad \text{and, hence,} \quad \varphi f \in W_2^2(\Omega) \tag{5.146}$$

as in (5.139). Assuming $\varphi f \in W_2^l(\Omega)$ for some $l \in \mathbb{N}$, $l \ge 2$, then one gets

$$\Delta(\varphi f) \in W_2^{l-1}(\Omega) \quad \text{and, hence,} \quad \varphi f \in W_2^{l+1}(\Omega). \tag{5.147}$$

Consequently, iteration and the embedding (4.87) imply that $f$ is a $C^\infty$ function on $\Omega$ with (5.144) and (5.145). Since $\Omega$ is connected it follows $f = |\Omega|^{-1/2}$. However, this contradicts $\mathrm{tr}_\Gamma f = 0$. □

**Remark 5.29.** Note that

$$\mathrm{ext}\, f(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^c = \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5.148}$$

is a linear and bounded extension operator from $\mathring{W}_2^1(\Omega)$ into $W_2^1(\mathbb{R}^n)$ and from $\mathring{W}_2^2(\Omega)$ into $W_2^2(\mathbb{R}^n)$.

**Exercise 5.30.** Let $k \in \mathbb{N}$. Prove that $\mathcal{D}(\Omega)$ is dense in

$$\mathring{W}_2^k(\Omega) = \left\{ f \in W_2^k(\Omega) : \operatorname{tr}_\Gamma \frac{\partial^l f}{\partial \nu^l} = 0 \text{ for } l = 0, \dots, k-1 \right\}, \tag{5.149}$$

and that ext $f$ in (5.148) is an extension operator from $\mathring{W}_2^k(\Omega)$ into $W_2^k(\mathbb{R}^n)$.
*Hint:* Use Exercise 4.27, compare it with Exercise 5.21.

After these preparations we come now to the counterpart of Theorem 5.22 for bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$ which may be considered as the main result of this Section 5.5. As before, $-\Delta$ in $\mathbb{R}^n$ is the Laplacian according to (5.112) and $\nu$ denotes the $C^\infty$ vector field of outer normals on $\Gamma = \partial\Omega$.

The *Dirichlet Laplacian* in the Hilbert space $H = L_2(\Omega)$ is given by $A_{\mathsf{D}}$,

$$A_{\mathsf{D}} u = -\Delta u \quad \text{with } \operatorname{dom}(A_{\mathsf{D}}) = \mathcal{D}(\Omega), \tag{5.150}$$

and the *Neumann Laplacian* by $A_{\mathsf{N}}$,

$$A_{\mathsf{N}} u = -\Delta u \quad \text{with } \operatorname{dom}(A_{\mathsf{N}}) = C^\infty(\Omega)^\nu, \tag{5.151}$$

where $C^\infty(\Omega)^\nu$ is defined in (5.140). As in (5.115) and in (5.116) one concludes for the scalar product $\langle \cdot, \cdot \rangle_\Omega$ in $L_2(\Omega)$ that

$$\langle A_{\mathsf{D}} u, v \rangle_\Omega = \int_\Omega (-\Delta u)(x) \overline{v(x)} \mathrm{d}x = \int_\Omega \sum_{k=1}^n \frac{\partial u}{\partial x_k}(x) \frac{\partial \overline{v}}{\partial x_k}(x) \mathrm{d}x \tag{5.152}$$

for $u \in \operatorname{dom}(A_{\mathsf{D}})$, $v \in \operatorname{dom}(A_{\mathsf{D}})$, and analogously,

$$\langle A_{\mathsf{N}} u, v \rangle_\Omega = \int_\Omega \sum_{k=1}^n \frac{\partial u}{\partial x_k}(x) \frac{\partial \overline{v}}{\partial x_k}(x) \mathrm{d}x. \tag{5.153}$$

This follows again by integration by parts according to Theorem A.7. In particular,

$$\langle A_{\mathsf{D}} u, u \rangle_\Omega \geq 0, \; u \in \operatorname{dom}(A_{\mathsf{D}}), \quad \text{and} \quad \langle A_{\mathsf{N}} u, u \rangle_\Omega \geq 0, \; u \in \operatorname{dom}(A_{\mathsf{N}}). \tag{5.154}$$

Hence both $A_{\mathsf{D}}$ and $A_{\mathsf{N}}$ are symmetric positive operators in $L_2(\Omega)$ according to Definition C.9. But compared with the corresponding operators in $\mathbb{R}^n_+$ resulting in Theorem 5.22 there are now some remarkable differences and the shifting (5.118) is no longer of any use. Otherwise we rely again on the notation and assertions in Appendix C. Let in particular $A_{\mathsf{D},F}$ and $A_{\mathsf{N},F}$ be the corresponding self-adjoint Friedrichs extensions according to Theorem C.13 and Remark C.17. Then $A_{\mathsf{D},F}$ and $A_{\mathsf{N},F}$ are positive operators in the understanding of Definition C.9 and one finds (at least) for their spectra,

$$\sigma(A_{\mathsf{D},F}) \subset [0, \infty) \quad \text{and} \quad \sigma(A_{\mathsf{N},F}) \subset [0, \infty). \tag{5.155}$$

Let $W^1_{2,0}(\Omega) = \mathring{W}^1_2(\Omega)$, $W^2_{2,0}(\Omega)$, and $W^{2,\nu}_2(\Omega)$ be as in Definition 4.30 and Proposition 5.28. Recall that $\lambda \in \mathbb{R}$ is called a *simple eigenvalue* of the self-adjoint operator $A$ if

$$\dim \ker(A - \lambda \, \mathrm{id}) = 1. \tag{5.156}$$

**Theorem 5.31.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3 and let $\nu$ be the $C^\infty$ vector field of outer normals.*

(i) *Let $A_{\mathrm{D},F}$ be Friedrichs extension of the Dirichlet Laplacian (5.150). Then $A_{\mathrm{D},F}$ is a self-adjoint positive-definite operator with pure point spectrum according to Definition C.7. Furthermore,*

$$H_{A_{\mathrm{D},F}} = \mathring{W}^1_2(\Omega), \tag{5.157}$$

$$A_{\mathrm{D},F}u = -\Delta u \quad \textit{with} \ \mathrm{dom}(A_{\mathrm{D},F}) = W^2_{2,0}(\Omega), \tag{5.158}$$

*and*

$$\sigma(A_{\mathrm{D},F}) \subset [c, \infty) \tag{5.159}$$

*with the same constant $c > 0$ as in (5.142).*

(ii) *Let $A_{\mathrm{N},F}$ be Friedrichs extension of the Neumann Laplacian (5.151). Then $A_{\mathrm{N},F}$ is a self-adjoint positive operator with pure point spectrum. Furthermore,*

$$H_{A_{\mathrm{N},F}} = W^1_2(\Omega), \tag{5.160}$$

$$A_{\mathrm{N},F}u = -\Delta u \quad \textit{with} \ \mathrm{dom}(A_{\mathrm{N},F}) = W^{2,\nu}_2(\Omega), \tag{5.161}$$

*and*

$$\sigma(A_{\mathrm{N},F}) \subset [0, \infty) \tag{5.162}$$

*where $0$ is a simple eigenvalue with the constant functions $u(x) = c \neq 0$ as the related eigenfunctions.*

*Proof. Step* 1. We conclude from (5.152) and (5.142) that $A_{\mathrm{D}}$ is positive-definite. Then (C.36) and Proposition 5.28 (ii) prove both (5.157) and (5.159) for the Dirichlet Laplacian. As for the Neumann Laplacian one obtains the corresponding assertions (5.160), (5.162) from (5.153), (C.36) (combined with Remark C.17) and Proposition 5.28 (i).

*Step* 2. As for (5.158) and (5.161) one can argue in the same way as in the Steps 2 and 3 of the proof of Theorem 5.22 relying on the one hand on the density assertions in Proposition 4.32, and on the other hand on Proposition 5.26.

*Step* 3. Theorem 4.17 and (5.157), (5.160) imply that the embeddings

$$\mathrm{id}\colon H_{A_{\mathrm{D},F}} \hookrightarrow L_2(\Omega), \quad \mathrm{id}\colon H_{A_{\mathrm{N},F}} \hookrightarrow L_2(\Omega), \tag{5.163}$$

are compact. Then it follows from Theorem C.15 that $A_{\mathsf{D},F}$, $A_{\mathsf{N},F} + \mathrm{id}$, and hence $A_{\mathsf{N},F}$ are operators with pure point spectrum. It remains to clarify what happens at $\lambda = 0$ in case of the Neumann Laplacian. Of course, $u(x) = c$ for $x \in \Omega$ belongs to $\mathrm{dom}(A_{\mathsf{N},F})$ and $\Delta u(x) = 0$. Hence 0 is an eigenvalue and we must prove that any eigenfunction is constant. Let $u$ be an eigenfunction for the eigenvalue 0. Then

$$0 = \langle -\Delta u, u \rangle_\Omega = \int_\Omega \sum_{j=1}^n \left| \frac{\partial u}{\partial x_j}(x) \right|^2 \mathrm{d}x. \tag{5.164}$$

Hence $\frac{\partial u}{\partial x_j}(x) = 0$ and one obtains by the same arguments as at the end of Step 2 of the proof of Proposition 5.28 that $u$ is constant. Since $\Omega$ is connected it follows that the eigenvalue 0 is simple and that the constant functions are the only eigenfunctions. □

**Remark 5.32.** The energy spaces and the domains of definition for the Dirichlet Laplacian and the Neumann Laplacian on $\mathbb{R}^n_+$ according to Theorem 5.22 on the one hand, and the corresponding energy spaces and domains of definitions for the Dirichlet Laplacian and the Neumann Laplacian on a bounded $C^\infty$ domain $\Omega$ as described in the above theorem on the other hand are similar. But otherwise there are some striking differences. As mentioned in (5.129), the operators $A^{\mathsf{D}}_F$ and $A^{\mathsf{N}}_F$ in $L_2(\mathbb{R}^n_+)$ have no eigenvalues at all, but the spectra of $A_{\mathsf{D},F}$ and $A_{\mathsf{N},F}$ in $L_2(\Omega)$ consist exclusively of eigenvalues of finite (geometric = algebraic) multiplicity. Hence both $A_{\mathsf{D},F}$ and $A_{\mathsf{N},F}$ in $L_2(\Omega)$ are outstanding examples of operators with pure point spectrum. We refer to Remark C.16 where we discussed some consequences. In Chapter 7 we return in detail to the study of the behaviour of the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ of such operators. Some comments may also be found in Note 5.12.5 below.

**Exercise 5.33.** In Exercise 4.8 we already considered Poincaré's inequality (4.28) for $1 \le p < \infty$ in an interval. We deal now with $p = 2$ and arbitrary bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$.

(a) Prove that

$$\|f \,|L_2(\Omega)\| \le c \left( \int_\Omega \sum_{j=1}^n \left| \frac{\partial f}{\partial x_j}(x) \right|^2 \mathrm{d}x \right)^{1/2} \tag{5.165}$$

for $f \in W_2^1(\Omega)$ with $\int_\Omega f(x)\mathrm{d}x = 0$ and some $c > 0$. Let

$$W^1_{2,M}(\Omega) = \left\{ f \in W^1_2(\Omega) : \int_\Omega f(x)\mathrm{d}x = 0 \right\},$$

normed by $\|f\,|W^1_{2,M}(\Omega)\| = \||\nabla f|\,|L_2(\Omega)\|$. Show that

$$W^1_2(\Omega) = W^1_{2,M}(\Omega) \oplus \{f \text{ is constant on } \Omega\},$$

and $A_{\mathsf{N},F}$ is positive-definite on $W^1_{2,M}(\Omega)$.
*Hint:* Modify the proof of (5.142). Use the last assertion of Theorem 5.31. Recall that $C^\infty$ domains are connected.

(b) Use (a) to show that

$$\left( \int_\Omega \sum_{j=1}^n \left| \frac{\partial f}{\partial x_j}(x) \right|^2 \mathrm{d}x \right)^{1/2} + \left| \int_\Omega f(x)\mathrm{d}x \right| \sim \|f\,|W^1_2(\Omega)\| \tag{5.166}$$

is an equivalent norm on $W^1_2(\Omega)$.
*Hint:* Modify Step 2 of the proof of Proposition 5.28.

## 5.6 Homogeneous boundary value problems

We always assume now that $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3 and that $A$, given by

$$(Au)(x) = -\sum_{j,k=1}^n a_{jk}(x)\,\frac{\partial^2 u}{\partial x_j\,\partial x_k}(x) + \sum_{l=1}^n a_l(x)\frac{\partial u}{\partial x_l}(x) + a(x)u(x), \tag{5.167}$$

is an elliptic differential expression according to Definition 5.1 and (5.65), hence

$$\|a_{jk}|C^1(\Omega)\| \le M, \quad \|a_l|C(\Omega)\| \le M, \quad \|a|C(\Omega)\| \le M \tag{5.168}$$

for all admitted $j,k,l$ and some $M > 0$,

$$a_{jk}(x) = a_{kj}(x) \in \mathbb{R}, \quad x \in \bar{\Omega},\ 1 \le j,k \le n, \tag{5.169}$$

such that

$$\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \ge E|\xi|^2, \quad x \in \bar{\Omega},\ \xi \in \mathbb{R}^n, \tag{5.170}$$

for some ellipticity constant $E > 0$. As for $C^1(\Omega)$, $C(\Omega)$ we refer to Definition A.1. We are now interested in the homogeneous Dirichlet problem according to Definition 5.3 (i) in $W^2_{2,0}(\Omega)$, where the latter has the same meaning as in (5.11). So far we got in Theorem 5.31 (i) a satisfactory theory for the Friedrichs extension $A_{\mathsf{D},F}$ of the Dirichlet Laplacian written now as $(-\Delta)_{\mathsf{D}}$,

$$(-\Delta)_{\mathsf{D}} u = -\Delta u, \quad \mathrm{dom}((-\Delta)_{\mathsf{D}}) = W^2_{2,0}(\Omega), \tag{5.171}$$

and considered as an unbounded operator in $L_2(\Omega)$. What follows is a typical bootstrapping procedure. We begin with (5.171) and climb up in finitely many steps from $-\Delta$ to $A$ in (5.167) using the a priori estimates according to Theorem 5.7 and Corollary 5.14 as an appropriate ladder. As suggested by Corollary 5.14, but also by the spectral assertions about the Dirichlet Laplacian in Theorem 5.31 (i) it is reasonable to deal not only with $A$ but with $A + \lambda\,\mathrm{id}$ where $\lambda \in \mathbb{C}$. The starting point of this procedure is the following perturbation assertion. Let, as usual,

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \tag{5.172}$$

where $1 \le j, k \le n$.

**Proposition 5.34.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $A$ be a differential expression according to (5.167) with bounded complex-valued coefficients such that*

$$\sum_{j,k=1}^{n} \sup_{x\in\bar\Omega} \left|a_{jk}(x) - \delta_{jk}\right| + \sum_{l=1}^{n} \sup_{x\in\bar\Omega} |a_l(x)| + \sup_{x\in\bar\Omega} |a(x)| \le \varepsilon \tag{5.173}$$

*for some $\varepsilon > 0$. If $\varepsilon$ is sufficiently small, then*

$$A\colon W^2_{2,0}(\Omega) \rightleftarrows L_2(\Omega) \text{ is an isomorphic map.} \tag{5.174}$$

*Furthermore,*

$$A^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega) \text{ is compact.} \tag{5.175}$$

*Proof. Step* 1. We write $A$ with $\mathrm{dom}(A) = W^2_{2,0}(\Omega)$ as

$$\begin{aligned} A &= (-\Delta)_{\mathsf{D}} + \tilde A, \\ \tilde A u &= -\sum_{j,k=1}^{n} \tilde a_{jk}(x)\,\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{l=1}^{n} a_l(x)\frac{\partial u}{\partial x_l} + a(x)u, \end{aligned} \tag{5.176}$$

where $\tilde a_{jk}(x) = a_{jk}(x) - \delta_{jk}$. Then

$$\|\tilde A u | L_2(\Omega)\| \le c\,\varepsilon \|u|W^2_2(\Omega)\|, \quad u \in \mathrm{dom}(A) = W^2_{2,0}(\Omega), \tag{5.177}$$

where $c$ is independent of $\varepsilon$ in (5.173). Theorem 5.31 (i) implies that the inverse $(-\Delta)^{-1}_{\mathsf{D}}$ is an isomorphic map of $L_2(\Omega)$ onto $W^2_{2,0}(\Omega)$. Choosing $\varepsilon > 0$ in (5.177) sufficiently small one obtains

$$\|B\| < 1 \quad \text{for } B = (-\Delta)^{-1}_{\mathsf{D}} \circ \tilde A\colon W^2_{2,0}(\Omega) \hookrightarrow W^2_{2,0}(\Omega). \tag{5.178}$$

Basic assertions of functional analysis tell us that $-1 \in \varrho(B) = \mathbb{C} \setminus \sigma(B)$, i.e., $-1$ belongs to the resolvent set of $B$ according to (C.7). Hence $B + \mathrm{id}$ is invertible in

$W^2_{2,0}(\Omega)$, that is, for any $f \in L_2(\Omega)$ there is a unique solution $u \in W^2_{2,0}(\Omega)$ such that

$$Bu + u = (-\Delta)^{-1}_{\mathsf{D}} f \in W^2_{2,0}(\Omega). \tag{5.179}$$

We apply the isomorphic map $(-\Delta)_{\mathsf{D}}$ from $W^2_{2,0}(\Omega)$ onto $L_2(\Omega)$ and conclude that

$$Au = (-\Delta)_{\mathsf{D}} u + \tilde{A}u = f \in L_2(\Omega) \tag{5.180}$$

has a unique solution. Now $A$ is an isomorphic map of $W^2_{2,0}(\Omega)$ onto $L_2(\Omega)$.

*Step* 2. We decompose $A^{-1}$ in (5.175) as

$$\begin{aligned} &A^{-1}(L_2(\Omega) \hookrightarrow L_2(\Omega)) \\ &\quad = \mathrm{id}(W^2_{2,0}(\Omega) \hookrightarrow L_2(\Omega)) \circ A^{-1}(L_2(\Omega) \hookrightarrow W^2_{2,0}(\Omega)) \end{aligned} \tag{5.181}$$

where the last operator is the above isomorphic map and id is the compact embedding according to Theorem 4.17. This proves that $A^{-1}$ in (5.175) is also compact. □

**Remark 5.35.** If $\varepsilon > 0$ in (5.173) is sufficiently small, then one has (5.170) for some $E > 0$. Hence $A$ according to (5.167) with (5.173) is elliptic, but in general no longer symmetric. In particular, the spectrum $\sigma(A^{-1})$ of $A^{-1}$ in (5.175) need not to be a subset of $\mathbb{R}$. But one has

$$\sigma(A^{-1}) = \{0\} \cup \sigma_{\mathsf{p}}(A^{-1}) \tag{5.182}$$

according to Theorem C.1. One can extend the definition of the resolvent set $\varrho(A)$, the spectrum $\sigma(A) = \mathbb{C} \setminus \varrho(A)$, the point spectrum $\sigma_{\mathsf{p}}(A)$ and the geometric multiplicity of eigenvalues from bounded operators $T$ in the Appendix C.1 to unbounded operators $A$ in a Hilbert space or Banach space, respectively, in an obvious way (avoiding the struggle with powers of unbounded operators in connection with algebraic multiplicities of eigenvalues, also discussed in Note 5.12.6). If $A^{-1}$ is considered as a compact map in $L_2(\Omega)$ according to (5.175) and $A$ as an unbounded operator in $L_2(\Omega)$ with domain of definition $W^2_{2,0}(\Omega)$, then the following assertions are true:

$$\lambda \in \varrho(A) \quad \text{if, and only if,} \quad \lambda^{-1} \in \varrho(A^{-1}) \quad \text{for } \lambda \neq 0, \tag{5.183}$$

and

$$\lambda \in \sigma_{\mathsf{p}}(A) \quad \text{if, and only if,} \quad \lambda^{-1} \in \sigma_{\mathsf{p}}(A^{-1}) \quad \text{for } \lambda \neq 0, \tag{5.184}$$

with

$$\dim\ker(A - \lambda\,\mathrm{id}) = \dim\ker(A^{-1} - \lambda^{-1}\mathrm{id}) < \infty. \tag{5.185}$$

As for (5.184), (5.185) it is sufficient to remark that

$$Au = \lambda u \in W^2_{2,0}(\Omega) \quad \text{if, and only if,} \quad u = \lambda A^{-1} u \in W^2_{2,0}(\Omega), \tag{5.186}$$

whereas (5.183) looks natural, but requires some extra care. Assuming that $0 \neq \lambda \in \varrho(A)$. Then the left-hand side of

$$(A - \lambda\,\mathrm{id})A^{-1} = -\lambda(A^{-1} - \lambda^{-1}\mathrm{id}) \tag{5.187}$$

is a one-to-one continuous map of $L_2(\Omega)$ onto itself (via $W^2_{2,0}(\Omega)$). Then it follows from the open mapping theorem in the version of [Rud91, Corollary 2.12(c), p. 50] that the left-hand side of (5.187) has a bounded inverse. Hence $\lambda^{-1} \in \varrho(A^{-1})$. Conversely, if $\lambda^{-1} \in \varrho(A^{-1})$, then (5.187) implies that $(A - \lambda\,\mathrm{id})$ is a one-to-one continuous map of $W^2_{2,0}(\Omega)$ onto $L_2(\Omega)$. Its inverse must be bounded from $L_2(\Omega)$ onto $W^2_{2,0}(\Omega)$ and hence into $L_2(\Omega)$ by the same reference as above. Consequently $\lambda \in \varrho(A)$. Furthermore, $0 \in \varrho(A)$ and $0 \in \sigma(A^{-1})$. Thus the spectrum of $A$ consists of eigenvalues of finite geometric multiplicity. Some further information may be found in Note 5.12.6.

**Theorem 5.36.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $A$ be an elliptic differential operator according to (5.167)–(5.170) with its domain of definition* $\mathrm{dom}(A) = W^2_{2,0}(\Omega)$. *Then there is a positive number $\lambda_0$ (depending only on $E$, $M$ and $\Omega$) such that*

$$A + \lambda\,\mathrm{id}\colon W^2_{2,0}(\Omega) \rightleftarrows L_2(\Omega) \text{ is an isomorphic map} \tag{5.188}$$

*for all $\lambda \in \mathbb{R}$ with $\lambda \geq \lambda_0$. Furthermore,*

$$(A + \lambda\,\mathrm{id})^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega) \text{ is compact} \tag{5.189}$$

*and*

$$\|(A + \lambda\,\mathrm{id})^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega)\| \leq \frac{c}{\lambda}, \quad \lambda \geq \lambda_0, \tag{5.190}$$

*for some $c > 0$ which depends only on $E$, $M$ and $\Omega$. The spectrum $\sigma(A)$ consists of isolated eigenvalues of finite geometric multiplicity $\mu = \xi + i\eta$ with $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}$, located within a parabola*

$$\{(\xi, \eta) \in \mathbb{R}^2 : \xi + \xi_0 \geq C\eta^2\} \tag{5.191}$$

*for some $C > 0$ and $\xi_0 \in \mathbb{R}$, see Figure 5.5 below, with no accumulation point in $\mathbb{C}$.*

*Proof. Step* 1. We use Corollary 5.14 (i) where $\lambda_0 > 0$ has the same meaning as there. Recall notation (5.172). We apply the so-called *continuity method*. For $0 \leq \theta \leq 1$ let $A_\theta$ be the family of elliptic operators

$$\begin{aligned} A_\theta u &= \theta A u + (1-\theta)(-\Delta)u \\ &= -\sum_{j,k=1}^n a^\theta_{jk}(x)\,\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{l=1}^n \theta a_l(x) \frac{\partial u}{\partial x_l} + \theta a(x) u, \end{aligned} \tag{5.192}$$

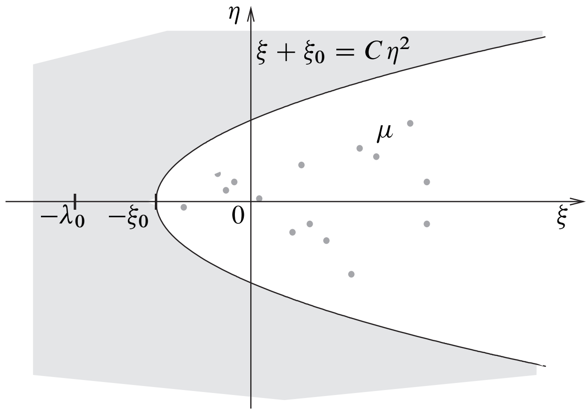



Figure 5.5

where

$$a^{\theta}_{jk}(x) = \theta a_{jk}(x) + (1-\theta)\delta_{jk}. \tag{5.193}$$

One may assume $0 < E \le 1$ in (5.170). Then

$$\sum_{j,k=1}^{n} a^{\theta}_{jk}(x)\xi_j\xi_k \;\ge\; E|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^n, \tag{5.194}$$

uniformly in $\theta$. Similarly one may assume that

$$\|a^{\theta}_{jk}|C^1(\Omega)\| \le M, \quad \theta\|a_l|C(\Omega)\| \le M, \quad \theta\|a|C(\Omega)\| \le M \tag{5.195}$$

for all admitted $j,k,l$ and some $M > 0$ as the uniform counterpart of (5.168). By (5.66) one has for some $c_1 > 0$ and $c_2 > 0$,

$$\|(A_\theta + \lambda\,\mathrm{id})u|L_2(\Omega)\| \ge c_1\|u|W^2_2(\Omega)\| + c_2\lambda\|u|L_2(\Omega)\|,\ u \in W^2_{2,0}(\Omega), \tag{5.196}$$

for all $\lambda \in \mathbb{R}$ with $\lambda \ge \lambda_0$ and all $\theta$ with $0 \le \theta \le 1$. Obviously, $A_0 = (-\Delta)_{\mathrm{D}}$ is the Dirichlet Laplacian in the notation (5.171). By Theorem 5.31 and (5.175) (based on (5.181)) one obtains for $\theta_0 = 0$ that

$$A_{\theta_0} + \lambda\,\mathrm{id}\colon W^2_{2,0}(\Omega) \rightleftarrows L_2(\Omega) \text{ is isomorphic for all } \lambda \ge \lambda_0, \tag{5.197}$$

$$(A_{\theta_0} + \lambda\,\mathrm{id})^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega) \text{ is compact}, \lambda \ge \lambda_0, \tag{5.198}$$

and as a consequence of (5.196),

$$c_2\lambda\|(A_{\theta_0} + \lambda\,\mathrm{id})^{-1}f|L_2(\Omega)\| \le \|f|L_2(\Omega)\|, \quad \lambda \ge \lambda_0. \tag{5.199}$$

This leads to (5.190). Now we assume that we have (5.197) for some $0 \le \theta_0 < 1$. We intend to apply the perturbation argument of the proof of Proposition 5.34 to

$$(A_\theta + \lambda \,\mathrm{id})u = A_{\theta_0}u + \lambda u + (\theta - \theta_0)\Big( - \sum_{j,k=1}^{n} b_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{l=1}^{n} a_l(x) \frac{\partial u}{\partial x_l} + a(x)u \Big) \tag{5.200}$$

where $b_{jk}(x) = a_{jk}(x) - \delta_{jk}$. If $0 \le \theta - \theta_0 \le \delta$ is sufficiently small, then there is a uniform counterpart of (5.173). Inequality (5.199) implies (5.190) with $A$ replaced by $A_{\theta_0}$ uniformly in $\theta_0$. Then one obtains (5.197) and also (5.198), (5.199) with $A_{\theta_0}$ replaced by $A_\theta$. Beginning with $\theta_0 = 0$ one arrives at $\theta = 1$ in finitely many steps. This proves (5.188)–(5.190).

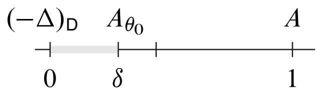


Figure 5.6

*Step* 2. It follows from the argument in Remark 5.35 that the spectrum $\sigma(A)$ consists of isolated eigenvalues of finite (geometric) multiplicity with no accumulation point in $\mathbb{C}$. It remains to prove that these eigenvalues are located within a parabola of type (5.191). Let $\mu = \xi + i\eta$ be an eigenvalue and $u \in W^2_{2,0}(\Omega)$ with $\|u|L_2(\Omega)\| = 1$ a related eigenfunction such that $\xi + i\eta = \langle Au, u\rangle_\Omega$. Then integration by parts leads to

$$\xi + i\eta = \int_\Omega \Big[ \sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial u}{\partial x_k} \, \frac{\partial \overline{u}}{\partial x_j} + \sum_{l=1}^{n} \widetilde{a}_l(x) \frac{\partial u}{\partial x_l} \, \overline{u} + \widetilde{a}(x)|u|^2 \Big] \mathrm{d}x, \tag{5.201}$$

where the first term on the right-hand side is real and can be estimated from below by $E|\nabla u|^2$ in view of (5.6). We use the standard notation

$$\nabla u = \Big( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \Big). \tag{5.202}$$

The real part of the remaining terms on the right-hand side can be estimated from below by

$$-\varepsilon \| |\nabla u| \, |L_2(\Omega)\|^2 - c_\varepsilon,$$

where $\varepsilon > 0$ is at our disposal. Consequently we obtain for the real parts,

$$c_1 + c_2 \int_\Omega |\nabla u(x)|^2 \mathrm{d}x \;\ge\; \xi \;\ge\; c_3 \int_\Omega |\nabla u(x)|^2 \mathrm{d}x - c_4, \tag{5.203}$$

and for the corresponding imaginary part that

$$|\eta| \le c_5 \Big( \int_\Omega |\nabla u(x)|^2 \mathrm{d}x \Big)^{1/2} + c_6. \tag{5.204}$$

Of interest are only eigenvalues $\mu = \xi + i\eta$ where $|\xi| + |\eta|$ is large. Then it follows by (5.203), (5.204) for $\xi > 0$ large that $|\eta| \le c_7 \sqrt{\xi}$. This proves (5.191) as illustrated in Figure 5.5. □

**Remark 5.37.** If $\lambda \in \varrho(A)$ belongs to the resolvent set of the above operator $A$, then for any $f \in L_2(\Omega)$ there is a unique solution of the homogeneous Dirichlet problem

$$(A - \lambda\,\mathrm{id})u = f, \quad u \in W_2^2(\Omega), \quad \mathrm{tr}_\Gamma\, u = 0, \tag{5.205}$$

according to (5.15), (5.16). In particular, for $\lambda \in \varrho(A)$ one has (5.188) with $A - \lambda\,\mathrm{id}$ in place of $A + \lambda\,\mathrm{id}$. In the next section we deal with corresponding inhomogeneous problems. But first we comment briefly on the homogeneous Neumann problem according to Definition 5.3 (ii). So far we have the satisfactory Theorem 5.31 (ii) for the Neumann Laplacian $A_{\mathrm{N},F} = (-\Delta)_{\mathrm{N}}$ with respect to the $C^\infty$ vector field $\nu$ of outer normals. To extend these assertions to arbitrary elliptic operators, say, of type (5.167)–(5.170) by the above method one would require counterparts of the a priori estimates in Theorem 5.7 and Corollary 5.14 (i). All this can be done, even for arbitrary non-tangential $C^\infty$ vector fields $\mu$ according to Remark 4.28. But it is not the subject of this book in which we try to avoid any additional technical complications. Some comments in connection with the above theorem may be found in Notes 5.12.1, 5.12.7, 5.12.8.

**Exercise 5.38.** Let $A$ be an elliptic differential operator with (5.167)–(5.170) such that its co-normal $\nu^A$ according to (5.20) coincides with the outer normal $\nu = \nu^A$. Prove that

$$A + \lambda\,\mathrm{id}\colon W_2^{2,\nu}(\Omega) \rightleftarrows L_2(\Omega) \text{ is an isomorphic map} \tag{5.206}$$

for $\lambda \ge \lambda_0$ where $\lambda_0 > 0$ has the same meaning as in (5.74). Formulate and verify the counterparts of the other assertions in Theorem 5.36.

*Hint:* Use Theorem 5.31 (ii) as a starter and proceed afterwards as in the proof of Theorem 5.36 based on (5.74).

## 5.7 Inhomogeneous boundary value problems

We deal with the inhomogeneous Dirichlet problem as introduced in Definition 5.3 (i) where $A$ is the elliptic differential expression according to (5.167)–(5.170). We denote now the operator $A$ as considered in Theorem 5.36 by $\mathring{A}$ to avoid misunderstandings, hence

$$\mathring{A}\colon \operatorname{dom}(\mathring{A}) = W^2_{2,0}(\Omega) \hookrightarrow L_2(\Omega). \tag{5.207}$$

Let $\varrho(\mathring{A})$ be its resolvent set. Then its spectrum $\sigma(\mathring{A}) = \mathbb{C} \setminus \varrho(\mathring{A})$ consists of isolated eigenvalues of finite multiplicity with no accumulation point in $\mathbb{C}$. We formalise the inhomogeneous Dirichlet problem by

$$T_\lambda = (A - \lambda\,\mathrm{id}, \mathrm{tr}_\Gamma)\colon \operatorname{dom}(T_\lambda) = W^2_2(\Omega) \hookrightarrow L_2(\Omega) \times W^{3/2}_2(\Gamma) \tag{5.208}$$

with $\Gamma = \partial\Omega$ and $\lambda \in \mathbb{C}$. Hence

$$T_\lambda u = ((Au)(x) - \lambda u(x), \mathrm{tr}_\Gamma\, u). \tag{5.209}$$

Furnished with the norm

$$\|(f,g)|L_2(\Omega) \times W^{3/2}_2(\Gamma)\| = (\|f|L_2(\Omega)\|^2 + \|g|W^{3/2}_2(\Gamma)\|^2)^{1/2}, \tag{5.210}$$

$L_2(\Omega) \times W^{3/2}_2(\Gamma)$ becomes a Hilbert space where $W^{3/2}_2(\Gamma)$ may be normed as in Definition 4.20.

**Theorem 5.39.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $T_\lambda$ be given by (5.208) where $Au$ is the elliptic differential expression according to (5.167)–(5.170). Let $\lambda \in \varrho(\mathring{A})$ where $\varrho(\mathring{A})$ is the resolvent set of $\mathring{A}$ in (5.207). Then*

$$T_\lambda\colon W^2_2(\Omega) \rightleftarrows L_2(\Omega) \times W^{3/2}_2(\Gamma) \tag{5.211}$$

*is an isomorphic map.*

*Proof.* Obviously, $T_\lambda$ is a continuous map from $W^2_2(\Omega)$ into $L_2(\Omega) \times W^{3/2}_2(\Gamma)$. But it is also a map *onto*: Let

$$f \in L_2(\Omega), \quad g \in W^{3/2}_2(\Gamma) \quad \text{and} \quad h \in W^2_2(\Omega) \quad \text{with } \mathrm{tr}_\Gamma\, h = g. \tag{5.212}$$

We used (4.101). Theorem 5.36 implies for $\lambda \in \varrho(\mathring{A})$ that there is a function

$$v \in W^2_{2,0}(\Omega) \quad \text{with } Av - \lambda v = f - Ah + \lambda h \in L_2(\Omega). \tag{5.213}$$

Then one has for $u = v + h \in W^2_2(\Omega)$ that

$$Au - \lambda u = f \quad \text{and} \quad \mathrm{tr}_\Gamma\, u = \mathrm{tr}_\Gamma\, h = g. \tag{5.214}$$

Assuming that for $w \in W^2_2(\Omega)$,

$$Aw - \lambda w = f \quad \text{and} \quad \mathrm{tr}_\Gamma\, w = g, \tag{5.215}$$

then

$$(A - \lambda \,\mathrm{id})(u - w) = 0 \quad \text{in } L_2(\Omega), \quad \mathrm{tr}_\Gamma (u - w) = 0. \tag{5.216}$$

Hence $u - w \in W^2_{2,0}(\Omega)$. Since $\lambda \in \varrho(\mathring{A})$, one obtains $u = w$. This shows that $T_\lambda$ is a continuous one-to-one map of $W^2_2(\Omega)$ onto $L_2(\Omega) \times W_2^{3/2}(\Gamma)$. Thus $T_\lambda^{-1}$ is also continuous and

$$\|T_\lambda u | L_2(\Omega) \times W_2^{3/2}(\Gamma)\| \sim \|u | W^2_2(\Omega)\|, \quad u \in W^2_2(\Omega). \tag{5.217}$$

We refer to [Rud91, Corollary 2.12(c), p. 50]. □

**Remark 5.40.** In other words, if $\lambda \in \varrho(\mathring{A})$, then the inhomogeneous Dirichlet problem

$$Au - \lambda u = f, \quad \mathrm{tr}_\Gamma \, u = g, \tag{5.218}$$

has for given $f \in L_2(\Omega)$ and $g \in W_2^{3/2}(\Gamma)$ a unique solution $u \in W^2_2(\Omega)$ and

$$\|u | W^2_2(\Omega)\| \sim \|f | L_2(\Omega)\| + \|g | W_2^{3/2}(\Gamma)\|. \tag{5.219}$$

**Exercise 5.41.** Prove the *inhomogeneous a priori estimate*

$$\|u | W^2_2(\Omega)\| \sim \|Au | L_2(\Omega)\| + \|u | L_2(\Omega)\| + \|\, \mathrm{tr}_\Gamma \, u \big| W_2^{3/2}(\Gamma)\| \tag{5.220}$$

for $u \in W^2_2(\Omega)$ which is a generalisation of the (homogeneous) a priori estimate (5.31).

*Hint:* Use (5.219).

**Exercise 5.42.** Let $A$ be the specific elliptic differential operator according to Exercise 5.38, hence $\nu^A = \nu$, and let $\lambda \in \varrho(\bar{A})$ be in the resolvent set of

$$\bar{A} \colon \mathrm{dom}(\bar{A}) = W_2^{2,\nu}(\Omega) \hookrightarrow L_2(\Omega), \tag{5.221}$$

and

$$U_\lambda = (A - \lambda \,\mathrm{id}, \mathrm{tr}_\Gamma \frac{\partial}{\partial \nu}) \colon \mathrm{dom}(U_\lambda) = W^2_2(\Omega) \hookrightarrow L_2(\Omega) \times W_2^{1/2}(\Gamma) \tag{5.222}$$

as the Neumann counterpart of (5.207), (5.208). Prove that

$$U_\lambda \colon W^2_2(\Omega) \rightleftarrows L_2(\Omega) \times W_2^{1/2}(\Gamma), \quad \lambda \in \varrho(\bar{A}), \tag{5.223}$$

is an isomorphic map. Show the *inhomogeneous a priori estimate*

$$\|u | W^2_2(\Omega)\| \sim \|Au | L_2(\Omega)\| + \|u | L_2(\Omega)\| + \left\| \mathrm{tr}_\Gamma \frac{\partial u}{\partial \nu} \left| W_2^{1/2}(\Gamma) \right.\right\| \tag{5.224}$$

for $u \in W^2_2(\Omega)$.

*Hint:* Consult Definition 5.3 (ii), use the Exercises 5.38, 5.41.

## 5.8 Smoothness theory

Theorem 5.39 may be considered as the main assertion of Chapter 5. It solves the boundary value problem (5.218) for $f \in L_2(\Omega)$ and $g \in W_2^{3/2}(\Gamma)$ in a satisfactory way, including the *stability assertion* (5.219) saying that small deviations of $f$ and $g$ in the respective spaces cause only small deviations of the solution $u$ in $W_2^2(\Omega)$. What can be said about $u$ if one knows more for the given data, typically like

$$f \in W_2^k(\Omega) \quad \text{and} \quad g \in W_2^{k+\frac{3}{2}}(\Gamma), \quad k \in \mathbb{N}_0\,? \tag{5.225}$$

The boundary data $g$ are harmless. By Theorem 4.24 (i) the corresponding inhomogeneous problem can easily be reduced to the related homogeneous problem in the same way as in the proof of Theorem 5.39. Hence it is sufficient to deal with homogeneous problems. Let

$$W_{2,0}^k(\Omega) = \{f \in W_2^k(\Omega) : \mathrm{tr}_\Gamma f = 0\}, \quad k \in \mathbb{N}, \tag{5.226}$$

as introduced in Definition 4.30. In the Definitions A.3 and A.1 we said what is meant by bounded $C^\infty$ domains and by the space $C^\infty(\Omega)$ in $\mathbb{R}^n$, respectively, where $n \in \mathbb{N}$.

**Proposition 5.43.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $A$ be an elliptic differential expression according to Definition 5.1 now assuming, in addition, that*

$$\{a_{jk}\}_{j,k=1}^n \subset C^\infty(\Omega), \quad \{a_l\}_{l=1}^n \subset C^\infty(\Omega), \quad a \in C^\infty(\Omega). \tag{5.227}$$

*Let $u \in W_{2,0}^2(\Omega)$ and $Au \in W_2^k(\Omega)$ where $k \in \mathbb{N}_0$. Then $u \in W_{2,0}^{k+2}(\Omega)$ and*

$$\|u|W_2^{k+2}(\Omega)\| \sim \|Au|W_2^k(\Omega)\| + \|u|L_2(\Omega)\|. \tag{5.228}$$

*Proof. Step* 1. If $u \in W_{2,0}^{k+2}(\Omega)$, then the right-hand side of (5.228) can be estimated from above by the left-hand side. Hence we have to prove that $u \in W_2^{k+2}(\Omega)$ and that there is a constant $c > 0$ such that

$$\|u|W_2^{k+2}(\Omega)\| \le c\|Au|W_2^k(\Omega)\| + c\|u|L_2(\Omega)\|. \tag{5.229}$$

*Step* 2. We wish to use the same reductions as in the proof of Theorem 5.7. But one has to act with caution since one knows only $u \in W_{2,0}^2(\Omega)$. At the end we argue by induction with respect to $k \in \mathbb{N}_0$. Let us assume that we already knew $u \in W_2^{k+1}(\Omega)$ in addition to $Au \in W_2^k(\Omega)$ where $k \in \mathbb{N}$. Then one can apply the localisation argument of Step 2 of the proof of Theorem 5.7. Hence the improved smoothness $u \in W_2^{k+2}(\Omega)$ and (5.229) is a local matter (under the hypothesis that induction applies). As in Step 3 of the proof of Theorem 5.7 one can straighten the problem as indicated there but only up to (5.41). The final reduction to elliptic expressions with constant coefficients would cause now some problems. In other words it is only justified to assume that

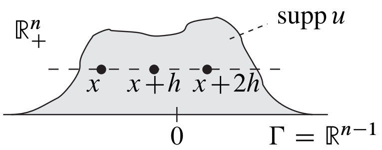



Figure 5.7

$$u \in W^2_{2,0}(\mathbb{R}^n_+),$$

$$\operatorname{supp} u \subset \{x \in \overline{\mathbb{R}^n_+} : |x| < 1\},$$

$$Au \in W^k_2(\mathbb{R}^n_+).$$

*Step* 3. Let $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ with $h_n = 0$ and let $\Delta^m_h$ be the iterated differences according to (3.41). Then $\Delta^m_h u \in W^2_{2,0}(\mathbb{R}^n_+)$ acts parallel to $\Gamma = \mathbb{R}^{n-1}$ and one obtains by Theorem 5.7 (applied to $\mathbb{R}^n_+$)

$$\|\Delta^m_h u | W^2_2(\mathbb{R}^n_+)\| \le c\|A(\Delta^m_h u)|L_2(\mathbb{R}^n_+)\| + c\|\Delta^m_h u|L_2(\mathbb{R}^n_+)\|. \tag{5.230}$$

Unfortunately $A$ has variable coefficients of type (5.227) with $\mathbb{R}^n_+$ in place of $\Omega$. Recall that for $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$,

$$(\Delta^m_h(fg))(x) = \sum_{r=0}^m \binom{m}{r} (\Delta^r_h f)(x)\, (\Delta^{m-r}_h g)(x + rh), \tag{5.231}$$

subject to Exercise 5.44 below, see also Exercise 3.19 (a). This implies

$$\begin{aligned} \Delta^m_h(Au)(x) =& A(\Delta^m_h u)(x) + \\ &+ \sum_{j,k=1}^n \sum_{r=1}^m \binom{m}{r} (\Delta^r_h a_{jk})(x) \Delta^{m-r}_h \frac{\partial^2 u}{\partial x_j \partial x_k}(x + rh) + \widetilde{R}_h u(x) \\ =& A(\Delta^m_h u)(x) + R_h u(x), \end{aligned} \tag{5.232}$$

where $\widetilde{R}_h u(x)$ collects terms with $\frac{\partial u}{\partial x_l}$ and $u$ instead of $\frac{\partial^2 u}{\partial x_j \partial x_k}$. Let $Au \in W^k_2(\mathbb{R}^n_+)$. We assume again that we already knew that $u \in W^{m+1}_2(\mathbb{R}^n_+)$ where $m \in \mathbb{N}$ and $m \le k$. Let $\mathrm{ext}^L$ be the extension operator according to (3.108) where $L$ is sufficiently large. Then

$$\mathrm{ext}^L u \in W^{m+1}_2(\mathbb{R}^n) \quad \text{and} \quad \mathrm{ext}^L(\Delta^m_h u) = \Delta^m_h(\mathrm{ext}^L u) \tag{5.233}$$

using that $\Delta^m_h$ is taken parallel to $\Gamma = \mathbb{R}^{n-1}$. By the same commutativity property it follows from (5.232) that

$$\Delta^m_h(\mathrm{ext}^L Au)(x) = \mathrm{ext}^L(A(\Delta^m_h u))(x) + \mathrm{ext}^L(R_h u)(x), \tag{5.234}$$

where $\mathrm{ext}^L(R_h u)$ preserves the structure of $R_h u$. Now (5.230) can be extended to $\mathbb{R}^n$,

$$\begin{aligned}\|\Delta_h^m(\operatorname{ext}^L u)|W_2^2(\mathbb{R}^n)\| \\ &\le c\|\Delta_h^m(\operatorname{ext}^L Au)|L_2(\mathbb{R}^n)\| + c\|\operatorname{ext}^L(R_h u)|L_2(\mathbb{R}^n)\| \\ &\quad + c\|\Delta_h^m(\operatorname{ext}^L u)|L_2(\mathbb{R}^n)\|. \qquad (5.235)\end{aligned}$$

By construction $\operatorname{ext}^L(Au) \in W_2^k(\mathbb{R}^n)$. We apply Proposition 3.28 and the technique developed there. With $f = \operatorname{ext}^L(Au)$ one obtains

$$\begin{aligned}|h|^{-2m}\|\Delta_h^m f|L_2(\mathbb{R}^n)\|^2 &= \int_{\mathbb{R}^n} \frac{|(e^{i\xi h}-1)^m|^2}{|h|^{2m}|\xi|^{2m}}\,|\xi|^{2m}|(\mathcal{F}f)(\xi)|^2\mathrm{d}\xi \\ &\le c\|f|W_2^m(\mathbb{R}^n)\|^2 \qquad (5.236)\end{aligned}$$

uniformly in $h$ with $|h| \le 1$. The structure of $R_h u$ implies the estimates

$$\begin{aligned}|h|^{-m}\|\operatorname{ext}^L(R_h u)|L_2(\mathbb{R}^n)\| &\le c\ \|u|W_2^{m+1}(\mathbb{R}^n_+)\| \\ &\sim \|\operatorname{ext}^L u|W_2^{m+1}(\mathbb{R}^n)\| \qquad (5.237)\end{aligned}$$

uniformly in $h$. This covers also the last term in (5.235) divided by $|h|^{-m}$. In view of (5.235) and the arguments of the proof of Proposition 3.28, especially (3.73), this leads to

$$\|\mathrm{D}^\alpha u|W_2^2(\mathbb{R}^n_+)\| \le c\|Au|W_2^m(\mathbb{R}^n_+)\| + c\|u|W_2^{m+1}(\mathbb{R}^n_+)\| \qquad (5.238)$$

for all $\alpha = (\alpha_1,\dots,\alpha_n)$ with $\alpha_n = 0$ and $|\alpha| \le m$. Consequently,

$$\mathrm{D}^\beta u \in L_2(\mathbb{R}^n_+) \quad \text{for all } \beta = (\beta_1,\dots,\beta_n),\ |\beta_n| \le 2,\ |\beta| \le m+2. \qquad (5.239)$$

Since $A$ is elliptic one has $a_{nn}(x) \ge E$, hence $a_{nn}^{-1}(x) \in C^\infty(\mathbb{R}^n_+)$. Then

$$\begin{aligned}\frac{\partial^2 u}{\partial x_n^2}(x) = a_{nn}^{-1}(x)\Big[Au(x) - {\sum_{1\le j,k\le n}}' a_{jk}(x)\frac{\partial^2 u}{\partial x_j\partial x_k}(x) \\ - \sum_{l=1}^n a_l(x)\frac{\partial u}{\partial x_l}(x) - a(x)u(x)\Big] \qquad (5.240)\end{aligned}$$

where $\sum'_{1\le j,k\le n}$ means the summation over all $j,k$ except $j = k = n$. By (5.239) one can apply $\mathrm{D}^\gamma$ with $|\gamma| \le m$ and $|\gamma_n| \le 1$ and one obtains (5.239) with $|\beta_n| \le 3$. Iteration gives (5.239) for all $|\beta| \le m+2$, and hence $u \in W_2^{m+2}(\mathbb{R}^n_+)$. Now (5.238) implies that

$$\begin{aligned}\|u|W_2^{m+2}(\mathbb{R}^n_+)\| &\le c\|Au|W_2^m(\mathbb{R}^n_+)\| + c\|u|W_2^{m+1}(\mathbb{R}^n_+)\| \\ &\le c'\|Au|W_2^k(\mathbb{R}^n_+)\| + c'\|u|L_2(\mathbb{R}^n_+)\|, \qquad (5.241)\end{aligned}$$

where the latter follows from (4.86).

*Step* 4. We justify the above induction. By Theorem 5.7 we have (5.228) for $k = 0$. Then the Steps 2 and 3 can be applied to $k = m = 1$, resulting in (5.241) with $m = 1$, and hence (5.228) with $k = 1$. Now it is clear that the above induction works. □

**Exercise 5.44.** Prove (5.231).

*Hint:* Either use induction or (more elegantly) shift the question to the Fourier side in view of

$$\mathcal{F}(\Delta_h^m(fg))(\xi) = c(e^{ih\xi} - 1)^m \int_{\mathbb{R}^n} \mathcal{F} f(\xi - \eta)\mathcal{F} g(\eta)\mathrm{d}\eta. \tag{5.242}$$

The question arises whether there is a counterpart of Proposition 5.43 related to the Neumann problem. Recall that $A = -\Delta$ is the Laplacian (5.5). Furthermore, $W_2^k(\Omega)$ and $W_2^{k,\nu}(\Omega)$ have the same meaning as in Theorem 4.1 and Definition 4.30 with the $C^\infty$ vector field of the outer normals $\mu = \nu$.

**Corollary 5.45.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$. Let $u \in W_2^{2,\nu}(\Omega)$ and $\Delta u \in W_2^k(\Omega)$ where $k \in \mathbb{N}_0$. Then $u \in W_2^{k+2,\nu}(\Omega)$ and*

$$\|u|W_2^{k+2}(\Omega)\| \sim \|\Delta u|W_2^k(\Omega)\| + \|u|L_2(\Omega)\|. \tag{5.243}$$

*Proof.* The case $k = 0$ is covered by Theorem 5.11. The proof of (5.59) (hence (5.243)) is reduced to the localised version in $\mathbb{R}^n_+$ according to (5.61), (5.62). But then one can argue as in the proof of Proposition 5.43. □

Proposition 5.43 and Corollary 5.45 pave the way to complement the homogeneous and inhomogeneous boundary value problems as considered in the Sections 5.6 and 5.7 by a corresponding smoothness theory in a satisfactory way. We always assume that $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3 and that $A$,

$$(Au)(x) = -\sum_{j,k=1}^n a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{l=1}^n a_l(x)\frac{\partial u}{\partial x_l}(x) + a(x)u(x), \tag{5.244}$$

is an elliptic differential expression as introduced in Definition 5.1 now with

$$\{a_{jk}\}_{j,k=1}^n \subset C^\infty(\Omega), \quad \{a_l\}_{l=1}^n \subset C^\infty(\Omega), \quad a \in C^\infty(\Omega), \tag{5.245}$$

where $C^\infty(\Omega)$ is given by (A.9). Let $W_2^s(\Gamma)$ with $\Gamma = \partial\Omega$ be the same spaces as in Definition 4.20. Otherwise we use the same notation as in Section 5.7. In particular,

$\varrho(\mathring{A})$ is the resolvent set of the operator $\mathring{A}$ with a reference to Theorem 5.36. Let again

$$T_\lambda u = ((Au)(x) - \lambda u(x), \operatorname{tr}_\Gamma u) \tag{5.246}$$

as in (5.209), but considered now for $k \in \mathbb{N}_0$ as a bounded map

$$T_\lambda = (A - \lambda \operatorname{id}, \operatorname{tr}_\Gamma) \colon \operatorname{dom}(T_\lambda) = W_2^{k+2}(\Omega) \hookrightarrow W_2^k(\Omega) \times W_2^{k+\frac{3}{2}}(\Gamma), \tag{5.247}$$

where the latter space, furnished with the norm

$$\|(f,g)|W_2^k(\Omega) \times W_2^{k+\frac{3}{2}}(\Gamma)\| = (\|f|W_2^k(\Omega)\|^2 + \|g|W_2^{k+\frac{3}{2}}(\Gamma)\|^2)^{1/2} \tag{5.248}$$

becomes a Hilbert space.

**Theorem 5.46.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $T_\lambda$ be given by (5.247) with $k \in \mathbb{N}_0$ where $A$ is the above elliptic differential operator (5.244) with (5.245). Let $\lambda \in \varrho(\mathring{A})$ where $\varrho(\mathring{A})$ is the resolvent set of $\mathring{A}$ in (5.207). Then*

$$T_\lambda \colon W_2^{k+2}(\Omega) \rightleftarrows W_2^k(\Omega) \times W_2^{k+\frac{3}{2}}(\Gamma) \tag{5.249}$$

*is an isomorphic map.*

*Proof.* We argue in the same way as in the proof of Theorem 5.39. Let

$$f \in W_2^k(\Omega), \quad g \in W_2^{k+\frac{3}{2}}(\Gamma) \quad \text{and} \quad h \in W_2^{k+2}(\Omega) \quad \text{with } \operatorname{tr}_\Gamma h = g, \tag{5.250}$$

where we used (4.101). By Theorem 5.36 and $\lambda \in \varrho(\mathring{A})$ there is a function

$$v \in W_{2,0}^2(\Omega) \quad \text{with } Av - \lambda v = f - Ah + \lambda h \in W_2^k(\Omega). \tag{5.251}$$

Proposition 5.43 (with $A$ replaced by $A - \lambda \operatorname{id}$) implies that $v \in W_{2,0}^{k+2}(\Omega)$. The rest is now the same as in the proof of Theorem 5.39. □

**Remark 5.47.** In other words, if $k \in \mathbb{N}_0$ and $\lambda \in \varrho(\mathring{A})$, then the inhomogeneous Dirichlet problem

$$Au - \lambda u = f, \quad \operatorname{tr}_\Gamma u = g, \tag{5.252}$$

where $f \in W_2^k(\Omega)$ and $g \in W_2^{k+\frac{3}{2}}(\Gamma)$ are given has a unique solution $u \in W_2^{k+2}(\Omega)$ and

$$\|u|W_2^{k+2}(\Omega)\| \sim \|f|W_2^k(\Omega)\| + \|g|W_2^{k+\frac{3}{2}}(\Gamma)\|. \tag{5.253}$$

Furthermore,

$$\|u|W_2^{k+2}(\Omega)\| \sim \|Au|W_2^k(\Omega)\| + \|u|L_2(\Omega)\| + \|\operatorname{tr}_\Gamma u|W_2^{k+\frac{3}{2}}(\Gamma)\| \tag{5.254}$$

for $u \in W_2^{k+2}(\Omega)$ in generalisation of Remark 5.40, Exercise 5.41 and the homogeneous a priori estimate (5.228).

**Exercise 5.48.** Prove (5.254).

*Hint:* Use (5.253) and (4.86).

**Remark 5.49.** In view of Theorem 5.46 and the comments in Remark 5.47 one has a perfect solution of the (homogeneous and inhomogeneous) Dirichlet problem in the spaces $W_2^s(\Omega)$ if $s = k \in \mathbb{N}_0$. What about an extension of this theory to arbitrary Sobolev spaces $W_2^s(\Omega)$, $s \ge 0$, as considered in Chapter 4, especially in the Theorems 4.1 and 4.24? This is possible. We return to this point in Note 5.12.9.

Let again $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and $\nu$ be the $C^\infty$ vector field of outer normals. So far we know according to Theorem 5.31 (ii) that the spectrum $\sigma(\bar{A})$ of the Neumann Laplacian $\bar{A}$,

$$\bar{A}u = -\Delta u\colon \operatorname{dom}(\bar{A}) = W_2^{2,\nu}(\Omega) \hookrightarrow L_2(\Omega) \tag{5.255}$$

consists of the simple eigenvalue 0 and positive eigenvalues $\lambda_j$ of finite multiplicity tending to infinity if $j \to \infty$. Let $\varrho(\bar{A})$ be the resolvent set. In particular, if $\lambda \in \varrho(\bar{A})$, then the homogeneous Neumann problem

$$\bar{A}u - \lambda u = f \in L_2(\Omega), \quad \operatorname{tr}_\Gamma \frac{\partial u}{\partial \nu} = 0 \tag{5.256}$$

has a unique solution in $W_2^{2,\nu}(\Omega)$. According to Definition 5.3 (ii) (and as used before in (5.221), (5.222)) the corresponding inhomogeneous Neumann problem can be reduced to

$$U_\lambda = \Big(-\Delta - \lambda\, \mathrm{id}, \operatorname{tr}_\Gamma \frac{\partial}{\partial \nu}\Big)\colon \operatorname{dom}(U_\lambda) = W_2^2(\Omega) \hookrightarrow L_2(\Omega) \times W_2^{1/2}(\Gamma) \tag{5.257}$$

with $\Gamma = \partial\Omega$ and $\lambda \in \mathbb{C}$, hence

$$U_\lambda u = \Big( - \Delta u(x) - \lambda u(x), \operatorname{tr}_\Gamma \frac{\partial u}{\partial \nu}\Big). \tag{5.258}$$

We ask for a counterpart of Theorem 5.46 and consider $U_\lambda$ now as a bounded map

$$U_\lambda = \Big( - \Delta - \lambda\, \mathrm{id}, \operatorname{tr}_\Gamma \frac{\partial}{\partial \nu}\Big)\colon \operatorname{dom}(U_\lambda) = W_2^{k+2}(\Omega) \hookrightarrow W_2^k(\Omega) \times W_2^{k+\frac{1}{2}}(\Gamma) \tag{5.259}$$

for $k \in \mathbb{N}_0$ where the latter space, furnished with the norm

$$\|(f,g)|W_2^k(\Omega) \times W_2^{k+\frac{1}{2}}(\Gamma)\| = (\|f|W_2^k(\Omega)\|^2 + \|g|W_2^{k+\frac{1}{2}}(\Gamma)\|^2)^{1/2} \tag{5.260}$$

becomes a Hilbert space.

**Theorem 5.50.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $\nu$ be the $C^\infty$ vector field of outer normals on $\Gamma = \partial\Omega$. Let $\lambda \in \varrho(\overline{A})$ where $\varrho(\overline{A})$ is the resolvent set of $\overline{A}$ in (5.255). Then*

$$U_\lambda : W_2^{k+2}(\Omega) \rightleftarrows W_2^k(\Omega) \times W_2^{k+\frac{1}{2}}(\Gamma), \tag{5.261}$$

*is an isomorphic map.*

*Proof.* Relying on (4.102) and Corollary 5.45 one can argue as in the proof of Theorem 5.46 reducing the problem to (5.256). □

**Remark 5.51.** The above result implies that for $k \in \mathbb{N}_0$ and $\lambda \in \varrho(\overline{A})$ the inhomogeneous Neumann problem

$$-\Delta u - \lambda u = f, \quad \mathrm{tr}_\Gamma \frac{\partial u}{\partial \nu} = g, \tag{5.262}$$

has for given $f \in W_2^k(\Omega)$ and $g \in W_2^{k+\frac{1}{2}}(\Gamma)$ a unique solution $u \in W_2^{k+2}(\Omega)$, and

$$\|u|W_2^{k+2}(\Omega)\| \sim \|f|W_2^k(\Omega)\| + \|g|W_2^{k+\frac{1}{2}}(\Gamma)\|. \tag{5.263}$$

Furthermore,

$$\|u|W_2^{k+2}(\Omega)\| \sim \|\Delta u|W_2^k(\Omega)\| + \|u|L_2(\Omega)\| + \left\| \mathrm{tr}_\Gamma \frac{\partial u}{\partial \nu} \Big| W_2^{k+\frac{1}{2}}(\Gamma)\right\| \tag{5.264}$$

for $u \in W_2^{k+2}(\Omega)$. This is the counterpart of Remark 5.47 and Exercise 5.48.

## 5.9 The classical theory

In Chapter 1 we dealt with harmonic functions $\Delta u = 0$ and inhomogeneous Dirichlet problems in bounded connected domains $\Omega$ in $\mathbb{R}^n$ according to Definition 1.43 for the Laplacian (called there *the Dirichlet problem for the Poisson equation* for historic reasons). But only in case of balls $\Omega = K_R$ we obtained in Theorem 1.48 a (more or less) satisfactory (i.e., explicit) solution. According to Remark 1.44 with a reference to Theorem 1.37 one has uniqueness for all admitted domains $\Omega$ in $\mathbb{R}^n$. We return now to the classical theory, more precisely, the $C^\infty$ theory as an after*math* of the above $L_2$ theory.

As in Section 5.8 we now assume that $A$,

$$(Au)(x) = -\sum_{j,k=1}^n a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{l=1}^n a_l(x)\frac{\partial u}{\partial x_l}(x) + a(x)u(x), \tag{5.265}$$

is an elliptic differential expression as introduced in Definition 5.1 with

$$\{a_{jk}\}_{j,k=1}^n \subset C^\infty(\Omega), \quad \{a_l\}_{l=1}^n \subset C^\infty(\Omega), \quad a \in C^\infty(\Omega) \tag{5.266}$$

in bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$ according to Definition A.3, whereas $C^\infty(\Omega)$ has the same meaning as in (A.9). If $A$ with $\operatorname{dom}(A) = \overset{\circ}{W}{}^2_{2,0}(\Omega)$ is considered as an (unbounded) operator in $L_2(\Omega)$, then we shall denote it by $\overset{\circ}{A}$,

$$\overset{\circ}{A}\colon \operatorname{dom}(\overset{\circ}{A}) = W^2_{2,0}(\Omega) \hookrightarrow L_2(\Omega) \tag{5.267}$$

as at the beginning of Section 5.7. Recall that the spectrum $\sigma(\overset{\circ}{A})$ consists of isolated eigenvalues of finite multiplicity located as shown in Figure 5.5. As before, $\varrho(\overset{\circ}{A}) = \mathbb{C} \setminus \sigma(\overset{\circ}{A})$ is the resolvent set. The spaces $C^l(\Gamma)$ with $l \in \mathbb{N}_0$ on the boundary $\Gamma = \partial\Omega$ can be introduced much as in Definition 4.20. Let

$$C^\infty(\Gamma) = \bigcap_{l=0}^{\infty} C^l(\Gamma). \tag{5.268}$$

**Theorem 5.52.** *Let $A$ be the above elliptic differential expression in a bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$.*

(i) *Let $f \in C^\infty(\Omega)$, $g \in C^\infty(\Gamma)$ and $\lambda \in \varrho(\overset{\circ}{A})$. Then (the classical inhomogeneous Dirichlet problem)*

$$Au - \lambda u = f \text{ in } \Omega \quad \text{and} \quad \operatorname{tr}_\Gamma u = g \text{ on } \Gamma \tag{5.269}$$

*has a unique solution $u \in C^\infty(\Omega)$.*

(ii) *Let $\lambda \in \sigma(\overset{\circ}{A})$ be an eigenvalue of $\overset{\circ}{A}$ according to (5.267) and let $u$ be a related eigenfunction,*

$$Au = \lambda u \text{ in } \Omega \quad \text{and} \quad \operatorname{tr}_\Gamma u = 0 \text{ on } \Gamma. \tag{5.270}$$

*Then $u \in C^\infty(\Omega)$.*

*Proof. Step* 1. Obviously $C^l(\Omega) \subset W^l_2(\Omega)$ and $C^l(\Gamma) \subset W^l_2(\Gamma)$ for any $l \in \mathbb{N}_0$. Then it follows from Theorem 5.46 and Remark 5.47 that (5.269) has a unique solution

$$u \in \bigcap_{k=0}^{\infty} W^k_2(\Omega) = \bigcap_{l=0}^{\infty} C^l(\Omega) = C^\infty(\Omega), \tag{5.271}$$

where we used Theorem 4.17 (ii).

*Step* 2. If $u$ is an eigenfunction of $\overset{\circ}{A}$ according to (5.270) and (5.267), then $u \in W^2_{2,0}(\Omega)$. Application of Proposition 5.43 with $Au \in W^2_2(\Omega)$ gives $u \in W^4_{2,0}(\Omega)$. Iteration results in $u \in C^\infty(\Omega)$. □

**Remark 5.53.** The Theorems 5.46 and 5.52 give satisfactory answers for the Dirichlet problem in $W^k_2(\Omega)$ and in $C^\infty(\Omega)$. One may ask for corresponding assertions in other spaces, for example, $W^k_p(\Omega)$ according to (4.1) or $C^k(\Omega)$ as introduced in

Definition A.1. We add a few comments in Note 5.12.10 as far as Sobolev spaces and Besov spaces are concerned. As for the spaces $C^k(\Omega)$, $k \in \mathbb{N}_0$, the situation is different from what would be expected at first glance. It turns out that the spaces $C^k(\Omega)$ do not fit very well in the above scheme. To get a theory comparable with the above assertions one must modify them by Hölder spaces as briefly mentioned in (3.44) and Exercise 3.20. However, this is not the subject of this book. But we add a comment on the dark side of $C^k(\Omega)$ in connection with elliptic differential equations. The natural counterpart of assumptions for $f$ and $g$ in the context of a $W_2^k$ theory according to Theorem 5.46 and Remark 5.47, respectively, would be $f \in C^k(\Omega)$ and $g \in C^k(\Gamma)$ for some $k \in \mathbb{N}_0$. If $k \geq 2$ and $l = k - 2$, then one can apply (5.252) to

$$\begin{aligned} f &\in C^k(\Omega) \hookrightarrow W_2^k(\Omega) \hookrightarrow W_2^l(\Omega), \\ g &\in C^k(\Gamma) \hookrightarrow W_2^k(\Gamma) \hookrightarrow W_2^{l+\frac{3}{2}}(\Gamma). \end{aligned} \tag{5.272}$$

One obtains a unique solution $u \in W_2^{l+2}(\Omega) = W_2^k(\Omega)$ of (5.252). But this is far from the desired outcome $u \in C^{k+2}(\Omega)$. Assuming that (5.252) with $f \in C^k(\Omega)$ and $g \in C^k(\Gamma)$ had always a solution $u \in C^{k+2}(\Omega)$, then one would get the counterpart of the a priori estimates (5.253), (5.254),

$$\|u|C^{k+2}(\Omega)\| \sim \|Au|C^k(\Omega)\| + \|u|C(\Omega)\| + \|\operatorname{tr}_\Gamma u|C^k(\Gamma)\|. \tag{5.273}$$

Questions of this type attracted a lot of attention in the 1960s and 1970s also in the framework of the theory of function spaces. As a consequence of (5.273) with $A = -\Delta$ and $k = 0$, $n = 2$ one would obtain

$$\begin{aligned} c\left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \Big| C(\mathbb{R}^2) \right\| \leq \left\| \frac{\partial^2 u}{\partial x_1^2} \Big| C(\mathbb{R}^2) \right\| + \left\| \frac{\partial^2 u}{\partial x_2^2} \Big| C(\mathbb{R}^2) \right\| + \left\| \frac{\partial u}{\partial x_1} \Big| C(\mathbb{R}^2) \right\| \\ + \left\| \frac{\partial u}{\partial x_2} \Big| C(\mathbb{R}^2) \right\| + \|u|C(\mathbb{R}^2)\|, \quad u \in \mathcal{D}(\mathbb{R}^2), \end{aligned} \tag{5.274}$$

for some $c > 0$. But this was disproved in [Bom72]. One may also consult [Bes74], [KJF77, Section 1.9, p. 52], [Tri78, Section 1.13.4, p. 86] and Note 5.12.11 below where we return to problems of this type.

**Exercise 5.54.** (a) Construct a function $u \in C(\mathbb{R}^2)$ such that all derivatives on the right-hand side of (5.274) are also elements of $C(\mathbb{R}^2)$ which disproves (5.274).

*Hint:* Rely on the same function

$$u(x_1, x_2) = x_1 x_2 \log\log\left(\frac{1}{\sqrt{x_1^2 + x_2^2}}\right) \text{ near the origin,} \tag{5.275}$$

as in the above-mentioned literature, see Figure 5.8 below.

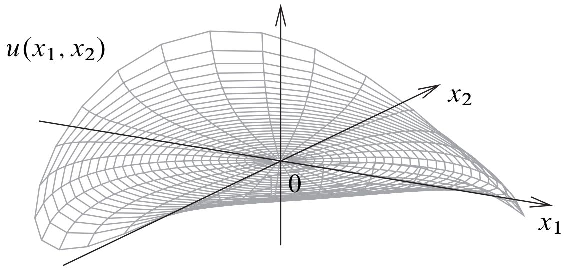



Figure 5.8

(b) Why can one not work with $\widetilde{u}(x_1, x_2) = x_1 x_2 \log\left(\frac{1}{\sqrt{x_1^2 + x_2^2}}\right)$?

*Hint:* Check the continuity of $\dfrac{\partial^2 \widetilde{u}}{\partial x_1^2}, \dfrac{\partial^2 \widetilde{u}}{\partial x_2^2}$ at the origin.

**Theorem 5.55.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $\nu$ be the $C^\infty$ vector field of outer normals on $\Gamma = \partial\Omega$. Let $\overline{A}$ according to (5.255) be the Neumann Laplacian with resolvent set $\varrho(\overline{A})$ and spectrum $\sigma(\overline{A})$.*

(i) *Let $f \in C^\infty(\Omega)$, $g \in C^\infty(\Gamma)$ and $\lambda \in \varrho(\overline{A})$. Then (the classical inhomogeneous Neumann problem)*

$$\Delta u + \lambda u = f \text{ in } \Omega \quad \text{and} \quad \mathrm{tr}_\Gamma \frac{\partial u}{\partial \nu} = g \text{ on } \Gamma \tag{5.276}$$

*has a unique solution $u \in C^\infty(\Omega)$.*

(ii) *Let $\lambda \in \sigma(\overline{A})$ be an eigenvalue of $\overline{A}$ and let $u$ be a related eigenfunction,*

$$-\Delta u = \lambda u \text{ in } \Omega \quad \text{and} \quad \mathrm{tr}_\Gamma \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma. \tag{5.277}$$

*Then $u \in C^\infty(\Omega)$.*

*Proof.* This is the counterpart of Theorem 5.52. One can follow the proof given there relying now on Theorem 5.50, Remark 5.51 and Corollary 5.45. □

**Exercise 5.56.** (a) Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$. Prove that there exist complete orthonormal systems $\{u_j\}_{j=1}^\infty \subset C^\infty(\Omega)$ in $L_2(\Omega)$.

*Hint:* Apply Theorem 5.52 or Theorem 5.55 to the (self-adjoint Dirichlet or Neumann) Laplacian.

(b) Apply (a) to prove that

$$\left\{\sqrt{\frac{2}{\pi}}\sin(mx)\right\}_{m=1}^\infty \quad \text{and} \quad \left\{\sqrt{\frac{1}{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}}\cos(mx)\right\}_{m=1}^\infty \tag{5.278}$$

are complete orthonormal systems in $L_2(I)$ with $I = (0, \pi)$.

## 5.10 Green's functions and Sobolev embeddings

One of the main problems in the classical theory of the Dirichlet Laplacian is the question of whether Green's functions according to Definition 1.10 exist. We discussed this point in Remark 1.11. So far we have a satisfactory answer in Theorem 1.12 only in case of balls. We refer also to Exercise 1.18. Recall our definition of $C^\infty(\Omega)$ in (A.9).

**Theorem 5.57.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3 where $n \geq 2$. Then there exists a (real uniquely determined) Green's function $g(x^0, x)$ according to Definition 1.10. Furthermore, for any $x^0 \in \Omega$ and $\varepsilon > 0$,*

$$g(x^0, \cdot) \in C^\infty(\Omega \setminus \overline{K_\varepsilon(x^0)}) \tag{5.279}$$

*if $K_\varepsilon(x^0) = \{y \in \mathbb{R}^n : |y - x^0| < \varepsilon\} \subset \Omega$, and*

$$g(x^1, x^2) = g(x^2, x^1), \quad x^1 \in \Omega,\ x^2 \in \Omega \text{ with } x^1 \neq x^2. \tag{5.280}$$

*If $n \geq 3$, then*

$$0 < g(x^0, x) < \frac{1}{(n-2)|\omega_n|} \frac{1}{|x - x^0|^{n-2}}, \quad x \in \Omega,\ x^0 \in \Omega,\ x \neq x^0. \tag{5.281}$$

*Proof.* If $\mathring{A} = -\Delta$ is the Dirichlet Laplacian in (5.267), then it follows from Theorem 5.31 (i) that $0 \in \varrho(\mathring{A})$. Theorem 5.52 (i) with $A = -\Delta$ and $\lambda = 0$ implies that (1.27) (with the usual modifications in case of $n = 2$) has a unique (and thus real) solution $\Phi \in C^\infty(\Omega)$. This proves the existence (and uniqueness) of the (real) Green's function and covers also (5.279). The remaining properties (5.280), (5.281) follow from Corollary 1.28 (in case of (5.280) extended to $n = 2$). $\square$

**Exercise* 5.58.** Is there a direct counterpart of (5.281) for $n = 2$?

The classical inhomogeneous Dirichlet problem for the Laplacian with $f \in C^\infty(\Omega)$ and $\varphi \in C^\infty(\Gamma)$, $\Gamma = \partial\Omega$, as formulated in (1.29), can now be solved by (1.28). In particular, if $\mathring{A} = -\Delta$ in (5.267) is the positive-definite self-adjoint operator with pure point spectrum according to Theorem 5.31 (i), then its compact inverse $(-\Delta)^{-1}$ in $L_2(\Omega)$ can be represented by

$$(-\Delta)^{-1} f(x) = u(x) = \int_\Omega g(x, y)\, f(y)\, \mathrm{d}y, \quad x \in \Omega, \tag{5.282}$$

at least if $f \in C^\infty(\Omega)$. Let $\chi$ be the characteristic function of a ball in $\mathbb{R}^n$ such that $\chi(x - y) = 1$ if $x \in \Omega$, $y \in \Omega$. We extend $f \in C^\infty(\Omega)$ outside $\Omega$ by zero.

Let $n \geq 3$; then (5.281) implies for $x \in \Omega$ that

$$|(-\Delta)^{-1} f(x)| \leq c \int_{\mathbb{R}^n} \frac{\chi(x-y)}{|x-y|^{n-2}} |f(y)| \mathrm{d}y$$
$$\leq c' \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n+\varepsilon-2}} |f(y)| \mathrm{d}y \tag{5.283}$$

for $0 \leq \varepsilon < 2$. We apply Theorem D.3, with $p = 2$, $\alpha = n - 2 + \varepsilon$, $\varepsilon$ near 2 and $q > p = 2$. Then one obtains for $f \in C^\infty(\Omega)$,

$$\|(-\Delta)^{-1} f | L_2(\Omega)\| \leq c \|(-\Delta)^{-1} f | L_q(\Omega)\| \leq c' \|f | L_2(\Omega)\|. \tag{5.284}$$

Hence the right-hand side of (5.282) is a bounded operator in $L_2(\Omega)$. Then it follows by completion that the inverse operator $(-\Delta)^{-1}$ can be represented for all $f \in L_2(\Omega)$ by (5.282). But (5.284) shows that one gets more.

**Theorem 5.59.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$. Let $p^*$, given by*

$$\frac{1}{p^*} = \frac{1}{2} - \frac{2}{n} \quad \textit{for } n \geq 5, \tag{5.285}$$

*be the* Sobolev exponent. *Then*

$$\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega) \tag{5.286}$$

*is compact if*

$$\begin{cases} 1 \leq p \leq \infty & \textit{for } n = 1, 2, 3, \\ 1 \leq p < \infty & \textit{for } n = 4, \\ 1 \leq p < p^* & \textit{for } n \geq 5. \end{cases} \tag{5.287}$$

*Furthermore,* id *in* (5.286) *is continuous, but not compact if*

$$p = p^* \quad \textit{for } n \geq 5. \tag{5.288}$$

*Proof. Step* 1. Let $n = 1$, 2 or 3. Then it follows from Theorem 4.17 (ii) with $s = 2$ and $l = 0$ that

$$\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow C(\Omega) \hookrightarrow L_p(\Omega), \quad 1 \leq p \leq \infty, \tag{5.289}$$

is compact where we used the boundedness of $\Omega$ in the last embedding. This covers the first line in (5.287).

*Step* 2. Let $n \geq 5$ and $p^*$ as in (5.285), hence $2 < p^* < \infty$, complemented by $p^* = \infty$ if $n = 4$. The continuity (compactness) of id in (5.286) can be reduced to the continuity (compactness) of

$$\mathrm{id}_0\colon W_{2,0}^2(\Omega) \hookrightarrow L_p(\Omega) \tag{5.290}$$

where $W^2_{2,0}(\Omega)$ has the same meaning as before, e.g., as in (5.11). This follows from the extension Theorem 4.1 where one may assume that the extension operator in (4.8) is multiplied with a suitable cut-off function in $\mathbb{R}^n$. We decompose $\mathrm{id}_0$ into

$$\begin{aligned}&\mathrm{id}_0(W^2_{2,0}(\Omega) \hookrightarrow L_p(\Omega))\\ &\quad = (-\Delta)^{-1}(L_2(\Omega) \hookrightarrow L_p(\Omega)) \circ (-\Delta)(W^2_{2,0}(\Omega) \hookrightarrow L_2(\Omega)) \qquad (5.291)\end{aligned}$$

where the latter is an isomorphic map according to the above consideration and we may assume that $(-\Delta)^{-1}$ is given by (5.282). We apply the Hardy–Littlewood–Sobolev inequality, Theorem D.3, in the same way as in (5.283), (5.284) with $\alpha = n-2+\varepsilon$, $0 \le \varepsilon < 2$, $p$ replaced by 2 and $q$ replaced by $p$, respectively, hence

$$2 < \frac{n}{2-\varepsilon}, \qquad \frac{1}{p} = \frac{n-2+\varepsilon}{n} - \frac{1}{2} = \frac{1}{p^*} + \frac{\varepsilon}{n}, \qquad (5.292)$$

and

$$\begin{cases} 0 < \varepsilon < 2,\ 1 \le p < p^* & \text{if } n = 4,\\ 0 \le \varepsilon < 2,\ 1 \le p \le p^* & \text{if } n \ge 5, \end{cases} \qquad (5.293)$$

using, in addition, that $\Omega$ is bounded. Then Theorem D.3 covers the continuity assertions in the above theorem if $n \ge 4$.

*Step* 3. We prove the compactness of id given by (5.286). Theorem 4.17 implies that the embedding

$$\mathrm{id}\colon W^2_2(\Omega) \hookrightarrow L_2(\Omega) \qquad (5.294)$$

is compact. Hence the unit ball $U$ in $W^2_2(\Omega)$ is precompact in $L_2(\Omega)$ and for any $\varepsilon > 0$ there exists a finite $\varepsilon$-net for the image of $U$ in $L_2(\Omega)$, that is, there exist finitely many elements $\{g_k\}_{k=1}^{K(\varepsilon)} \subset U$ such that for any $g \in U$ there is at least one $k$ with

$$\|g - g_k | L_2(\Omega)\| \le \varepsilon. \qquad (5.295)$$

Let $2 < p < q < p^*$ as indicated in Figure 5.9 aside with

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}$$

for some suitable $\theta \in (0,1)$.

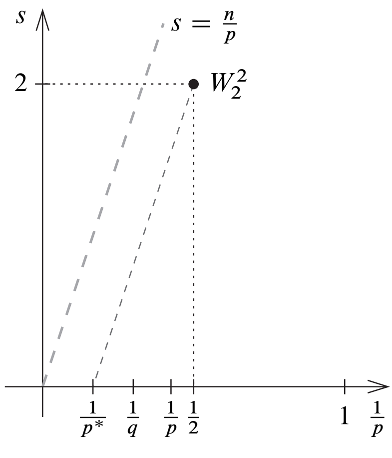



Figure 5.9

Since (5.286) with $q$ in place of $p$ is continuous, Hölder's inequality implies

$$\|g - g_k | L_p(\Omega)\| \le c\|g - g_k | L_q(\Omega)\|^{1-\theta} \|g - g_k | L_2(\Omega)\|^{\theta} \le c'\varepsilon^{\theta} \qquad (5.296)$$

for any $g \in U$ and appropriately chosen $g_k$. Hence $\{g_k\}_{k=1}^{K(\varepsilon)}$ is an $\widetilde{\varepsilon} = c'\varepsilon^\theta$-net in $L_p(\Omega)$. This proves that id is compact in all cases covered by (5.293) (using again the boundedness of $\Omega$).

*Step* 4. It remains to show that id in (5.286) is not compact when $p = p^*$ and $n \geq 5$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be a not identically vanishing function and

$$\varphi_j(x) = 2^{j(\frac{n}{2}-2)}\varphi(2^j x - x^j), \quad x^j \in \Omega, \ j \in \mathbb{N}, \tag{5.297}$$

such that

$$\operatorname{supp}\varphi_j \subset \Omega, \quad \text{and} \quad \operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k = \emptyset, \ j \neq k. \tag{5.298}$$

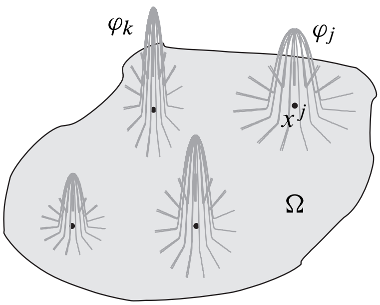



Figure 5.10

Then one obtains

$$\|\varphi_j | W_2^2(\Omega)\| \leq c \quad \text{and} \quad \|\varphi_j - \varphi_k | L_{p^*}(\Omega)\| \geq c' \tag{5.299}$$

for some $c > 0$ and $c' > 0$ and all $j, k \in \mathbb{N}$ with $j \neq k$. This shows that $\{\varphi_j\}_j$ is bounded in $W_2^2(\Omega)$, but not precompact in $L_{p^*}(\Omega)$. □

**Remark 5.60.** Continuity and compactness of the embedding in (5.286) are special examples of the famous *Sobolev embedding*, the never-ending bargain

*'Give smoothness and you get integrability.'*

It goes back to S. L. Sobolev [Sob38] and as far as limiting embeddings of type (5.285) with $p = p^*$ in (5.288) are concerned to [Kon45]. We refer also to [Sob91, §6] including Sobolev's own remarks concerning the limiting case. We add a few comments about these embeddings in Note 5.12.14.

## 5.11 Degenerate elliptic operators

So far we considered the homogeneous Dirichlet problem for elliptic differential expressions $A$ of second order,

$$(Au)(x) = -\sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{l=1}^{n} a_l(x)\frac{\partial u}{\partial x_l}(x) + a(x)u(x), \quad (5.300)$$

according to Definition 5.1 in bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$ (as introduced in Definition A.3) as (unbounded) operators in $L_2(\Omega)$ with domain of definition

$$\operatorname{dom}(A) = W^2_{2,0}(\Omega) = \{f \in W^2_2(\Omega) : \operatorname{tr}_\Gamma f = 0\} \quad (5.301)$$

and $\Gamma = \partial\Omega$. If we strengthen (5.1) by (5.168), that is,

$$\{a_{jk}\}^n_{j,k=1} \subset C^1(\Omega), \quad \{a_l\}^n_{l=1} \subset C^1(\Omega), \quad a \in C(\Omega), \quad (5.302)$$

then we can apply Theorem 5.36 where we now assume (without restriction of generality) that $0 \in \varrho(A)$. In particular,

$$\begin{aligned} A^{-1}\colon\ & L_2 \rightleftarrows W^2_{2,0}(\Omega) && \text{is isomorphic,} \\ A^{-1}\colon\ & L_2(\Omega) \hookrightarrow L_2(\Omega) && \text{is compact.} \end{aligned} \quad (5.303)$$

There are several good mathematical and physical reasons to have a closer look at *degenerate elliptic operators* typically of type

$$d(x)\,A, \quad A\,d(x), \quad \text{or} \quad d_1(x)\,A\,d_2(x), \quad (5.304)$$

where $d$, $d_1$ and $d_2$ are singular functions, for example, like $d(x) = |x - x^0|^\varkappa$, $x^0 \in \Omega$, $\varkappa \in \mathbb{R}$ (including $\varkappa = 2 - n$ as in the Newtonian potential (1.77)), $d(x) = \operatorname{dist}(x,\Gamma)^\varkappa$, $x \in \Omega$, $\varkappa \in \mathbb{R}$, or some singular potentials of quantum mechanics. We return to related questions later on in the Chapters 6 and 7 in greater detail.

At this moment we wish to demonstrate how the results of the preceding Section 5.10 including the Sobolev embeddings can be used to say something about degenerate elliptic operators of type (5.304). If $d_1$, $d_2$ are 'rough', then there is a problem with the domain of definition. This suggests to deal with the 'inverse',

$$B = b_2\,A^{-1}\,b_1 \quad \text{with } A^{-1} \text{ as in (5.303)}, \quad (5.305)$$

where $b_1, b_2$ are (singular) functions. One may think about $d_1 b_1 = d_2 b_2 = 1$ in the context of (5.304).

**Theorem 5.61.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$ and let $A$ be the above elliptic operator according to* (5.300)–(5.303).

(i) *Let* $1 \le n \le 3$, $2 \le p \le \infty$, *and*

$$b_1 \in L_q(\Omega) \ \ with \ \ \frac{1}{q} = \frac{1}{2} - \frac{1}{p} \ \ and \ \ b_2 \in L_p(\Omega). \tag{5.306}$$

*Then* $B$,

$$B = b_2\, A^{-1}\, b_1 : \ L_p(\Omega) \hookrightarrow L_p(\Omega) \tag{5.307}$$

*is compact.*

(ii) *Let* $n \ge 4$ *and* $p^*$ *be given by* $\frac{1}{p^*} = \frac{1}{2} - \frac{2}{n}$. *Assume* $2 \le p < p^*$, $1 \le r_1 \le \infty$, $1 \le r_2 \le \infty$, *and*

$$b_1 \in L_{r_1}(\Omega),\ b_2 \in L_{r_2}(\Omega) \quad with\ \frac{1}{r_1} = \frac{1}{2} - \frac{1}{p}\ and\ \frac{1}{r_1} + \frac{1}{r_2} < \frac{2}{n}. \tag{5.308}$$

*Then* $B$ *according to* (5.307) *is compact.*

*Proof.* Let $n \ge 4$; then we have the situation as indicated in Figure 5.11 below.

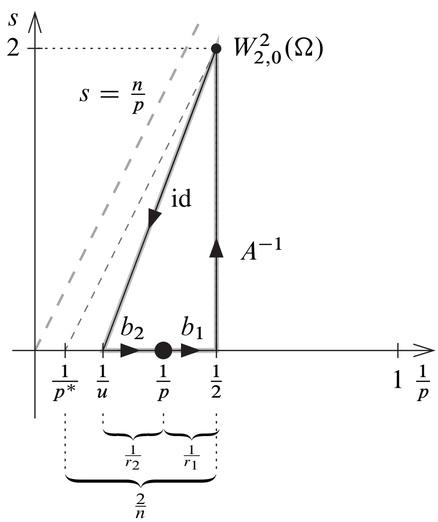



Figure 5.11

It follows from Hölder's inequality, (5.303) and Theorem 5.59 that

$$B = b_2 \circ \mathrm{id} \circ A^{-1} \circ b_1 \tag{5.309}$$

with

$$\begin{aligned} b_1 &: \ L_p(\Omega) \hookrightarrow L_2(\Omega), \\ A^{-1} &: \ L_2(\Omega) \hookrightarrow W^2_{2,0}(\Omega), \\ \mathrm{id} &: \ W^2_{2,0}(\Omega) \hookrightarrow L_u(\Omega), \\ b_2 &: \ L_u(\Omega) \hookrightarrow L_p(\Omega). \end{aligned} \tag{5.310}$$

is compact since id is compact. If $1 \le n \le 3$, then one may choose $u = \infty$ and (5.306) implies that $B$ is compact. □

**Remark 5.62.** According to the Riesz Theorem C.1 the spectrum of the compact operator $B$ in $L_p(\Omega)$ consists of the origin and at most countably many eigenvalues different from zero. Let $0 \ne \mu \in \sigma_{\mathrm{p}}(B)$ and $f \in L_p(\Omega)$ be a related eigenfunction,

$$b_2\, A^{-1}\, b_1\, f = Bf = \mu f. \tag{5.311}$$

Assume that $b_2 \ne 0$, then substituting $f = b_2 g$, $b = b_1 b_2$, $\lambda\mu = 1$ leads formally to

$$Ag = \lambda b(\cdot) g. \tag{5.312}$$

This type of modified eigenvalue problem attracted some attention originating from physical questions. One may consider the above theorem (combined with Theorem C.1) as a rigorous reformulation of (5.312) trying to compose given $b = b_1 b_2$ optimally and looking for a suitable $p$ such that Theorem 5.61 and (5.311) can be applied. We add a few comments in Note 5.12.15.

## 5.12 Notes

**5.12.1.** Chapter 5 dealt with Dirichlet and Neumann boundary value problems for elliptic differential expressions of second order according to the Definitions 5.1 and 5.3 in the framework of an $L_2(\Omega)$ theory where $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$. In case of the Neumann problem we mostly restricted our considerations to the Laplacian (5.5) and the $C^\infty$ vector field $\mu = \nu$ in (5.17) of outer normals on $\Gamma = \partial\Omega$. This theory can be extended to other basic spaces than $L_2(\Omega)$ and to more general (elliptic) differential operators, say, of order $2m$ with $m \in \mathbb{N}$,

$$Au = \sum_{|\alpha| \le 2m} a_\alpha(x) \mathrm{D}^\alpha u = f \text{ in } \Omega, \quad a_\alpha \in C^\infty(\Omega), \tag{5.313}$$

called *properly elliptic* if (5.4) is replaced by

$$\sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \neq 0 \quad \text{for all } x \in \bar{\Omega},\ 0 \neq \xi \in \mathbb{R}^n. \tag{5.314}$$

Typically the boundary conditions in (5.15), (5.17) are generalised by

$$B_j u = \sum_{|\beta| \le k_j} b_{j\beta}(\gamma) \operatorname{tr}_\Gamma \mathrm{D}^\beta u = g_j \text{ on } \Gamma, \quad b_{j\beta} \in C^\infty(\Gamma), \tag{5.315}$$

where $j = 1, \ldots, m$, and $k_j \in \mathbb{N}_0$ with

$$0 \le k_1 < k_2 < \cdots < k_m < 2m. \tag{5.316}$$

Several other conditions both for $A$ in (5.313), the boundary operators $B_j$ in (5.315) and, in particular, their interplay are needed to obtain a satisfactory theory generalising the Theorems 5.39, 5.46, 5.50. This is one of the major subjects of research in analysis since the late 1950s up to our time. As far as an $L_2$ theory is concerned we refer to the celebrated book by S. Agmon [Agm65]. The extension of this theory to $L_p$ spaces with $1 < p < \infty$ is more difficult and attracted a lot of attention. It may be found in [Tri78] including many references, especially to the original papers. One can replace $L_p$ with $1 < p < \infty$ by Hölder–Zygmund spaces $\mathcal{C}^s$ or spaces of type $B^s_{p,q}$, $F^s_{p,q}$ as mentioned briefly in Notes 3.6.1, 3.6.3 (restricted to $\Omega$). The corresponding theory in full generality has been developed in [FR95] and may be found in [RS96, Chapter 3]. We restrict ourselves here to an outstanding example which goes back to [Agm62] (we refer for formulations also to [Tri78, Sections 4.9.1, 5.2.1]):

*Let $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k \le m$. Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and $\mu$ be a non-tangential $C^\infty$ vector field on $\Gamma = \partial\Omega$. Let $A$ in (5.313) and $B_j$ in (5.315) be specified by*

$$Au = (-\Delta)^m u, \quad B_j u = \mathrm{tr}_\Gamma \left( \frac{\partial^{k+j-1} u}{\partial \mu^{k+j-1}} + \sum_{|\beta|<k+j-1} b_{j\beta}(\gamma) \mathrm{D}^\beta u \right), \tag{5.317}$$

*where $j = 1, \dots, m$ and $b_{j\beta} \in C^\infty(\Gamma)$. Then one has full counterparts of the appropriately modified Theorems 5.36, 5.46 in the framework of an $L_p$ theory with $1 < p < \infty$ (may be with exception of (5.191)).*

**5.12.2.** Proposition 5.19 plays a crucial rôle in our arguments. The density assertions proved there can be extended in several directions. Let $W^l_p(\mathbb{R}^n_+)$ with $l \in \mathbb{N}$ and $1 < p < \infty$ be the spaces considered in Theorem 3.41. Then $\mathcal{D}(\mathbb{R}^n_+)$ is dense in

$$\mathring{W}^l_p(\mathbb{R}^n_+) = \left\{ f \in W^l_p(\mathbb{R}^n_+) : \mathrm{tr}_\Gamma \frac{\partial^k f}{\partial \nu^k} = 0 \text{ with } k = 1, \dots, l-1 \right\} \tag{5.318}$$

where we used the same notation as in connection with (5.99), in particular,

$$\mathrm{tr}_\Gamma \frac{\partial^k f}{\partial \nu^k} = \frac{\partial^k f}{\partial x_n^k}(x', 0) = 0, \quad k = 1, \dots, l-1. \tag{5.319}$$

The case $p = 2$ is covered by Exercise 5.21. Otherwise we refer to [Tri78, Section 2.9.1, p. 211] where one finds also further assertions of this type.

**5.12.3.** In Appendix C, especially in the Sections C.2, C.3 we collected some assertions about self-adjoint and positive-definite operators in Hilbert spaces where the energy spaces and Friedrichs extension according to Theorem C.13 and Remark C.14 were of special interest for us. For operators $A_F$ as in (C.34) there is an elaborated spectral theory which, in particular, gives the possibility to introduce fractional powers $A_F^\varkappa$, $\varkappa \in \mathbb{R}$, of $A_F$ with their domains of definition $\mathrm{dom}(A_F^\varkappa)$. Of special interest is the observation that

$$\mathrm{dom}\big(\sqrt{A_F}\big) = H_A \quad \text{(the energy space)}. \tag{5.320}$$

We refer for details again to [Tri92a, Chapter 4]. In the concrete case as considered in Theorem 5.22 and the Remarks 5.23, 5.24 one gets for the shifted Dirichlet Laplacian and Neumann Laplacian,

$$\mathrm{dom}\Big(\sqrt{A_F^{\mathrm{D}}}\Big) = \mathring{W}^1_2(\mathbb{R}^n_+), \quad \mathrm{dom}\Big(\sqrt{A_F^{\mathrm{N}}}\Big) = W^1_2(\mathbb{R}^n_+). \tag{5.321}$$

**5.12.4.** In connection with the (abstract) Theorem C.3 and the concrete assertions about the spectrum of the Dirichlet Laplacian and the Neumann Laplacian, respectively, in Remark 5.24 and Exercise 5.25 we add a comment about the resolvent set $\varrho(A)$ and the spectrum $\sigma(A) = \mathbb{C} \setminus \varrho(A)$ of a self-adjoint operator $A$ in a Hilbert space. We prove in Theorem 6.8 below that $\lambda \in \varrho(A)$ if, and only if, there is a number $c > 0$ such that

$$\|Ah - \lambda h|H\| \geq c\|h|H\| \quad \text{for all } h \in \operatorname{dom}(A). \tag{5.322}$$

Hence $\lambda \in \sigma(A)$ if, and only if, there is no such $c > 0$ with (5.322). In other words, $\lambda \in \sigma(A)$ if, and only if, there is a sequence

$$\{h_j\}_{j=1}^{\infty} \subset \operatorname{dom}(A), \quad \|h_j|H\| = 1, \quad Ah_j - \lambda h_j \to 0 \text{ if } j \to \infty. \tag{5.323}$$

If there is a converging subsequence $\{\widetilde{h}_j\}_{j=0}^{\infty}$ of $\{h_j\}_{j=1}^{\infty}$, then $h = \lim_{j\to\infty} \widetilde{h}_j$ is an eigenelement of $A$ and, hence, $\lambda \in \sigma_{\mathsf{p}}(A)$ belongs to the point spectrum. On the other hand, a sequence $\{h_j\}_j$ according to (5.323) is called a *Weyl sequence* (of $A$ corresponding to $\lambda \in \mathbb{C}$) if it does not contain a converging subsequence and

$$\sigma_{\mathsf{e}}(A) = \{\lambda \in \mathbb{C} : \text{ there is a Weyl sequence of } A \text{ corresponding to } \lambda\} \tag{5.324}$$

is called the *essential spectrum* of $A$. Then

$$\sigma(A) = \sigma_{\mathsf{p}}(A) \,\cup\, \sigma_{\mathsf{e}}(A) \subset \mathbb{R}. \tag{5.325}$$

In particular, by (5.128), (5.129) one has

$$\sigma(A_F^{\mathsf{D}}) = \sigma_{\mathsf{e}}(A_F^{\mathsf{D}}) = [1,\infty), \quad \sigma(A_F^{\mathsf{N}}) = \sigma_{\mathsf{e}}(A_F^{\mathsf{N}}) = [1,\infty) \tag{5.326}$$

for the shifted Dirichlet Laplacian and Neumann Laplacian in $\mathbb{R}^n_+$, respectively. If one replaces $\mathbb{R}^n_+$ by a bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$, then Theorem 5.31 implies a totally different assertion,

$$\sigma_{\mathsf{e}}(A_{\mathsf{D},F}) = \sigma_{\mathsf{e}}(A_{\mathsf{N},F}) = \emptyset. \tag{5.327}$$

A discussion of various types of spectra in the context of quasi-Banach spaces may be found in [ET96, Section 1.2]. We return to problems of this type in the Chapters 6 and 7 in greater detail.

**5.12.5.** Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3 which means in particular that $\Omega$ is connected. According to Theorem 5.31 (ii) the Neumann Laplacian (5.161) is a positive operator in $L_2(\Omega)$ with pure point spectrum. Its smallest eigenvalue is 0 and this eigenvalue is simple in the understanding of (5.156), the related eigenfunctions are constant in $\Omega$. By the same theorem the Dirichlet Laplacian

$$A_{\mathsf{D},F}u = -\Delta u, \quad \operatorname{dom}(A_{\mathsf{D},F}) = \mathring{W}^2_{2,0}(\Omega), \tag{5.328}$$

is a positive-definite operator in $L_2(\Omega)$ with pure point spectrum. Its smallest eigenvalue, denoted by $\lambda$, is positive (obviously) and simple (remarkably). Furthermore, according to Theorem 5.52 one has

$$u \in C^\infty(\Omega) \text{ for the eigenfunctions } -\Delta u = \lambda u \text{ in } \Omega. \tag{5.329}$$

It has been observed by R. Courant in 1924, [CH53, pp. 398/399] that any such (non-trivial) eigenfunction (5.329) has no zeros in $\Omega$ (called *'Nullstellenfreiheit'* by him) and, hence,

$$u(x) = c\, U(x), \quad c \in \mathbb{C},\ c \neq 0,\ U(x) > 0 \text{ in } \Omega. \tag{5.330}$$

Courant's strikingly short elegant proof of (5.329), (5.330) on less than one page entitled

*'Charakterisierung der ersten Eigenfunktion durch ihre Nullstellenfreiheit'*

indicates what follows in a few lines. Based on quadratic forms Courant relies (as we would say nowadays) on $W_2^1$ arguments. But he did not bother very much about the technical rigour of his proof. A more recent version may be found in [Tay96, pp. 315/316].

**5.12.6.** The Riesz theory as presented and discussed in Theorem C.1 and Remark C.2 stresses the *algebraic multiplicity* of the non-zero eigenvalues of compact operators in (quasi-) Banach spaces. Formally one can extend the notion of algebraic multiplicity according to (C.11) to unbounded operators $A$ in Hilbert spaces and Banach spaces, respectively. But then one may have some trouble with the domains of definition of the powers of $A$ whereas the *geometric multiplicity* does not cause any problems at all. We discussed this point in Remark 5.35 where $A$ is the elliptic operator (5.167) underlying Proposition 5.34 and Theorem 5.36. As for the algebraic multiplicity of $\mu \in \sigma_{\mathrm{p}}(A)$ it might be better to shift this question to its compact inverse $T = A^{-1}$ and $\mu^{-1} = \lambda \in \sigma_{\mathrm{p}}(T)$ in (C.11) (assuming $0 \in \varrho(A)$). One may replace $A^{-1}$ by any $(A - \varkappa\, \mathrm{id})^{-1}$ with $\varkappa \in \varrho(A)$. A detailed discussion about these questions may be found in [Agm65, Section 12, especially pp. 179–181]. Recall that

$$u \in \bigcup_{k=1}^{\infty} \ker(A^{-1} - \lambda^{-1}\mathrm{id})^k, \quad \lambda \in \sigma_{\mathrm{p}}(A), \tag{5.331}$$

is called an *associated* (or *generalised*) *eigenelement* with respect to $A$ and $\lambda$. As discussed in Theorem C.15 and Remark C.16 the eigenelements of self-adjoint (positive-definite) operators with pure point spectrum span the underlying Hilbert space. One may ask for conditions ensuring that associated eigenelements of non-self-adjoint operators span the underlying Hilbert space. The abstract theory has been developed in [GK65]. Some information may also be found in [Tri78, Section 5.6.1, pp. 394/395]. Corresponding assertions with respect to elliptic differential operators of order $2m$ as briefly mentioned in Note 5.12.1 (covering, in particular, elliptic operators of second order as treated in this chapter) may be found in [Agm65, Section 16] and also in [Agm62]. We return later on in Section 7.5 to this point.

**5.12.7.** Let $A$ be a (closed) densely defined operator in a Hilbert space or a Banach space such that $(-\infty, \lambda_0] \subset \varrho(A)$ for some $\lambda_0 \le -1$. If there is a $d > 0$ such that

$$\|(A - \lambda\,\mathrm{id})^{-1}\| \le \frac{d}{|\lambda|} \quad \text{for all } \lambda \le \lambda_0, \tag{5.332}$$

then one says that the resolvent $R_\lambda = (A - \lambda\,\mathrm{id})^{-1}$ has *minimal growth*. A typical example in our context is the operator $A$ in Theorem 5.36 with (5.190). There is no better decay than in (5.332). We must even have $d \ge 1$. We prove this assertion by contradiction assuming that we have (5.332) with $d < 1$. Obviously

$$A - \mu\,\mathrm{id} = (A - \lambda\,\mathrm{id})[\mathrm{id} - (\mu - \lambda)R_\lambda] \quad \text{for } \mu \in \mathbb{C}. \tag{5.333}$$

However, both operators on the right-hand side are invertible for some $\lambda \le \lambda_0$, the second one according to the Neumann series applied to

$$|\mu - \lambda|\|R_\lambda\| \le d\,\frac{|\mu - \lambda|}{|\lambda|} \longrightarrow d < 1 \quad \text{if } |\lambda| \to \infty. \tag{5.334}$$

Furthermore,

$$\|R_\mu\| \le C\,\|R_\lambda\| \longrightarrow 0 \quad \text{if } |\lambda| \to \infty, \tag{5.335}$$

where $C$ may be chosen independently of $\mu$. Hence $R_\mu = 0$ which is a contradiction. Operators $A$ in Hilbert spaces and Banach spaces with resolvents having minimal growth according to (5.332) are the best possible generalisations of (unbounded) self-adjoint operators in Hilbert spaces. Several properties of self-adjoint operators in Hilbert spaces can be extended to this distinguished class of (unbounded) operators, including integral representations (as a weak version of spectral representations), fractional powers and their domains of definition in terms of (real and complex) interpolation spaces. One may consult [Tri78, Sections 1.14, 1.15] where one finds also the necessary references to the original papers.

**5.12.8.** In Step 1 of the proof of Theorem 5.36 we relied on the so-called *continuity method* which has been used before in [LU64, Chapter III, §§1,3] and [GT01, Section 6.3] for similar purposes.

**5.12.9.** For a bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ we have by Theorem 4.24 that for $s \ge 0$,

$$\mathrm{tr}_\Gamma : W_2^{s+2}(\Omega) \hookrightarrow W_2^{s+\frac{3}{2}}(\Gamma), \quad \mathrm{tr}_\Gamma\, W_2^{s+2}(\Omega) = W_2^{s+\frac{3}{2}}(\Gamma). \tag{5.336}$$

We used this observation with $s = k \in \mathbb{N}_0$ in the proof of Theorem 5.46. In particular $T_\lambda$, given by (5.246), generates the isomorphic map according to (5.249). It is quite natural to ask whether this assertion can be extended from the classical Sobolev spaces $W_2^k(\Omega)$ with $k \in \mathbb{N}_0$ to arbitrary Sobolev spaces $W_2^s(\Omega)$ with $s \geq 0$. By Theorem 4.1 and Proposition 4.22 we have for all spaces $W_2^s(\Omega)$ and $W_2^s(\Gamma)$ with $s \geq 0$ natural intrinsic norms. One can extend (5.249) from $k \in \mathbb{N}_0$ to $s \geq 0$ by real or complex interpolation. The corresponding theory is beyond the scope of this book, but all that one needs can be found in [Tri78]. An interpolation method, say, the so-called *complex method* $[\cdot,\cdot]_\theta$, constructs a new Banach space

$$X_\theta = [X_0, X_1]_\theta, \quad 0 < \theta < 1, \tag{5.337}$$

from two given (complex) Banach spaces $X_0$, $X_1$, say, with $X_1 \subset X_0$. In case of the above Sobolev spaces one obtains for $0 \leq s_0 < s_1 < \infty$,

$$W_2^s(\Omega) = [W_2^{s_0}(\Omega), W_2^{s_1}(\Omega)]_\theta, \quad s = (1-\theta)s_0 + \theta s_1, \tag{5.338}$$

hence an 'intermediate' space. One can replace $\Omega$ in (5.338) by $\Gamma = \partial\Omega$. Then the so-called *interpolation property* extends immediately (without any further considerations) the isomorphism (5.249) from $0 \leq s_0 = k_0 \in \mathbb{N}_0$ and $s_0 < s_1 = k_1 \in \mathbb{N}$ to all $s \geq 0$. In other words one gets the following assertion:

*Under the hypotheses of Theorem 5.46 the operator $T_\lambda$ given by (5.246) generates for all $s \geq 0$ an isomorphic map*

$$T_\lambda : W_2^{s+2}(\Omega) \rightleftarrows W_2^s(\Omega) \times W_2^{s+\frac{3}{2}}(\Gamma). \tag{5.339}$$

**5.12.10.** We asked in Remark 5.53 for extensions and modifications of the $L_2$ theory subject of this chapter and also of the preceding Note 5.12.9. Recall that we have for the spaces $W_p^k(\Omega)$ with $k \in \mathbb{N}_0$ and $1 < p < \infty$ satisfactory intrinsic norms according to Theorem 4.1 (ii). An extension of the classical Sobolev spaces $W_p^k(\Omega)$ to the Sobolev spaces $H_p^s(\Omega)$ with $1 < p < \infty$ and $s \geq 0$ has been indicated in Note 4.6.4 as the restriction of the corresponding spaces on $\mathbb{R}^n$ according to (3.140)–(3.142). In particular,

$$H_p^k(\Omega) = W_p^k(\Omega) \quad \text{if } k \in \mathbb{N}_0 \text{ and } 1 < p < \infty. \tag{5.340}$$

As for traces of $H_p^s(\Omega)$ on $\Gamma = \partial\Omega$ we recall (4.153). Then (5.336) can be generalised by

$$\mathrm{tr}_\Gamma : H_p^{s+2}(\Omega) \hookrightarrow B_{p,p}^{s+2-\frac{1}{p}}(\Gamma), \quad \mathrm{tr}_\Gamma\, H_p^{s+2}(\Omega) = B_{p,p}^{s+2-\frac{1}{p}}(\Gamma), \tag{5.341}$$

where $B_{p,p}^\sigma(\Gamma)$ are the same Besov spaces on $\Gamma$ as in Note 4.6.4. One can extend the isomorphism (5.339) as follows:

*Under the hypotheses of Theorem 5.46 the operator $T_\lambda$ given by (5.246) generates for all $s \ge 0$ and all $1 < p < \infty$ an isomorphic map*

$$T_\lambda\colon H_p^{s+2}(\Omega) \rightleftarrows H_p^s(\Omega) \times B_{p,p}^{s+2-\frac{1}{p}}(\Gamma). \tag{5.342}$$

For a proof of this assertion, diverse modifications and also the generalisations to elliptic differential equations of higher order as indicated in Note 5.12.1 we refer to [Tri78]. In case of $p = \infty$ one must replace the Sobolev–Besov spaces in (5.342) by respective Hölder–Zygmund spaces $\mathcal{C}^s$ on $\Omega$ and on $\Gamma$ as restrictions of corresponding spaces $\mathcal{C}^s(\mathbb{R}^n)$ according to Note 3.6.1, especially (3.146), to $\Omega$ and $\Gamma$. Then one gets that $T_\lambda$ is an isomorphic map

$$T_\lambda\colon \mathcal{C}^{s+2}(\Omega) \rightleftarrows \mathcal{C}^s(\Omega) \times \mathcal{C}^{s+2}(\Gamma) \tag{5.343}$$

for all $s > 0$. One may consult [Tri78, Section 5.7.3] or, better, [Tri83, Section 4.3.4].

**5.12.11.** Whereas the assertions of the preceding Note 5.12.10 for the Sobolev spaces $H_p^s(\Omega)$ and the Hölder–Zygmund spaces $\mathcal{C}^s(\Omega)$ are satisfactory we indicated in Remark 5.53 that nothing of this type can be expected in terms of the spaces $C^k(\Omega)$ with $k \in \mathbb{N}_0$. Let

$$\Omega = K = K_1(0) = \{x \in \mathbb{R}^2 : |x| < 1\} \tag{5.344}$$

be the unit circle in the plane $\mathbb{R}^2$ and let $\Gamma = \partial\Omega = \{y \in \mathbb{R}^2 : |y| = 1\}$ be its boundary. Let $\psi \in \mathcal{D}(K)$ with $\psi(0) \neq 0$ and $\psi(y) = 0$ if $|y| > \varepsilon$. Then

$$f(x) = \Delta(u\psi)(x) \in C(K), \tag{5.345}$$

where $u$ is the same function as in (5.275), assuming that $\varepsilon > 0$ is sufficiently small. Then the homogeneous Dirichlet problem

$$\Delta v(x) = f(x) \text{ if } x \in K, \quad v(y) = 0 \text{ if } |y| = 1, \tag{5.346}$$

has a unique solution which belongs to $W_{2,0}^2(K)$ as a consequence of Theorem 5.31. By Exercise 5.54 the unique solution $v = u\psi$ of (5.346) does not belong to $C^2(K)$. This disproves (5.273) with $k = 0$ and makes clear that nothing like (5.343) can be expected. Essentially, the end of Remark 5.53 and Exercise 5.54 are reformulations of this negative assertion in terms of function spaces. There one finds also a few related references. As a consequence the Dirichlet problem for the Poisson equation according to Definition 1.43 has not always a classical solution $u \in C(\Omega) \cap C^{2,\mathrm{loc}}(\Omega)$ for given $f \in C(\Omega)$ and $\varphi \in C(\Gamma)$. This nasty effect is known for a long time and usually discussed in literature in terms of counter-examples. Almost the same counter-example as above was used in [LL97, p. 223], whereas different ones may be found in [GT01, Problem 4.9, p. 71] and [Fra00, Exercise A.29, pp. 217/218]. Closely related to assertions of type (5.343) for the Laplacian in terms of Hölder–Zygmund spaces, but also to the questions discussed above is the problem of the smoothness of the Newtonian potential $\mathcal{N} f$ according to (1.77) in dependence on the smoothness of $f$. This was the decisive ingredient in Theorem 1.48. So far we proved in Theorem 1.45 that $u = \mathcal{N} f \in C^{2,\mathrm{loc}}(\mathbb{R}^n)$ if $f \in C^2(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$. But this assertion can be strengthened (with some additional efforts) in a natural way as follows. Let $\mathcal{C}^s(\mathbb{R}^n)$ be again the Hölder–Zygmund spaces as used in Note 5.12.10 with a reference to Note 3.6.1. Then

$$u = \mathcal{N} f \in \mathcal{C}^{2+s}(\mathbb{R}^n) \text{ locally, if } f \in \mathcal{C}^s(\mathbb{R}^n), \tag{5.347}$$

$0 < s < 1$, and supp $f$ compact. Furthermore,

$$\Delta u(x) = f(x), \quad x \in \mathbb{R}^n. \tag{5.348}$$

A proof of this well-known assertion may be found in [Fra00, Theorem A.16, p. 211]. Moreover, the above-mentioned counter-example [Fra00, Exercise A.29, pp. 217/218] makes also clear that there are compactly supported functions $f \in C(\mathbb{R}^2)$ such that $u = \mathcal{N} f$ satisfies (5.348), but does not belong to $C^{2,\mathrm{loc}}(\mathbb{R}^2)$.

**5.12.12.** Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$. Then it follows from Friedrichs's inequality in Proposition 5.28 (iii) that

$$\langle u, v \rangle_{\mathring{H}^1(\Omega)} = \sum_{j=1}^n \int_\Omega \frac{\partial u}{\partial x_j}(x) \, \frac{\partial \overline{v}}{\partial x_j}(x) \, \mathrm{d}x \tag{5.349}$$

is a scalar product generating an equivalent norm in $\mathring{H}^1(\Omega) = \mathring{W}_2^1(\Omega)$. One obtains by the same inequality that for given $f \in L_2(\Omega)$,

$$v \in \mathring{H}^1(\Omega) \longmapsto \langle f, \overline{v} \rangle_{L_2(\Omega)} = \int_\Omega f(x) v(x) \mathrm{d}x, \tag{5.350}$$

is a linear and bounded functional on $\mathring{H}^1(\Omega)$. Hence there is a uniquely determined $u \in \mathring{H}^1(\Omega)$ such that

$$\langle u, \varphi \rangle_{\mathring{H}^1(\Omega)} = \langle f, \varphi \rangle_{L_2(\Omega)}, \quad \varphi \in \mathcal{D}(\Omega). \tag{5.351}$$

Here we used that $\mathcal{D}(\Omega)$ is dense in $\mathring{H}^1(\Omega)$ which is covered by Proposition 5.28 (ii). By (5.349) with $\overline{v} = \varphi \in \mathcal{D}(\Omega)$, and standard notation of the theory of distributions according to (2.40) one gets

$$(-\Delta u)(\varphi) = f(\varphi) \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{5.352}$$

Usually one calls the uniquely determined $u$ originating from (5.351) and (5.352) a *weak solution* of

$$-\Delta u = f \in L_2(\Omega), \quad u \in \mathring{H}^1(\Omega). \tag{5.353}$$

But one can say more than this. First we remark that by Proposition 5.28 (ii) or Remark 5.29, any $u \in \mathring{H}^1(\Omega)$, extended by zero outside of $\Omega$, belongs to $H^1(\mathbb{R}^n) = W_2^1(\mathbb{R}^n)$. Then it follows easily from Definition 3.13 that $\Delta u \in H^{-1}(\mathbb{R}^n)$ and by restriction as in Definition 3.37 that $\Delta u \in H^{-1}(\Omega)$. One obtains

$$-\Delta \colon \mathring{H}^1(\Omega) \hookrightarrow H^{-1}(\Omega). \tag{5.354}$$

Let $\omega$ be an arbitrary bounded domain in $\mathbb{R}^n$ (that is, an arbitrary open set in $\mathbb{R}^n$). Let $H^1(\omega)$ and $H^{-1}(\omega)$ be defined by restriction of the corresponding spaces on $\mathbb{R}^n$ to $\omega$ as in Definition 3.37, and let $\mathring{H}^1(\omega)$ be the completion of $\mathcal{D}(\omega)$ in $H^1(\omega)$. Then it follows by standard arguments that $\mathring{H}^1(\omega)$ can be equivalently normed by

$$\|u|\mathring{H}^1(\omega)\| = \Big( \sum_{j=1}^n \Big\| \frac{\partial u}{\partial x_j} \,|L_2(\omega)\Big\|^2 \Big)^{1/2} \tag{5.355}$$

generated by the scalar product (5.349) with $\omega$ in place of $\Omega$. Within the dual pairing $(\mathcal{D}(\omega), \mathcal{D}'(\omega))$ one gets for the dual space of $\mathring{H}^1(\omega)$ that

$$(\mathring{H}^1(\omega))' = H^{-1}(\omega). \tag{5.356}$$

We refer for details, proofs and explanations to [Tri01, Proposition 20.3, pp. 296–298].

Let $\Omega$ be again a bounded $C^\infty$ domain in $\mathbb{R}^n$. Then one can extend the arguments in (5.351)–(5.353) from $L_2(\Omega)$ to $H^{-1}(\Omega)$. Together with (5.354) one gets that

$$-\Delta \colon \mathring{H}^1(\Omega) \hookrightarrow H^{-1}(\Omega) \text{ is an isomorphic map.} \tag{5.357}$$

This complements the previous assertion that

$$-\Delta \colon W^2_{2,0}(\Omega) \hookrightarrow L_2(\Omega) \text{ is an isomorphic map,} \tag{5.358}$$

which is covered by Theorem 5.31 (i) and which can also be obtained from the above considerations if one applies, in addition, Proposition 5.26 (i). However, the main advantage of the method of weak solutions is not so much that one can complement (5.358) by (5.357), but that it can be applied to more general situations.

Let now $\omega$ be an arbitrary bounded domain in $\mathbb{R}^n$ and let $\{a_{jk}(x)\}_{j,k=1}^n \subset L_\infty(\omega)$ with

$$a_{jk}(x) = a_{kj}(x) \in \mathbb{R}, \quad x \in \overline{\omega},\ 1 \le j,k \le n, \tag{5.359}$$

such that

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \;\geq\; E|\xi|^2, \quad x \in \overline{\omega},\ \xi \in \mathbb{R}^n, \tag{5.360}$$

for some $E > 0$ as in (5.169), (5.170). Then for given $f \in L_2(\omega)$ there is a unique $u \in \mathring{H}^1(\omega)$, called *weak solution*, such that

$$\int_\omega \sum_{j,k=1}^{n} a_{jk}(x)\,\frac{\partial u}{\partial x_j}(x)\,\frac{\partial \overline{\varphi}}{\partial x_k}(x)\,\mathrm{d}x = \int_\omega f(x)\overline{\varphi}(x)\mathrm{d}x \tag{5.361}$$

for all $\varphi \in \mathcal{D}(\omega)$. This follows from the above considerations and the observation that the left-hand side of (5.361) is a scalar product generating a norm which is equivalent to the norm in (5.355). The above arguments and (5.356) imply that one can replace the right-hand side of (5.361) by $f(\overline{\varphi})$ with $f \in H^{-1}(\omega)$ and $\varphi \in \mathcal{D}(\omega)$. One gets again a uniquely determined weak solution $u \in \mathring{H}^1(\omega)$. Some additional smoothness assumptions for the coefficients $a_{jk}$ ensure also a counterpart of (5.354). Altogether one obtains the following assertion:

*Let $\omega$ be an arbitrary bounded domain in $\mathbb{R}^n$. Let $A$,*

$$(Au)(x) = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{jk}(x)\frac{\partial u}{\partial x_k}\Big), \quad u \in \mathring{H}^1(\omega), \tag{5.362}$$

*be an elliptic differential operator with $\{a_{jk}\}_{j,k=1}^n \subset C^1(\omega)$, (5.359), and (5.360). Then*

$$A\colon \mathring{H}^1(\omega) \rightleftarrows H^{-1}(\omega) \text{ is an isomorphic map.} \tag{5.363}$$

By the above comments it remains to justify the counterpart of (5.354). This follows from $a_{jk}v \in H^{-1}(\omega)$ if $v \in H^{-1}(\omega)$ obtained by the above duality (5.356) from $a_{jk}u \in \mathring{H}^1(\omega)$ if $u \in \mathring{H}^1(\omega)$. Finally, we refer to [Tri92a, Section 6.2] dealing in detail with weak solutions for boundary value problems for second order elliptic operators.

**5.12.13.** It is quite natural to ask how (*strong*) *solutions* of second order elliptic equations with (5.358) as a proto-type, *weak solutions* as indicated in the preceding Note 5.12.12, and *classical solutions* as briefly mentioned in the Notes 1.7.1 and 1.7.2 are related to each other. This is not the subject of this book, but a few comments and references may be found in [Tri01, Section 20.14, pp. 309/310].

**5.12.14.** So far we discussed in Theorem 4.17 (ii) compact embeddings of $W_2^s(\Omega)$ in $C^l(\Omega)$ and in Theorem 5.59 Sobolev embeddings in a rather specific case. These are two examples of a far-reaching theory of embeddings between function spaces, one of the major topics of the theory of function spaces as it may be found in [Tri78], [Tri83], [Tri92b]. We restrict ourselves to a few assertions which are directly related to the above spaces. In particular, let $H^s_p(\Omega)$ with $s > 0$, $1 < p < \infty$, and $\mathcal{C}^s(\Omega)$ with $s > 0$ be the same spaces as in Note 5.12.10 where again $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$. Then one has the following assertions which generalise and modify the above-mentioned results.

(i) *Let* $1 < p < \infty$, $s > 0$, *and* $s - \frac{n}{p} > \sigma > 0$, $1 \le q \le \infty$. *Then both*

$$\mathrm{id}\colon H^s_p(\Omega) \hookrightarrow \mathcal{C}^\sigma(\Omega) \quad \textit{and} \quad \mathrm{id}\colon H^s_p(\Omega) \hookrightarrow L_q(\Omega) \tag{5.364}$$

*are compact.*

(ii) *Let* $1 < p < \infty$, $s > 0$, *and* $s - \frac{n}{p} = 0$, $1 \le q < \infty$. *Then*

$$\mathrm{id}\colon H^s_p(\Omega) \hookrightarrow L_q(\Omega) \tag{5.365}$$

*is compact.*

(iii) *Let* $1 < p < \infty$, $s > 0$, *and* $s - \frac{n}{p} = -\frac{n}{p^*} < 0$. *Then*

$$\mathrm{id}\colon H^s_p(\Omega) \hookrightarrow L_q(\Omega), \quad 1 \le q < p^*, \tag{5.366}$$

*is compact and*

$$\mathrm{id}\colon H^s_p(\Omega) \hookrightarrow L_{p^*}(\Omega) \tag{5.367}$$

*is continuous, but not compact.*

Although these are special cases of a more general embedding theory, they complement and illustrate the specific assertion in the Theorems 4.17 and 5.59. In connection with $l = 0$ in (4.87) and also $n = 4$ in (5.285) one may ask whether (5.367) can be extended to the limiting situation

$$\mathrm{id}\colon H^{n/p}_p(\Omega) \hookrightarrow L_\infty(\Omega), \quad 1 < p < \infty, \tag{5.368}$$

which is also illustrated in Figure 5.9. But this is not the case, recall also Exercises 3.6 and 3.33. Problems of this type have been studied in detail in literature and resulted finally in the theory of envelopes as it may be found in [Har07].

**5.12.15.** The interest in eigenvalue problems of type (5.312) where $A$ might be an elliptic operator of second order (or higher order as outlined briefly in Note 5.12.1) and $b$ is a singular function comes from physics. For example, the eigenfrequencies $e^{i\lambda t}$ (where $t$ represents the time) of a vibrating drum (or membrane) in a bounded domain $\Omega \subset \mathbb{R}^2$ can be characterised as the eigenvalues $\lambda^2$ of the boundary value problem

$$-\Delta u(x) = \lambda^2 m(x) u(x), \quad x \in \Omega, \quad \mathrm{tr}_\Gamma\, u = 0, \tag{5.369}$$

where $m(x)$ is the mass density. We return in Example 6.1 below to this point. If the mass is unevenly or even discontinuously distributed on the membrane, then one gets a problem of type (5.312) or (5.369) with non-smooth $m(\cdot)$. Other examples come from quantum mechanics where $m(\cdot)$ stands for (singular) potentials. One may consult Note 6.7.1. The first comprehensive study may be found in [BS72], [BS73]. Theorem 5.61 might be considered as an example in the context of this book. Otherwise we refer to [ET96] dealing systematically with problems of this type. Recently there is a growing interest in the replacement of (possibly singular) functions $b(\cdot)$ in (5.312) or $m(\cdot)$ in (5.369) by finite Radon measures in $\mathbb{R}^n$. This comes again from quantum mechanics but also from fractal analysis, resulting in the fractal counterpart of (5.305), say,

$$B = (-\Delta)^{-1} \circ \mu, \quad \mu \text{ finite Radon measure.} \tag{5.370}$$

Of course, first one has to clarify what this means. We refer to [Tri97], [Tri01], [Tri06] where problems of this type have been considered systematically.

# Chapter 6

# Spectral theory in Hilbert spaces and Banach spaces

## 6.1 Introduction and examples

So far we got in Chapter 5 a satisfactory $L_2$ theory for (Dirichlet and Neumann) boundary value problems for second order elliptic differential equations in bounded smooth domains. This covers first *qualitative* assertions about the spectrum of the operators considered. Of special interest is the question whether these operators have a pure point spectrum or whether related inverse operators are compact. We refer to the Theorems 5.31 (Laplace operator), 5.36 (second order operators, not necessarily self-adjoint), and 5.61 (degenerate operators). In Note 5.12.4 we discussed some types of spectra, mostly to illuminate what had been said before. Now we return to these questions in greater detail, mainly interested in *quantitative* assertions, especially the

*distribution of eigenvalues.*

This Chapter 6 deals with the abstract background, especially approximation numbers, entropy numbers and their relations to spectra. This will be used in Chapter 7 to discuss the spectral behaviour of elliptic operators.

The distribution of eigenvalues of elliptic operators is one of the outstanding problems of mathematics in the last century up to our time. This interest comes not only from challenging mathematical questions, but even more from its numerous applications in physics. We give some examples for both.

**Example 6.1** (Vibrating membrane). We suppose that a membrane fills a bounded, say, $C^\infty$ domain $\Omega$ in the plane $\mathbb{R}^2$, fixed at its boundary $\Gamma = \partial\Omega$ and buckling under the influence of a force with the continuous *density* $p(x)$, $x \in \Omega$, in vertical directions, see Figure 6.1 below.

Let $v \in C^2(\Omega)$ with $\mathrm{tr}_\Gamma\, v = 0$ be the elongation resulting in a surface $F$ which can be described as

$$x_3 = v(x), \ x = (x_1, x_2) \in \bar{\Omega} \subset \mathbb{R}^2$$

with $v(x) = 0$ if $x \in \Gamma$. Let $|F|$ be the surface area and

$$\Delta F = |F| - |\Omega|$$

be the enlargement of the membrane under the influence of $p$.

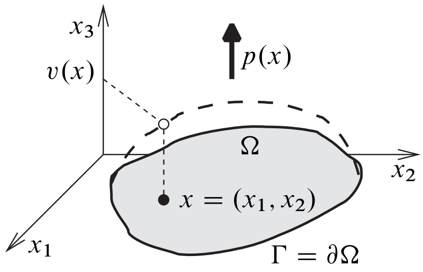



Figure 6.1

The corresponding potential $J(v)$ is given by

$$\begin{aligned} J(v) &= \Delta F + \int\limits_{\Omega} p(x)v(x)\mathrm{d}x \\ &= \int\limits_{\Omega} \left( \sqrt{1 + \sum_{j=1}^{2} \left( \frac{\partial v}{\partial x_j}(x) \right)^2} - 1 + p(x)v(x) \right) \mathrm{d}x \\ &\sim \int\limits_{\Omega} \left( \frac{1}{2} |\nabla v(x)|^2 + p(x)v(x) \right) \mathrm{d}x = \mathring{J}(v), \end{aligned} \tag{6.1}$$

where we first used Theorem A.8 and afterwards assumed that the elongation is so small that $v$ and its first derivatives are small. The *wisdom of nature* (i.e., to be as stable as possible) or the *calculus of variations* (resulting in the Euler–Lagrange equations) propose that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathring{J}(v + \varepsilon\varphi)_{|\varepsilon=0} = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega), \tag{6.2}$$

where $\varepsilon \geq 0$. Then (6.1) and integration by parts imply

$$\begin{aligned} 0 &= \int\limits_{\Omega} \Big( \sum_{j=1}^{2} \frac{\partial v}{\partial x_j}(x) \, \frac{\partial \varphi}{\partial x_j}(x) + p(x)\varphi(x) \Big) \mathrm{d}x \\ &= \int\limits_{\Omega} (-\Delta v(x) + p(x))\varphi(x)\mathrm{d}x \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \end{aligned} \tag{6.3}$$

Using Proposition 2.7 (ii) it follows that $v$ is the solution of the Dirichlet problem

$$\Delta v(x) = p(x), \; x \in \Omega, \quad v_{|\partial\Omega} = 0. \tag{6.4}$$

Assume now that the membrane is vibrating in the vertical elongation and that $v = v(x, t)$ with $x \in \overline{\Omega}$ and the time $t \geq 0$. Then one has to replace $p(x)$ in (6.4) by the acceleration

$$m(x)\frac{\partial^2 v}{\partial t^2}(x,t), \quad x \in \Omega,\ t \geq 0,$$

where $m(x)$ is the *mass density* of the membrane at the point $x \in \Omega$. Consequently one obtains for $t \geq 0$,

$$\Delta v(x,t) = m(x)\,\frac{\partial^2 v}{\partial t^2}(x,t),\ x \in \Omega, \quad \text{and} \quad v(y,t) = 0,\ y \in \Gamma. \tag{6.5}$$

Of interest are *eigenfrequencies*, hence non-trivial solutions $v(x,t) = e^{i\lambda t}u(x)$, $\lambda \in \mathbb{R}$, of (6.5). This results in the eigenvalue problem for the (degenerate) Dirichlet Laplacian,

$$-\Delta u(x) = \lambda^2\, m(x)\, u(x),\ x \in \Omega, \quad \operatorname{tr}_\Gamma u = 0, \tag{6.6}$$

and fits in the scheme of Theorem 5.61 and of Note 5.12.15 where we discussed problems of this type. If the mass density is constant, say, $m(x) = 1$, $x \in \Omega$, then one has Theorem 5.31 (i) and Courant's remarkable observation described in Note 5.12.5. By the above considerations it is clear that the distribution of the eigenvalues of the Dirichlet Laplacian is not only of mathematical interest, but also of physical relevance. It will be one of the main concerns in Chapter 7, we refer in particular to Theorem 7.13.

**Example 6.2** (The hydrogen atom and semi-classical limits). The classical Hamiltonian function for the (neutral) hydrogen atom $H$ with its nucleus fixed at the origin in $\mathbb{R}^3$ and a revolving electron having mass $m$ and charge $e$ is given by

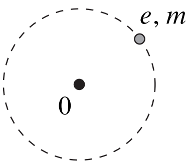



Figure 6.2

$$\Phi(x,p) = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - \frac{e^2}{|x|}, \tag{6.7}$$

where $x \in \mathbb{R}^3$ is the position of the electron and $p \in \mathbb{R}^3$ its momentum, see Figure 6.2 aside. Recall that $-e^2|x|^{-1}$ is the Coulomb potential. Quantisation requires the replacement

$$x_j \mapsto x_j \cdot \text{ (multiplication operator)} \quad \text{and} \quad p_j \mapsto \frac{\hbar}{i}\,\frac{\partial}{\partial x_j} \tag{6.8}$$

in (6.7) resulting in the hydrogen operator

$$\mathcal{H}_H f = -\frac{\hbar^2}{2m}\Delta f - \frac{e^2}{|x|}f, \quad \operatorname{dom}(\mathcal{H}_H) = W_2^2(\mathbb{R}^3), \tag{6.9}$$

in $L_2(\mathbb{R}^3)$ where $\hbar = \frac{h}{2\pi}$ and $h$ is *Planck's quantum of action*. $\mathcal{H}_H$ is called the *Hamiltonian operator* of the hydrogen atom. In Note 6.7.1 we add a few further comments about the quantum-mechanical background. We compare $\mathcal{H}_H$ in (6.9) with

$$\mathcal{H}_\beta f = -\Delta f + f + \beta V(x) f, \quad \beta \sim h^{-2}, \quad \operatorname{dom}(\mathcal{H}_\beta) = W_2^2(\mathbb{R}^3), \tag{6.10}$$

where we transferred $\frac{\hbar^2}{2m}$ in (6.9) to the potential $V(x) = |x|^{-1}$ and shifted the outcome by id. Only the dependence of $\beta$ on $h$ will be of interest. As indicated so far in Exercise 5.25 and Note 5.12.4 (and considered later on in Section 7.7) we know that $A$, given by

$$Af = -\Delta f + f, \quad \operatorname{dom}(A) = W_2^2(\mathbb{R}^3), \tag{6.11}$$

is self-adjoint, positive-definite in $L_2(\mathbb{R}^3)$ and

$$\sigma(A) = \sigma_{\mathsf{e}}(A) = [1, \infty), \quad \sigma_{\mathsf{p}}(A) = \emptyset. \tag{6.12}$$

Furthermore, if for real $V(x)$ the multiplication operator $B$,

$$Bf = V(x) f, \quad \operatorname{dom}(B) \supset \operatorname{dom}(A), \tag{6.13}$$

is *relatively compact* with respect to $A$, that is, $BA^{-1}$ is compact, then

$$\mathcal{H}_\beta = A + \beta B, \quad \operatorname{dom}(\mathcal{H}_\beta) = \operatorname{dom}(A) = W_2^2(\mathbb{R}^3), \tag{6.14}$$

is also self-adjoint and

$$\sigma_{\mathsf{e}}(\mathcal{H}_\beta) = \sigma_{\mathsf{e}}(A) = [1, \infty), \tag{6.15}$$

where $\beta > 0$ is the coupling constant. One asks for the behaviour of possible negative eigenvalues if $\beta \to \infty$, in particular, for the cardinal number

$$\#\{\sigma(\mathcal{H}_\beta) \cap (-\infty, 0]\}. \tag{6.16}$$

This is the problem of the *negative spectrum* we are dealing with in Section 6.5 on an abstract level and returning in Section 7.7 to operators of type (6.10). The interest in these questions comes from quantum mechanics. Planck's quantum of action $h$ is so small that it developed (by the *wisdom of physicists*) the ability of tending to zero, what means that the coupling constant $\beta \sim h^{-2}$ in (6.10) tends to infinity. This is called the *semi-classical limit* and the physicists extract information from the cardinality of the set (6.16). But the physical side of this problems is not the subject of this book. We add some references in Note 6.7.10.

**Example 6.3** (Weyl exponent). Let

$$Au = -\Delta u, \quad \operatorname{dom}(A) = W_{2,0}^2(\Omega), \tag{6.17}$$

be the Dirichlet Laplacian in a bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$ as considered in Theorem 5.31 (i) and let

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lambda_j \to \infty \text{ if } j \to \infty, \tag{6.18}$$

be its eigenvalues repeated according to their (geometric = algebraic) multiplicities. (By Note 5.12.5 the first eigenvalue is simple.) Of interest is the distribution of these eigenvalues, also for physical reasons as outlined in Example 6.1. To smooth out possible local irregularities in the behaviour of $\lambda_j$ it is usual to consider the *spectral counting function*

$$N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j < \lambda\}, \quad \lambda > 0. \tag{6.19}$$

The first systematic treatment of problems of this type goes back to H. Weyl in 1912 [Wey12a], [Wey12b] resulting in the question under which circumstances one has

$$N(\lambda) = (2\pi)^{-n}|\omega_n||\Omega|\lambda^{\frac{n}{2}} - \varkappa_n|\partial\Omega|_{n-1}\lambda^{\frac{n-1}{2}}(1 + \mathbf{o}(1)) \quad \text{for } \lambda \to \infty. \tag{6.20}$$

Here $|\omega_n|$ is the volume of the unit ball in $\mathbb{R}^n$ mentioned in (1.18), $\varkappa_n$ is some positive number depending only on $n$, $|\Omega|$ is the Lebesgue measure of $\Omega$ and $|\partial\Omega|_{n-1}$ is the surface area of $\Gamma = \partial\Omega$ which can be calculated according to Theorem A.8 (for smooth surfaces). As usual, $\mathbf{o}(1)$ indicates a remainder term tending to zero if $\lambda \to \infty$. Generations of mathematicians dealt with this problem up to our time. The state-of-the-art at the end of the 1990s may be found in [SV97]. We add a few further comments in Note 6.7.2. Sharp asymptotic assertions of type (6.20) are beyond the scope of this book. We are interested in the simple consequence

$$\lambda_j \sim j^{2/n}, \quad j \in \mathbb{N}, \tag{6.21}$$

inserting $\lambda = \lambda_j$ and $N(\lambda_j) = j$ in (6.20). Sometimes $\frac{n}{2}$ in (6.20) or $\frac{2}{n}$ in (6.21), respectively, is called *Weyl exponent*.

**Remark 6.4** (Our method, generalisations). Our proof of (6.21) will be based on approximation numbers and entropy numbers of compact embeddings as considered in Theorems 4.17, 5.59 on the one hand, and isomorphic mappings of elliptic operators according to Theorems 5.31, 5.36 or continuous mappings as in Theorem 5.61 on the other hand. These are qualitative assertions and nothing like (6.20) can be obtained in this way. However, this type of arguments can be applied to all the operators in the just-mentioned theorems, hence self-adjoint, not self-adjoint, regular and degenerate elliptic operators. As said, Chapter 6 provides the abstract background, whereas Chapter 7 deals with applications to function spaces and elliptic operators including the above examples.

## 6.2 Spectral theory of self-adjoint operators

In this book we encounter three types of linear operators in complex Banach spaces and Hilbert spaces. First there are bounded operators *in* or *between* Banach spaces. If such an operator acts compactly in a given Banach space, then the related spectral theory is covered by the Riesz Theorem C.1. Secondly we deal with (unbounded) self-adjoint operators in (complex separable) Hilbert spaces. Thirdly, in cases where the operator $A$ considered in a Hilbert space is unbounded and not necessarily self-adjoint, then we know in addition that its inverse $A^{-1}$ (or the inverse $(A - \lambda\,\mathrm{id})^{-1}$ for some $\lambda \in \mathbb{C}$) is compact which gives the possibility to reduce spectral assertions to compact operators in Hilbert spaces. Typical examples are the operators $A$ in Proposition 5.34 and Theorem 5.36. We discussed the somewhat delicate question of related spectral assertions in Remark 5.35 and Note 5.12.6. We return later on briefly to this point in connection with the density of the linear hull of associated eigenvectors in the Sections 6.6 and 7.5. At this moment it is sufficient for us to collect and complement what had been said so far about the spectrum of (unbounded) self-adjoint operators.

As always we assume that the reader is familiar with the basic elements of operator theory in Hilbert spaces. We collected some related notation and assertions in Appendix C, especially in Section C.2. In particular, we recalled in (C.14)–(C.19) under which conditions a (linear, densely defined, not necessarily bounded) operator $A$ is called self-adjoint. As in Section C.1 we let $\mathcal{L}(H)$ be the space of all linear and bounded operators in the Hilbert space $H$. Let id be the identity in $H$ according to (C.4) with $X = Y = H$.

**Definition 6.5.** Let $A$ be a self-adjoint operator in the (complex, separable) Hilbert space $H$.

(i) Then

$$\varrho(A) = \{\lambda \in \mathbb{C} \,:\, (A - \lambda\,\mathrm{id})^{-1} \text{ exists and belongs to } \mathcal{L}(H)\} \tag{6.22}$$

is the *resolvent set* of $A$ and

$$\sigma(A) = \mathbb{C} \setminus \varrho(A) \tag{6.23}$$

is called the *spectrum* of $A$.

(ii) A number $\lambda \in \mathbb{C}$ is an *eigenvalue* of $A$ if there exists an element $h$,

$$h \in \mathrm{dom}(A), \quad h \neq 0, \quad \text{with } Ah = \lambda h. \tag{6.24}$$

Furthermore,

$$\ker(A - \lambda\,\mathrm{id}) = \{h \in \mathrm{dom}(A) : (A - \lambda\,\mathrm{id})h = 0\} \tag{6.25}$$

is the *kernel* (or *null space*) of $A - \lambda\,\mathrm{id}$, and

$$\dim\ker(A - \lambda\,\mathrm{id}) \tag{6.26}$$

the *multiplicity* of the eigenvalue $\lambda$. The *point spectrum* $\sigma_{\mathsf{p}}(A)$ is the collection of all eigenvalues of $A$.

(iii) For $\lambda \in \mathbb{C}$ a sequence

$$\{h_j\}_{j=1}^{\infty} \subset \mathrm{dom}(A), \quad \|h_j|H\| \le 1, \quad Ah_j - \lambda h_j \to 0 \text{ if } j \to \infty, \tag{6.27}$$

is called a *Weyl sequence* (with respect to $A$ and $\lambda$) if $\{h_j\}_j$ has no convergent subsequence. The *essential* (*or continuous*) *spectrum* $\sigma_{\mathsf{e}}(A)$ is the collection of all $\lambda \in \mathbb{C}$ for which such a Weyl sequence exists.

**Remark 6.6.** We collected what we need in the sequel. Otherwise we refer again to Appendix C, where now Sections C.1, C.2 are of relevance. Furthermore, one may consult Note 6.7.3 below for additional information especially about Weyl sequences which will be of some use for us later on. In connection with (C.10), (C.11), naturally extended to (not necessarily bounded) self-adjoint operators (what does not cause any problems), we recall that

$$\ker(A - \lambda\,\mathrm{id})^k = \ker(A - \lambda\,\mathrm{id}) \quad \text{for any } k \in \mathbb{N}, \tag{6.28}$$

what makes clear that there is no need to distinguish between the *algebraic* multiplicity and the *geometric* multiplicity of an eigenvalue of a self-adjoint operator (speaking simply of *multiplicity*). In Note 5.12.4 we discussed briefly the point spectrum, the essential spectrum and Weyl sequences in connection with the Dirichlet Laplacian and the Neumann Laplacian. Now we return to these questions in greater detail. One comment seems to be appropriate. In abstract spectral theory (of self-adjoint operators) it is desirable to collect in the point spectrum $\sigma_{\mathsf{p}}(A)$ only eigenvalues of *finite* multiplicity. This is supported by Rellich's Theorem C.15. If $\lambda \in \sigma_{\mathsf{p}}(A)$ is a (real) eigenvalue of infinite multiplicity of a self-adjoint operator $A$, then $\lambda \in \sigma_{\mathsf{e}}(A)$ (subject to Exercise 6.7 below). However, self-adjoint elliptic differential operators (of second order) do not have eigenvalues of infinite multiplicity. This may justify that we stick at the above definition of the point spectrum $\sigma_{\mathsf{p}}(A)$.

**Exercise* 6.7.** Prove that eigenvalues of infinite multiplicity of a self-adjoint operator $A$ belong to $\sigma_{\mathsf{e}}(A)$. Construct operators having eigenvalues of infinite multiplicity.

*Hint:* Use orthonormal bases in $\ker(A - \lambda\,\mathrm{id})$ and determine the spectrum of $A_d = d \cdot \mathrm{id}$ with $d \in \mathbb{R}$.

**Theorem 6.8.** *Let $A$ be a self-adjoint operator in a Hilbert space $H$. Let $\varrho(A)$, $\sigma(A)$, $\sigma_{\mathsf{p}}(A)$ and $\sigma_{\mathsf{e}}(A)$ be as in Definition 6.5. Then*

$$\lambda \in \varrho(A) \quad \text{if, and only if,} \quad \|(A-\lambda\,\mathrm{id})h|H\| \geq c\|h|H\| \tag{6.29}$$

*for some $c > 0$ and all $h \in \mathrm{dom}(A)$. Furthermore,*

$$\sigma(A) \subset \mathbb{R} \quad \text{and} \quad \sigma(A) = \sigma_{\mathsf{p}}(A) \cup \sigma_{\mathsf{e}}(A). \tag{6.30}$$

*Proof. Step* 1. If $\lambda \in \varrho(A)$, then there is some $C > 0$ such that

$$\|(A-\lambda\mathrm{id})^{-1}h|H\| \leq C\|h|H\|, \quad h \in H. \tag{6.31}$$

This implies the right-hand side of (6.29) with $c = C^{-1}$. Conversely we rely on Theorem C.3 (i). In particular, any $\mu \in \mathbb{C}$ with $\mathrm{Im}\,\mu \neq 0$ belongs to $\varrho(A)$, also subject of Exercise 6.9 below. It remains to check that the right-hand side of (6.29) implies $\lambda \in \varrho(A)$ for $\lambda \in \mathbb{R}$.

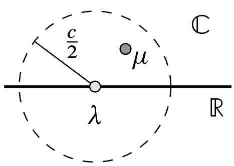


Figure 6.3

We choose $\mu \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda - \mu| < \frac{c}{2}$ as indicated in Figure 6.3 aside. Then $\mu \in \varrho(A)$ and

$$\|(A-\mu\,\mathrm{id})h|H\| \geq \frac{c}{2}\|h|H\|, \quad h \in \mathrm{dom}(A). \tag{6.32}$$

For some $g \in H$ the question whether there exists a (unique) $h \in \mathrm{dom}(A)$ with

$$Ah - \lambda h = g \tag{6.33}$$

is equivalent to the question whether there is an $h \in H$ with

$$(\mathrm{id} + T)h = (A-\mu\,\mathrm{id})^{-1}g \quad \text{with } T = (\mu-\lambda)(A-\mu\,\mathrm{id})^{-1}. \tag{6.34}$$

Since (6.32) implies $\|T\| < 1$ one can solve (6.34), and hence (6.33) uniquely (Neumann series) and obtains $\lambda \in \varrho(A)$.

*Step* 2. We prove (6.30) where, as said, the assertion $\sigma(A) \subset \mathbb{R}$ is taken for granted. If $\lambda \in \sigma(A)$, then it follows by (6.29) that there is a sequence

$$\{h_j\}_{j=1}^{\infty} \subset \mathrm{dom}(A), \quad \|h_j|H\| = 1, \quad Ah_j - \lambda h_j \to 0 \text{ if } j \to \infty. \tag{6.35}$$

When there is a converging subsequence of $\{h_j\}_j$, identified with $\{h_j\}_j$ for convenience, then we have $h_j \to h$ in $H$ for some $h \in H$, $\|h|H\| = 1$, and for any $v \in \mathrm{dom}(A)$,

$$\langle Av, h\rangle = \lim_{j\to\infty}\langle Av, h_j\rangle = \lim_{j\to\infty}\langle v, Ah_j\rangle = \langle v, \lambda h\rangle. \tag{6.36}$$

Thus $h \in \mathrm{dom}(A^*) = \mathrm{dom}(A)$, $Ah = \lambda h$, and hence $\lambda \in \sigma_{\mathsf{p}}(A)$.

If $\{h_j\}_j$ is not precompact, then we find for some $\varepsilon > 0$ a subsequence of $\{h_j\}_j$, again identified with $\{h_j\}_j$ for convenience, such that $\|h_j - h_k|H\| \geq \varepsilon$ if $j \neq k$. Consequently $\{h_j\}_j$ is a Weyl sequence and hence $\lambda \in \sigma_e(A)$. □

**Exercise* 6.9.** Let $A$ be a self-adjoint operator in a Hilbert space $H$. Prove that

$$H = \overline{\mathrm{range}(A - \lambda\,\mathrm{id})} \oplus \ker(A - \overline{\lambda}\,\mathrm{id}), \quad \lambda \in \mathbb{C}, \tag{6.37}$$

$$\ker(A - \lambda\,\mathrm{id}) = \ker(A - \overline{\lambda}\,\mathrm{id}) = \{0\}, \quad \lambda \in \mathbb{C},\ \mathrm{Im}\,\lambda \neq 0, \tag{6.38}$$

and

$$\|(A - \lambda\,\mathrm{id})h|H\| \geq |\mathrm{Im}\,\lambda| \|h|H\|, \quad h \in \mathrm{dom}(A),\ \lambda \in \mathbb{C}. \tag{6.39}$$

As a consequence of (6.37)–(6.39) show that $\sigma(A) \subset \mathbb{R}$.

## 6.3 Approximation numbers and entropy numbers: definition and basic properties

As indicated in Remark 6.4 we rely on approximation numbers and entropy numbers to get assertions of type (6.21). This Section 6.3 deals with basic properties of these numbers in Banach spaces. Spectral properties are shifted to the next Section 6.4. Recall that all Banach spaces considered are complex. Let

$$U_X = \{x \in X : \|x|X\| \leq 1\} \tag{6.40}$$

be the (closed) unit ball in the Banach space $X$.

**Definition 6.10.** Let $X$ and $Y$ be Banach spaces and let $T \in \mathcal{L}(X, Y)$, $k \in \mathbb{N}$.

(i) The *$k$th (dyadic) entropy number $e_k(T)$ of $T$* is defined as the infimum of all $\varepsilon > 0$ such that

$$T(U_X) \subset \bigcup_{i=1}^{2^{k-1}} \{y_i + \varepsilon U_Y\} \quad \text{for some } y_1, \ldots, y_{2^{k-1}} \in Y. \tag{6.41}$$

(ii) The *$k$th approximation number $a_k(T)$ of $T$* is defined by

$$a_k(T) = \inf\{\|T - S\| : S \in \mathcal{L}(X, Y),\ \mathrm{rank}\,S < k\}, \tag{6.42}$$

where $\mathrm{rank}\,S = \dim \mathrm{range}\,S$.

**Remark 6.11.** Obviously $\{y + \varepsilon U_Y\} = K_\varepsilon(y)$ is a ball in $Y$ centred at $y$ and of radius $\varepsilon > 0$. In the Notes 6.7.5 and 6.7.6 we give some references and add a few historical comments. Here we restrict ourselves to basic properties. We follow essentially [ET96].

**Theorem 6.12.** *Let $X$, $Y$, $Z$ be Banach spaces and $T \in \mathcal{L}(X,Y)$. Let $h_k$ stand for either the entropy numbers, i.e., $h_k = e_k$, or the approximation numbers, i.e., $h_k = a_k$, respectively.*

(i) (Monotonicity) *Then*

$$\|T\| = h_1(T) \geq h_2(T) \geq \cdots \geq 0. \tag{6.43}$$

(ii) (Additivity) *Let $S \in \mathcal{L}(X,Y)$. Then for all $k, m \in \mathbb{N}$,*

$$h_{k+m-1}(S+T) \leq h_k(S) + h_m(T), \tag{6.44}$$

*in particular,*

$$|h_k(S) - h_k(T)| \leq \|S - T\|, \quad k \in \mathbb{N}. \tag{6.45}$$

(iii) (Multiplicativity) *Let $R \in \mathcal{L}(Y,Z)$. Then for all $k, m \in \mathbb{N}$,*

$$h_{k+m-1}(R \circ T) \leq h_k(R) h_m(T). \tag{6.46}$$

*Proof. Step* 1. The monotonicity (6.43) and $a_1(T) = \|T\|$ are obvious. Definition 6.10 (i) with $y_1 = 0$ gives $e_1(T) \leq \|T\|$. As for the converse, assume that $\varepsilon > e_1(T) \geq 0$ and let $y \in Y$ be such that $T(U_X) \subset \{y + \varepsilon U_Y\}$. Then for arbitrary $x \in U_X$ there are $z_1, z_2 \in U_Y$ such that $Tx = y + \varepsilon z_1$, $T(-x) = y + \varepsilon z_2$ which leads to $\|Tx|Y\| \leq \varepsilon$. Taking the supremum over $U_X$ and afterwards the infimum over all $\varepsilon > e_1(T)$ results in $\|T\| \leq e_1(T)$.

*Step* 2. We first consider approximation numbers, i.e., $h_k = a_k$. The additivity (6.44) is a consequence of

$$(S - L) + (T - M) = S + T - N \tag{6.47}$$

and optimally chosen finite rank operators $L$ and $M$. More precisely, let $\lambda > a_k(S)$, $\mu > a_m(T)$, and $L \in \mathcal{L}(X,Y)$, $M \in \mathcal{L}(X,Y)$ be such that

$$\operatorname{rank} L \leq k-1, \ \|S - L\| < \lambda, \quad \text{and} \quad \operatorname{rank} M \leq m-1, \ \|T - M\| < \mu.$$

Since $N \in \mathcal{L}(X,Y)$, $\operatorname{rank} N \leq k + m - 2$, and $\|(S+T) - N\| < \lambda + \mu$, (6.44) follows. In a similar way one can prove the multiplicativity of approximation numbers (6.46) (with $h_k = a_k$), this time using the counterpart of (6.47) in the form

$$(R - L) \circ (T - M) = R \circ T - N \tag{6.48}$$

for optimally chosen finite rank operators $L$ and $M$ related to $a_k(R)$ and $a_m(T)$, respectively.

*Step* 3. We prove the additivity of entropy numbers, i.e., (6.44) with $h_k = e_k$. Let $\varepsilon > 0$, then there exist elements $\{y_1, \dots, y_{2^{k-1}}\} \subset Y$ and $\{z_1, \dots, z_{2^{m-1}}\} \subset Y$ such that

$$S(U_X) \subset \bigcup_{i=1}^{2^{k-1}} \{y_i + (\varepsilon + e_k(S))U_Y\}, \quad T(U_X) \subset \bigcup_{j=1}^{2^{m-1}} \{z_j + (\varepsilon + e_m(T))U_Y\},$$

according to (6.41). Hence for any $x \in U_X$ there are $y_r \in \{y_1, \dots, y_{2^{k-1}}\}$ and $z_l \in \{z_1, \dots, z_{2^{m-1}}\}$ such that

$$\|(S+T)x - y_r - z_l|Y\| \le e_k(S) + e_m(T) + 2\varepsilon, \tag{6.49}$$

that is, $(S+T)(U_X)$ can be covered by $2^{k-1} \cdot 2^{m-1}$ balls of radius $e_k(S) + e_m(T) + 2\varepsilon$, where $\varepsilon > 0$ can be chosen arbitrarily small. This proves (6.44). As for the multiplicativity property (6.46) with $h_k = e_k$, one first covers $T(U_X)$ by $2^{m-1}$ balls $\{z_j + (\varepsilon + e_m(T))U_Y\}$ in $Y$ and afterwards each of their images $R(\{y_i + (\varepsilon + e_m(T))U_Y\})$ in $Z$ by $2^{k-1}$ balls of radius $(\varepsilon + e_m(T))(\varepsilon + e_k(R))$. This gives a covering of $(R \circ T)(U_X)$ with $2^{k+m-2}$ balls in $Z$ of radius $e_k(R)e_m(T) + \varepsilon'$ where $\varepsilon' > 0$ can be chosen arbitrarily small. This concludes the proof of (6.46). □

**Remark 6.13.** Let $T \in \mathcal{L}(X,Y)$. Then

$$T \text{ is compact} \quad \text{if, and only if,} \quad \lim_{k\to\infty} e_k(T) = 0. \tag{6.50}$$

This is obvious since $T(U_X)$ is precompact in $Y$ if, and only if, one finds for any $\varepsilon > 0$ a finite $\varepsilon$-net which can be taken as centres of $\varepsilon$-balls.

If $T \in \mathcal{L}(X,Y)$ is an *operator of finite rank*, i.e., rank $T$ = dim range $T < \infty$, then

$$a_m(T) = 0 \quad \text{if, and only if,} \quad \operatorname{rank} T < m. \tag{6.51}$$

Furthermore, if $T \in \mathcal{L}(X,Y)$, then

$$\lim_{k\to\infty} a_k(T) = 0 \text{ implies that } T \text{ is compact.} \tag{6.52}$$

This is a consequence of the approximation of $T$ by finite rank operators. However, in contrast to (6.50) the converse is not true in general, but for Hilbert spaces subject to Exercise 6.14 (c). We add also a corresponding comment in Note 6.7.9.

**Exercise* 6.14.** (a) Let $X$ be a Banach space with dim $X \ge m$, and $T = \mathrm{id}_X \in \mathcal{L}(X)$ the identity in $X$. Prove the *norm property* of approximation numbers,

$$a_k(\mathrm{id}_X) = 1, \quad k = 1, \dots, m. \tag{6.53}$$

*Hint:* Apply (6.43) and the fact (proof ?) that for any $L \in \mathcal{L}(X)$ with rank $L < m$ there exists an $x_0 \in X$ with $L(x_0) = 0$ and $x_0 \neq 0$.

(b) Prove the *rank property* of approximation numbers (6.51).

*Hint:* The if-part is obvious; for the converse use (6.46), (6.53) and the fact that rank $T \geq m$ implies the existence of a Banach space $Z$ with $\dim Z = m$, and of operators $S \in \mathcal{L}(Z, X)$, $R \in \mathcal{L}(Y, Z)$ such that $RTS = \mathrm{id}_Z$ is the identity in $Z$, cf. [CS90, Lemma 2.1.2].

(c) Let $H_1$, $H_2$ be (separable complex) Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Prove that

$$T \text{ is compact} \quad \text{if, and only if,} \quad \lim_{k\to\infty} a_k(T) = 0. \tag{6.54}$$

*Hint:* Use the orthogonal projections on subspaces spanned by finite $\varepsilon$-nets mentioned in Remark 6.13 or consult [EE87, Theorem II.5.7], [Tri92a, Theorem 2.2.6, p. 97].

**Exercise* 6.15.** (a) Let $X$ and $Y$ be real Banach spaces and $T \in \mathcal{L}(X, Y)$. Prove that

$$\operatorname{rank} T = m \quad \text{if, and only if,} \quad c\, 2^{-\frac{k-1}{m}} \leq e_k(T) \leq 4\|T\| 2^{-\frac{k-1}{m}} \tag{6.55}$$

for some $c > 0$ and all $k \in \mathbb{N}$. This implies for a finite-dimensional real Banach space $X$ with $\dim X = m < \infty$ and $T = \mathrm{id}_X \in \mathcal{L}(X)$ that

$$e_k(\mathrm{id}_X) \sim 2^{-\frac{k-1}{m}}, \quad k \in \mathbb{N}. \tag{6.56}$$

(b) Prove that for a finite-dimensional complex Banach space $X$ with $\dim X = m < \infty$ the equivalence (6.56) must be replaced by

$$e_k(\mathrm{id}_X) \sim 2^{-\frac{k-1}{2m}}, \quad k \in \mathbb{N}. \tag{6.57}$$

*Hint:* Reduce the complex case to the real one in the same way as in the proof of Theorem 6.25 below. Compare this result with the *rank property* (6.51) and the *norm property* of approximation numbers (6.53).

**Example 6.16.** Let $\ell_p$, $1 \leq p \leq \infty$, be given by (3.152), (3.153). We consider the diagonal operator $D_\sigma : \ell_p \longrightarrow \ell_p$, defined by

$$D_\sigma : x = (\xi_k)_k \longmapsto (\sigma_k \xi_k)_k \tag{6.58}$$

where $(\sigma_k)_{k\in\mathbb{N}}$ is a monotonically decreasing sequence of non-negative numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. For convenience, let $\ell_p$ be real. Then

$$a_k(D_\sigma) = \sigma_k, \quad k \in \mathbb{N}, \tag{6.59}$$

and

$$e_k(D_\sigma) \sim \sup_{m\in\mathbb{N}} 2^{-\frac{k-1}{m}} (\sigma_1 \cdots \sigma_m)^{\frac{1}{m}}, \quad k \in \mathbb{N}. \tag{6.60}$$

We refer to [CS90, Proposition 1.3.2, (1.5.11)] and to [Pie78] for a proof of (6.60) and to Exercise 6.17 below concerning (6.59). Obviously, (6.53) and (6.55) coincide with (6.59) and (6.60), respectively, for $X = \ell_p$.

**Exercise* 6.17.** Prove (6.59).

*Hint:* For the estimate from above, $a_k(D_\sigma) \le \sigma_k$, one may approximate $D_\sigma$ by $D_\tau$ with $\tau_j = \sigma_j$ for $j \le k-1$ and $\tau_j = 0$ for $j \ge k$. Conversely, when $\sigma_k > 0$, the idea is to consider first the $k$-dimensional matrix operator $\boldsymbol{D}_\sigma^k$ corresponding to the 'upper left corner' suggested by $D_\sigma$ and to prove $a_k(\boldsymbol{D}_\sigma^k) \ge \sigma_k$, using (6.46) and (6.53). Afterwards, represent $\boldsymbol{D}_\sigma^k$ as $P_k \circ D_\sigma \circ \mathrm{id}_k$, where $P_k$ and $\mathrm{id}_k$ are a suitable projection and identity, respectively, such that (6.46) concludes the argument.

**Exercise 6.18.** Of interest are *universal* estimates between entropy numbers and approximation numbers. Prove that in general there cannot exist constants $c > 0$ or $C > 0$ such that for arbitrary operators $T \in \mathcal{L}(X,Y)$ in some Banach spaces $X$ and $Y$ the inequalities

$$e_k(T) \le c\, a_k(T),\ k \in \mathbb{N}, \quad \text{or} \quad a_k(T) \le C\, e_k(T),\ k \in \mathbb{N}, \tag{6.61}$$

are true, respectively.

*Hint:* To disprove the first estimate, recall either (6.51) in connection with (6.55), or use (6.59), (6.60) for an appropriately chosen sequence $(\sigma_k)_{k\in\mathbb{N}}$. As far as the second estimate is concerned, review Remark 6.13 together with Note 6.7.9.

**Remark 6.19.** Though *universal* estimates of type (6.61) cannot be true, there exist rather general inequalities using weighted means of entropy numbers and approximation numbers, respectively. We refer to Note 6.7.8, in particular, to (6.155) and (6.156).

**Exercise* 6.20.** According to Exercise 6.18 there cannot exist general term-wise estimates of type (6.61). However, for arbitrary Banach spaces $X$ and $Y$ and $T \in \mathcal{L}(X,Y)$ it is not difficult to prove that

$$\lim_{k\to\infty} e_k(T) \le a_n(T) \quad \text{for all } n \in \mathbb{N}. \tag{6.62}$$

*Hint:* Approximate $T$ by finite rank operators and use their compactness.

## 6.4 Approximation numbers and entropy numbers: spectral assertions

As indicated in Remark 6.4 we wish to discuss the distribution of eigenvalues of elliptic differential operators, where (6.21) may serve as a proto-type, by reducing these problems to the study of approximation numbers and entropy numbers, respectively, of compact embeddings between function spaces. In this Section 6.4 we develop the necessary abstract background.

*Notational agreement.* If the compact self-adjoint operator $T$ in the Hilbert space $H$ has only finitely many non-vanishing eigenvalues $\lambda_1(T), \dots, \lambda_k(T)$ according to Theorem C.5, then we put $\lambda_m(T) = 0$ for $m > k$.

**Theorem 6.21.** *Let $T$ be a compact self-adjoint operator in the* (*complex, separable, infinite-dimensional*) *Hilbert space $H$. Let for $k \in \mathbb{N}$, $\lambda_k(T)$ be the eigenvalues of $T$ according to Theorem C.5 and let $a_k(T)$ be the corresponding approximation numbers as introduced in Definition 6.10* (ii). *Then*

$$|\lambda_k(T)| = a_k(T), \quad k \in \mathbb{N}. \tag{6.63}$$

*Proof. Step* 1. We apply Theorem C.5, in particular (C.27), and represent $T$ as

$$Th = \sum_{j=1}^{\infty} \lambda_j \langle h, h_j \rangle h_j, \quad h \in H, \tag{6.64}$$

where $\lambda_j = \lambda_j(T)$ and $\{h_j\}_j$ is a corresponding orthonormal system of eigenelements, $Th_j = \lambda_j h_j$. Plainly, $a_1(T) = \|T\| = |\lambda_1|$ in view of (6.43) for $a_k$ and Remark C.6. Let $k \geq 2$. Then

$$a_k(T) \leq \Big\| T - \sum_{j=1}^{k-1} \lambda_j \langle \cdot, h_j \rangle h_j \Big\| = \Big\| \sum_{j=k}^{\infty} \lambda_j \langle \cdot, h_j \rangle h_j \Big\| \leq |\lambda_k|. \tag{6.65}$$

*Step* 2. It remains to show the converse, i.e., $|\lambda_k| \leq a_k(T)$, $k \in \mathbb{N}$, $k \geq 2$. Let $S \in \mathcal{L}(H)$ with rank $S \leq k-1$. We shall prove that there exists an $h^\circ \in H$ such that $Sh^\circ = 0$ and $\|h^\circ | H\| = 1$. Let $\{v_1, \dots, v_{k-1}\}$ be an orthonormal system spanning range $S$ such that

$$Sh = \sum_{j=1}^{k-1} \langle Sh, v_j \rangle v_j, \quad h \in H. \tag{6.66}$$

Let $\{h_j\}_j$ be the above orthonormal system of eigenvectors, $Th_j = \lambda_j h_j$; then

$$S\Big(\sum_{r=1}^{k} \mu_r h_r\Big) = \sum_{r=1}^{k} \mu_r S h_r = \sum_{j=1}^{k-1} \Big(\sum_{r=1}^{k} \mu_r c_{jr}\Big) v_j \tag{6.67}$$

with $\mu_r \in \mathbb{C}$ and $c_{jr} = \langle Sh_r, v_j \rangle$. Since there is a non-trivial solution $\{\mu_1^\circ, \dots, \mu_k^\circ\}$ of

$$\sum_{r=1}^{k} \mu_r c_{jr} = 0, \quad j = 1, \dots, k-1,$$

we have found $h^\circ = \sum_{r=1}^{k} \mu_r^\circ h_r$ with $Sh^\circ = 0$ and $h^\circ \neq 0$. Without restriction of generality we may assume that $\|h^\circ|H\| = \big(\sum_{r=1}^{k} |\mu_r^\circ|^2\big)^{1/2} = 1$. Consequently,

$$\|Th^\circ|H\| = \Big\| \sum_{r=1}^{k} \mu_r^\circ \, \lambda_r h_r|H \Big\| = \Big( \sum_{r=1}^{k} |\mu_r^\circ|^2 \, |\lambda_r|^2 \Big)^{1/2} \geq \, |\lambda_k|. \tag{6.68}$$

Moreover, $\|h^\circ|H\| = 1$ and $Sh^\circ = 0$ imply that

$$\|T - S\| \geq \big\|Th^\circ - Sh^\circ|H\big\| = \big\|Th^\circ|H\big\| \, \geq \, |\lambda_k|, \tag{6.69}$$

and finally taking the infimum over all admitted $S$ concludes the argument. □

**Remark 6.22.** The proof uses that $H$ is a Hilbert space and that $T$ is compact and self-adjoint. Otherwise an assertion of type (6.63) cannot be expected. This can be illustrated by the following example due to [EE87, pp. 59/60].

**Example 6.23.** Let $X = \mathbb{C}^2$ and $T \in \mathcal{L}(X)$ be connected with the matrix

$$\boldsymbol{T} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{having eigenvalues } \lambda_1(T) = 2, \ \lambda_2(T) = 1.$$

However, according to [EE87, pp. 59/60] the corresponding approximation numbers satisfy

$$a_1(T) = \sqrt{3 + \sqrt{5}} > \lambda_1(T) \quad \text{and} \quad a_2(T) = \sqrt{3 - \sqrt{5}} < \lambda_2(T).$$

**Remark 6.24.** In other words, a result like (6.63) cannot be expected in general. Some further comments may be found in Note 6.7.5. On the other hand, for arbitrary compact operators in Banach spaces there is a remarkable relation between entropy numbers and eigenvalues we are going to discuss now.

Let $T \in \mathcal{L}(X)$ be a compact operator in the infinite-dimensional (complex) Banach space $X$. By the Riesz Theorem C.1 the spectrum of $T$, apart from the origin, consists solely of eigenvalues of finite algebraic multiplicity according to (C.11), denoted by $\{\lambda_k(T)\}_k$, repeated according to their algebraic multiplicity and ordered so that

$$\|T\| \geq |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k| \geq \cdots > 0. \tag{6.70}$$

Recall our *notational agreement* on p. 195, i.e., if $T$ has only finitely many distinct eigenvalues different from zero and $k$ is the sum of their algebraic multiplicities, then we put again $\lambda_m(T) = 0$ for $m > k$.

**Theorem 6.25** (Carl's inequality). *Let $T \in \mathcal{L}(X)$ be a compact operator in the infinite-dimensional* (*complex*) *Banach space $X$. Let $\{\lambda_k(T)\}_k$ be its eigenvalue sequence as described above and let $e_k(T)$ be the related entropy numbers according to Definition 6.10* (i). *Then*

$$\Big( \prod_{j=1}^{k} |\lambda_j(T)| \Big)^{1/k} \le \inf_{m \in \mathbb{N}} 2^{\frac{m}{2k}} \, e_m(T), \quad k \in \mathbb{N}. \tag{6.71}$$

*Proof. Step* 1. We begin with a preparation and assume that $\lambda \neq 0$ is an eigenvalue of $T$ with algebraic multiplicity $m$. Let $b$ be an (associated) eigenvector such that

$$(T - \lambda\,\mathrm{id})^{r-1}\, b \neq 0, \quad \text{and} \quad (T - \lambda\,\mathrm{id})^{r}\, b = 0 \text{ for } r \in \mathbb{N},\ r \le m. \tag{6.72}$$

Then the elements $\{b_1, \ldots, b_r\}$,

$$b_j = (T - \lambda\,\mathrm{id})^{j-1} b, \quad j = 1, \ldots, r, \tag{6.73}$$

are linearly independent. Since

$$b_{j+1} = (T - \lambda\,\mathrm{id}) b_j \quad \text{for } j = 1, \ldots, r-1, \tag{6.74}$$

one obtains

$$T b_j = b_{j+1} + \lambda b_j \text{ for } j = 1, \ldots, r-1, \quad \text{and} \quad T b_r = \lambda b_r. \tag{6.75}$$

In particular, for $a = \sum_{j=1}^{r} \gamma_j b_j$ with $\gamma_j \in \mathbb{C}$ one gets for $r \ge 2$,

$$Ta = \sum_{j=1}^{r-1} \gamma_j (b_{j+1} + \lambda b_j) + \lambda \gamma_r b_r = \gamma_1 \lambda b_1 + \sum_{j=2}^{r} (\gamma_{j-1} + \gamma_j \lambda) b_j. \tag{6.76}$$

Let $\boldsymbol{T}$ be the related matrix so that

$$\boldsymbol{T} \begin{pmatrix} \gamma_1 \\ \\ \vdots \\ \\ \gamma_r \end{pmatrix} = \begin{pmatrix} \lambda & 0 & & \cdots & 0 \\ 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \\ \vdots \\ \\ \gamma_r \end{pmatrix} : \mathbb{C}^r \longrightarrow \mathbb{C}^r. \tag{6.77}$$

In particular, $\Lambda = \mathrm{span}\{b_1, \ldots, b_r\}$ is an invariant subspace of $T$ with $T\Lambda = \Lambda$, since $\lambda \neq 0$. We interpret $\mathbb{C}^r$ as $\mathbb{R}^{2r}$ equipped with the Lebesgue measure and decompose $\lambda$ and $\gamma_j$ into their real and imaginary parts, respectively. Let $\boldsymbol{T}^{\mathbb{R}}$ be the corresponding matrix, then

$$
\boldsymbol{T}^{\mathbb{R}} \begin{pmatrix} \operatorname{Re}\gamma_1 \\ \operatorname{Im}\gamma_1 \\ \\ \vdots \\ \\ \operatorname{Re}\gamma_r \\ \operatorname{Im}\gamma_r \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} \operatorname{Re}\lambda & -\operatorname{Im}\lambda \\ \operatorname{Im}\lambda & \operatorname{Re}\lambda \end{matrix}} & 0 & \cdots & 0 \\ 1 & \boxed{\phantom{xx}} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots \; 0 & 1 & \boxed{\begin{matrix} \operatorname{Re}\lambda & -\operatorname{Im}\lambda \\ \operatorname{Im}\lambda & \operatorname{Re}\lambda \end{matrix}} \end{pmatrix} \begin{pmatrix} \operatorname{Re}\gamma_1 \\ \operatorname{Im}\gamma_1 \\ \\ \vdots \\ \\ \operatorname{Re}\gamma_r \\ \operatorname{Im}\gamma_r \end{pmatrix} \tag{6.78}
$$

as a map in $\mathbb{R}^{2r}$. If $M$ is a bounded set in $\Lambda$, interpreted in $\mathbb{R}^{2r}$, one obtains

$$\operatorname{vol}(T(M)) = |\lambda|^{2r} \operatorname{vol}(M). \tag{6.79}$$

Now let

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq |\lambda_k(T)| > 0. \tag{6.80}$$

Then the counterparts of (6.77), (6.78) have a block structure with (6.77), (6.78) as blocks. We add a comment about this argument in Remark 6.26 below. Parallel to (6.79) one obtains

$$\operatorname{vol}(T(M)) = \prod_{j=1}^{k} |\lambda_j(T)|^2 \operatorname{vol}(M), \tag{6.81}$$

where $M$ is a bounded set in the span of the (associated) eigenelements corresponding to $\lambda_1(T), \ldots, \lambda_k(T)$.

*Step* 2. Let $\Lambda = \operatorname{span}\{b_1, \ldots, b_k\}$ be the span of the (associated) eigenelements of $\lambda_1(T), \ldots, \lambda_k(T)$ and let $T(U_X)$ be covered by $2^{m-1}$ balls of radius $ce_m(T)$ for some $c > 1$ where we may assume that those balls having non-empty intersection with $\Lambda$ are centred in $\Lambda$ (at the expense of $c$). We apply (6.81) to $M = U_X \cap \Lambda$, see Figure 6.4 aside. Since

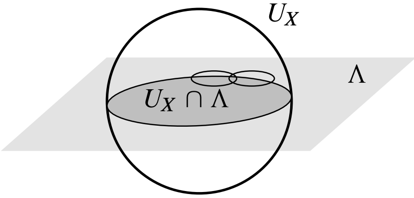



Figure 6.4

$$T(M) \subset T(U_X) \cap T(\Lambda) = T(U_X) \cap \Lambda,$$

one obtains by (6.81) that

$$
\begin{aligned}
\prod_{j=1}^{k} |\lambda_j(T)|^2 \operatorname{vol}(U_X \cap \Lambda) &\leq \operatorname{vol}(T(U_X) \cap \Lambda) \\
&\leq 2^{m-1} \, (c \, e_m(T))^{2k} \operatorname{vol}(U_X \cap \Lambda).
\end{aligned} \tag{6.82}
$$

This leads to

$$\Big(\prod_{j=1}^{k} |\lambda_j(T)|\Big)^{\frac{1}{k}} \le c\, 2^{\frac{m-1}{2k}}\, e_m(T) \le c\, 2^{\frac{m}{2k}}\, e_m(T). \tag{6.83}$$

*Step* 3. It remains to care about the constant $c$ in (6.83). We apply (6.83) to $S = T^r$ with $r \in \mathbb{N}$. Note that $\lambda_m(T^r) = \lambda_m^r(T)$ (including algebraic multiplicities) which can be justified by looking at the canonical situation in (6.77) (or (6.86) below), where $\boldsymbol{T}^r$ is again the triangular matrix with $\lambda^r$ in place of $\lambda$. Inserted in (6.83) with $m$ replaced by $m' = rm$ one obtains

$$\Big(\prod_{j=1}^{k} |\lambda_j(T)|\Big)^{\frac{r}{k}} \le c\, 2^{\frac{rm}{2k}}\, e_{rm}(T^r) \le c\, 2^{\frac{rm}{2k}}\, e_m^r(T), \tag{6.84}$$

where we used (6.46) with $h_k = e_k$. This implies

$$\Big(\prod_{j=1}^{k} |\lambda_j(T)|\Big)^{\frac{1}{k}} \le c^{\frac{1}{r}}\, 2^{\frac{m}{2k}}\, e_m(T), \tag{6.85}$$

and finally (6.71) when $r \to \infty$. □

**Remark 6.26.** We proved (6.77) under the assumption (6.72) and obtained the so-called *Jordan canonical form* for the restriction of $T$ to $\Lambda = \operatorname{span}\{b_1, \ldots, b_r\}$. As indicated, but not proved in detail, this can be extended to the first $k$ eigenvalues with (6.80) (always counted with respect to their algebraic multiplicities). One obtains

$$\boldsymbol{T}\begin{pmatrix} \gamma_1 \\ \vdots \\ \vdots \\ \vdots \\ \gamma_k \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 0 & & \\ 1 & \lambda_2 & 0 & \\ & \ddots & \ddots & \ddots \\ & & 1 & \lambda_l \end{matrix}} & 0 & \cdots & 0 \\ 0 & \boxed{\begin{matrix} \lambda_{l+1} & & 0 \\ 1 & \ddots & \\ & \ddots & \lambda_s \end{matrix}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \boxed{\begin{matrix} \lambda_r & 0 & & \\ 1 & \lambda_{r+1} & 0 & \\ & \ddots & \ddots & \ddots \\ & & 1 & \lambda_k \end{matrix}} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \vdots \\ \vdots \\ \gamma_k \end{pmatrix} \tag{6.86}$$

with the Jordan $\lambda$-blocks as in (6.77). A detailed proof may be found in [HS74, Chapter 6, §4]. One may also consult [Gre81, Chapter 13]. This sharp Jordan canonical form is not needed. It is sufficient to know that $T$ in (6.86) is a triangular matrix. This is the version used in [CS90, Theorem 4.2.1, Lemma 4.2.1, Supplement 1, pp. 139–143] where it comes out as a refinement of the Riesz Theorem C.1.

**Corollary 6.27.** *Let $T \in \mathcal{L}(X)$ be a compact operator in the infinite-dimensional (complex) Banach space $X$. Let $\{\lambda_k(T)\}_k$ be its eigenvalue sequence ordered by (6.70) and let $e_k(T)$ be the related entropy numbers according to Definition 6.10* (i). *Then*

$$|\lambda_k(T)| \le \sqrt{2}\, e_k(T), \quad k \in \mathbb{N}. \tag{6.87}$$

*Proof.* This follows immediately from (6.71) due to the monotonicity (6.70) on the left-hand side, and with $m = k$ on the right-hand side. □

**Remark 6.28.** In Note 6.7.7 we add a few comments about the history of Theorem 6.25 and Corollary 6.27.

## 6.5 The negative spectrum

In Example 6.2 we discussed the problem of the so-called *negative spectrum* and its relations to physics. Now we deal with the abstract background and return later on in Section 7.7 to some applications.

Again we assume that the reader is familiar with basic operator theory in Hilbert spaces collected in Appendix C, especially in Sections C.2 and C.3. Furthermore, we rely on Section 6.2, in particular Definition 6.5 and Theorem 6.8.

**Definition 6.29.** Let $A$ be a self-adjoint positive-definite operator according to (C.19) and Definition C.9 in a (complex, infinite-dimensional, separable) Hilbert space $H$ and $B$ be a symmetric operator in $H$ with $\mathrm{dom}(A) \subset \mathrm{dom}(B)$. Then $B$ is said to be *relatively compact* (with respect to $A$) if $BA^{-1} \in \mathcal{L}(H)$ is compact.

**Remark 6.30.** One can replace $A^{-1}$ in this definition by any other resolvent $R_\lambda = (A - \lambda\, \mathrm{id})^{-1}$ with $\lambda \in \varrho(A)$. This follows from the so-called *resolvent equation* which will also play a rôle in what follows. Let $\lambda \in \varrho(A)$ and $\mu \in \varrho(A)$. Then (5.333) implies that

$$\mathrm{id} = (A - \lambda\, \mathrm{id})[\mathrm{id} - (\mu - \lambda) R_\lambda] R_\mu \tag{6.88}$$

and

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu. \tag{6.89}$$

In particular, $R_\lambda R_\mu = R_\mu R_\lambda$. Furthermore, for $\lambda \in \varrho(A)$ and $\mu \in \varrho(A)$,

$$BR_\lambda \text{ is compact} \quad \text{if, and only if,} \quad BR_\mu \text{ is compact.} \tag{6.90}$$

The resolvent equation (6.89) suggests that $R_\lambda$ is an analytic function in $\varrho(A)$ and that (formally)

$$\frac{\mathrm{d}R_\lambda}{\mathrm{d}\lambda} = R_\lambda^2, \quad \lambda \in \varrho(A). \tag{6.91}$$

The following exercise clarifies what is meant by (6.91).

**Exercise 6.31.** A vector-valued function $\lambda \mapsto h(\lambda) \in H$ defined in an open set $\Lambda \subset \mathbb{C}$ is called *analytic* in $\Lambda$ if for any $\lambda \in \Lambda$ there exists an element $h'(\lambda) \in H$ such that

$$\frac{h(\lambda) - h(\mu)}{\lambda - \mu} \longrightarrow h'(\lambda) \quad \text{in } H \text{ for } \mu \to \lambda. \tag{6.92}$$

The above vector-valued function $h(\lambda)$ is called *weakly analytic* in $\Lambda$ if for any $\lambda \in \Lambda$ there is an element $h'(\lambda) \in H$ such that for any $g \in H$,

$$\left\langle \frac{h(\lambda) - h(\mu)}{\lambda - \mu}, g \right\rangle \longrightarrow \langle h'(\lambda), g \rangle \quad \text{in } \mathbb{C} \text{ for } \mu \to \lambda. \tag{6.93}$$

Prove that $h(\lambda) = R_\lambda h$ is an analytic function and a weakly analytic function in $\varrho(A)$ and that

$$\frac{R_\lambda h - R_\mu h}{\lambda - \mu} \longrightarrow R_\lambda^2 h = h'(\lambda) \quad \text{for } \mu \to \lambda \in \varrho(A). \tag{6.94}$$

*Hint:* Use (6.89) and similar arguments as in (5.333), (5.334).

**Theorem 6.32.** *Let $A$ be a self-adjoint positive-definite operator and let $B$ be a symmetric relatively compact operator according to Definition 6.29 in the Hilbert space $H$. Let $\sigma_\mathrm{p}$ be the point spectrum and $\sigma_\mathrm{e}$ be the essential spectrum as introduced in Definition 6.5. Then the eigenvalues $\{\mu_k\}_k$ of $BA^{-1}$ are real, and $(BA^{-1})^* = A^{-1}B$ is the adjoint operator after extension by continuity from* $\mathrm{dom}(B)$ *to $H$. Furthermore,*

$$C = A + B, \quad \mathrm{dom}(C) = \mathrm{dom}(A), \tag{6.95}$$

*is self-adjoint,*

$$\sigma_\mathrm{e}(C) = \sigma_\mathrm{e}(A), \tag{6.96}$$

*and*

$$\begin{aligned} \#\{\sigma_\mathrm{p}(C) \cap (-\infty, 0]\} &= \#\{\sigma(C) \cap (-\infty, 0]\} \\ &= \#\{k \in \mathbb{N} : \mu_k(BA^{-1}) \le -1\} < \infty. \end{aligned} \tag{6.97}$$

*Proof. Step* 1. If $\mu$ is an eigenvalue of $BA^{-1}$, then one gets from $BA^{-1}v = \mu v$, $v \neq 0$, that

$$Bw = \mu Aw \quad \text{with } w = A^{-1}v \in \mathrm{dom}(A), \ w \neq 0.$$

By $\langle Aw, w\rangle > 0$ it follows from

$$\langle Bw, w\rangle = \mu\langle Aw, w\rangle, \tag{6.98}$$

that $\mu$ is real. Since

$$\langle BA^{-1}v, w\rangle = \langle A^{-1}v, Bw\rangle = \langle v, A^{-1}Bw\rangle, \quad v \in H,\ w \in \mathrm{dom}(B), \tag{6.99}$$

this leads to $(BA^{-1})^* = A^{-1}B$ (recall that $\mathrm{dom}(B)$ is dense in $H$).

*Step* 2. Theorem C.3 (ii) implies that the symmetric operator $C$ is self-adjoint if for any $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$,

$$\mathrm{range}(C + \lambda\,\mathrm{id}) = H. \tag{6.100}$$

In other words, one has to ask whether one finds for given $v \in H$ an element $u \in \mathrm{dom}(A) = \mathrm{dom}(C)$ such that

$$Au + Bu + \lambda u = v. \tag{6.101}$$

Since $-\lambda \in \varrho(A)$ the question (6.101) can be reduced to

$$w + B(A + \lambda\,\mathrm{id})^{-1}w = v \quad \text{with } (A + \lambda\,\mathrm{id})u = w. \tag{6.102}$$

According to (6.90) the operator $B(A + \lambda\,\mathrm{id})^{-1}$ is compact. By Theorem C.1 the equation (6.102) is uniquely solvable if, and only if, $-1$ is not an eigenvalue of the related operator. We proceed by contradiction. Let us assume that there exists a non-trivial solution of (6.102) with $v = 0$. This implies the existence of a non-trivial solution $u \in \mathrm{dom}(C)$ of (6.101) with $v = 0$. However, this is not possible since $\operatorname{Im}\lambda \neq 0$ and

$$\langle Cu, u\rangle + \lambda\|u|H\|^2 = 0. \tag{6.103}$$

*Step* 3. We prove (6.96) and assume $\lambda \in \sigma_{\mathrm{e}}(A)$. Thus one finds a Weyl sequence according to (6.27). By the spectral theory of self-adjoint operators it follows that one may assume in addition that both

$$\{h_j\}_{j=1}^\infty \text{ and } \{Ah_j\}_{j=1}^\infty \text{ are orthogonal and } \|h_j|H\| \geq c \tag{6.104}$$

for some $c > 0$. We refer to [Tri92a, Section 4.3.7, especially Remark 1, p. 252]. Some additional explanations may be found in Note 6.7.3 below. We rely on this refinement and ask whether

$$Ah_j + Bh_j - \lambda h_j \longrightarrow 0 \quad \text{if } j \to \infty. \tag{6.105}$$

This is equivalent to the question whether

$$g_j + BA^{-1}g_j - \lambda A^{-1}g_j \longrightarrow 0 \quad \text{if } j \to \infty \text{ where } g_j = Ah_j. \tag{6.106}$$

Since $BA^{-1}$ is compact we may assume in addition that

$$BA^{-1}g_j \longrightarrow g \quad \text{in } H. \tag{6.107}$$

Let $u_j = g_{2j} - g_{2j-1}$, $j \in \mathbb{N}$, then

$$\{u_j\}_{j=1}^{\infty} \text{ is orthogonal} \quad \text{and} \quad BA^{-1}u_j \longrightarrow 0 \text{ in } H. \tag{6.108}$$

Hence,

$$u_j + BA^{-1}u_j - \lambda A^{-1}u_j \longrightarrow 0 \quad \text{if } j \to \infty, \tag{6.109}$$

and one obtains (6.105) for $\tilde{h}_j = h_{2j} - h_{2j-1}$ instead of $h_j$. This proves that $\{\tilde{h}_j\}_j$ is a Weyl sequence for $C = A + B$ if $\lambda \in \sigma_{\mathrm{e}}(A)$. Consequently $\sigma_{\mathrm{e}}(A) \subseteq \sigma_{\mathrm{e}}(C)$ and by a parallel argument, interchanging the rôles of $A$ and $C$, also (6.96).

*Step* 4. Next we prove that

$$\#\{\sigma_{\mathrm{p}}(C) \cap (-\infty, 0]\} < \infty. \tag{6.110}$$

We proceed by contradiction and assume that $C$ has infinitely many eigenvalues $\lambda_j \le 0$. Let $\{u_j\}_j$ be a related orthonormal system of eigenelements,

$$(A + B)u_j = \lambda_j u_j, \quad \lambda_j \le 0, \ j \in \mathbb{N}. \tag{6.111}$$

Then

$$u_j + A^{-1}Bu_j = \lambda_j A^{-1}u_j, \quad j \in \mathbb{N}, \tag{6.112}$$

and

$$1 + \langle A^{-1}Bu_j, u_j\rangle = \langle u_j + A^{-1}Bu_j, u_j\rangle = \lambda_j\langle A^{-1}u_j, u_j\rangle \le 0 \tag{6.113}$$

since $A$ is positive-definite. Recall that $A^{-1}B$ is the adjoint operator of $BA^{-1}$, hence it is compact and we may assume $A^{-1}Bu_j \to u$ in $H$ such that

$$|\langle A^{-1}Bu_j, u_j\rangle| \le \varepsilon + |\langle u, u_j\rangle|, \quad j \ge j_0(\varepsilon). \tag{6.114}$$

However, since $\{\langle u, u_j\rangle\}_j \in \ell_2$ are the Fourier coefficients of $u \in H$, this leads to a contradiction with (6.113),

$$0 \ge 1 + \langle A^{-1}Bu_j, u_j\rangle \to 1 \quad \text{for } j \to \infty.$$

*Step* 5. It remains to prove (6.97). We begin with a preparation. We replace $C = A + B$ by the family of operators $C_\varepsilon = A + \varepsilon B$ for $\varepsilon \in \mathbb{R}$ with $\mathrm{dom}(C_\varepsilon) = \mathrm{dom}(A)$. Of course, one has the obvious counterparts of (6.96) and (6.110). If $c > 0$ has the same meaning as in (C.29), then $A - d$ id with $0 \le d < c$ is also positive-definite. This shows that (6.110) can be strengthened by

$$\#\{\sigma_{\mathrm{p}}(C_\varepsilon) \cap (-\infty, d]\} < \infty, \quad \varepsilon \in \mathbb{R}. \tag{6.115}$$

If $\lambda_j(\varepsilon) \le d$ is a possible eigenvalue of $C_\varepsilon$, ordered by $\lambda_1(\varepsilon) \le \lambda_2(\varepsilon) \le \cdots$, then it follows from the Max–Min principle as it may be found in [EE87, Section XI.1, pp. 489–490] that

$$\lambda_j(\varepsilon) = \sup \inf \big\langle (A + \varepsilon B)h, h \big\rangle, \quad j \in \mathbb{N}, \tag{6.116}$$

where the supremum is taken over all linear subspaces $M_{j-1}$ in $H$ of dimension at most $j-1$ and the infimum is taken over

$$\{h \in \mathrm{dom}(A) : h \perp M_{j-1},\ \|h|H\| = 1\}. \tag{6.117}$$

From this observation it follows easily that possible eigenvalues $\lambda_j(\varepsilon) < c$ depend continuously on $\varepsilon$ (including multiplicities). In particular, if $|\varepsilon|$ is small, then $C_\varepsilon$ has no negative eigenvalues. We assume $\lambda_j(\varepsilon) \le 0$.

We wish to prove that

$$\lambda_j(\eta) \le \lambda_j(\varepsilon) < 0 \text{ if } \eta \ge \varepsilon \quad \text{and} \quad \lim_{\eta \uparrow \varepsilon} \lambda_j(\eta) = \lambda_j(\varepsilon), \tag{6.118}$$

where the latter again means continuity. Let $\varepsilon > 0$ and $0 < \varkappa \le 1$. Then one has for $\langle Bh, h\rangle \le 0$ and $\langle Ah, h\rangle > 0$ in (6.116) with $\lambda_j(\varepsilon) < 0$,

$$\varepsilon \langle Bh, h\rangle \le -\langle Ah, h\rangle \le -\varkappa \langle Ah, h\rangle \tag{6.119}$$

and with $\eta = \varepsilon/\varkappa$ that

$$\eta \langle Bh, h\rangle \le -\langle Ah, h\rangle \quad \text{if} \quad \varepsilon \langle Bh, h\rangle \le -\langle Ah, h\rangle. \tag{6.120}$$

This proves by (6.116) the first assertion in (6.118). As for the second assertion one may insert an optimal system $M_{j-1}$ (the orthonormal eigenvectors for $C_\varepsilon$). Letting $\eta \to \varepsilon$ one gets $\lim_{\eta \uparrow \varepsilon} \lambda_j(\eta) \ge \lambda_j(\varepsilon)$. Now the second assertion in (6.118) follows from the first one.

*Step* 6. We prove (6.97). For small $\varepsilon > 0$ we know that $C_\varepsilon = A + \varepsilon B$ has no negative eigenvalues. If $\lambda_j < 0$ is an eigenvalue of $C = A + B$, then (6.118) implies that there must be an $\varepsilon$ with $0 < \varepsilon \le 1$ such that $\lambda_j(\varepsilon) = 0$. (One is sitting at the origin and observes what is passing by when $\varepsilon \uparrow 1$, see also Figure 6.5 aside.) Hence

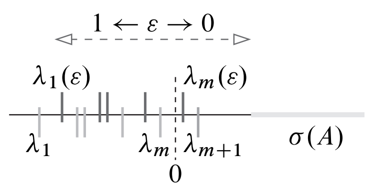



Figure 6.5

$$\#\{\sigma(A+B) \cap (-\infty, 0]\} = \#\{j \in \mathbb{N} : \lambda_j(\varepsilon) = 0 \text{ for some } 0 < \varepsilon \le 1\}. \tag{6.121}$$

However, if for some $u \in \mathrm{dom}(A)$,

$$Au + \varepsilon Bu = 0, \text{ then } v + \varepsilon BA^{-1}v = 0 \text{ where } v = Au, \tag{6.122}$$

and vice versa. Hence (6.121) coincides with the number of eigenvalues of $BA^{-1}$ which are smaller than or equal to $-1$. This concludes the proof of (6.97) and the theorem. □

**Remark 6.33.** In Note 6.7.10 we give some references and add a few comments.

## 6.6 Associated eigenelements

In Note 5.12.6 we discussed the question of associated eigenelements of unbounded operators $A$ in Hilbert spaces or Banach spaces. Under the assumption that $\varrho(A) \neq \emptyset$, say, $0 \in \varrho(A)$, we advocated to shift the question of associated eigenelements to $A^{-1}$ as done in (5.331). However, under some restriction the more direct definition of an associated eigenelement is successful.

An element

$$0 \neq v \in \bigcap_{j=1}^{\infty} \operatorname{dom}(A^j)$$

of a linear operator $A$ in a (complex) Banach space is called *an associated eigenelement* if

$$(A - \lambda \operatorname{id})^k v = 0 \quad \text{for some } \lambda \in \mathbb{C} \text{ and } k \in \mathbb{N}. \tag{6.123}$$

Then $\lambda$ is an eigenvalue. Similarly as in (C.11),

$$\dim \bigcup_{k=1}^{\infty} \ker(A - \lambda \operatorname{id})^k \quad \text{with } \lambda \in \mathbb{C} \tag{6.124}$$

is called the *algebraic multiplicity* of the eigenvalue $\lambda$. If $A$ is a self-adjoint operator in a Hilbert space $H$, then one has again

$$\ker(A - \lambda \operatorname{id})^k = \ker(A - \lambda \operatorname{id}), \quad k \in \mathbb{N}. \tag{6.125}$$

In particular, the algebraic multiplicity coincides with the geometric multiplicity (the dimension of the null space). If, in addition, $A$ is a self-adjoint (positive-definite) operator with pure point spectrum, then the corresponding eigenelements span $H$. We refer to Section C.3, especially, Remark C.16. Of course, the assumption that $A$ is positive-definite is immaterial. One may ask whether this assertion can be extended to (unbounded) operators and their associated eigenelements. In general, this is impossible, but we formulate an interesting result which fits in the scheme of the above considerations. Recall that $a_k(T)$ are the approximation numbers according to Definition 6.10 of an operator $T$, in our case $T \in \mathcal{L}(H)$.

**Theorem 6.34.** *Let $A$ be a self-adjoint operator in a Hilbert space $H$ with pure point spectrum according to Definition C.7. Let $B$ be a linear operator in $H$ with*

$$\operatorname{dom}(A) \subset \operatorname{dom}(B) \quad \textit{and} \quad \sum_{k=1}^{\infty} a_k(BR_\lambda)^p < \infty \tag{6.126}$$

*for some* $\lambda \in \varrho(A)$, $R_\lambda = (A - \lambda\, \mathrm{id})^{-1}$, *and some* $1 \le p < \infty$. *Then the spectrum of*

$$C = A + B, \quad \mathrm{dom}(C) = \mathrm{dom}(A), \tag{6.127}$$

*consists of isolated eigenvalues of finite algebraic multiplicity and the linear hull of corresponding associated eigenelements is dense in* $H$.

**Remark 6.35.** Algebraic multiplicity must be understood as explained above in (6.123), (6.124). It follows from the resolvent equation (6.89) and Theorem 6.12 that the assumption (6.126) is independent of the chosen point $\lambda \in \varrho(A)$. The above formulation has been taken over from [Tri78, Theorem 3, pp. 394/395] with a reference to [GK65, Chapter V, § 10] for a proof. We apply later on in Section 7.5 this assertion to elliptic differential operators, complementing Theorem 5.36.

## 6.7 Notes

**6.7.1.** The application of the quantisation rules (6.8) to (6.7) results in the Hamiltonian $\mathcal{H}_H$ in (6.9) for the (non-relativistic) hydrogen atom (without spin). This is the most distinguished example of a so-called *Schrödinger operator*. The underlying mathematical foundation of quantum mechanics was formulated in the 1920s, see, for example, [Hei26], [HvNN28], [vN32], and can be found nowadays in several books, e.g., in [Tri92a, Chapter 7] (and in a more extended version in its German original, 1972). The operator in (6.11) is self-adjoint. One can prove that $B$,

$$(Bf)(x) = |x|^{-1} f(x), \quad \mathrm{dom}(B) = W_2^2(\mathbb{R}^3), \tag{6.128}$$

is a symmetric operator in $L_2(\mathbb{R}^3)$, which is relatively compact with respect to $A$ according to Definition 6.29, hence

$$|x|^{-1}(-\Delta f + f)^{-1} \text{ is compact in } L_2(\mathbb{R}^3). \tag{6.129}$$

Then Theorem 6.32 implies that $\mathcal{H}_H$ is self-adjoint and

$$\sigma_{\mathrm{e}}(\mathcal{H}_H) = \sigma_{\mathrm{e}}(-\Delta) = [0, \infty). \tag{6.130}$$

Furthermore, the point spectrum $\sigma_{\mathrm{p}}(\mathcal{H}_H)$ consists of negative eigenvalues,

$$\sigma_{\mathrm{p}}(\mathcal{H}_H) = \{E_N\}_{N=1}^{\infty} \quad \text{with } E_N = -\frac{me^4}{2\hbar^2 N^2}, \; N \in \mathbb{N}, \tag{6.131}$$

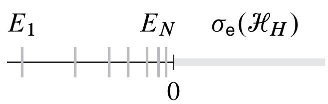


Figure 6.6

of (geometric) multiplicity $N^2$, see Figure 6.6 aside, where $m$, $e$, and $\hbar$ have the same meaning as in Example 6.2. A detailed proof may be found in [Tri92a]. In physics eigenvalues of Hamiltonians are identified with the energy of related stationary states. By *Bohr's postulate* a quantum mechanical system jumping from one stationary state of energy $E$ into another stationary state of energy $E^*$ can emit or absorb electromagnetic radiation of frequency

$$\nu = \frac{|E - E^*|}{h}, \tag{6.132}$$

where $h$ is Planck's quantum of action. This underlines the importance of eigenvalues and their distributions also from a physical point of view. In case of the hydrogen atom one gets one of the most famous formulas of quantum mechanics,

$$\nu_{N,M} = \frac{|E_N - E_M|}{h} = R\Big(\frac{1}{N^2} - \frac{1}{M^2}\Big), \quad M > N \geq 1, \tag{6.133}$$

where $R = 2\pi^2 me^4 h^{-3}$ is the *Rydberg constant*. Assuming that all constants are fixed (the usual destiny of constants), but that Planck's constant $h$ tends to zero, $h \downarrow 0$, one obtains

$$\#\{\sigma_{\mathsf{e}}(\mathcal{H}_H) \cap (-\infty, -1]\} \sim h^{-3} \tag{6.134}$$

(with equivalence constants which are independent of $h$). This is a concrete example of the *negative spectrum* as considered in (6.14)–(6.16) with Theorem 6.32 as the abstract background. We return to problems of this type in Section 7.7. Further references may be found in Note 6.7.10.

**6.7.2.** Let $(-\Delta)_{\mathsf{D}} = A_{\mathsf{D},F}$ and $(-\Delta)_{\mathsf{N}} = A_{\mathsf{N},F}$ be the Dirichlet Laplacian and Neumann Laplacian, respectively, in a bounded $C^\infty$ domain $\Omega$ in $\mathbb{R}^n$ according to Theorem 5.31. Then the spectral counting function $N(\lambda)$ in (6.19), (6.20) can be detailed by

$$N(\lambda) = (2\pi)^{-n}|\omega_n||\Omega|\lambda^{\frac{n}{2}} \pm \frac{(2\pi)^{-n+1}}{4}|\omega_{n-1}||\partial\Omega|_{n-1}\lambda^{\frac{n-1}{2}}(1+\mathbf{o}(1)) \tag{6.135}$$

for $\lambda \to \infty$ with the same explanations as in connection with (6.20). Here the '$-$' corresponds to $(-\Delta)_{\mathsf{D}}$ and the '$+$' to $(-\Delta)_{\mathsf{N}}$. We refer to [SV97, Example 1.6.16, p. 47], where a corresponding assertion is formulated for the Laplace–Beltrami operator with respect to an $n$-dimensional Riemannian manifold. One may ask what happens if one replaces the Laplacian $-\Delta$ by a more general self-adjoint operator of second order, typically of type

$$Au = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{jk}(x)\frac{\partial u}{\partial x_k}\Big) + a(x)u, \quad \operatorname{dom}(A) = W^2_{2,0}(\Omega), \tag{6.136}$$

where $a_{jk}$ and $a$ are $C^\infty$ coefficients satisfying (5.169), (5.170), (5.227), or of higher order as discussed briefly in Note 5.12.1, or of even more general fractional powers of elliptic operators or pseudo-elliptic operators. Problems of this type have been considered with great intensity since a long time beginning with H. Weyl

[Wey12a], [Wey12b]. A detailed account and recent techniques may be found in [SV97] and [Sog93, Section 4.2]. In case of (6.136) we are typically led to

$$N(\lambda) = c_0 \lambda^{\frac{n}{2}} + \mathcal{O}(\lambda^{\frac{n-1}{2}}) \quad \text{as } \lambda \to \infty, \tag{6.137}$$

with

$$c_0 = (2\pi)^{-n} \Big| \Big\{ (x,\xi) \in \Omega \times \mathbb{R}^n : \sum\nolimits_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \le 1 \Big\} \Big|, \tag{6.138}$$

and

$$\lim_{\lambda \to \infty} \lambda^{-\frac{n-1}{2}} |\mathcal{O}(\lambda^{\frac{n-1}{2}})| < \infty. \tag{6.139}$$

This coincides in the case of the Dirichlet Laplacian with the main term in (6.135). Assertions of type (6.135)–(6.139) are beyond the scope of this book. We are interested in the Weyl exponent as indicated in (6.21), subject of Chapter 7. On the other hand, our method relies on entropy numbers and approximation numbers of compact embeddings between Sobolev spaces. This applies also to rough elliptic operators and even degenerate elliptic operators where nothing like the sharp assertions (6.135) or (6.137) can be expected.

**6.7.3.** Let $A = (a_{jk})_{j,k=1}^n$ be a real symmetric matrix which can be interpreted as a self-adjoint mapping $A$ in the complex $n$-dimensional Hilbert space $H = \mathbb{C}^n$. Let

$$-\infty < \lambda_1 \le \cdots \le \lambda_n < \infty, \quad Ae_j = \lambda_j e_j, \quad e_j \in H,$$

be the ordered eigenvalues $\lambda_j$ and the related orthonormal eigenelements $e_j$. Then $A$ can be written formally (in a strong or weak sense similarly as explained in connection with (6.93), (6.94)) as a (vector-valued or scalar) Riemann–Stieltjes integral

$$Ah = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E_\lambda h = \sum_{j=1}^{n} \lambda_j \langle \cdot, e_j \rangle e_j, \quad h \in H, \tag{6.140}$$

where $E_\lambda$ is the projection of $H$ onto $\operatorname{span}\{e_j : \lambda_j < \lambda\}$. This is illustrated formally in Figure 6.7 (a) below. It is one of the most spectacular (and most beautiful) achievements of the analysis of the last century that there is a counterpart of (6.140) for arbitrary self-adjoint operators $A$ in (infinite-dimensional) complex Hilbert spaces $H$,

$$Ah = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E_\lambda h, \quad h \in \operatorname{dom}(A), \tag{6.141}$$

where $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is a so-called *spectral family* consisting of projections such that

$$E_\lambda h \to 0 \quad \text{if } \lambda \to -\infty, \quad E_\lambda h \to h \quad \text{if } \lambda \to \infty, \quad h \in H, \tag{6.142}$$

and

$$E_\lambda E_\mu = E_\mu E_\lambda = E_{\min(\lambda,\mu)}, \quad \lambda \in \mathbb{R},\ \mu \in \mathbb{R}, \tag{6.143}$$

illustrated in Figure 6.7 (b) below.

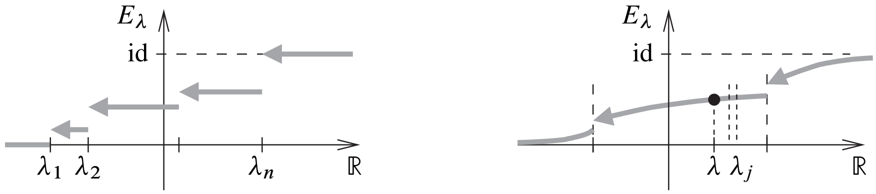



Figure 6.7 (a) Figure 6.7 (b)

This may be found in many books, for example in [Rud91] or [Tri92a, Sections 4.3, 4.4]. Roughly speaking, $\lambda \in \sigma_{\mathrm{p}}(A)$ if, and only if, $E_\mu$ is discontinuous at $\lambda$ and, more precisely,

$$\lambda \in \sigma_{\mathrm{e}}(A) \quad \text{if, and only if,} \quad \dim[(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon})H] = \infty \tag{6.144}$$

for any $\varepsilon > 0$. If we assume (without restriction of generality) that for $\lambda \in \sigma_{\mathrm{e}}(A)$,

$$\dim[(E_{\lambda_j} - E_{\lambda_{j+1}})H] = \infty, \quad \lambda_j = \lambda + 2^{-j},\ j \in \mathbb{N}, \tag{6.145}$$

then there are orthonormal sequences $h_j \in (E_{\lambda_j} - E_{\lambda_{j+1}})H$ such that

$$\{Ah\}_{j=1}^\infty \text{ is orthogonal and } Ah_j - \lambda h_j \to 0. \tag{6.146}$$

Hence $\{h_j\}_j$ is a special Weyl sequence according to Definition 6.5 (iii). For details we refer to [Tri92a, Section 4.3.7, Remark 1, p. 252]. We used these special Weyl sequences in Step 3 of the proof of Theorem 6.32.

**6.7.4.** Some (apparently) typical assertions of spectral theory in Hilbert spaces can be extended to quasi-Banach spaces. This applies in particular to (6.29) and to the second half of (6.30). This is of some use for a theory of elliptic operators in quasi-Banach spaces. We refer to [ET96, Section 1.2] for the abstract part and subsequent applications.

**6.7.5.** An early proof of (6.63) may be found in [GK65, Chapter II, § 2.3, Theorem 3.1] with a reference to [All57]. The approximation numbers in Banach spaces according to (6.42) had been introduced in [Pie63]. Afterwards it turned out that there are many other useful numbers to characterise subclasses of compact operators in Banach spaces, called *s-numbers* (including, e.g., Kolmogorov numbers, Weyl numbers, Gel'fand numbers, ...). All this has been studied with great intensity from the middle of the 1960s up to the end of the 1980s (with some modest contributions of the second-named author of this book). The standard references of the abstract theory are [Pie78], [Pie87], [Kön86], [CS90], [EE87]. They are in common use up to our time. The step from Hilbert spaces to Banach spaces is crucial in this context and the diverse $s$-numbers coincide when reduced to Hilbert spaces. Then one is back to H. Weyl's seminal paper [Wey49]. There one finds what has been called later on *Weyl's inequalities*:

*Let $T \in \mathcal{L}(H)$ be a compact operator in an (infinite-dimensional complex) Hilbert space and let $\{\lambda_k(T)\}_k$ be its eigenvalues (counted with respect to their algebraic multiplicity) and let $a_k(T)$ be the corresponding approximation numbers as introduced in Definition 6.10* (ii). *Then*

$$\prod_{k=1}^{m} |\lambda_k(T)| \le \prod_{k=1}^{m} a_k(T), \quad m \in \mathbb{N}, \tag{6.147}$$

*and*

$$\sum_{k=1}^{m} |\lambda_k(T)|^p \le \sum_{k=1}^{m} a_k(T)^p, \quad m \in \mathbb{N},\ 0 < p < \infty. \tag{6.148}$$

Of course, if $T$ is self-adjoint, then both (6.147) and (6.148) follow from (6.63). It is one of the main subjects of the above-mentioned books to study in detail generalisations of these results, for example, replacing $\ell_p$ in (6.148) by other sequence spaces and, in particular, to extend this theory to Banach spaces. For example, if $X$ is an arbitrary Banach space and $T \in \mathcal{L}(X)$ compact, then (6.63) can always be replaced by the weaker assertion that

$$|\lambda_k(T)| = \lim_{m\to\infty} a_k(T^m)^{1/m}, \quad k \in \mathbb{N},$$

see [Kön79].

The entropy numbers do not fit perfectly in the scheme of $s$-numbers, although they have a lot of properties in common, recall Theorem 6.12. On the other hand, in contrast to approximation numbers they obviously do not satisfy one of the constitutive features of $s$-numbers, that is, the *rank property* (6.51), recall Exercise 6.15. But they have their own history which we outline next.

**6.7.6.** The idea to measure compactness in terms of $\varepsilon$-*entropy* goes back to [PS32] and, in particular, to [KT59]. Let $M$ be a precompact set in a Banach space $Y$. Let $N(\varepsilon, M)$ denote the finite minimal number of balls of radius $\varepsilon > 0$ in $Y$ needed to cover $M$. Then

$$H(\varepsilon, M) = \log N(\varepsilon, M), \tag{6.149}$$

is called the $\varepsilon$-*entropy*, where log is taken to base 2. The idea to use the *inverse functions* came up in [MP68] and in [Tri70]. Based on [MP68] the $m$th entropy number $\varepsilon_m(M, Y)$ of the precompact set $M$ in the Banach space $Y$ had been introduced in

[Tri70] as the infimum of all $\varepsilon > 0$ such that

$$M \subset \bigcup_{i=1}^{m} \{y_i + \varepsilon U_Y\} \quad \text{for some } y_1, \ldots, y_m \in Y,\ m \in \mathbb{N}, \tag{6.150}$$

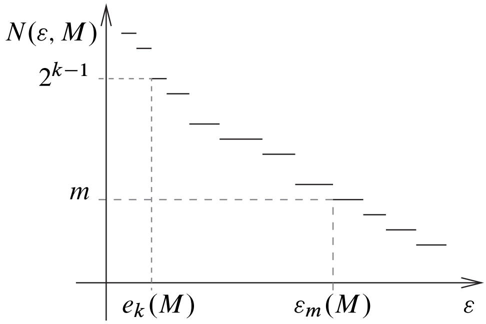



Figure 6.7

where we used the same notation as in connection with (6.41). In [Pie78, Section 12.1] it had been suggested to work only with the dyadic sequence $m = 2^{k-1}$, $k \in \mathbb{N}$, hence

$$e_k(M, Y) = \varepsilon_{2^{k-1}}(M, Y), \quad k \in \mathbb{N}. \tag{6.151}$$

With $M = T(U_X)$ one gets the numbers $e_k(T) = e_k(T(U_X), Y)$ according to Definition 6.10 (i). For a while they had been denoted as *dyadic* entropy numbers. But especially after the discovery of the spectral assertions (6.71), (6.87) in 1979 it became clear that $e_k$ might be the better choice for many purposes. The motivation in [Tri70] to deal with the (original) entropy numbers

$$\varepsilon_m(T) = \varepsilon_m(T(U_X), Y), \quad T \in \mathcal{L}(X, Y), \quad m \in \mathbb{N}, \tag{6.152}$$

using the same notation as in Definition 6.10, came from the proposal to study so-called *entropy ideals* $E_{p,q}$ with $0 < p < \infty$, $0 < q \le \infty$,

$$\begin{aligned} E_{p,q}(X,Y) &= \begin{cases} \Big\{ T \in \mathcal{L}(X,Y) : \sum_{m=2}^{\infty} \varepsilon_m^q(T)(\log m)^{\frac{q}{p}-1} m^{-1} < \infty \Big\}, & q < \infty, \\ \Big\{ T \in \mathcal{L}(X,Y) : \sup_{m \ge 2} \varepsilon_m(T)(\log m)^{\frac{1}{p}} < \infty \Big\}, & q = \infty, \end{cases} \\ &= \begin{cases} \Big\{ T \in \mathcal{L}(X,Y) : \sum_{k=1}^{\infty} e_k^q(T) k^{\frac{q}{p}-1} < \infty \Big\}, & q < \infty, \\ \Big\{ T \in \mathcal{L}(X,Y) : \sup_{k \ge 1} e_k(T) k^{\frac{1}{p}} < \infty \Big\}, & q = \infty, \end{cases} \end{aligned} \tag{6.153}$$

where the second line makes clear again that the use of $e_k$ simplifies the matter considerably. These *entropy ideals* (also called *entropy classes*) attracted afterwards some attention especially the distinguished case $0 < p = q < \infty$, hence $E_{p,p}$ selecting those compact operators $T \in \mathcal{L}(X,Y)$ for which $\{e_k(T)\}_k \in \ell_p$ (and their weighted counterparts). We refer in particular to [Pie78, Section 14.3.1, p. 197], [Car81b] and [CS90, Section 1.5].

**6.7.7.** In 1979 B. Carl discovered (6.87) in the even a little bit sharper version

$$|\lambda_k(T)| \leq \inf_{m \in \mathbb{N}} 2^{\frac{m-1}{2k}} e_m(T), \quad k \in \mathbb{N}, \tag{6.154}$$

published in [Car81b] as suggested by the first inequality in (6.83). Discussing this remarkable formula with the second-named author of this book it came out more or less instantaneously (within hours) that this assertion can be improved by (6.71) using geometric arguments, published in [CT80]. The proof given there became standard (with some modifications). It is not only the same as the above one in connection with Theorem 6.25, but was taken over in all books known to us dealing with this subject, including [CS90], [EE87], [ET96], [Kön86], [Pis89].

**6.7.8.** The systematic study of entropy numbers in Banach spaces is not subject of this book. We refer to [Pie78], [Kön86] and in particular to [CS90] which is the standard reference in this field of research. Some generalisations to quasi-Banach spaces may be found in [ET96]. One may ask how the entropy numbers are related to other $s$-numbers, in particular, to approximation numbers. We restrict ourselves to the typical, but rather useful observations,

$$\sup_{k=1,\dots,m} k^\nu e_k(T) \leq C \sup_{k=1,\dots,m} k^\nu a_k(T), \quad m \in \mathbb{N}, \tag{6.155}$$

and

$$\sup_{k=1,\dots,m} (\log(1+k))^\nu e_k(T) \leq C \sup_{k=1,\dots,m} (\log(1+k))^\nu a_k(T), \quad m \in \mathbb{N}, \tag{6.156}$$

where $\nu > 0$ and $C$ is some positive constant (which may depend on $\nu$, but not on $m$). This goes back to [Car81b], [CS90, p. 96] (for Banach spaces) and [ET96, p. 17].

**6.7.9.** One has (6.50) as a characterisation of compact operators $T \in \mathcal{L}(X,Y)$ acting between Banach spaces $X$ and $Y$ in terms of their corresponding entropy numbers $e_k(T)$. In case of approximation numbers $a_k(T)$ there is (6.52) based on the observation (6.51) for *finite rank operators*. Is there a converse or an extension of (6.54) from couples of Hilbert spaces to Banach spaces? The answer is negative. We illustrate the situation and first recall the notion of the *approximation property* in Banach spaces.

A Banach space $X$ is said to have the *approximation property* if for every precompact set $M \subset X$, and every $\varepsilon > 0$ there is a finite rank operator $L \in \mathcal{L}(X)$ such that

$$\|x - Lx|X\| \leq \varepsilon \quad \text{for every } x \in M. \tag{6.157}$$

One gets as an immediate consequence of [LT77, Theorem 1.e.4, p. 32] that the following two assertions are equivalent to each other:

(i) *$X$ has the approximation property,*

(ii) *for any Banach space $Y$ and any $T \in \mathcal{L}(Y, X)$,*

$$T \text{ is compact} \quad \text{if, and only if,} \quad \lim_{k\to\infty} a_k(T) = 0. \tag{6.158}$$

This is the converse of (6.52) and the perfect counterpart of (6.50). The question whether each Banach space has the approximation property was one of the most outstanding problems of Banach space theory for a long time. It was solved finally by P. Enflo in [Enf73] negatively: *there exist separable Banach spaces which do not have the approximation property*. We refer to [LT77, Section 1.e] and [Pie78, Chapter 10] for further information.

**6.7.10.** In Example 6.2 we discussed the problem of the negative spectrum and its physical relevance. Theorem 6.32 is the abstract background. In particular, (6.97) is called the *Birman–Schwinger principle*. It goes back to [Bir61], [Sch61]. Proofs may be found in [Sim79, Chapter 7] and [Sch86, Chapter 8, § 5]. A short description has also been given in [ET96, Section 5.2.1, p. 186]. Our formulation is different and adapted to our later needs. Nevertheless a few decisive ideas of the proof have been borrowed from [Sim79, p. 87]. This applies in particular to the continuously moving eigenvalues as described in (6.118). But our justifications via (6.116) and the use of the Max–Min principle might be new. Using (6.87) with $T = BA^{-1}$ one obtains by (6.97) that

$$\#\{\sigma(C) \cap (-\infty, 0]\} \le \#\{k \in \mathbb{N} : \sqrt{2} e_k(BA^{-1}) \ge 1\}. \tag{6.159}$$

This entropy version of the Birman–Schwinger principle appeared first in [HT94a, Theorem 2.4], cf. also [ET96, Corollary, p. 186].

**6.7.11.** There are further interesting connections between (the limits of) entropy numbers and approximation numbers on the one hand, and spectral assertions as well as famous inequalities on the other hand. This concerns, for example, the first eigenvalues of $(-\Delta)_{\mathrm{D}}$ and $(-\Delta)_{\mathrm{N}}$ in a bounded domain $\Omega$, the (essential) spectral radius of compact embeddings of type $\mathrm{id}: W_2^1(\Omega) \hookrightarrow L_2(\Omega)$, as well as connections to Poincaré's inequality (5.165) and Friedrichs's inequality (5.142). First results may be found in [Ami78] and [Zem80], we refer to [EE87, Section V.5] and [EE04, Section 4.1] for a comprehensive account on this topic.

# Chapter 7

# Compact embeddings, spectral theory of elliptic operators

## 7.1 Introduction

In Section 5.6 we dealt with the homogeneous Dirichlet problem for *regular elliptic differential operators* $A$,

$$Au(x) = -\sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{l=1}^{n} a_l(x)\frac{\partial u}{\partial x_l}(x) + a(x)u(x), \qquad (7.1)$$

in $L_2(\Omega)$, where $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$ and $A$ is interpreted as an unbounded operator in $L_2(\Omega)$ with

$$\mathrm{dom}(A) = W^2_{2,0}(\Omega) = \{f \in W^2_2(\Omega) : \mathrm{tr}_\Gamma\, f = 0\} \qquad (7.2)$$

according to (5.11) as its domain of definition. Theorem 5.36 solves this problem in a satisfactory way including an assertion about the spectrum $\sigma(A)$ of $A$, consisting of isolated eigenvalues of finite multiplicities located as indicated in Figure 5.5. One may ask whether one can say more about the distribution of eigenvalues and the (associated) eigenelements. For self-adjoint operators $A$,

$$Au = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{jk}(x)\frac{\partial u}{\partial x_k}\Big) + a(x)u, \quad \mathrm{dom}(A) = W^2_{2,0}(\Omega), \qquad (7.3)$$

according to (6.136), we described in Note 6.7.2 and in Example 6.3 far-reaching assertions about the distribution of eigenvalues of these operators with pure point spectrum according to Definition C.7 and Theorem C.15.

The most distinguished case is the Dirichlet Laplacian $A = (-\Delta)_{\mathrm{D}}$. In Section 6.1 we stressed the physical relevance of eigenvalue problems. In Remark 6.4 we outlined our method to get assertions of type (6.21) (the *Weyl exponent*) based on approximation numbers and entropy numbers. To apply the corresponding abstract theory as developed in Chapter 6 we rely on Theorem 4.13 reducing Sobolev spaces to weighted $\ell_2$ spaces. This may justify our dealing first in Section 7.2 with approximation numbers and entropy numbers for compact embeddings between weighted sequence spaces. This will be employed in Section 7.3 to study corresponding problems in function spaces. Equipped with these assertions we discuss afterwards in the Sections 7.4–7.6 eigenvalue problems for *self-adjoint*, *regular*, and *degenerate* elliptic operators. Finally we deal in Section 7.7 with the problem of the negative spectrum as considered so far in Example 6.2 (physical relevance) and in Section 6.5 (abstract background).

## 7.2 Compact embeddings: sequence spaces

Let $M \in \mathbb{N}$ and $1 \le p \le \infty$. By $\ell_p^M$ we shall mean the linear Banach space of all complex $M$-tuples $\xi = (\xi_1, \dots, \xi_M) \in \mathbb{C}^M$ such that

$$\|\xi|\ell_p^M\| = \begin{cases} \Big(\sum_{j=1}^M |\xi_j|^p\Big)^{1/p}, & 1 \le p < \infty, \\ \max_{j=1,\dots,M} |\xi_j|, & p = \infty, \end{cases} \tag{7.4}$$

is finite. We further need $\ell_p$ sequence spaces consisting of weighted blocks of the above type (7.4).

**Definition 7.1.** Let $d > 0, \delta \ge 0, 1 \le p \le \infty$. Let $M_j \in \mathbb{N}$ be such that $M_j \sim 2^{jd}$ for $j \in \mathbb{N}_0$. For

$$b = \{b_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0,\ m = 1, \dots, M_j\} \tag{7.5}$$

we introduce

$$\big\| b \,\big|\ell_p(2^{j\delta}\ell_p^{M_j})\big\| = \begin{cases} \Big(\sum_{j=0}^\infty 2^{j\delta p} \sum_{m=1}^{M_j} |b_{j,m}|^p\Big)^{1/p}, & p < \infty, \\ \sup_{j\in\mathbb{N}_0} 2^{j\delta} \sup_{m=1,\dots,M_j} |b_{j,m}|, & p = \infty. \end{cases} \tag{7.6}$$

Then

$$\ell_p(2^{j\delta}\ell_p^{M_j}) = \big\{b : \big\| b\,\big|\ell_p(2^{j\delta}\ell_p^{M_j})\big\| < \infty\big\}. \tag{7.7}$$

**Remark 7.2.** Recall that $M_j \sim 2^{jd}$ means that there are two constants $0 < c_1 \le c_2 < \infty$ such that

$$c_1\, 2^{jd} \le M_j \le c_2\, 2^{jd} \quad \text{for all } j \in \mathbb{N}_0. \tag{7.8}$$

If $\delta = 0$, then $\ell_p = \ell_p(2^{j\delta}\ell_p^{M_j})$ are the usual $\ell_p$ spaces. In any case, $\ell_p(2^{j\delta}\ell_p^{M_j})$ is a Banach space. These are special cases of the larger scale of spaces $\ell_q(2^{j\delta}\ell_p^{M_j})$ with $0 < q \le \infty$, $0 < p \le \infty$, where (7.7) is generalised by

$$\big\| b \,\big|\ell_q(2^{j\delta}\ell_p^{M_j})\big\| = \begin{cases} \Big\|\Big\{2^{j\delta}\Big(\sum_{m=1}^{M_j} |b_{j,m}|^p\Big)^{1/p}\Big\}_{j\in\mathbb{N}_0}\Big|\ell_q\Big\|, & p < \infty, \\ \Big\|\Big\{2^{j\delta} \sup_{m=1,\dots,M_j} |b_{j,m}|\Big\}_{j\in\mathbb{N}_0}\Big|\ell_q\Big\|, & p = \infty. \end{cases} \tag{7.9}$$

These sequence spaces play a crucial rôle in the recent theory of function spaces of type $B^s_{p,q}(\mathbb{R}^n)$ as briefly mentioned in the Notes 3.6.1, 3.6.3 and in Appendix E. Here we are mainly interested in the case $q = p$ and, in particular, $q = p = 2$. But we add a few comments in Note 7.8.3 below about the more general spaces.

Since particular attention should be paid to approximation numbers and entropy numbers for compact embeddings between the spaces according to Definition 7.1, we deal first with the above spaces $\ell^M_p$. Approximation numbers and entropy numbers have been introduced in Definition 6.10.

**Proposition 7.3.** *Let $1 \le p \le \infty$ and $M \in \mathbb{N}$. Let $\ell^M_p$ be the above (complex) spaces and let*

$$\mathrm{id}\colon \ell^M_p \hookrightarrow \ell^M_p \tag{7.10}$$

*be the identity. Then*

$$a_k(\mathrm{id}\colon \ell^M_p \hookrightarrow \ell^M_p) = \begin{cases} 1 & \text{if } 1 \le k \le M, \\ 0 & \text{if } k > M, \end{cases} \tag{7.11}$$

*and*

$$e_k(\mathrm{id}\colon \ell^M_p \hookrightarrow \ell^M_p) \sim \begin{cases} 1 & \text{if } 1 \le k \le 2M, \\ 2^{-\frac{k}{2M}} & \text{if } k > 2M, \end{cases} \tag{7.12}$$

*where the equivalence constants are independent of $k \in \mathbb{N}$, $M$ and $p$.*

*Proof. Step* 1. Plainly, (7.11) coincides with (6.53) in Exercise 6.14 (a).

*Step* 2. One may interpret the complex space $\ell^M_p$ as $\mathbb{R}^{2M}$ furnished with the norm generated by (7.4). Let $|U|$ be the volume of the unit ball $U$ in $\ell^M_p$ (in the above interpretation). Then $2^{-2M}|U|$ is the volume of a ball of radius $1/2$; hence (6.43) (with $h_k = e_k$) implies $e_{2M}(\mathrm{id}\colon \ell^M_p \hookrightarrow \ell^M_p) \ge \frac{1}{2}$ and the first line in (7.12). Let $k > 2M$ and let $K_\varepsilon$ be the maximal number of points $y^r \in U$ with

$$\|y^r - y^l|\ell^M_p\| > \varepsilon = 2^{-\frac{k}{2M}}, \quad r \ne l. \tag{7.13}$$

Since the balls centred at $y^r$, $r = 1, \dots, K_\varepsilon$, and of radius $\varepsilon$ cover $U$,

$$1 \le K_\varepsilon\, \varepsilon^{2M} = K_\varepsilon 2^{-k}, \quad \text{thus,} \quad K_\varepsilon \ge 2^k. \tag{7.14}$$

On the other hand, balls of radius $\varepsilon/2$ are pairwise disjoint and contained in $2U$, such that

$$K_\varepsilon \Big(\frac{\varepsilon}{2}\Big)^{2M} \le 2^{2M}, \quad \text{that is,} \quad K_\varepsilon \le 2^{k+4M}. \tag{7.15}$$

Consequently, (7.14) and (7.15) lead to

$$e_{k+4M+1}\big(\mathrm{id}\colon \ell_p^M \hookrightarrow \ell_p^M\big) \le 2^{-\frac{k}{2M}} \le e_k\big(\mathrm{id}\colon \ell_p^M \hookrightarrow \ell_p^M\big). \tag{7.16}$$

This proves the second line in (7.12). □

**Remark 7.4.** This is a simplified version of a corresponding proof in [ET96, pp. 98/99] dealing with the more complicated situation of embeddings (7.10) between $\ell_p^M$ spaces with different $p$-parameters for source and target space, see also Exercise 7.5 below. We return to this point in Note 7.8.2 where we also give some further references.

**Exercise* 7.5.** Prove the following (partial) counterpart of Proposition 7.3 in the more general situation

$$\mathrm{id}\colon \ell_{p_1}^M \hookrightarrow \ell_{p_2}^M, \quad 0 < p_1 \le p_2 \le \infty, \quad M \in \mathbb{N}. \tag{7.17}$$

Then

$$e_k(\mathrm{id}\colon \ell_{p_1}^M \hookrightarrow \ell_{p_2}^M) \sim 1, \quad k \le \log(2M), \tag{7.18}$$

where the equivalence constants are independent of $k \in \mathbb{N}$, $M \in \mathbb{N}$. Here log is taken to base 2.

*Hint:* Concerning the lower estimate in (7.18) one may consider the $2M$ 'corner' points $(0, \dots, \pm 1, \dots, 0)$ of $U$ and estimate the radius that is necessary to cover these points with $2^{k-1} \le 2M$ balls. Further details are given in Note 7.8.2.

**Theorem 7.6.** *Let $d > 0$, $\delta > 0$, $1 \le p \le \infty$, and let $\ell_p(2^{j\delta}\ell_p^{M_j})$ be the spaces according to Definition 7.1 with $M_j$ as in (7.8). Then*

$$\mathrm{id}_{(p)}\colon \ell_p(2^{j\delta}\ell_p^{M_j}) \hookrightarrow \ell_p \tag{7.19}$$

*is compact and*

$$e_k(\mathrm{id}_{(p)}\colon \ell_p(2^{j\delta}\ell_p^{M_j}) \hookrightarrow \ell_p) \sim k^{-\delta/d}, \quad k \in \mathbb{N}. \tag{7.20}$$

*Furthermore, if in addition $p = 2$, then*

$$a_k(\mathrm{id}_{(2)}\colon \ell_2(2^{j\delta}\ell_2^{M_j}) \hookrightarrow \ell_2) \sim k^{-\delta/d}, \quad k \in \mathbb{N}. \tag{7.21}$$

*Proof. Step* 1. The compactness of $\mathrm{id}_{(p)}$ follows from $\delta > 0$.

*Step* 2. First we show (7.21). Let

$$D_{\pm} = \begin{pmatrix} \underbrace{\boxed{\begin{matrix} 2^{\pm\delta} & \ddots \\ \ddots & 2^{\pm\delta} \end{matrix}}}_{M_1 \sim 2^d} & 0 & \cdots & 0 & \cdots \\ 0 & \ddots & & \vdots & \\ \vdots & 0 & \underbrace{\boxed{\begin{matrix} 2^{\pm j\delta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 2^{\pm j\delta} \end{matrix}}}_{M_j \sim 2^{jd}} & 0 & \cdots \\ \vdots & & & \ddots & \end{pmatrix}$$

be the indicated diagonal operators in $\ell_2$ inverse to each other. Then $D_-$ is a compact self-adjoint operator in $\ell_2$ with eigenvalues

$$\lambda_k(D_-) \sim 2^{-j\delta} \sim k^{-\delta/d} \quad \text{if } k \in \mathbb{N},\ k \sim 2^{jd}, \tag{7.22}$$

since

$$D_{-} = \begin{pmatrix} \boxed{\begin{matrix} 2^{-\delta} & \ddots \\ \ddots & 2^{-\delta} \end{matrix}} & 0 & \cdots & 0 & \cdots \\ 0 & \ddots & & \vdots & \\ \vdots & 0 & \underbrace{\boxed{\begin{matrix} 2^{-j\delta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 2^{-j\delta} \end{matrix}}}_{M_j \sim 2^{jd}} & 0 & \cdots \\ \vdots & & & \ddots & \end{pmatrix} \; k$$

Thus (6.63) implies

$$a_k(D_-\colon \ell_2 \hookrightarrow \ell_2) \sim k^{-\delta/d}, \quad k \in \mathbb{N}, \tag{7.23}$$

see also (6.59). We use the factorisations

$$D_-(\ell_2 \hookrightarrow \ell_2) = \mathrm{id}_{(2)}\big(\ell_2(2^{j\delta}\ell_2^{M_j}) \hookrightarrow \ell_2\big) \circ D_-\big(\ell_2 \hookrightarrow \ell_2(2^{j\delta}\ell_2^{M_j})\big) \tag{7.24}$$

and

$$\mathrm{id}_{(2)}\big(\ell_2(2^{j\delta}\ell_2^{M_j}) \hookrightarrow \ell_2\big) = D_-(\ell_2 \hookrightarrow \ell_2) \circ D_+\big(\ell_2(2^{j\delta}\ell_2^{M_j}) \hookrightarrow \ell_2\big), \tag{7.25}$$

where the second operators on the corresponding right-hand sides are isomorphic maps. Consequently, by (6.46),

$$a_k\big(\mathrm{id}_{(2)}\colon \ell_2\big(2^{j\delta}\ell_2^{M_j}\big) \hookrightarrow \ell_2\big) \sim a_k(D_-\colon \ell_2 \hookrightarrow \ell_2) \sim k^{-\delta/d}, \quad k \in \mathbb{N}. \tag{7.26}$$

This proves (7.21).

*Step* 3. We split the proof of (7.20) into two steps. For $j \in \mathbb{N}$ we decompose

$$\mathrm{Id}_j\big(\ell_p^{M_j} \hookrightarrow \ell_p^{M_j}\big) = \mathrm{id}_j \circ \mathrm{id}_{(p)} \circ \mathrm{id}^j, \quad j \in \mathbb{N}_0, \tag{7.27}$$

as represented in the following commutative diagram,

$$\begin{array}{ccc} \ell_p^{M_j} & \xrightarrow{\ \mathrm{id}^j\ } & \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \\ {\scriptstyle \mathrm{Id}_j}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}_{(p)}} \\ \ell_p^{M_j} & \xleftarrow[\ \mathrm{id}_j\ ]{} & \ell_p \end{array} \tag{7.28}$$

where

$$\mathrm{id}^j : \ell_p^{M_j} \hookrightarrow \ell_p\big(2^{j\delta}\ell_p^{M_j}\big),$$
$$(x_{j,1},\dots,x_{j,M_j}) \mapsto (0,\dots,0,x_{j,1},\dots,x_{j,M_j},0,\dots),$$

$$\mathrm{id}_j : \ell_p \hookrightarrow \ell_p^{M_j},$$
$$(x_{0,1},x_{1,1},\dots,x_{j,1},\dots,x_{j,M_j},x_{j+1,1},\dots) \mapsto (x_{j,1},\dots,x_{j,M_j}).$$

Plainly,

$$\big\|\mathrm{id}^j : \ell_p^{M_j} \hookrightarrow \ell_p\big(2^{j\delta}\ell_p^{M_j}\big)\big\| = 2^{j\delta} \quad \text{and} \quad \big\|\mathrm{id}_j : \ell_p \hookrightarrow \ell_p^{M_j}\big\| = 1. \tag{7.29}$$

Then by (6.46),

$$e_k\big(\mathrm{Id}_j : \ell_p^{M_j} \hookrightarrow \ell_p^{M_j}\big) \le 2^{j\delta} e_k\big(\mathrm{id}_{(p)}\colon \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p\big), \quad k \in \mathbb{N}, \tag{7.30}$$

and (7.12) with $k \sim 2^{jd} \sim M_j$ implies for some $c > 0$ and $c' > 0$ that

$$1 \le c\, 2^{j\delta} e_k\big(\mathrm{id}_{(p)}\colon \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p\big)$$

and hence

$$e_k\big(\mathrm{id}_{(p)}\colon \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p\big) \ge c'\, 2^{-j\delta} \sim k^{-\delta/d}. \tag{7.31}$$

This is the estimate from below in (7.20).

*Step* 4. It remains to prove the estimate from above in (7.20). For this it is sufficient to show that there are two positive constants $c$ and $c'$ such that

$$e_{c2^{Ld}}\big(\mathrm{id}_{(p)} : \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p\big) \le c' 2^{-L\delta} \quad \text{for all } L \in \mathbb{N}. \tag{7.32}$$

Let $e_\lambda(\cdot) = e_{[\lambda]}(\cdot)$ for $\lambda \in \mathbb{R}$, $\lambda \ge 1$, for simplicity, where $[\lambda] = \max\{k \in \mathbb{Z} : k \le \lambda\}$. We decompose

$$\mathrm{id}_{(p)}\big(\ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p\big) = \sum_{j=0}^{L} \mathrm{id}_j + \sum_{j=L+1}^{\infty} \mathrm{id}_j, \quad L \in \mathbb{N}, \tag{7.33}$$

where

$$\mathrm{id}_j : \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p, \quad x \longmapsto (0,\dots,0,x_{j,1},\dots,x_{j,M_j},0,\dots), \quad j \in \mathbb{N}_0.$$

Since $\|\mathrm{id}_j : \ell_p(2^{j\delta}\ell_p^{M_j}) \hookrightarrow \ell_p\| \le 2^{-j\delta}$, one obtains

$$\Big\| \sum_{j=L+1}^{\infty} \mathrm{id}_j : \ell_p(2^{j\delta}\ell_p^{M_j}) \hookrightarrow \ell_p \Big\| \le c\, 2^{-L\delta}, \quad L \in \mathbb{N}. \tag{7.34}$$

By an argument similar to Step 3 we may decompose $\mathrm{id}_j$ as $\mathrm{id}_j = \widetilde{\mathrm{id}}_j \circ \mathrm{Id}_j \circ \widetilde{\mathrm{id}}^j$,

$$\begin{array}{ccc}
\ell_p^{M_j} & \xleftarrow{\ \widetilde{\mathrm{id}}^j\ } & \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \\
{\scriptstyle \mathrm{Id}_j}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_j} \\
\ell_p^{M_j} & \xrightarrow[\ \widetilde{\mathrm{id}}_j\ ]{} & \ell_p
\end{array}$$

with

$$\widetilde{\mathrm{id}}^j : \ \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p^{M_j}, \quad x \mapsto (x_{j,1},\dots,x_{j,M_j}),$$

$$\widetilde{\mathrm{id}}_j : \ \ell_p^{M_j} \hookrightarrow \ell_p, \quad (x_{j,1},\dots,x_{j,M_j}) \mapsto (0,\dots,0,x_{j,1},\dots,x_{j,M_j},0,\dots),$$

such that

$$\big\|\widetilde{\mathrm{id}}^j : \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p^{M_j}\big\| = 2^{-j\delta} \quad \text{and} \quad \big\|\widetilde{\mathrm{id}}_j : \ell_p^{M_j} \hookrightarrow \ell_p\big\| = 1, \ j \in \mathbb{N}_0.$$

Thus (6.46) (with $h_k = e_k$) implies

$$\begin{aligned}
e_m\big(\mathrm{id}_j : \ell_p\big(2^{j\delta}\ell_p^{M_j}\big) \hookrightarrow \ell_p\big) &= e_m\big(\widetilde{\mathrm{id}}_j \circ \mathrm{Id}_j(\ell_p^{M_j} \hookrightarrow \ell_p^{M_j}) \circ \widetilde{\mathrm{id}}^j\big) \\
&\le c\, 2^{-j\delta}\, e_m\big(\mathrm{Id}_j : \ell_p^{M_j} \hookrightarrow \ell_p^{M_j}\big), \ m \in \mathbb{N}.
\end{aligned} \tag{7.35}$$

Let $0 < \varepsilon < d$ and

$$k_j = 2^{Ld}\, 2^{-(L-j)\varepsilon}, \quad j = 0, \dots, L, \tag{7.36}$$

i.e.,

$$k_j = 2^{jd}\, 2^{(L-j)(d-\varepsilon)} \ge 2^{jd} \sim M_j, \quad j = 0, \dots, L, \tag{7.37}$$

and

$$k = \sum_{j=0}^{L} k_j \sim 2^{Ld}, \quad L \in \mathbb{N}. \tag{7.38}$$

We apply (7.35) and the second line in (7.12) with $k_j \ge M_j \sim 2^{jd}$ and obtain

$$\begin{aligned} \sum_{j=0}^{L} e_{k_j}(\mathrm{id}_j) &\le c_1 \sum_{j=0}^{L} 2^{-j\delta}\, 2^{-\frac{k_j}{2M_j}} \\ &\le c_2\, 2^{-L\delta} \sum_{j=0}^{L} 2^{(L-j)\delta}\, 2^{-2^{(L-j)(d-\varepsilon)}} \le c_3\, 2^{-L\delta}, \end{aligned} \tag{7.39}$$

where $c_1$, $c_2$, $c_3$ are positive constants which are independent of $L$. Now (7.32) follows from (6.44) (with $h_k = e_k$), (7.38), together with (7.34) and (7.39). □

**Remark 7.7.** The proof of (7.20) is a simplified version of corresponding assertions in [Tri97, Theorem 8.2, p. 39]. There we dealt with

$$\mathrm{id}\colon \ell_{q_1}\big(2^{j\delta}\ell_{p_1}^{M_j}\big) \hookrightarrow \ell_{q_2}\big(\ell_{p_2}^{M_j}\big), \quad \delta > 0, \tag{7.40}$$

for $0 < p_1 \le p_2 \le \infty$, $0 < q_1, q_2 \le \infty$, $d > 0$, and $M_j \sim 2^{jd}$, $j \in \mathbb{N}_0$, using the notation (7.9). The space on the right-hand side refers to the unweighted case, hence 1 in place of $2^{j\delta}$ in (7.9). If $p_1 = p_2 = p$, then one can easily replace the outer $p$'s in (7.19) by $q_1$ and $q_2$ (in the interpretation of (7.40)). The case $p_1 \ne p_2$ is more difficult, but of great use in the theory of function spaces as indicated in Remark 7.2. We return to theses questions in Note 7.8.3 below where we give also some references.

## 7.3 Compact embeddings: function spaces

Theorems 4.13 and 4.17 open the possibility to transfer problems of compact embeddings

$$\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow L_2(\Omega), \quad s > 0, \tag{7.41}$$

to corresponding questions for sequence spaces of the above type. We use the same notation as in Section 4.4, in particular,

$$\mathbb{Q}^n = (-\pi, \pi)^n = \{x \in \mathbb{R}^n : -\pi < x_j < \pi,\ j = 1, \dots, n\}, \tag{7.42}$$

and

$$K = K_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}. \tag{7.43}$$

We always assume that $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$. If, in addition,

$$\Omega \subset \left\{x \in \mathbb{R}^n : |x| \le \frac{1}{2}\right\}, \tag{7.44}$$

then we can apply the common extension operator

$$\mathrm{ext}^L_\Omega \colon W^s_2(\Omega) \hookrightarrow \widetilde{W}^s_2(\bar{K}), \quad 0 \le s < L, \tag{7.45}$$

according to (4.90) with $\widetilde{W}^s_2(\bar{K})$ as in (4.70). Now we can complement Theorem 4.17 as follows.

**Theorem 7.8.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3 and let $W^s_2(\Omega)$ with $s > 0$ be the Sobolev spaces as in (4.2) (with a reference to Definition 3.37). Then the embedding*

$$\mathrm{id} \colon W^s_2(\Omega) \hookrightarrow L_2(\Omega) \tag{7.46}$$

*is compact. Let $a_k(\mathrm{id} \colon W^s_2(\Omega) \hookrightarrow L_2(\Omega))$ and $e_k(\mathrm{id} \colon W^s_2(\Omega) \hookrightarrow L_2(\Omega))$ be the corresponding approximation numbers and entropy numbers according to Definition 6.10. Then*

$$a_k(\mathrm{id} \colon W^s_2(\Omega) \hookrightarrow L_2(\Omega)) \sim e_k(\mathrm{id} \colon W^s_2(\Omega) \hookrightarrow L_2(\Omega)) \sim k^{-s/n}, \quad k \in \mathbb{N}, \tag{7.47}$$

*where the equivalence constants are independent of $k$.*

*Proof. Step* 1. By Theorem 4.17 we know that id is compact.

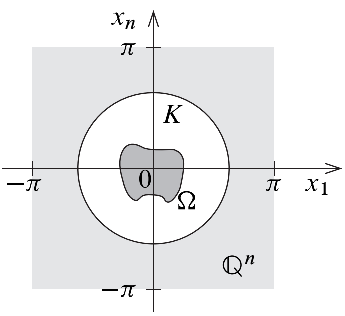



Figure 7.1

Without restriction of generality we may assume that $\Omega$ satisfies (7.44) as shown in Figure 7.1. Let $\ell^s_2$ be the space of all sequences $b = \{b_m \in \mathbb{C} : m \in \mathbb{Z}^n\}$ normed by

$$\|b|\ell^s_2\| = \Big(\sum_{m \in \mathbb{Z}^n} (1 + |m|^2)^s |b_m|^2\Big)^{\frac{1}{2}}, \tag{7.48}$$

used (implicitly) in Theorem 4.13. Then

$$\widetilde{\mathrm{id}} \colon \ell^s_2 \hookrightarrow \ell^0_2 = \ell_2 \text{ is compact.} \tag{7.49}$$

Furthermore, id according to (7.46) can be factorised as

$$\mathrm{id}(W_2^s(\Omega) \hookrightarrow L_2(\Omega)) = \mathrm{re}_\Omega \circ T \circ \widetilde{\mathrm{id}}(\ell_2^s \hookrightarrow \ell_2) \circ S \circ \mathrm{ext}_\Omega^L \tag{7.50}$$

with $\mathrm{ext}_\Omega^L$ given by (7.45),

$$S\colon \widetilde{W}_2^s(\bar{K}) \hookrightarrow \ell_2^s, \qquad f \mapsto \{b_m(f)\}_{m\in\mathbb{Z}^n} \tag{7.51}$$

$$\text{with } b_m(f) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{Q}^n} f(x) e^{-imx} \mathrm{d}x, \; m \in \mathbb{Z}^n, \tag{7.52}$$

$$T\colon \; \ell_2 \hookrightarrow L_2(\mathbb{Q}^n), \qquad \{b_m\}_{m\in\mathbb{Z}^n} \;\mapsto\; f = (2\pi)^{-\frac{n}{2}} \sum_{m\in\mathbb{Z}^n} b_m e^{imx}, \tag{7.53}$$

and the restriction

$$\mathrm{re}_\Omega\colon L_2(\mathbb{Q}^n) \hookrightarrow L_2(\Omega). \tag{7.54}$$

By the above comments, (4.59) and Theorem 4.13 it follows that all operators are bounded. With $h_k$ replaced by either $a_k$ or $e_k$, respectively,

$$h_k(\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow L_2(\Omega)) \le c\, h_k(\widetilde{\mathrm{id}}\colon \ell_2^s \hookrightarrow \ell_2), \quad k \in \mathbb{N}, \tag{7.55}$$

is a consequence of Theorem 6.12.

*Step* 2. We wish to prove the converse of (7.55) assuming without restriction of generality that $\mathbb{Q}^n$ and $\Omega$ are located as indicated in Figure 7.2. There is a linear and bounded extension operator from $W_2^s(\mathbb{Q}^n)$ according to Definition 3.37 into $W_2^s(\Omega)$. But we did not prove this ($\mathbb{Q}^n$ is not a $C^\infty$ domain, unfortunately). However, one can circumvent this problem as follows.

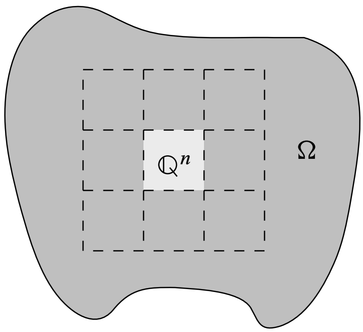



Figure 7.2

Let $0 < s < 1$ and $f$ be given by

$$f(x) = (2\pi)^{-\frac{n}{2}} \sum_{m\in\mathbb{Z}^n} b_m e^{imx}, \quad x \in \mathbb{Q}^n, \; b = \{b_m\}_{m\in\mathbb{Z}^n} \in \ell_2^s, \tag{7.56}$$

normed as in (4.65) by

$$\|f\,|W_2^s(\mathbb{Q}^n)^\pi\| = \Bigg( \|f\,|L_2(\mathbb{Q}^n)\|^2 + \iint\limits_{\mathbb{Q}^n\times\mathbb{Q}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \Bigg)^{1/2}, \tag{7.57}$$

where $\pi$ refers to *periodic*. (One may first assume that only finitely many coefficients $b_m$ are different from zero, the rest is afterwards a matter of completion.)

Then we extend $f$ periodically into adjacent cubes and multiply the outcome with a cut-off function $\psi \in \mathcal{D}(\Omega)$ with $\psi \equiv 1$ on $\mathbb{Q}^n$. The resulting function $f^\Omega$,

$$f^\Omega(x) = \psi(x)(2\pi)^{-\frac{n}{2}} \sum_{m\in\mathbb{Z}^n} b_m e^{imx}, \quad x \in \Omega,\ b \in \ell_2^s, \tag{7.58}$$

belongs to $W_2^s(\Omega)$ and one gets by the same arguments as in (4.65)–(4.67) that

$$\| f^\Omega | W_2^s(\Omega)\| \sim \| f | W_2^s(\mathbb{Q}^n)^\pi \| \sim \|b|\ell_2^s\|. \tag{7.59}$$

These arguments can be extended to $W_2^k(\mathbb{Q}^n)$, $k \in \mathbb{N}$, and then to all $W_2^s(\mathbb{Q}^n)$, $s > 0$. In particular, at least the periodic subspace $W_2^s(\mathbb{Q}^n)^\pi$ of $W_2^s(\mathbb{Q}^n)$ spanned by (7.56) has the desired extension operator. Now one can prove the converse of (7.55) factorising $\widetilde{\mathrm{id}}$ in (7.49) as

$$\widetilde{\mathrm{id}}(\ell_2^s \hookrightarrow \ell_2) = V \circ \mathrm{re}_{\mathbb{Q}^n} \circ \mathrm{id}(W_2^s(\Omega) \hookrightarrow L_2(\Omega)) \circ U, \tag{7.60}$$

with

$$U : \ell_2^s \hookrightarrow W_2^s(\Omega) \quad \text{according to (7.58)}, \tag{7.61}$$

the restriction $\mathrm{re}_{\mathbb{Q}^n}$ from $L_2(\Omega)$ to $L_2(\mathbb{Q}^n)$, and the isomorphic map $V$ in (4.59). In that way one obtains the converse of (7.55) and hence

$$h_k(\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow L_2(\Omega)) \sim h_k(\widetilde{\mathrm{id}}\colon \ell_2^s \hookrightarrow \ell_2), \quad k \in \mathbb{N}. \tag{7.62}$$

*Step* 3. For $j \in \mathbb{N}$ one has

$$\#\{m \in \mathbb{Z}^n : 2^j \le |m| < 2^{j+1}\} \sim M_j \sim 2^{jn} \tag{7.63}$$

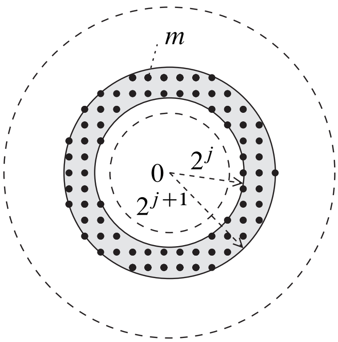


Figure 7.3

as indicated in Figure 7.3. Then it follows from (7.48) and Definition 7.1 that (after a suitable re-numbering)

$$\ell_2^s = \ell_2(2^{j\delta}\ell_2^{M_j}) \tag{7.64}$$

with $d = n$, $\delta = s$. Now (7.62) and Theorem 7.6 prove (7.47). □

**Exercise* 7.9.** Consider the embedding

$$\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow W_2^t(\Omega), \quad 0 < t < s < \infty, \tag{7.65}$$

and show that

$$a_k(\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow W_2^t(\Omega)) \sim e_k(\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow W_2^t(\Omega)) \sim k^{-\frac{s-t}{n}}, \quad k \in \mathbb{N}, \tag{7.66}$$

for the corresponding approximation numbers and entropy numbers.
*Hint:* The extension of $V$ from $L_2$ to $W_2^t$ causes some trouble. Use

$$W_2^s(\Omega) \hookrightarrow W_2^t(\Omega) \hookrightarrow L_2(\Omega) \tag{7.67}$$

and what is already known.

**Remark 7.10.** The above theorem is a rather special case of a far-reaching theory of approximation numbers and entropy numbers for compact embeddings between function spaces. One may consult Note 7.8.5 below where we also give some references. But we formulate a specific result which will be of some use for us later on as a corollary of the compact embedding in Theorem 5.59.

**Corollary 7.11.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ where $n \in \mathbb{N}$. Let*

$$\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega) \tag{7.68}$$

*be the compact embedding* (5.286), (5.287). *Then*

$$e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega)) \sim k^{-2/n}, \quad k \in \mathbb{N}. \tag{7.69}$$

*Proof* (of a weaker assertion). A full proof of (7.69) is beyond the scope of this book. It is a special case of more general properties mentioned in Note 7.8.5. But one can prove some weaker estimates supporting (7.69).

*Step* 1. Let, in addition, $p \geq 2$. Then

$$e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega)) \geq c\, k^{-2/n} \tag{7.70}$$

for some $c > 0$ and all $k \in \mathbb{N}$ as a consequence of (7.47) and

$$W_2^2(\Omega) \hookrightarrow L_p(\Omega) \hookrightarrow L_2(\Omega). \tag{7.71}$$

Let $n \geq 5$ and $2 < p < p^*$ with $p^*$ as in (5.285). For some $c > 1$ one finds for all $k \in \mathbb{N}$ elements $\{g_j\}_{j=1}^{2^{k-1}} \subset W_2^2(\Omega)$ with $\|g_j | W_2^2(\Omega)\| \leq 1$ such that

$$\min_j \|f - g_j | L_2(\Omega)\| \leq c\, e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_2(\Omega)) \quad \text{for all } f \in W_2^2(\Omega),\ \|f | W_2^2(\Omega)\| \leq 1. \tag{7.72}$$

Using

$$\|g | L_p(\Omega)\| \leq \|g | L_2(\Omega)\|^{1-\theta} \|g | L_{p^*}(\Omega)\|^{1-\theta}, \quad \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p^*}, \tag{7.73}$$

and the continuity of the embedding (7.68) with $p$ replaced by $p^*$ according to Theorem 5.59, (7.72) and (7.47) imply that

$$\begin{aligned} e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega)) &\le c\, e_k^{1-\theta}(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_2(\Omega)) \\ &\le c'\, k^{-\frac{2}{n}+\frac{1}{2}-\frac{1}{p}}, \quad k \in \mathbb{N}. \end{aligned} \tag{7.74}$$

The exponent can be calculated directly or obtained by arguing that it must depend linearly on $\frac{1}{p}$ with $-\frac{2}{n}$ if $p = 2$ and 0 if $p = p^*$, given by (5.285). Obviously this is a weaker estimate than the desired one (7.69).

*Step* 2. Let $1 \le p < 2$. Then one obtains

$$e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega)) \le c\, k^{-2/n} \tag{7.75}$$

for some $c > 0$ and all $k \in \mathbb{N}$ as a consequence of (7.47) and

$$W_2^2(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow L_p(\Omega). \tag{7.76}$$

We prove the converse and choose a number $r > 2$ satisfying (5.287) with $p$ replaced by $r$. Let

$$\frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{r}, \quad 0 < \theta < 1. \tag{7.77}$$

We cover the unit ball $U$ in $W_2^s(\Omega)$ with $2^{k-1}$ balls $K_j$ in $L_r(\Omega)$ having radius $(1+\varepsilon)e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_r(\Omega))$ for given $\varepsilon > 0$. Each of the sets $K_j \cap U$ can be covered by $2^{k-1}$ balls in $L_p(\Omega)$ of radius $2(1+\varepsilon)e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega))$ with centres in $K_j \cap U$ (the number 2 comes from the triangle inequality and the assumption that the centres should lie in $K_j \cap U$). There are $2^{2k-2}$ such centres $g_l$. Hence for given $f \in U$ there is one such centre $g_l$ with

$$\| f - g_l | L_r(\Omega)\| \le c\, e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_r(\Omega)) \tag{7.78}$$

and

$$\| f - g_l | L_p(\Omega)\| \le c\, e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega)). \tag{7.79}$$

Using (7.77) and the counterpart of (7.73) one gets by (7.47) that

$$\begin{aligned} k^{-2/n} &\le c\, e_{2k-1}(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_2(\Omega)) \\ &\le c'\, e_k^{1-\theta}(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega)) e_k^{\theta}(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_r(\Omega)) \end{aligned} \tag{7.80}$$

for some $c > 0$ and $c' > 0$ and all $k \in \mathbb{N}$. Now we take (7.69) for $r = p > 2$ for granted and insert it in (7.80). Hence,

$$e_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_p(\Omega)) \ge c\, k^{-2/n}, \quad k \in \mathbb{N}, \tag{7.81}$$

for some $c > 0$. This is the converse of (7.75) and proves (7.69) for $1 \le p \le 2$. □

**Remark 7.12.** This is a special case of a more general assertion mentioned in Note 7.8.5. The arguments in Step 2 reflect the so-called *interpolation property* of entropy numbers. We return to this point in Note 7.8.4 including some references.

## 7.4 Spectral theory of elliptic operators: the self-adjoint case

Equipped with Theorem 7.8 and the assertions about approximation numbers and entropy numbers as obtained in Chapter 6 on an abstract level we can now complement the theory of elliptic differential operators as developed in Chapter 5 by some spectral properties. We outlined the programme in Section 7.1. For sake of convenience we recall and refine some basic notation.

We always assume that $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$ according to Definition A.3 and that $C(\Omega)$ and $C^1(\Omega)$ are the spaces as introduced in Definition A.1. Let $A$,

$$(Au)(x) = -\sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{l=1}^{n} a_l(x)\frac{\partial u}{\partial x_l}(x) + a(x)u(x), \quad (7.82)$$

be an elliptic differential operator according to Definition 5.1 now with

$$\{a_{jk}\}_{j,k=1}^n \subset C^1(\Omega), \quad \{a_l\}_{l=1}^n \subset C(\Omega), \quad a \in C(\Omega), \quad (7.83)$$

and

$$a_{jk}(x) = a_{kj}(x) \in \mathbb{R}, \quad x \in \overline{\Omega},\ j,k = 1,\ldots,n, \quad (7.84)$$

such that

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \ \geq\ E|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^n, \quad (7.85)$$

for some ellipticity constant $E > 0$. The spaces $W_2^2(\Omega)$, $W_{2,0}^2(\Omega)$, and $W_2^{2,\nu}(\Omega)$ have the same meaning as in Definitions 3.37 and 4.30 where $\nu$ is the $C^\infty$ vector field of outer normals on $\Gamma = \partial\Omega$. As in Theorem 5.36 elliptic expressions of type (7.82)–(7.85) are considered as unbounded operators in $L_2(\Omega)$ with $W_{2,0}^2(\Omega)$ or $W_2^{2,\nu}(\Omega)$ as their respective domains of definition.

**Theorem 7.13.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$.*

(i) *Let $A$,*

$$(Au)(x) = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}(x)\right) + a(x)u(x), \quad (7.86)$$

*be an elliptic operator with real coefficients,*

$$\{a_{jk}\}_{j,k=1}^n \subset C^1(\Omega), \quad a \in C(\Omega), \quad a(x) \geq 0, \quad (7.87)$$

(7.84), (7.85) *and* $\operatorname{dom}(A) = W^2_{2,0}(\Omega)$ *as its domain of definition. Then* $A$ *is a positive-definite self-adjoint operator with pure point spectrum according to Definitions* C.7 *and* C.9. *Let*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty, \tag{7.88}$$

*be its ordered eigenvalues repeated according to their multiplicities. Then there are two constants* $0 < c_1 \leq c_2 < \infty$ *such that*

$$c_1\, k^{2/n} \leq \lambda_k \leq c_2\, k^{2/n}, \quad k \in \mathbb{N}. \tag{7.89}$$

(ii) *Let* $A$,

$$(Au)(x) = -\Delta u(x), \quad \operatorname{dom}(A) = W_2^{2,\nu}(\Omega), \tag{7.90}$$

*be the Neumann Laplacian. Then* $A$ *is a self-adjoint positive operator with pure point spectrum according to Definitions* C.7 *and* C.9. *Let*

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty, \tag{7.91}$$

*be its ordered eigenvalues repeated according to their multiplicities. Then* $\lambda_0 = 0$ *is a simple eigenvalue and* $\lambda_k$ *with* $k \in \mathbb{N}$ *satisfy* (7.89).

*Proof. Step* 1. We prove (i) in two steps. Let $u \in C^\infty(\Omega)$, $v \in C^\infty(\Omega)$, and $\operatorname{tr}_\Gamma u = \operatorname{tr}_\Gamma v = 0$ for $\Gamma = \partial\Omega$. Integration by parts implies

$$\begin{aligned}
\langle Au, v\rangle &= -\int_\Omega \sum_{j,k=1}^n \frac{\partial}{\partial x_j}\Big(a_{jk}(x)\frac{\partial u}{\partial x_k}(x)\Big)\,\overline{v(x)}\mathrm{d}x + \int_\Omega a(x)u(x)\overline{v(x)}\mathrm{d}x \\
&= \int_\Omega \Big(\sum_{j,k=1}^n a_{jk}(x)\frac{\partial u}{\partial x_k}(x)\,\frac{\partial \overline{v}}{\partial x_j}(x) + a(x)u(x)\overline{v}(x)\Big)\mathrm{d}x \\
&= \langle u, Av\rangle
\end{aligned} \tag{7.92}$$

since $a_{jk}(x)$ and $a(x)$ are real. By the density assertion of Proposition 4.32 (i) we can argue by completion that

$$\langle Au, v\rangle = \langle u, Av\rangle, \quad u \in \operatorname{dom}(A),\ v \in \operatorname{dom}(A). \tag{7.93}$$

Hence $A$ is a symmetric operator. By the Theorems 5.36 and C.3 (ii) it follows that $A$ is self-adjoint. Furthermore, due to the ellipticity condition (7.85), $a \geq 0$, and Friedrichs's inequality (5.142) one gets for $u \in \operatorname{dom}(A)$ that

$$\langle Au, u\rangle \geq c\int_\Omega \sum_{j=1}^n \Big|\frac{\partial u}{\partial x_j}(x)\Big|^2\mathrm{d}x \;\geq\; c'\int_\Omega |u(x)|^2\mathrm{d}x, \tag{7.94}$$

for some positive constants $c$ and $c'$. Hence $A$ is a positive-definite self-adjoint operator. Following the same arguments as in Step 3 of the proof of Theorem 5.31 one concludes that $A$ is an operator with pure point spectrum and we have (7.88) for its eigenvalues.

*Step* 2. To complete the proof of (i) it remains to verify (7.89). It follows by Theorem 7.8 and the same restriction and extension arguments as used there that

$$a_k(\mathrm{id}\colon W_2^2(\Omega) \hookrightarrow L_2(\Omega)) \sim a_k(\mathrm{id}\colon W_{2,0}^2(\Omega) \hookrightarrow L_2(\Omega)) \sim k^{-2/n} \tag{7.95}$$

for the approximation numbers of the corresponding embeddings. We factorise the inverse $A^{-1}$ of the above operator $A$ as

$$\begin{aligned} &A^{-1}(L_2(\Omega) \hookrightarrow L_2(\Omega)) \\ &\quad = \mathrm{id}(W_{2,0}^2(\Omega) \hookrightarrow L_2(\Omega)) \circ A^{-1}(L_2(\Omega) \hookrightarrow W_{2,0}^2(\Omega)) \end{aligned} \tag{7.96}$$

where the latter is an isomorphic map according to Theorem 5.36. Application of Theorem 6.12 and (7.95) leads to

$$a_k(A^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega)) \le c\, k^{-2/n}, \quad k \in \mathbb{N}. \tag{7.97}$$

The factorisation

$$\begin{aligned} &\mathrm{id}(W_{2,0}^2(\Omega) \hookrightarrow L_2(\Omega)) \\ &\quad = A^{-1}(L_2(\Omega) \hookrightarrow L_2(\Omega)) \circ A(W_{2,0}^2(\Omega) \hookrightarrow L_2(\Omega)) \end{aligned} \tag{7.98}$$

results in the converse of (7.97). Hence,

$$a_k(A^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega)) \sim k^{-2/n}, \quad k \in \mathbb{N}. \tag{7.99}$$

Now (7.89) is a consequence of (7.99) and (6.63).

*Step* 3. The proof of (ii) follows from Theorem 5.31 (ii) and the same type of arguments as in Step 2. □

**Remark 7.14.** We obtained the *Weyl exponent* $\frac{2}{n}$ as discussed in (6.21). Otherwise we refer to Example 6.3 and Note 6.7.2 for further comments and sharper results.

## 7.5 Spectral theory of elliptic operators: the regular case

For self-adjoint elliptic operators (7.86) we got the satisfactory Theorem 7.13 including the assertion (7.89) about the distribution of eigenvalues. Furthermore, according to Remark C.16 the corresponding eigenfunctions span $L_2(\Omega)$ (there is even a complete orthonormal system of eigenfunctions in $H = L_2(\Omega)$). The question arises whether there are similar properties for the more general *regular* elliptic operators (7.82)–(7.85) with $\operatorname{dom}(A) = W^2_{2,0}(\Omega)$. So far we have Theorem 5.36. We may assume without restriction of generality that 0 belongs to the resolvent set $\varrho(A)$ of $A$. Then

$$A^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega) \text{ is compact} \tag{7.100}$$

and one can apply the Riesz Theorem C.1. We discussed in Remark 5.35 how the spectra $\sigma(A), \sigma(A^{-1})$ and the point spectra $\sigma_{\mathrm{p}}(A), \sigma_{\mathrm{p}}(A^{-1})$ are related to each other with a satisfactory outcome as far as the geometric multiplicities of eigenvalues are concerned. Furthermore, the spectrum is located as indicated in Figure 5.5. In Note 5.12.6 we dealt with the more delicate question of the algebraic multiplicity of $\lambda \in \sigma_{\mathrm{p}}(A)$ advocating that it might be better (at least in an abstract setting) to shift this question to the algebraic multiplicity of $\lambda^{-1} \in \sigma_{\mathrm{p}}(A^{-1})$. However, if one has additional information, then it is reasonable to define the *algebraic multiplicity* of $\lambda \in \sigma_{\mathrm{p}}(A)$ as

$$\dim \Big( \operatorname{dom}(A^\infty) \cap \bigcup_{k=1}^{\infty} \ker(A - \lambda \,\mathrm{id})^k \Big) \quad \text{with } \operatorname{dom}(A^\infty) = \bigcap_{j=1}^{\infty} \operatorname{dom}(A^j). \tag{7.101}$$

In other words, only

$$u \in \operatorname{dom}(A^\infty) \quad \text{with } (A - \lambda\,\mathrm{id})^k u = 0 \text{ for some } k \in \mathbb{N} \tag{7.102}$$

are admitted. On $\operatorname{dom}(A^\infty)$ one can freely operate with powers of $A$ and $A^{-1}$. In particular, if $u \in \operatorname{dom}(A^\infty)$ and

$$(A - \lambda\,\mathrm{id})^k u = 0, \quad \text{then} \quad (A^{-1} - \lambda^{-1}\mathrm{id})^k u = 0, \tag{7.103}$$

leading to

$$\dim \Big( \operatorname{dom}(A^\infty) \cap \bigcup_{k=1}^{\infty} \ker(A - \lambda\,\mathrm{id})^k \Big) \le \dim \Big( \bigcup_{k=1}^{\infty} \ker(A^{-1} - \lambda^{-1}\mathrm{id})^k \Big) \tag{7.104}$$

where the latter is the algebraic multiplicity of $\lambda^{-1}$ with respect to the bounded operator $A^{-1}$ according to (C.11). In other words, we adopt now the same point of view as in Section 6.6 with the possibility to apply Theorem 6.34. We complement Theorem 5.36 as follows.

**Theorem 7.15.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $A$ be an elliptic operator according to (7.82)–(7.85) with its domain of definition* $\operatorname{dom}(A) = W^2_{2,0}(\Omega)$. *Then the spectrum $\sigma(A)$ consists of isolated eigenvalues $\lambda = \xi + i\eta$ with $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}$, of finite algebraic multiplicity according to (7.101) located in a parabola*

$$\{(\xi, \eta) \in \mathbb{R}^2 : \xi + \xi_0 \ge C\eta^2\} \quad \textit{for some } C > 0,\ \xi_0 \in \mathbb{R}, \tag{7.105}$$

*see Figure 5.5. Let*

$$0 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_j| \le \cdots, \quad |\lambda_j| \to \infty \text{ as } j \to \infty, \tag{7.106}$$

*be the ordered eigenvalues repeated according to their algebraic multiplicities. Then there is a positive number c such that*

$$|\lambda_k| \ \ge \ c\, k^{2/n}, \quad k \in \mathbb{N}. \tag{7.107}$$

*Furthermore, the linear hull of all associated eigenfunctions is dense in $L_2(\Omega)$.*

*Proof. Step* 1. In view of Theorem 5.36 it remains to prove (7.106), (7.107) and the density of the linear combinations of all associated eigenfunctions in $L_2(\Omega)$. The inverse $A^{-1}$ can be factorised by

$$\begin{aligned} &A^{-1}(L_2(\Omega) \hookrightarrow L_2(\Omega)) \\ &\quad = \mathrm{id}(W^2_{2,0}(\Omega) \hookrightarrow L_2(\Omega)) \circ A^{-1}(L_2(\Omega) \hookrightarrow W^2_{2,0}(\Omega)) \end{aligned} \tag{7.108}$$

where the latter is an isomorphic map. Using (7.95) with the entropy numbers $e_k$ in place of the approximation numbers $a_k$, Theorem 7.8 and (6.46) with $h_k = e_k$, leads to

$$e_k(A^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega)) \le c\, k^{-2/n}, \quad k \in \mathbb{N}. \tag{7.109}$$

Application of (6.87) to $A^{-1}$ and its eigenvalues $\mu_k \neq 0$ (counted with respect to their algebraic multiplicities) gives

$$|\mu_k| \le c\, k^{-2/n}, \quad k \in \mathbb{N}, \tag{7.110}$$

for some $c > 0$. By (7.104) one obtains $|\lambda_k|^{-1} \le |\mu_k|$, thus leading to (7.107).

*Step* 2. The operator $A$ can be written as

$$\begin{aligned} (Au)(x) \ &= -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{jk}(x)\frac{\partial u}{\partial x_k}(x)\Big) + \sum_{l=1}^{n} \tilde{a}_l(x)\,\frac{\partial u}{\partial x_l}(x) + a(x)u(x) \\ &= (\mathring{A}u)(x) + (Bu)(x), \end{aligned} \tag{7.111}$$

where $\mathring{A}$ refers to the first sum with $\mathrm{dom}(\mathring{A}) = \mathrm{dom}(B) = W^2_{2,0}(\Omega)$. Theorem 7.13 implies that $\mathring{A}$ is a self-adjoint positive-definite operator with pure point spectrum. We may assume that $0 \in \varrho(\mathring{A})$. Then $B\mathring{A}^{-1}$ can be decomposed into

$$\begin{aligned} &B\mathring{A}^{-1}(L_2(\Omega) \hookrightarrow L_2(\Omega)) \\ &= \mathrm{id}(W^1_2(\Omega) \hookrightarrow L_2(\Omega)) \circ B(W^2_{2,0}(\Omega) \hookrightarrow W^1_2(\Omega)) \circ \mathring{A}^{-1}(L_2(\Omega) \hookrightarrow W^2_{2,0}(\Omega)). \end{aligned} \tag{7.112}$$

The last operator is an isomorphic map, $B$ is bounded and one can apply (7.47) to $\mathrm{id}\colon W_2^1(\Omega) \hookrightarrow L_2(\Omega)$. Together with the multiplicativity of approximation numbers (6.46) (with $h_k = a_k$) this results in

$$a_k(B\mathring{A}^{-1}\colon L_2(\Omega) \hookrightarrow L_2(\Omega)) \le c\, k^{-1/n}, \quad k \in \mathbb{N}. \tag{7.113}$$

Using Theorem 6.34 gives both (7.106) as far as the existence of infinitely many eigenvalues is concerned and the density of the linear hull of all associated eigenfunctions in $L_2(\Omega)$. □

**Remark 7.16.** In Note 7.8.7 we give some references and add a few comments.

## 7.6 Spectral theory of elliptic operators: the degenerate case

Although we did not give a complete proof of Corollary 7.11 we apply this result now to the degenerate elliptic operator $B$ in Theorem 5.61.

**Theorem 7.17.** *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ where $n \ge 4$ and let $A$ be an elliptic operator according to (7.82)–(7.85) such that $0 \in \varrho(A)$. Let $\frac{1}{p^*} = \frac{1}{2} - \frac{2}{n}$ and $2 \le p < p^*$. Let $1 \le r_1 \le \infty$, $1 \le r_2 \le \infty$ and*

$$b_1 \in L_{r_1}(\Omega), \quad b_2 \in L_{r_2}(\Omega) \quad \text{with } \frac{1}{r_1} = \frac{1}{2} - \frac{1}{p}, \ \frac{1}{r_1} + \frac{1}{r_2} < \frac{2}{n}. \tag{7.114}$$

*Then $B$,*

$$B = b_2 A^{-1} b_1\colon L_p(\Omega) \hookrightarrow L_p(\Omega) \tag{7.115}$$

*is compact. If there are infinitely many non-vanishing eigenvalues $\mu_k$ which are counted with respect to their algebraic multiplicities and ordered by*

$$|\mu_1| \ge |\mu_2| \ge \dots \ge |\mu_k| \ge \dots > 0, \tag{7.116}$$

*then there is a positive number $c$ such that*

$$|\mu_k| \le c\, k^{-2/n}, \quad k \in \mathbb{N}. \tag{7.117}$$

*Proof.* According to Theorem 5.61 (and Theorem C.1) it remains to prove (7.117). We use the decomposition (5.309), (5.310), that is,

$$B = b_2 \circ \mathrm{id} \circ A^{-1} \circ b_1$$

with

$$\begin{aligned} b_1\colon\ & L_p(\Omega) \hookrightarrow L_2(\Omega), \\ A^{-1}\colon\ & L_2(\Omega) \hookrightarrow W^2_{2,0}(\Omega), \\ \mathrm{id}\colon\ & W^2_{2,0}(\Omega) \hookrightarrow L_u(\Omega), \\ b_2\colon\ & L_u(\Omega) \hookrightarrow L_p(\Omega). \end{aligned} \tag{7.118}$$

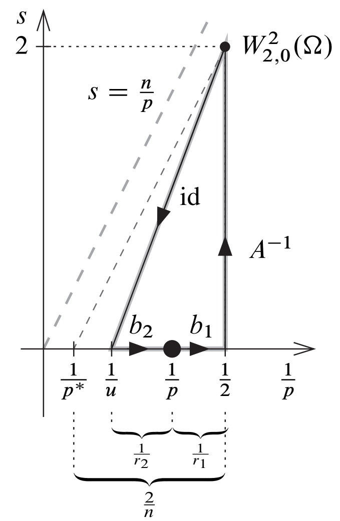



Figure 7.4

By Corollary 7.11 we have for the embedding $\mathrm{id}\colon W^2_{2,0}(\Omega) \hookrightarrow L_u(\Omega)$ in (7.118) that

$$e_k(\mathrm{id}\colon W^2_{2,0}(\Omega) \hookrightarrow L_u(\Omega)) \sim k^{-\frac{2}{n}} \tag{7.119}$$

for $k \in \mathbb{N}$. Consequently, Theorem 6.12 (iii) (with $h_k = e_k$) implies that

$$e_k(B\colon L_p(\Omega) \hookrightarrow L_p(\Omega)) \le c\, k^{-\frac{2}{n}} \tag{7.120}$$

for $k \in \mathbb{N}$; hence (7.117) follows from (6.87). □

**Remark 7.18.** If $A$ is the positive-definite self-adjoint operator studied in Theorem 7.13, then one obtains

$$\mu_k(A^{-1}) \sim k^{-2/n}, \quad k \in \mathbb{N}, \tag{7.121}$$

for its eigenvalues. If $B$ is given by (7.115) with (7.114), then one has at least the estimate (7.117) from above with the same Weyl exponent $\frac{2}{n}$. This somewhat surprising assertion is a consequence of the miraculous properties of entropy numbers with (7.69) as a special case and their relations to spectral theory according to Theorem 6.25 and Corollary 6.27. We add a few comments in Note 7.8.8 below.

**Exercise* 7.19.** Formulate and prove the counterpart of Theorem 7.17 for the dimensions $n = 1, 2, 3$.

*Hint:* Rely on Theorem 5.59 and Corollary 7.11.

**Exercise 7.20.** Let

$$b \in L_\infty(\Omega) \text{ be real with } 0 < c_1 \le b(x) \le c_2, \; x \in \Omega, \tag{7.122}$$

for some $0 < c_1 \le c_2 < \infty$. Let $A$ be the same operator as in Theorem 7.13 and Remark 7.18. Prove that $B$, given by

$$B = b \circ A^{-1} \circ b \colon L_2(\Omega) \hookrightarrow L_2(\Omega), \tag{7.123}$$

is a self-adjoint, positive, compact operator with

$$\mu_k(B) \sim k^{-2/n}, \quad k \in \mathbb{N}, \tag{7.124}$$

for its eigenvalues. The corresponding eigenfunctions span $L_2(\Omega)$.

*Hint:* Prove

$$a_k(A^{-1}) \sim a_k(B), \quad k \in \mathbb{N}, \tag{7.125}$$

and use Theorem 6.21.

## 7.7 The negative spectrum

In Example 6.2 we discussed the physical relevance of the so-called negative spectrum. The abstract foundation of this theory was subject of Theorem 6.32. Now we are in $\mathbb{R}^n$ and it appears reasonable to illuminate what follows by glancing first at the Laplacian $A$,

$$A = -\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \quad \operatorname{dom}(A) = W_2^2(\mathbb{R}^n), \tag{7.126}$$

as an unbounded operator in $L_2(\mathbb{R}^n)$. Throughout the text we scattered comments about this operator, but not in a very systematic way (Remark 5.24, Exercise 5.25, (6.11), (6.12)). Recall that $W_2^2(\mathbb{R}^n)$ is the Sobolev space according to Definition 3.1. Otherwise we use the same notation as in Appendix C. In particular, we fixed in Definition C.9 what is meant by a positive operator in a Hilbert space $H$, here $H = L_2(\mathbb{R}^n)$. Furthermore, the resolvent set $\varrho(A)$, the spectrum $\sigma(A)$, the point spectrum $\sigma_{\mathsf{p}}(A)$, and the essential spectrum $\sigma_{\mathsf{e}}(A)$ of a self-adjoint operator $A$ have the same meaning as in Definition 6.5. Recall Theorem 6.8.

**Proposition 7.21.** *The Laplacian $A$ according to (7.126) is a self-adjoint positive operator in $L_2(\mathbb{R}^n)$. Furthermore,*

$$\sigma(A) = \sigma_{\mathsf{e}}(A) = [0, \infty), \quad \sigma_{\mathsf{p}}(A) = \emptyset. \tag{7.127}$$

*Proof.* Let $L_2(\mathbb{R}^n, w_s)$ be the weighted $L_2$ spaces as considered in Definition 3.8, Remark 3.9 and Proposition 3.10. By Theorem 3.11 the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ generate unitary maps

$$\mathcal{F} W_2^2(\mathbb{R}^n) = \mathcal{F}^{-1} W_2^2(\mathbb{R}^n) = L_2(\mathbb{R}^n, w_2). \tag{7.128}$$

For $f \in W_2^2(\mathbb{R}^n)$ one obtains by (2.139) that

$$Af = \mathcal{F}^{-1}\mathcal{F} Af = \mathcal{F}^{-1}(|\xi|^2 \mathcal{F} f). \tag{7.129}$$

Hence $A$ is a unitary equivalent to the multiplication operator $B$ in $L_2(\mathbb{R}^n)$,

$$(Bg)(x) = |x|^2 g(x), \quad x \in \mathbb{R}^n, \ \mathrm{dom}(B) = L_2(\mathbb{R}^n, w_2), \tag{7.130}$$

and it is sufficient to prove the proposition for $B$ in place of $A$. Of course, $B$ is a symmetric, positive operator in $L_2(\mathbb{R}^n)$. If $\lambda < 0$ and $f \in L_2(\mathbb{R}^n)$, then

$$g \in \mathrm{dom}(B) \quad \text{where } g(x) = \frac{1}{|x|^2 - \lambda} f(x), \ \ x \in \mathbb{R}^n. \tag{7.131}$$

Furthermore, $(B - \lambda\,\mathrm{id})g = f$ and hence $\mathrm{range}(B - \lambda\,\mathrm{id}) = L_2(\mathbb{R}^n)$. By Theorem C.3 the operator $B$ is self-adjoint. If for some $g \in \mathrm{dom}(B)$ and $\lambda \geq 0$

$$(Bg)(x) = |x|^2 g(x) = \lambda g(x), \tag{7.132}$$

then $g = 0$ (in $L_2(\mathbb{R}^n)$). Hence $\sigma_{\mathrm{p}}(B) = \emptyset$. If $\lambda \geq 0$, then one finds a Weyl sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$ of functions with pairwise disjoint supports, $\|\varphi_j | L_2(\mathbb{R}^n)\| = 1$, and

$$\|B\varphi_j - \lambda\varphi_j | L_2(\mathbb{R}^n)\|^2 = \int_{\mathbb{R}^n} \Big| |x|^2 - \lambda \Big|^2 |\varphi_j(x)|^2 \,\mathrm{d}x \to 0 \quad \text{if } j \to \infty, \tag{7.133}$$

as indicated in Figure 7.5 (a) below (for $n = 1$), see also Figure 7.5 (b) below.

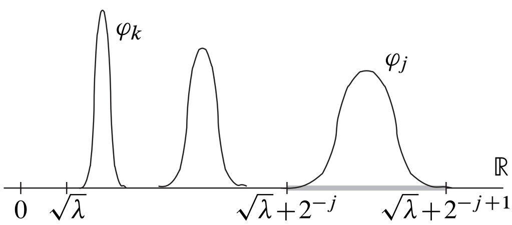



Figure 7.5 (a)

Hence $\lambda \in \sigma_{\mathrm{e}}(B)$ according to Definition 6.5 (iii). This proves (7.127) for $B$ and hence for $A$. □

**Exercise* 7.22.** Let $\lambda \geq 0$. Construct such a sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} \varphi_j \cap \operatorname{supp} \varphi_k = \emptyset$ for $j \neq k$, $\|\varphi_j | L_2(\mathbb{R}^n)\| = 1$, $j \in \mathbb{N}$, and (7.133).

*Hint:* One may take Figures 7.5 (a) (for $n = 1$) and 7.5 (b) as inspiration.

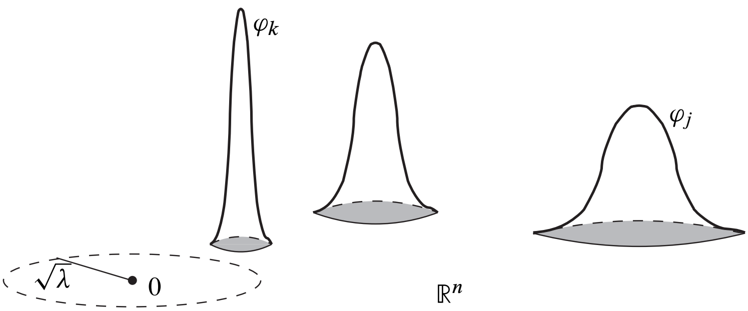



Figure 7.5 (b)

After this preparation we deal with the negative spectrum of the operator $\mathcal{H}_\beta$,

$$\begin{aligned} \mathcal{H}_\beta f &= (-\Delta + \mathrm{id}) f + \beta V(\cdot) f \\ &= \mathcal{H}_0 f + \beta V(\cdot) f, \qquad \operatorname{dom}(\mathcal{H}_\beta) = W_2^2(\mathbb{R}^n), \end{aligned} \tag{7.134}$$

where $V$ is a suitable real potential and $\beta \geq 0$ is a parameter. Of course, $\mathcal{H}_0 = A + \mathrm{id}$ is the shifted Laplacian $A = -\Delta$ as considered in Proposition 7.21.

**Theorem 7.23.** *Let $n \in \mathbb{N}$, $r \geq 2$, $0 \leq \frac{1}{r} < \frac{2}{n}$, and let*

$$V \in L_r(\mathbb{R}^n) \text{ be real with } \operatorname{supp} V \text{ compact.} \tag{7.135}$$

*Then the multiplication operator $B$,*

$$(Bf)(x) = V(x) f(x), \quad \operatorname{dom}(B) = W_2^2(\mathbb{R}^n), \tag{7.136}$$

*is relatively compact with respect to $\mathcal{H}_0$ according to Definition 6.29. Furthermore, $\mathcal{H}_\beta$ given by (7.134) with $\beta \geq 0$, is a self-adjoint operator in $L_2(\mathbb{R}^n)$, with*

$$\sigma_{\mathrm{e}}(\mathcal{H}_\beta) = \sigma_{\mathrm{e}}(\mathcal{H}_0) = [1, \infty), \tag{7.137}$$

*and*

$$\#\{\sigma(\mathcal{H}_\beta) \cap (-\infty, 0]\} \leq c\, \beta^{n/2} \tag{7.138}$$

*for some $c > 0$ and all $\beta \geq 0$.*

*Proof. Step* 1. First we prove that $B$ makes sense and that $B\mathcal{H}_0^{-1}$ is compact in $L_2(\mathbb{R}^n)$. We rely on the same arguments as in connection with Theorem 5.59 and Figure 5.9 with $p = 2$ now being Figure 7.6. In particular, for any ball $K$ with

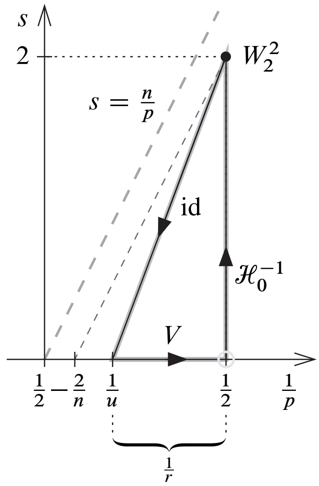



Figure 7.6

supp $V \subset K$ the embedding

$$\mathrm{id}\colon W_2^2(K) \hookrightarrow L_u(\mathbb{R}^n), \quad \frac{1}{u} = \frac{1}{2} - \frac{1}{r}, \tag{7.139}$$

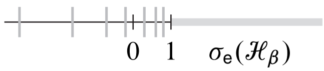
is compact. Then one obtains by Hölder's inequality that

$$B\colon W_2^2(\mathbb{R}^n) \hookrightarrow L_2(\mathbb{R}^n) \tag{7.140}$$

makes sense and is compact. Since $V$ is real it follows that $B$ is symmetric. Let $\psi$ be a smooth cut-off function with respect to the above ball $K \supset \operatorname{supp} V$. Then the factorisation

$$\begin{aligned} &B\mathcal{H}_0^{-1}(L_2(\mathbb{R}^n) \hookrightarrow L_2(\mathbb{R}^n)) \\ &\quad = V(L_u(\mathbb{R}^n) \hookrightarrow L_2(\mathbb{R}^n)) \circ \mathrm{id}(W_2^2(K) \hookrightarrow L_u(\mathbb{R}^n)) \\ &\qquad \circ \psi(W_2^2(\mathbb{R}^n) \hookrightarrow W_2^2(K)) \circ \mathcal{H}_0^{-1}(L_2(\mathbb{R}^n) \hookrightarrow W_2^2(\mathbb{R}^n)) \end{aligned} \tag{7.141}$$

and (7.139) show that $B\mathcal{H}_0^{-1}$ is compact. Now it follows from Definition 6.29 and Theorem 6.32 that $\mathcal{H}_\beta$ is self-adjoint and




$$\sigma_e(\mathcal{H}_\beta) = \sigma_e(\mathcal{H}_0) = [1, \infty). \tag{7.142}$$

Figure 7.7

*Step* 2. We apply Corollary 7.11 to $\mathrm{id}\colon W_2^2(K) \hookrightarrow L_u(\mathbb{R}^n)$ in (7.139) such that (7.141) implies for some $c > 0$,

$$e_k(B\mathcal{H}_0^{-1}\colon L_2(\mathbb{R}^n) \hookrightarrow L_2(\mathbb{R}^n)) \le c\, k^{-2/n}, \quad k \in \mathbb{N}. \tag{7.143}$$

Using again Corollary 6.27 leads to

$$|\mu_k(B\mathcal{H}_0^{-1})| \le c'\, k^{-2/n}, \quad k \in \mathbb{N}, \tag{7.144}$$

for the ordered eigenvalues of $B\mathcal{H}_0^{-1}$. Application of (6.97) to $\beta B\mathcal{H}_0^{-1}$ results in the question for which $k \in \mathbb{N}$,

$$\beta k^{-2/n} \ge c, \quad \text{that is,} \quad k \le c'\beta^{n/2} \tag{7.145}$$

for some $c > 0$ and $c' > 0$. This proves (7.138). □

**Remark 7.24.** A comment is added in Note 7.8.9 below. We return to the hydrogen operator $\mathcal{H}_H$ according to (6.9). The Coulomb potential $c\,|x|^{-1}$ in $\mathbb{R}^3$ fits in the above scheme at least locally if one chooses $V(x) = |x|^{-1}\psi(x)$ in (7.134) where $\psi(x)$ is an appropriate cut-off function. Then one obtains by (7.138) that

$$\#\{\sigma(\mathcal{H}_\beta) \cap (-\infty, 0]\} \le c\, \beta^{3/2}. \tag{7.146}$$

But this is just what one would expect according to (6.134) with $\beta \sim \hbar^{-2}$ as suggested by (6.10). If one wishes to deal with $V(x) = |x|^{-1}$ instead of $|x|^{-1}\psi(x)$ one needs some splitting arguments. This may be found in [HT94b] and [ET96, Sections 5.4.8, 5.4.9].

## 7.8 Notes

**7.8.1.** A *quasi-norm* on a complex linear space $X$ is a map $\|\cdot|X\|$ from $X$ to the non-negative reals such that

$$\|x|X\| = 0 \quad \text{if, and only if,} \quad x = 0, \tag{7.147}$$

$$\|\lambda x|X\| = |\lambda|\|x|X\| \quad \text{for all } \lambda \in \mathbb{C} \text{ and all } x \in X, \tag{7.148}$$

and there exists a constant $C \ge 1$ such that for all $x_1 \in X$, $x_2 \in X$,

$$\|x_1 + x_2|X\| \le C(\|x_1|X\| + \|x_2|X\|). \tag{7.149}$$

If $C = 1$ is admitted, then $\|\cdot|X\|$ is a norm. $X$ is called a *quasi-Banach space* (Banach space if $C = 1$) if any Cauchy sequence in the quasi-normed space $X$ converges (to an element in $X$). In this book we dealt mainly with Hilbert spaces and occasionally with Banach spaces. Recall that for given $p$ with $0 < p \le 1$, a *$p$-norm* on a complex linear space $X$ is a map $\|\cdot|X\|$ from $X$ to the non-negative reals satisfying (7.147), (7.148) and

$$\|x_1 + x_2|X\|^p \le \|x_1|X\|^p + \|x_2|X\|^p, \quad x_1 \in X,\ x_2 \in X, \tag{7.150}$$

instead of (7.149). Of course, any $p$-norm is a quasi-norm. There is a remarkable converse. The equivalence of norms according to (C.1) can be extended to quasi-Banach spaces verbatim. It can be shown that for any quasi-norm $\|\cdot|X\|_1$ on a quasi-Banach space $X$ there is an equivalent $p$-norm $\|\cdot|X\|_2$ for some $p$ with $0 < p \leq 1$. We refer to [Kön86, p. 47], [Köt69, §15.10] or [DL93, Chapter 2, Theorem 1.1]. But otherwise the theory of abstract quasi-Banach spaces is rather poor, compared with the rich theory of abstract Banach spaces. However, in case of the function spaces

$$B^s_{p,q}(\mathbb{R}^n),\ F^s_{p,q}(\mathbb{R}^n) \quad \text{with } s \in \mathbb{R},\ 0 < p \leq \infty,\ 0 < q \leq \infty, \tag{7.151}$$

according to Appendix E (with $p < \infty$ for the $F$-spaces) and briefly mentioned in Note 3.6.3 and their restrictions to domains $\Omega$ in $\mathbb{R}^n$,

$$B^s_{p,q}(\Omega),\ F^s_{p,q}(\Omega) \quad \text{with } s \in \mathbb{R},\ 0 < p \leq \infty,\ 0 < q \leq \infty, \tag{7.152}$$

as in Definition 3.37 (with $p < \infty$ for the $F$-spaces) the situation is completely different. These are quasi-Banach spaces and $s$, $p$, $q$ as above are the natural restrictions. Another useful extension from Banach spaces to quasi-Banach spaces are the sequence spaces

$$\ell^M_p \quad \text{and} \quad \ell_q(2^{j\delta}\ell^{M_j}_p), \quad 0 < p \leq \infty,\ 0 < q \leq \infty, \tag{7.153}$$

according to (7.4) and (7.9) naturally extended to all $p$ and $q$ as above. Obviously there is no problem to extend the Definition 6.10 of entropy numbers and approximation numbers from Banach spaces to quasi-Banach spaces.

**7.8.2.** Proposition 7.3 can be extended from $1 \leq p \leq \infty$ to $0 < p \leq \infty$. But the situation is more complicated if one asks for the entropy numbers $e_k(\mathrm{id})$ of the compact embedding

$$\mathrm{id}\colon \ell^M_{p_1} \hookrightarrow \ell^M_{p_2}, \quad 0 < p_1 \leq p_2 \leq \infty,\ M \in \mathbb{N}, \tag{7.154}$$

for the above complex quasi-Banach spaces. One obtains for $k \in \mathbb{N}$,

$$e_k(\mathrm{id}\colon \ell^M_{p_1} \to \ell^M_{p_2}) \sim \begin{cases} 1, & 1 \leq k \leq \log(2M), \\ \left(k^{-1}\log(1+\frac{2M}{k})\right)^{\frac{1}{p_1}-\frac{1}{p_2}}, & \log(2M) \leq k \leq 2M, \\ 2^{-\frac{k}{2M}}\,(2M)^{\frac{1}{p_2}-\frac{1}{p_1}}, & k \geq 2M, \end{cases} \tag{7.155}$$

where log is taken to base 2 and the equivalence constants are independent of $M$ and $k$. Recall Exercise 7.5. Obviously, (7.12) follows from (7.155) with $p_1 = p_2$. The case $1 \leq p_1 \leq p_2 \leq \infty$ is due to Schütt [Sch84]. The extension to all parameters $0 < p_1 \leq p_2 \leq \infty$ was done in [ET96, Section 3.2.2, pp. 98–101], [Tri97, Theorem 7.3, p. 37]. But there remained a gap as far as the estimate from below in the middle line in (7.155) is concerned. This was finally sealed in [Küh01].

**7.8.3.** Theorem 7.6 can be easily extended from $1 \le p \le \infty$ to $0 < p \le \infty$. Furthermore, (7.155) is the main ingredient to study the behaviour of the entropy numbers of the compact embedding

$$\mathrm{id}\colon \ell_{q_1}(2^{j\delta}\ell_{p_1}^{M_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{M_j}), \tag{7.156}$$

where the first space is quasi-normed by (7.9) and the latter refers to a corresponding space with 1 in place of the weight factors $2^{j\delta}$. Otherwise we again assume as in (7.8) that

$$\delta > 0, \quad d > 0, \quad M_j \sim 2^{jd} \quad \text{for } j \in \mathbb{N}_0. \tag{7.157}$$

*Let*

$$0 < p_1 \le \infty, \quad \frac{1}{p_*} = \frac{1}{p_1} + \frac{\delta}{d}, \quad p_* < p_2 \le \infty, \tag{7.158}$$

*and* $0 < q_1 \le \infty$, $0 < q_2 \le \infty$. *Then* $\mathrm{id}\colon \ell_{q_1}(2^{j\delta}\ell_{p_1}^{M_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{M_j})$ *is compact with*

$$e_k(\mathrm{id}\colon \ell_{q_1}(2^{j\delta}\ell_{p_1}^{M_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{M_j})) \sim k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}, \quad k \in \mathbb{N}. \tag{7.159}$$

This is a rather sharp and final assertion. The first step was taken in [Tri97, Theorem 8.2, p. 39]. The above version is due to [HT05, Theorem 3.5, p. 115] and may also be found in [Tri06, Theorem 6.20, p. 274]. More general situations were studied in [KLSS06a], [KLSS06b], [KLSS07], whereas corresponding assertions for approximation numbers $a_k(\mathrm{id})$ of id given by (7.156) were obtained in [Skr05].

The main interest in the above sequence spaces and assertions of type (7.159) comes from the possibility to reduce compact embeddings between the function spaces in (7.152) to these sequence spaces, for example via wavelet expansions. But this is beyond the scope of this book. Details may be found in [Tri97], [Tri01], [Tri06]. As far as further types of useful sequence spaces are concerned one may consult [Tri06, Section 6.3] where one finds also relevant references.

**7.8.4.** In Step 2 of the proof of Corollary 7.11 we used the interpolation property of entropy numbers in a special situation. Recall that two complex quasi-Banach spaces $X_0$ and $X_1$ are called an *interpolation couple* $\{X_0, X_1\}$ if they are (linearly and continuously) embedded in a linear Hausdorff spaces $\mathfrak{X}$ which may be identified (afterwards) with the quasi-Banach spaces $X_0 + X_1$, consisting of all $x \in \mathfrak{X}$ such that $x = x_0 + x_1$ for some $x_0 \in X_0$, $x_1 \in X_1$, and quasi-normed by *Peetre's* $K$*-functional*

$$K(t,x) = K(t,x;X_0,X_1) = \inf(\|x_0|X_0\| + t\|x_1|X_1\|) \tag{7.160}$$

for some fixed $t > 0$, where the infimum is taken over all representations $x = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$. Plainly, $K(t_1,x) \sim K(t_2,x)$ for fixed $0 < t_1 \le t_2 < \infty$. As mentioned in Note 7.8.1 any quasi-Banach space is also a $p$-Banach space for some $p$ with $0 < p \leq 1$. Then the following assertions are called the *interpolation property for entropy numbers* . As usual, $X_0 \cap X_1$ is quasi-normed by $\|x|X_0 \cap X_1\| = \max(\|x|X_0\|, \|x|X_1\|)$.

(i) *Let $X$ be a quasi-Banach space and let $\{Y_0, Y_1\}$ be an interpolation couple of $p$-Banach spaces. Let $0 < \theta < 1$ and let $Y_\theta$ be a quasi-Banach space such that $Y_0 \cap Y_1 \hookrightarrow Y_\theta \hookrightarrow Y_0 + Y_1$ and*

$$\|y|Y_\theta\| \leq \|y|Y_0\|^{1-\theta} \|y|Y_1\|^\theta \quad \textit{for all } y \in Y_0 \cap Y_1. \tag{7.161}$$

*Let $T \in \mathcal{L}(X, Y_0 \cap Y_1)$. Then for all $k_0, k_1 \in \mathbb{N}$,*

$$\begin{aligned} &e_{k_0+k_1-1}(T : X \hookrightarrow Y_\theta) \\ &\quad \leq 2^{1/p}\, e_{k_0}^{1-\theta}(T : X \hookrightarrow Y_0)\, e_{k_1}^\theta(T : X \hookrightarrow Y_1). \end{aligned} \tag{7.162}$$

(ii) *Let $\{X_0, X_1\}$ be an interpolation couple of quasi-Banach spaces and let $Y$ be a $p$-Banach space. Let $0 < \theta < 1$ and let $X_\theta$ be a quasi-Banach space such that $X_\theta \hookrightarrow X_0 + X_1$ and*

$$t^{-\theta} K(t, x) \leq \|x|X_\theta\| \quad \textit{for all } x \in X_\theta \textit{ and all } 0 < t < \infty. \tag{7.163}$$

*Let $T : X_0 + X_1 \hookrightarrow Y$ be a linear operator such that its restrictions to $X_0$ and $X_1$, respectively, are continuous. Then its restriction to $X_\theta$ is also continuous and for all $k_0, k_1 \in \mathbb{N}$,*

$$\begin{aligned} &e_{k_0+k_1-1}(T : X_\theta \hookrightarrow Y) \\ &\quad \leq 2^{1/p}\, e_{k_0}^{1-\theta}(T : X_0 \hookrightarrow Y)\, e_{k_1}^\theta(T : X_1 \hookrightarrow Y). \end{aligned} \tag{7.164}$$


Figure 7.8

Figure 7.8 illustrates the situations. By (7.73) one obtains (7.74) as a special case of (7.161), (7.162). The above assertions have a little history. The first step was taken by J. Peetre in [Pee68] which corresponds (after reformulation in terms of entropy numbers) to the cases $X = Y_0$ in (7.162) and $X_0 = Y$ in (7.164). A complete proof of the above assertions (again after reformulation in terms of entropy numbers) for Banach spaces was given in [Tri70]. One may also consult [Tri78, Section 1.16.2, pp. 112–115] for further results and references (at that time). The extension to quasi-Banach spaces including the above constants goes back to [HT94a] and this formulation coincides with [ET96, Section 1.3.2, pp. 13/14]. From the point of view of interpolation theory both (i) and (ii) as illustrated in Figure 7.8 are special situations.

*Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be two interpolation couples. What can be said about the entropy numbers of*

$$T: X_\theta \hookrightarrow Y_\theta, \quad 0 < \theta < 1, \tag{7.165}$$

*in dependence on the entropy numbers of*

$$T: X_0 \hookrightarrow Y_0 \quad \textit{and} \quad T: X_1 \hookrightarrow Y_1 \ ? \tag{7.166}$$

This is an open problem.

**7.8.5.** As just explained, the interest in the entropy numbers of compact embeddings between sequence spaces according to (7.156) comes mainly from the possibility to transfer these results to function spaces of type (7.152). We formulate here a key result of this theory and add afterwards a few comments. As far as the function spaces $B^s_{p,q}(\Omega)$ are concerned one may consult Appendix E.

*Let $\Omega$ be a bounded domain (i.e., a bounded open set) in $\mathbb{R}^n$. Let $s_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}$,*

$$0 < p_1 \le \infty, \quad 0 < p_2 \le \infty, \quad 0 < q_1 \le \infty, \quad 0 < q_2 \le \infty, \tag{7.167}$$

*and*

$$s_1 - s_2 > \max\Big(0, \frac{n}{p_1} - \frac{n}{p_2}\Big). \tag{7.168}$$

*Then*

$$\mathrm{id}\colon B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega) \tag{7.169}$$

*is compact and*

$$e_k\big(\mathrm{id}\colon B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)\big) \sim k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}. \tag{7.170}$$

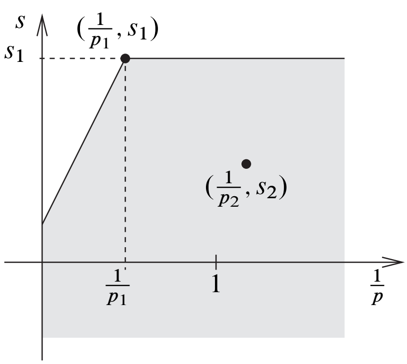



Figure 7.9

Let $W^s_p(\Omega)$ with $1 < p < \infty$ and $s \in \mathbb{N}$ be the classical Sobolev spaces on $\Omega$ according to Definition 3.37 and Theorem 4.1. Then it follows from the above assertion and the embeddings mentioned in Theorem E.8 (ii) below that

$$\mathrm{id}\colon W^s_p(\Omega) \hookrightarrow L_q(\Omega), \quad 1 < q < \infty,\ s > n\big(\tfrac{1}{p} - \tfrac{1}{q}\big), \tag{7.171}$$

is compact and

$$e_k(\mathrm{id}) \sim k^{-s/n}, \quad k \in \mathbb{N}. \tag{7.172}$$

Corollary 7.11 based on Theorem 5.59 is a special case of (7.171), (7.172). Also the second equivalence in (7.47) is covered by (7.170) since $W_2^s = B_{2,2}^s$. Assertions of the above type have a long and substantial history. First of all we mention that (7.171), (7.172) is due to [BS67], [BS72] using sophisticated piecewise polynomial approximations in the spaces under consideration, extending (7.171), (7.172) also to fractional $s > 0$, $s \notin \mathbb{N}$, where $W_p^s = B_{p,p}^s$. The first proof of (7.168)–(7.170) for bounded $C^\infty$ domains in $\mathbb{R}^n$ and

$$1 < p_1 < \infty, \quad 1 < p_2 < \infty, \quad 1 \le q_1 \le \infty, \quad 1 \le q_2 \le \infty, \tag{7.173}$$

was given in [Tri78, Theorem 4.10.3, p. 355] based on [Tri70], [Tri75] using (7.171), (7.172) and interpolation for entropy numbers as indicated in Note 7.8.4. In [Tri78] one finds also further references to related papers at that time. Restricted to $n = 1$ and an interval the above assertions had been extended in [Car81a] to $1 \le p_1 \le \infty$, $1 \le p_2 \le \infty$ reducing this problem for the $B_{p,q}^s$ spaces to corresponding sequence spaces of the same type as in (7.156) using the so-called *Ciesielski isomorphism* in terms of splines. We refer in this context also to [Kön86, Section 3.c, especially Proposition 3.c.9, p. 191]. On the one hand, the Ciesielski isomorphism gives the possibility to reduce some Besov spaces $B_{p,q}^s$ to sequence spaces introduced in Remark 7.2, but on the other hand, this method is rather limited and there is no hope to prove (7.169) for all parameters according to (7.167), (7.168). This was done in [ET89], [ET92] by a direct approach using the Fourier-analytical definition of the $B_{p,q}^s$ spaces as indicated in Appendix E. The rather long proof (14 pages) for bounded $C^\infty$ domains $\Omega$ in $\mathbb{R}^n$ may also be found in [ET96, Sections 3.3.1–3.3.5]. Finally one can remove the smoothness assumption for $\Omega$. This was indicated in [ET96, Section 3.5] and detailed (based on a new method) in [Tri97, Section 23]. However, the main advantage of [Tri97] compared with [ET96] and the underlying papers was the observation that there are constructive elementary building blocks, called *quarks*, which allows us to reduce problems of type (7.167)–(7.170) to their sequence counterparts (7.156)–(7.159). This technique has been elaborated over the years. It is quite standard nowadays employed in many papers and also in the books [Tri97], [Tri01], [Tri06]. Moreover, it can be used for function spaces on rough structures such as fractals and quasi-metric spaces. This was even the main motivation to look for such possibilities. In case of $\mathbb{R}^n$ or domains in $\mathbb{R}^n$ one can use nowadays also *wavelet isomorphisms* as explained in [Tri06] and the references given there.

**7.8.6.** For entropy numbers one has the final satisfactory assertion (7.167)–(7.170), whereas the outcome for approximation numbers is more complicated. For our purpose, (7.47) is sufficient. Nevertheless, we formulate a partial counterpart of Corollary 7.11 and (7.167)–(7.170), complementing Theorem 7.8.

*Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let*

$$s > 0, \quad 1 \le p \le \infty, \quad s - \frac{n}{2} > -\frac{n}{p}. \tag{7.174}$$

*Then*

$$\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow L_p(\Omega) \tag{7.175}$$

*is compact and for $k \in \mathbb{N}$,*

$$a_k(\mathrm{id}\colon W_2^s(\Omega) \hookrightarrow L_p(\Omega)) \sim \begin{cases} k^{-\frac{s}{n}} & \text{if } p \le 2, \\ k^{-\frac{s}{n}+\frac{1}{2}-\frac{1}{p}} & \text{if } p > 2. \end{cases} \tag{7.176}$$

By (7.170) entropy numbers and approximation numbers behave differently if $p_2 > p_1 = 2$. For general compact embeddings of type (7.167)–(7.169) the behaviour of approximation numbers is rather complicated. The (almost) final outcome may be found in [Tri06, Theorem 1.107, pp. 67/68] going back to [ET89], [ET92], [ET96, Section 3.3.4, p. 119] and [Cae98].

**7.8.7.** The self-adjoint operator $A$ in Theorem 7.13 has a pure point spectrum. In particular, its eigenelements span $L_2(\Omega)$, their linear hull is dense in $L_2(\Omega)$. This applies also to the more general elliptic operators of higher order mentioned briefly in Note 5.12.1 as long as they are self-adjoint. We reduced the corresponding question for regular non-self-adjoint second order elliptic equations in Theorem 7.15 to the abstract Theorem 6.34. This can be done for higher order elliptic operators, too. We refer to [Tri78, Theorem 5.6.3, p. 396]. The same arguments apply also to several types of degenerate higher order elliptic operators, [Tri78, Theorems 6.6.2, 7.5.1, pp. 425, 449]. A different approach to problems of this type using the analyticity of the resolvent $R_\lambda$ and its minimal growth as considered in Remark 6.30, Exercise 6.31 and Note 5.12.7 was given in [Agm62] and [Agm65, Section 16]. One may also consult Note 5.12.6.

**7.8.8.** The distribution (7.117) of the eigenvalues $\mu_j$ of the degenerate operator $B$ based on the elliptic operator $A$ of second order according to (7.82)–(7.85) might be considered as a typical example of a more general theory. One can replace $A$ by higher order elliptic operators, their fractional powers or pseudodifferential operators and one can rely on more general assertions for entropy numbers of related embeddings of type (7.170). This theory started in [ET94], [HT94a], [HT94b] and has been presented in detail in [ET96, Chapter 5].

**7.8.9.** Much as in the preceding Note 7.8.8 one may consider Theorem 7.23 with $\mathcal{H}_\beta$ as in (7.134)–(7.136) as a typical example which fits in the context of this book. But again there are many generalisations. We refer to [HT94a], [HT94b] and [ET96, Sections 5.4.7–5.4.9], also for some new aspects.

# Appendix A
# Domains, basic spaces, and integral formulae

## A.1 Basic notation and basic spaces

We fix some basic notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean $n$-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$ whereas $\mathbb{C}$ is the complex plane and $\mathbb{C}^n$ stands for the complex $n$-space. As usual, $\mathbb{Z}$ is the collection of all integers, and $\mathbb{Z}^n$, where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$, $j = 1, \ldots, n$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$ and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \alpha! = \alpha_1! \cdots \alpha_n!. \tag{A.1}$$

As usual, derivatives are abbreviated by

$$\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \alpha \in \mathbb{N}_0^n, \ x \in \mathbb{R}^n, \tag{A.2}$$

and

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \quad \alpha \in \mathbb{N}_0^n, \ \xi \in \mathbb{R}^n. \tag{A.3}$$

We shall often use the notation

$$\langle \xi \rangle = (1 + |\xi|^2)^{1/2}, \quad \xi \in \mathbb{R}^n. \tag{A.4}$$

For $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, let

$$xy = \langle x, y \rangle = \sum_{j=1}^{n} x_j y_j, \quad x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n). \tag{A.5}$$

If $a \in \mathbb{R}$, then

$$a_+ = \max(a, 0) = \begin{cases} a, & a \geq 0, \\ 0, & a < 0. \end{cases} \tag{A.6}$$

For a set $M$ of finitely many elements, we denote by $\#M$ its cardinality, i.e., the number of its elements.

An open set in $\mathbb{R}^n$ is called a *domain*. If it is necessary or desirable that the domain considered is *connected*, then this will be mentioned explicitly. The *boundary* of a domain $\Omega$ in $\mathbb{R}^n$ is denoted by $\partial\Omega$, whereas $\overline{\Omega}$ stands for its *closure*.

Let $\Omega$ be an (arbitrary) domain in $\mathbb{R}^n$. Then $C^{\mathrm{loc}}(\Omega) = C^{0,\mathrm{loc}}(\Omega)$ is the collection of all complex-valued continuous functions in $\Omega$. For $m \in \mathbb{N}$ let $C^{m,\mathrm{loc}}(\Omega)$ be the collection of all functions $f \in C^{\mathrm{loc}}(\Omega)$ having all classical derivatives $\mathrm{D}^\alpha f \in C^{\mathrm{loc}}(\Omega)$ with $|\alpha| \le m$, and let

$$C^{\infty,\mathrm{loc}}(\Omega) = \bigcap_{m=0}^{\infty} C^{m,\mathrm{loc}}(\Omega) \tag{A.7}$$

be the collection of all $C^\infty$ functions in $\Omega$.

**Definition A.1.** Let $\Omega$ be an (arbitrary) domain in $\mathbb{R}^n$ and let $m \in \mathbb{N}_0$. Then $C^m(\Omega)$ is the collection of all $f \in C^{m,\mathrm{loc}}(\Omega)$ such that any function $\mathrm{D}^\alpha f$ with $|\alpha| \le m$ can be extended continuously to $\overline{\Omega}$ and

$$\|f \,|C^m(\Omega)\| = \sum_{|\alpha|\le m} \sup_{x\in\Omega} |\mathrm{D}^\alpha f(x)| < \infty. \tag{A.8}$$

Furthermore, $C(\Omega) = C^0(\Omega)$ and

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega). \tag{A.9}$$

**Remark A.2.** Recall that $C^m(\Omega)$ with $m \in \mathbb{N}_0$ normed by (A.8) is a Banach space. Details may be found in [Tri92a]. Some other notation are in common use in literature, especially, if $\Omega$ is unbounded and, in particular, if $\Omega = \mathbb{R}^n$. We are mostly interested in bounded smooth connected domains.

## A.2 Domains

Recall that domain means open set.

**Definition A.3.** (i) Let $n \in \mathbb{N}$, $n \ge 2$, and $k \in \mathbb{N}$. Then a *special $C^k$ domain* in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ such that

$$h(x') < x_n < \infty, \tag{A.10}$$

where $h \in C^k(\mathbb{R}^{n-1})$ according to Definition A.1.

(ii) Let $n \in \mathbb{N}$, $n \ge 2$, and $k \in \mathbb{N}$. Then a *bounded $C^k$ domain in $\mathbb{R}^n$* is a bounded connected domain $\Omega$ in $\mathbb{R}^n$ where the boundary $\partial\Omega$ can be covered by finitely many open balls $K_j$ in $\mathbb{R}^n$, $j = 1, \dots, J$, centred at $\partial\Omega$ such that

$$K_j \cap \Omega = K_j \cap \Omega_j \quad \text{with } j = 1, \dots, J, \tag{A.11}$$

where $\Omega_j$ are rotations of suitable special $C^k$ domains in $\mathbb{R}^n$.

(iii) Let $n \in \mathbb{N}$, $n \geq 2$. If $\Omega$ is a bounded $C^k$ domain for every $k \in \mathbb{N}$, then it is called a *bounded $C^\infty$ domain*.

(iv) If $n = 1$, then *bounded $C^\infty$ domain* simply means open bounded interval.

**Remark A.4.** In other words, for $n \geq 2$ we have the illustrated situation

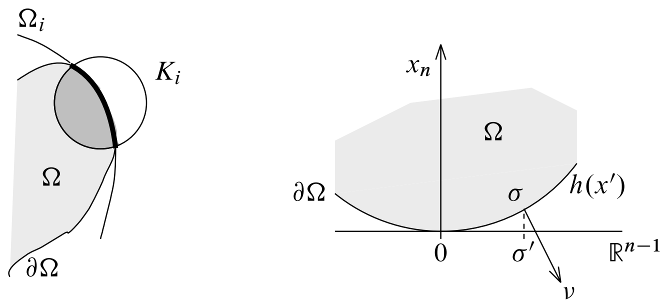



Figure A.1

where one may assume that $\partial\Omega \cap K_i$ can be represented in local coordinates by

$$x_n = h(x') \text{ with } h(0) = 0 \quad \text{and} \quad \frac{\partial h}{\partial x_r}(0) = 0,\ r = 1, \dots, n-1. \tag{A.12}$$

Using these local coordinates the outer normal $\nu = \nu(\sigma)$ at a point $(\sigma', \sigma_n) = \sigma \in \partial\Omega$ is given by

$$\begin{aligned} \nu(\sigma) &= \big(\nu_1(\sigma), \dots, \nu_n(\sigma)\big) \\ &= \frac{1}{\sqrt{1 + \sum_{r=1}^{n-1} |\frac{\partial h}{\partial x_r}(\sigma')|^2}} \left(\frac{\partial h}{\partial x_1}(\sigma'), \dots, \frac{\partial h}{\partial x_{n-1}}(\sigma'), -1\right). \end{aligned} \tag{A.13}$$

## A.3 Integral formulae

Let $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^n$ according to Definition A.3 and let $f \in C^1(\overline{\Omega})$ as introduced in Definition A.1. Then

$$\frac{\partial f}{\partial \nu}(\sigma) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\sigma)\nu_j(\sigma), \quad \sigma \in \partial\Omega, \tag{A.14}$$

is the normal derivative at the point $\sigma \in \partial\Omega$, where $\nu$ is the outer normal (A.13). Let $d\sigma$ be the surface element on $\partial\Omega$ (in the usual naïve understanding).

**Theorem A.5** (Gauß's formula). *Let $n \geq 2$, $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^n$, and $f \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega \frac{\partial f}{\partial x_j}(x)\mathrm{d}x = \int_{\partial\Omega} f(\sigma)\nu_j(\sigma)\mathrm{d}\sigma, \quad j = 1, \dots, n. \tag{A.15}$$

**Remark A.6.** Usually the above assertion is formulated for more general domains, called *normal domains* or *standard domains*, as it can be found in Calculus books, cf. [Cou36, Chapter V, Section 5]. As for a short proof one may consult [Tri92a, Appendix A.3].

More or less as a corollary one gets the following assertions. Recall that

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian.

**Theorem A.7** (Green's formulae). *Let $n \geq 2$, $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^n$, and $f \in C^2(\Omega)$.*

(i) *Let $g \in C^1(\Omega)$. Then*

$$\begin{aligned} &\int_\Omega g(x)(\Delta f)(x)\mathrm{d}x \\ &\qquad = -\sum_{j=1}^{n} \int_\Omega \frac{\partial g}{\partial x_j}(x)\frac{\partial f}{\partial x_j}(x)\mathrm{d}x + \int_{\partial\Omega} g(\sigma)\frac{\partial f}{\partial \nu}(\sigma)\mathrm{d}\sigma. \end{aligned} \tag{A.16}$$

(ii) *Let $g \in C^2(\Omega)$. Then*

$$\begin{aligned} &\int_\Omega (g(x)(\Delta f)(x) - (\Delta g)(x) f(x))\mathrm{d}x \\ &\qquad = \int_{\partial\Omega} \Big( g(\sigma)\frac{\partial f}{\partial \nu}(\sigma) - \frac{\partial g}{\partial \nu}(\sigma) f(\sigma)\Big)\mathrm{d}\sigma. \end{aligned} \tag{A.17}$$

## A.4 Surface area

We have a closer look at the *area* (*volume*) of smooth $(n-1)$-dimensional surfaces in $\mathbb{R}^n$ again adopting the usual naïve point of view in the Riemannian spirit. Similarly as in (A.12) and in modification of (A.13) we assume that the surface $\Phi$ is given by

$$x_n = h(x'), \quad x' \in \mathbb{R}^{n-1},\ h \in C^1(\omega), \tag{A.18}$$

where $\omega \subset \mathbb{R}^{n-1}$ is a smooth bounded domain as indicated in Figure A.2 below, that is, $x = (x', h(x')) \in \Phi$ for $x' \in \omega$, and

$$\widetilde{\nu}(x') = \Big( -\frac{\partial h}{\partial x_1}(x'), \ldots, -\frac{\partial h}{\partial x_{n-1}}(x'), 1\Big), \quad x' \in \omega, \tag{A.19}$$

is the modified normal to $\Phi$ of length

$$|\widetilde{\nu}(x')| = \sqrt{1 + \sum_{j=1}^{n-1} \left| \frac{\partial h}{\partial x_j}(x') \right|^2}, \quad x' \in \omega. \tag{A.20}$$

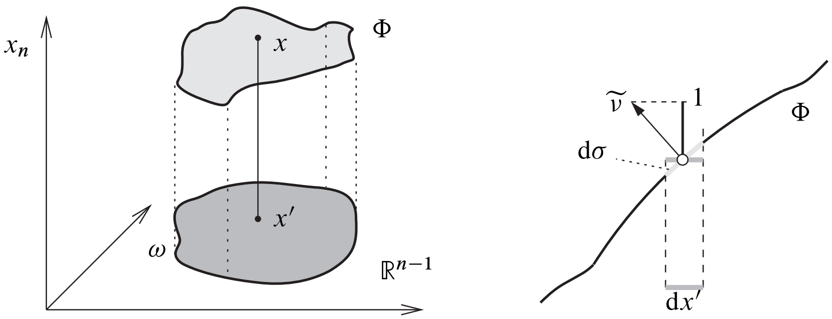



Figure A.2

**Theorem A.8.** *Let the smooth surface $\Phi$ be given by* (A.18). *Then*

$$|\Phi| = \int_\omega |\widetilde{\nu}(x')| dx' = \int_\omega \sqrt{1 + \sum_{j=1}^{n-1} \left| \frac{\partial h}{\partial x_j}(x') \right|^2} \, dx' \tag{A.21}$$

*is its surface area* (*volume*).

*Proof.* Let $d\sigma$ be the surface element at $x = (x', x_n) \in \Phi$. Then one has – as indicated in Figure A.2 – that $d\sigma = |\widetilde{\nu}(x')| dx'$. Using (A.20) one obtains (A.21) by Riemannian arguments. □

# Appendix B
# Orthonormal bases of trigonometric functions

Let $n \in \mathbb{N}$, and

$$\mathbb{Q}^n = (-\pi, \pi)^n = \{x \in \mathbb{R}^n : -\pi < x_j < \pi,\ j = 1, \dots, n\}, \tag{B.1}$$

where $x = (x_1, \dots, x_n)$. Let $L_2(\mathbb{Q}^n)$ be the usual complex Hilbert space according to (2.16) where $p = 2$, furnished with the scalar product

$$\langle f, g \rangle_{L_2} = \int_{\mathbb{Q}^n} f(x)\overline{g(x)}\mathrm{d}x. \tag{B.2}$$

**Theorem B.1.** *Let $h_m(x) = (2\pi)^{-\frac{n}{2}}\, e^{imx}$, $m \in \mathbb{Z}^n$, $x \in \mathbb{Q}^n$. Then*

$$\{h_m(\cdot)\ :\ m \in \mathbb{Z}^n\} \tag{B.3}$$

*is a complete orthonormal system in $L_2(\mathbb{Q}^n)$.*

*Proof. Step* 1. One checks immediately that (B.3) is an orthonormal system in $L_2(\mathbb{Q}^n)$. It remains to prove that this system spans $L_2(\mathbb{Q}^n)$. If one knows this assertion for $n = 1$, then it follows for $n \geq 2$ by standard arguments of Hilbert space theory.

*Step* 2. Hence it remains to show that the (one-dimensional) trigonometric polynomials

$$p(x) = \sum_{m=-L}^{L} a_m \frac{1}{\sqrt{2\pi}}\, e^{imx}, \quad x \in \mathbb{Q} = (-\pi, \pi), \tag{B.4}$$

are dense in $L_2(\mathbb{Q})$. By Proposition 2.7 (i) it is sufficient to prove that any $f \in \mathcal{D}(\mathbb{Q}) = C_0^\infty(\mathbb{Q})$ can be represented in $L_2(\mathbb{Q})$ by

$$f(x) = \sum_{m=-\infty}^{\infty} a_m \frac{1}{\sqrt{2\pi}}\, e^{imx}, \quad x \in \mathbb{Q}, \tag{B.5}$$

with

$$a_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-imx}\mathrm{d}x, \quad m \in \mathbb{Z}. \tag{B.6}$$

By

$$a_m = \frac{1}{\sqrt{2\pi}\,(im)^k} \int_{-\pi}^{\pi} \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x) e^{-imx} \mathrm{d}x, \quad m \in \mathbb{Z} \setminus \{0\},\ k \in \mathbb{N}, \tag{B.7}$$

it follows that $|a_m|$ decreases rapidly. In particular,

$$g(x) = \sum_{m=-\infty}^{\infty} a_m \frac{1}{\sqrt{2\pi}} e^{imx}, \quad x \in \mathbb{Q}, \tag{B.8}$$

converges absolutely and $g(x)$ is a continuous function. We prove $f(x) = g(x)$ by contradiction, assuming that $h(x) = f(x) - g(x)$ does not vanish everywhere. The functions $f$ and $g$ have the same Fourier coefficients and, hence,

$$\int_{-\pi}^{\pi} h(x) p(x) \mathrm{d}x = 0 \tag{B.9}$$

for any $p$ with (B.4). Then $\overline{h}(x)$, and, consequently, $\mathrm{Re}\, h(x)$, $\mathrm{Im}\, h(x)$, possess the same property. In other words, if $h = f - g$ is not identically zero, then there is a real continuous function $h_0$ with (B.9) and $h_0(x_0) > 0$ for some $x_0 \in \mathbb{Q}$. For $\delta > 0$ sufficiently small, let

$$p_0(x) = 1 + \cos(x - x_0) - \cos\delta, \quad x \in \mathbb{Q}. \tag{B.10}$$

Then $p_0(x) \geq 1$ if, and only if, $|x - x_0| \leq \delta$. One obtains

$$\int_{-\pi}^{\pi} h_0(x) p_0^k(x) \mathrm{d}x \to \infty \quad \text{if } k \in \mathbb{N} \text{ and } k \to \infty, \tag{B.11}$$

what contradicts (B.9) with $p = p_0^k$ and $h = h_0$. □

**Remark B.2.** Basic properties for trigonometric functions may be found in [Edw79]. As for a theory of function spaces on the $n$-torus parallel to the Euclidean $n$-space we refer to [ST87].

# Appendix C
# Operator theory

## C.1 Operators in Banach spaces

We assume that the reader is familiar with basic elements of functional analysis, in particular, of operator theory in Banach spaces and in Hilbert spaces. Here we fix some notation and formulate a few key assertions needed in this book. More specific notation and properties which are beyond standard courses of functional analysis will be explicated in the text, especially in Chapter 6.

All Banach spaces and Hilbert spaces considered in this book are complex. The norm in a Banach space $Y$ is denoted by $\| \cdot |Y\|$. Usually we do not distinguish between *equivalent norms* in a given Banach space $Y$, that is, where

$$\|y|Y\|_1 \sim \|y|Y\|_2 \quad \text{means} \quad c_1\|y|Y\|_1 \le \|y|Y\|_2 \le c_2\|y|Y\|_1 \tag{C.1}$$

for some numbers $0 < c_1 \le c_2 < \infty$ and all $y \in Y$.

Let $X$ and $Y$ be two complex Banach spaces. Then $\mathcal{L}(X, Y)$ is the Banach space of all linear and bounded operators acting from $X$ into $Y$ furnished with the norm

$$\|T\| = \sup\{\|Tx|Y\| : \|x|X\| \le 1,\ x \in X\}, \quad T \in \mathcal{L}(X, Y). \tag{C.2}$$

If $X = Y$, then we put $\mathcal{L}(Y) = \mathcal{L}(Y, Y)$. As usual nowadays,

$$T : X \hookrightarrow Y \quad \text{stands for} \quad T \in \mathcal{L}(X, Y). \tag{C.3}$$

If for $X \subset Y$ the continuous embedding of $X$ into $Y$ is considered as a map, then this will be indicated by the *identity* (*operator*) id,

$$\mathrm{id} : X \hookrightarrow Y, \quad \text{hence } \mathrm{id}\, x = x \text{ for all } x \in X \tag{C.4}$$

and

$$\|x|Y\| \le c\|x|X\| \quad \text{for all } x \in X \text{ and some } c \ge 0. \tag{C.5}$$

If $T \in \mathcal{L}(X, Y)$ is one-to-one, hence $Tx_1 = Tx_2$ if, and only if, $x_1 = x_2$, then $T^{-1}$ stands for its *inverse*,

$$Tx = y \quad \Longleftrightarrow \quad x = T^{-1}y, \quad x \in X. \tag{C.6}$$

Of course, $T^{-1}$ is linear on its domain of definition $\mathrm{dom}(T^{-1})$; this is the range of $T$, $\mathrm{range}(T)$. But $T^{-1}$ need not be bounded.

For $T \in \mathcal{L}(Y)$ the *resolvent set* $\varrho(T)$ of $T$ is the set

$$\varrho(T) = \{\lambda \in \mathbb{C} : (T - \lambda \mathrm{id})^{-1} \text{ exists and belongs to } \mathcal{L}(Y)\}. \tag{C.7}$$

Here id stands for the identity of $Y$ to itself, hence (C.4) with $X = Y$. As usual,

$$\sigma(T) = \mathbb{C} \setminus \varrho(T) \tag{C.8}$$

is called *spectrum* of $T$. By the *point spectrum* $\sigma_{\mathrm{p}}(T)$ we mean the set of *eigenvalues* of $T$; that is, $\lambda \in \sigma_{\mathrm{p}}(T)$ if, and only if, $\lambda \in \mathbb{C}$ and

$$Ty = \lambda y \quad \text{for some } y \neq 0. \tag{C.9}$$

Then

$$\ker(T - \lambda \mathrm{id}) = \{y \in Y : (T - \lambda \mathrm{id})y = 0\} \tag{C.10}$$

is called the *kernel* or *null space* of $T - \lambda\mathrm{id}$. It is a linear subspace of $Y$ and its dimension, $\dim \ker(T - \lambda \mathrm{id})$, is the *geometric multiplicity* of the eigenvalue $\lambda$ of $T$. Furthermore,

$$\dim \bigcup_{k=1}^{\infty} \ker(T - \lambda \mathrm{id})^k \quad \text{with } \lambda \in \mathbb{C} \tag{C.11}$$

is denoted as the *algebraic multiplicity* of $\lambda$. It is at least 1 if, and only if, $\lambda \in \sigma_{\mathrm{p}}(T)$.

An operator $T \in \mathcal{L}(X, Y)$ is called *compact* if the image $TU_X$ of the unit ball

$$U_X = \{x \in X : \|x|X\| \leq 1\} \tag{C.12}$$

in $Y$ is pre-compact (its closure is compact). The following assertion is a cornerstone of the famous Fredholm–Riesz–Schauder theory of compact operators in Banach spaces.

**Theorem C.1.** *Let $Y$ be a* (*complex*) *infinite-dimensional Banach space and let $T \in \mathcal{L}(Y)$ be compact. Then*

$$\sigma(T) = \{0\} \cup \sigma_{\mathrm{p}}(T). \tag{C.13}$$

*Furthermore, $\sigma(T) \setminus \{0\}$ consists of an at most countably infinite number of eigenvalues of finite algebraic multiplicity which may accumulate only at the origin.*

**Remark C.2.** Detailed presentations of the Fredholm–Riesz–Schauder theory may be found in [EE87, pp. 1–12] and [Rud91, Chapter 4]. For a short proof of the above theorem and further discussions about the spectral theory of compact operators in quasi-Banach spaces one may consult [ET96, Section 1.2, especially p. 5]. As for the Riesz–Schauder theory in Hilbert spaces we refer also to [Tri92a, Section 2.4]. It is remarkable that some basic assertions of the above theorem go back to F. Riesz in 1918 ([Rie18]) more than ten years before the theory of Banach spaces was established formally, [Ban32].

## C.2 Symmetric and self-adjoint operators in Hilbert spaces

We collect some assertions about bounded and (preferably) unbounded operators in complex separable infinite-dimensional Hilbert spaces $H$ furnished in the usual way with a scalar product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H$ and a norm $\| f \,|H\| = \sqrt{\langle f, f \rangle}$, $f \in H$. Again we assume that the reader is familiar with basic Hilbert space theory as it may be found in many books, for example [Tri92a, Chapter 2].

Up to the end of this Section C we now follow [Tri92a, Chapter 4] essentially; there one finds further details, explanations and proofs. We also refer to [RS75, Chapter X.1, 3] in this context.

We say that $A$ is a *linear operator* in $H$ if it is defined on a linear subset of $H$, denoted by $\mathrm{dom}(A)$, the *domain of definition* of $A$, and

$$A(\lambda_1 h_1 + \lambda_2 h_2) = \lambda_1 A h_1 + \lambda_2 A h_2, \quad \lambda_1, \lambda_2 \in \mathbb{C}, \ h_1, h_2 \in \mathrm{dom}(A), \tag{C.14}$$

and $Ah \in H$ for $h \in \mathrm{dom}(A)$. The *range* (or *image*) of $A$ is denoted by

$$\mathrm{range}(A) = \{g \in H : \text{ there is an } h \in \mathrm{dom}(A) \text{ with } Ah = g\}. \tag{C.15}$$

We shall always assume that $\mathrm{dom}(A)$ is dense in $H$ unless otherwise expressly agreed. Then the *adjoint operator* $A^*$ makes sense, defined on

$$\begin{aligned} \mathrm{dom}(A^*) = \{g \in H : \ & \text{there is } g^* \in H \text{ such that} \\ & \text{for all } h \in \mathrm{dom}(A)\text{: } \langle Ah, g \rangle = \langle h, g^* \rangle\}, \end{aligned} \tag{C.16}$$

and $A^* g = g^*$. In particular,

$$\langle Ah, g \rangle = \langle h, A^* g \rangle \quad \text{for all } h \in \mathrm{dom}(A) \text{ and } g \in \mathrm{dom}(A^*). \tag{C.17}$$

A linear operator $A$ is called *symmetric* if, again, $\mathrm{dom}(A)$ is dense in $H$, and

$$\langle Ah, g \rangle = \langle h, Ag \rangle \quad \text{for all } h \in \mathrm{dom}(A) \text{ and } g \in \mathrm{dom}(A). \tag{C.18}$$

In particular, the adjoint operator $A^*$ of a symmetric operator $A$ is an *extension* of $A$, written as $A \subset A^*$. A (densely defined) linear operator

$$A \text{ is called } \textit{self-adjoint} \text{ if } A = A^*. \tag{C.19}$$

Hence any self-adjoint operator is symmetric. The converse is not true. It is just one of the major topics of operator theory in Hilbert spaces to find criteria ensuring that a symmetric operator is self-adjoint. The notion of the *resolvent set* in (C.7), the *spectrum* (C.8), the *point spectrum* and also of (C.9), (C.10) are extended obviously to arbitrary linear operators $A$; one may consult also Section 6.2.

**Theorem C.3.** (i) *Let $A$ be a self-adjoint operator in $H$. Then*

$$\sigma(A) \subset \mathbb{R} \quad \text{and} \quad \operatorname{range}(A - \lambda \mathrm{id}) = H \quad \text{if } \lambda \in \mathbb{C},\ \operatorname{Im}\lambda \neq 0. \tag{C.20}$$

(ii) *A symmetric operator $A$ is self-adjoint if, and only if, there is a number $\lambda \in \mathbb{C}$ such that*

$$\operatorname{range}(A - \lambda \mathrm{id}) = \operatorname{range}(A - \overline{\lambda}\mathrm{id}) = H. \tag{C.21}$$

**Remark C.4.** If $\lambda \in \varrho(A)$, then exists, by definition, $(A - \lambda \mathrm{id})^{-1} \in \mathcal{L}(H)$. In particular,

$$\operatorname{range}(A - \lambda \mathrm{id}) = H \quad \text{if } \lambda \in \varrho(A). \tag{C.22}$$

Hence the second part of (C.20) follows from the first one. Furthermore, for a self-adjoint operator $A$ any eigenvalue $\lambda \in \sigma_{\mathrm{p}}(A)$ is real and its algebraic multiplicity coincides with the geometric multiplicity,

$$\dim \bigcup_{k=1}^{\infty} \ker(A - \lambda \mathrm{id})^k = \dim \ker(A - \lambda \mathrm{id}), \quad \lambda \in \sigma_{\mathrm{p}}(A). \tag{C.23}$$

We combine Theorems C.1 and C.3.

**Theorem C.5.** *Let $A$ be a compact self-adjoint operator in $H$. Then*

$$\sigma(A) \subset \big[-\|A\|, \|A\|\big] \quad \text{and} \quad 0 \in \sigma(A). \tag{C.24}$$

*Furthermore, $\sigma(A) \setminus \{0\}$ consists of an at most countably infinite number of eigenvalues $\lambda_j$ of finite multiplicity which can be ordered by magnitude including their multiplicity,*

$$|\lambda_1| \geq |\lambda_2| \geq \cdots, \quad \lambda_j \to 0 \text{ for } j \to \infty, \tag{C.25}$$

*(if there are infinitely many eigenvalues). There is an orthonormal system $\{h_j\}_j$ of related eigenelements,*

$$\langle h_j, h_k \rangle = 0 \text{ if } j \neq k, \quad \|h_j | H\| = 1, \quad A h_j = \lambda_j h_j. \tag{C.26}$$

*Furthermore,*

$$A h = \sum_j \lambda_j \langle h, h_j \rangle h_j \quad \text{for any } h \in H. \tag{C.27}$$

**Remark C.6.** One obtains (C.24) from Theorem C.1 and (C.20). Moreover, $|\lambda_1| = \|A\|$. Here we have the situation as indicated in Figure C.1.

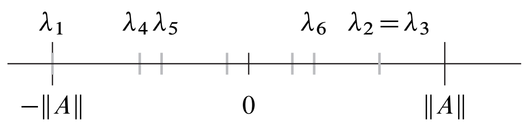



Figure C.1

By (C.23) there is no need to distinguish between geometric and algebraic multiplicity.

**Definition C.7.** A self-adjoint operator is called an *operator with a pure point spectrum* if its spectrum consists of eigenvalues with finite (geometric = algebraic) multiplicity.

**Proposition C.8.** *An operator $A$ with pure point spectrum does not belong to $\mathcal{L}(H)$ (it is not bounded). Furthermore, if $\lambda \in \varrho(A)$, then $(A - \lambda \mathrm{id})^{-1}$ is compact. The eigenvalues have no accumulation point in $\mathbb{C}$.*

## C.3 Semi-bounded and positive-definite operators in Hilbert spaces

Not every symmetric operator can be extended to a self-adjoint operator. But this is the case for an important sub-class we are going to discuss now. First we remark that for symmetric operators $A$,

$$\langle Ah, h\rangle = \langle h, Ah\rangle = \overline{\langle Ah, h\rangle}, \quad h \in \mathrm{dom}(A), \tag{C.28}$$

is real.

**Definition C.9.** A (linear, densely defined) symmetric operator in $H$ is called *semi-bounded* (or *bounded from below*) if there is a constant $c \in \mathbb{R}$ such that

$$\langle Ah, h\rangle \geq c\|h|H\|^2 \quad \text{for } h \in \mathrm{dom}(A). \tag{C.29}$$

If $c = 0$ in (C.29), then $A$ is called *positive*, if $c > 0$ in (C.29), then $A$ is called *positive-definite*.

**Remark C.10.** If $A$ is semi-bounded, then $A+\lambda\mathrm{id}$ is positive definite for $\lambda+c > 0$. Hence, at least in the framework of the abstract theory one may assume without restriction of generality that $A$ is positive-definite.

**Definition C.11.** Let $A$ be a positive-definite operator according to Definition C.9. Then the *energy space* $H_A$ is the completion of $\mathrm{dom}(A)$ in the norm

$$\|h|H_A\| = \sqrt{[h,h]_A} \quad \text{where } [h,g]_A = \langle Ah, g\rangle \tag{C.30}$$

for $h \in \mathrm{dom}(A)$ and $g \in \mathrm{dom}(A)$.

**Remark C.12.** The idea is to collect all elements $h \in H$ for which there is a Cauchy sequence $\{h_j\}_{j=1}^{\infty} \subset \mathrm{dom}(A)$ in the norm $\|\cdot|H_A\|$ (which is also a Cauchy sequence in $H$). As mentioned above, we closely followed [Tri92a]; in this case one may consult [Tri92a, Sections 4.1.8, 4.1.9, 4.4.3]. Furthermore,

$$\mathrm{dom}(A) \hookrightarrow H_A \hookrightarrow H, \tag{C.31}$$

which is even a continuous embedding (hence '$\hookrightarrow$'), if one furnishes $\mathrm{dom}(A)$ with the norm

$$\|h|\mathrm{dom}(A)\| = \sqrt{\|Ah|H\|^2 + \|h|H\|^2} \sim \|Ah|H\|, \quad h \in \mathrm{dom}(A). \tag{C.32}$$

If $A_0$ and $A_1$ are symmetric operators, then it follows from

$$A_0 \subset A_1 \quad \text{that} \quad A_1^* \subset A_0^*. \tag{C.33}$$

In particular, if one looks for a self-adjoint extension $A_1$ of the symmetric operator $A_0$, hence $A_1 = A_1^*$, then it must be a restriction of $A_0^*$. It turns out that in case of positive-definite operators $A = A_0$ (and hence also for semi-bounded operators) there exist restrictions of $A^*$ which are self-adjoint extensions of $A$.

**Theorem C.13** (Friedrichs extension). *Let $A$ be a positive-definite operator in the Hilbert space $H$ with* (C.29) *for some $c > 0$. Let $H_A$ be the energy space according to Definition* C.11. *Then*

$$A_F h = A^* h, \quad \mathrm{dom}(A_F) = H_A \cap \mathrm{dom}(A^*), \tag{C.34}$$

*is a self-adjoint extension of $A$ and*

$$\langle A_F h, h\rangle \ge c\|h|H\|^2, \quad h \in \mathrm{dom}(A_F), \tag{C.35}$$

*with the same constant $c$ as in* (C.29). *Furthermore,*

$$\sigma(A_F) \subset [c, \infty) \quad \textit{and} \quad H_{A_F} = H_A. \tag{C.36}$$

**Remark C.14.** By (C.31) and $A \subset A^*$ it is clear that $A_F$ is an extension of $A$, that is, $A \subset A_F$. Moreover, (C.36) implies that $0 \in \varrho(A_F)$; in particular, $A_F^{-1} \in \mathcal{L}(H)$ exists. Furthermore,

$$[h, g]_A = \langle A_F h, g\rangle \quad \text{if } h \in \mathrm{dom}(A_F),\ g \in \mathrm{dom}(A_F). \tag{C.37}$$

**Theorem C.15** (Rellich's criterion). *A self-adjoint positive-definite operator according to Definition* C.9 *is an operator with pure point spectrum in the sense of Definition* C.7 *if, and only if, the embedding*

$$\mathrm{id}\colon H_A \hookrightarrow H \tag{C.38}$$

*of the energy space as introduced in Definition* C.11 *is compact.*

**Remark C.16.** In view of (C.35) the spectrum of a positive-definite self-adjoint operator $A$ with pure point spectrum consists of isolated positive eigenvalues $\lambda_j$ of finite (geometric = algebraic) multiplicity,

$$0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lambda_j \to \infty \text{ for } j \to \infty, \tag{C.39}$$

where the latter assertion follows from $A \notin \mathcal{L}(H)$. Hence

$$Au_j = \lambda_j u_j, \quad j \in \mathbb{N}, \tag{C.40}$$

and $u_j \in \mathrm{dom}(A)$ are related eigenelements spanning $H$. In particular one may assume that $\{u_j\}_{j=1}^{\infty}$ is an orthonormal basis in $H$. Furthermore, $A^{-1}$ is compact,

$$A^{-1}u_j = \lambda_j^{-1} u_j, \quad j \in \mathbb{N}, \tag{C.41}$$

and

$$\sigma(A^{-1}) = \{0\} \cup \{\lambda_j^{-1}\}_{j=1}^{\infty}. \tag{C.42}$$

The latter result follows from Theorem C.1.

**Remark C.17.** The restriction to positive-definite operators in Theorem C.13 is convenient but not necessary. If $A$ is semi-bounded according to Definition C.9 and $\lambda + c > 0$, then $A + \lambda\mathrm{id}$ is positive-definite and $A_F$,

$$A_F h = (A + \lambda\mathrm{id})_F h - \lambda h, \quad h \in \mathrm{dom}(A + \lambda\mathrm{id})_F, \tag{C.43}$$

is a self-adjoint extension of $A$ which is independent of $\lambda$ with the same bound $c$ as in (C.29) (in analogy to (C.35)).

# Appendix D
# Some integral inequalities

Integral inequalities for convolution operators play a decisive rôle in the theory of function spaces. Not so in this book where they are needed only in connection with a few complementing considerations. We collect very few assertions which are of interest for us in this context. The spaces $L_p(\mathbb{R}^n)$ with $1 \le p \le \infty$ have the same meaning as in Section 2.2. Let, as usual, $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \le p \le \infty$.

**Theorem D.1.** *Let $k \in L_r(\mathbb{R}^n)$ where $1 \le r \le \infty$. Let*

$$1 \le p \le r' \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{r'} = \frac{1}{r} - \frac{1}{p'} = \frac{1}{r} + \frac{1}{p} - 1. \tag{D.1}$$

*Then the convolution operator $K$,*

$$(Kf)(x) = \int_{\mathbb{R}^n} k(x-y) f(y) \mathrm{d}y = \int_{\mathbb{R}^n} k(y) f(x-y) \mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{D.2}$$

*maps $L_p(\mathbb{R}^n)$ continuously into $L_q(\mathbb{R}^n)$,*

$$\|Kf \,|L_q(\mathbb{R}^n)\| \le \|k\,|L_r(\mathbb{R}^n)\| \; \|f\,|L_p(\mathbb{R}^n)\|. \tag{D.3}$$

**Remark D.2.** This well-known assertion, often called *Young's inequality*, follows from Hölder's inequality. We refer, for example, to [Tri78, Section 1.18.9, p. 139]; see also Exercise 2.70 (a). Combining this inequality with some real interpolation, then one obtains the following famous (and deeper) *Hardy–Littlewood–Sobolev inequality*.

**Theorem D.3.** *Let*

$$0 < \alpha < n, \quad 1 < p < \frac{n}{n-\alpha} \quad \text{and} \quad \frac{1}{q} = \frac{\alpha}{n} - \frac{1}{p'} = \frac{1}{p} + \frac{\alpha}{n} - 1. \tag{D.4}$$

*Then $K$, given by*

$$(Kf)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\alpha} \, \mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{D.5}$$

*maps $L_p(\mathbb{R}^n)$ continuously into $L_q(\mathbb{R}^n)$,*

$$\|Kf\,|L_q(\mathbb{R}^n)\| \le c \|f\,|L_p(\mathbb{R}^n)\|. \tag{D.6}$$

**Remark D.4.** The integral $(Kf)(x)$ in (D.5) is called the *Riesz potential* of $f$.

This inequality has some history. The case $n = 1$ goes back to Hardy and Littlewood, [HL28], [HL32]. This was extended by Sobolev in [Sob38] to $n \in \mathbb{N}$, and may also be found in [Sob91, §6] (first edition 1950). Furthermore, one can prove (D.6) by real interpolation of (D.3). This was observed by Peetre in [Pee66]. Short proofs (on two pages) of both theorems using interpolation may be found in [Tri78, Section 1.18.9, pp. 139/140].

# Appendix E
# Function spaces

## E.1 Definitions, basic properties

This book deals with the Sobolev spaces

$$W_p^k(\mathbb{R}^n), \quad H^s(\mathbb{R}^n), \quad W_2^s(\mathbb{R}^n), \tag{E.1}$$

as introduced in the Definitions 3.1, 3.13, 3.22 and their restrictions

$$W_p^k(\Omega), \quad W_2^s(\Omega), \tag{E.2}$$

to domains $\Omega$ in $\mathbb{R}^n$ according to Definition 3.37. However, in the Notes we hint(ed) occasionally at more general spaces covering the above spaces and their properties as special cases. To provide a better understanding what is meant there we collect some basic definitions and assertions, and list a few special cases.

We use the same basic notation as in Section A.1. In particular, arbitrary open sets $\Omega$ in $\mathbb{R}^n$ are called domains. We extend the definition of the complex Lebesgue space $L_p(\Omega)$ as introduced at the beginning of Section 2.2 naturally from $1 \le p \le \infty$ (Banach spaces) to $0 < p \le \infty$ (quasi-Banach spaces according to Note 7.8.1 consisting of equivalence classes (2.18) quasi-normed by (2.16), (2.17) now for $0 < p \le \infty$). Let $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ as in the Definitions 2.32, 2.43 furnished with the Fourier transform and its inverse according to Definitions 2.36, 2.58. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge \frac{3}{2}, \tag{E.3}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n,\ k \in \mathbb{N}, \tag{E.4}$$

see also Figure E.1 below.

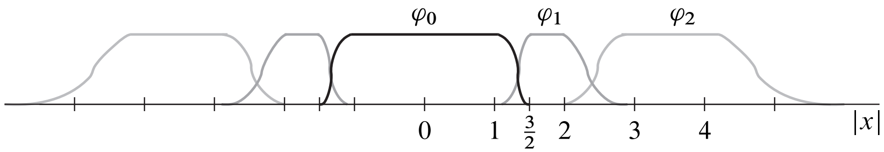



Figure E.1

Since

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for } x \in \mathbb{R}^n,$$

the $\{\varphi_j\}_{j=0}^{\infty}$ form a *dyadic resolution of unity*. The entire analytic functions $(\varphi_j \widehat{f})^\vee(x)$ make sense pointwise for any $f \in \mathcal{S}'(\mathbb{R}^n)$.

**Definition E.1.** Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}. \tag{E.5}$$

Then $B^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\,|B^s_{p,q}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^\vee |L_p(\mathbb{R}^n)\|^q\Big)^{1/q} < \infty \tag{E.6}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}. \tag{E.7}$$

Then $F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\,|F^s_{p,q}(\mathbb{R}^n)\|_\varphi = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q\Big)^{1/q} \Big|L_p(\mathbb{R}^n)\Big\| < \infty \tag{E.8}$$

(with the usual modification if $q = \infty$).

**Remark E.2.** It is not our aim to give a brief survey of the above spaces. We wish to support some Notes in the main body of this book where we hinted on spaces of the above type and to provide some background information. We refer, in particular, to the Notes 3.6.1–3.6.3 where we gave also a list of relevant books for further reading and to Note 7.8.5.

**Theorem E.3.** *The spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ are independent of $\varphi$ (in the sense of equivalent quasi-norms). They are quasi-Banach spaces. Furthermore,*

$$\mathcal{S}(\mathbb{R}^n) \subset B^s_{p,q}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \tag{E.9}$$

$$\mathcal{S}(\mathbb{R}^n) \subset F^s_{p,q}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \tag{E.10}$$

*and*

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n) \tag{E.11}$$

*for all admitted parameters.*

Let $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ be as in Definitions 2.2 and 2.5. Furthermore, $g|_\Omega \in \mathcal{D}'(\Omega)$ for $g \in \mathcal{S}'(\mathbb{R}^n)$ means

$$(g|_\Omega)(\varphi) = g(\varphi) \quad \text{for } \varphi \in \mathcal{D}(\Omega). \tag{E.12}$$

**Definition E.4.** Let $\Omega$ be a domain in $\mathbb{R}^n$. Let $A^s_{p,q}(\mathbb{R}^n)$ be either $B^s_{p,q}(\mathbb{R}^n)$ with (E.5) or $F^s_{p,q}(\mathbb{R}^n)$ with (E.7). Then

$$A^s_{p,q}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \text{there exists } g \in A^s_{p,q}(\mathbb{R}^n) \text{ with } g|_\Omega = f\}, \tag{E.13}$$

quasi-normed by

$$\|f|A^s_{p,q}(\Omega)\| = \inf \|g|A^s_{p,q}(\mathbb{R}^n)\|, \tag{E.14}$$

where the infimum is taken over all $g \in A^s_{p,q}(\mathbb{R}^n)$ with $g|_\Omega = f$ in $\mathcal{D}'(\Omega)$.

**Remark E.5.** This is a generalisation of Definition 3.37.

**Theorem E.6.** *Let $\Omega$ be a domain in $\mathbb{R}^n$. Then*

$$B^s_{p,q}(\Omega), \quad 0 < p \le \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}, \tag{E.15}$$

*and*

$$F^s_{p,q}(\Omega), \quad 0 < p < \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}, \tag{E.16}$$

*are quasi-Banach spaces. Furthermore,*

$$\mathcal{D}(\Omega) \subset B^s_{p,q}(\Omega) \subset \mathcal{D}'(\Omega), \tag{E.17}$$

$$\mathcal{D}(\Omega) \subset F^s_{p,q}(\Omega) \subset \mathcal{D}'(\Omega), \tag{E.18}$$

*and*

$$B^s_{p,\min(p,q)}(\Omega) \hookrightarrow F^s_{p,q}(\Omega) \hookrightarrow B^s_{p,\max(p,q)}(\Omega) \tag{E.19}$$

*for all admitted parameters.*

**Remark E.7.** For bounded domains $\Omega$ the assertion (7.170) is independent of $q_1$ and $q_2$ in (7.169). Then it follows from (E.19) that one can replace there $B$ by $F$ on one side or on both sides. In particular, with the special cases listed in Section E.2 below one gets for

$$s > 0, \quad 1 < p < \infty, \quad s - \frac{n}{2} > -\frac{n}{p}, \tag{E.20}$$

that

$$e_k(\mathrm{id}\colon W^s_2(\Omega) \hookrightarrow L_p(\Omega)) \sim k^{-\frac{s}{n}}, \quad k \in \mathbb{N}, \tag{E.21}$$

complementing Corollary 7.11.

**Theorem E.8.** *Let $A^s_{p,q}$ be either $B^s_{p,q}$ with (E.5) or $F^s_{p,q}$ with (E.7).*

(i) *Let*

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}, \quad p_2 \geq p_1. \tag{E.22}$$

*Then*

$$\mathrm{id}\colon A^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n) \tag{E.23}$$

*is continuous, but not compact.*

(ii) *Let $\Omega$ be a bounded domain. Then*

$$\mathrm{id}\colon A^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2,q_2}(\Omega) \tag{E.24}$$

*is compact if, and only if,*

$$s_1 - s_2 > \max\left(0, \frac{n}{p_1} - \frac{n}{p_2}\right). \tag{E.25}$$

**Remark E.9.** Below we have sketched the different situations for $\mathbb{R}^n$ and for a bounded domain $\Omega$ in Figure E.2 (i) and (ii), respectively.

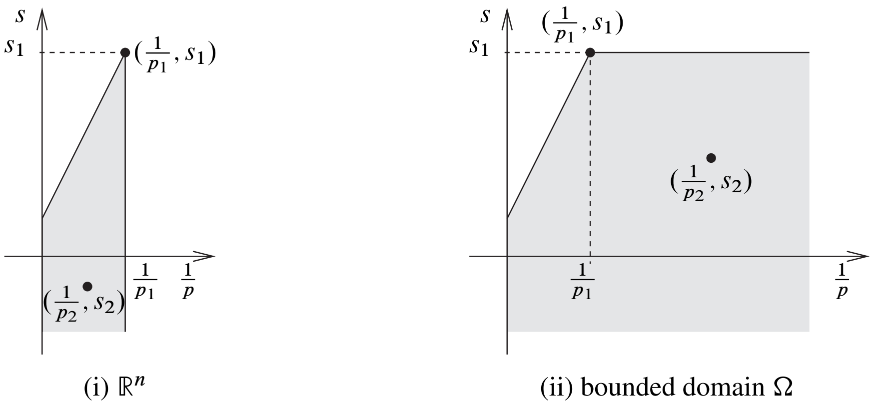



Figure E.2

By (E.19) the assertions (7.167)–(7.170) strengthen part (ii) of the theorem, specifying the degree of compactness.

## E.2 Special cases, equivalent norms

Although we discussed in Note 3.6.1 some generalisations of the spaces (E.1), (E.2) it seems reasonable to complement the preceding Section E.1 by a few properties, special cases and equivalent quasi-norms.

**Lifting**. For $\sigma \in \mathbb{R}$ let

$$I_\sigma f = \mathcal{F}^{-1} \langle \xi \rangle^\sigma \mathcal{F} f, \quad f \in \mathcal{S}'(\mathbb{R}^n), \tag{E.26}$$

as in (2.148), recall notation (2.83). According to Proposition 2.63 the operator $I_\sigma$ maps $\mathcal{S}(\mathbb{R}^n)$ onto itself and $\mathcal{S}'(\mathbb{R}^n)$ onto itself. Let $A^s_{p,q}(\mathbb{R}^n)$ be either $B^s_{p,q}(\mathbb{R}^n)$ with (E.5) or $F^s_{p,q}(\mathbb{R}^n)$ with (E.7). Then

$$I_\sigma A^s_{p,q}(\mathbb{R}^n) = A^{s-\sigma}_{p,q}(\mathbb{R}^n), \quad \sigma \in \mathbb{R}. \tag{E.27}$$

We used assertions of this type in Section 3.2 in connection with the spaces $H^s(\mathbb{R}^n)$.

**Spaces of regular distributions**. According to Definition 2.10 a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is called *regular* if, in addition, $f \in L^{\mathrm{loc}}_1(\mathbb{R}^n)$ (in the interpretation given there). With $A^s_{p,q}(\mathbb{R}^n)$ as above one has

$$A^s_{p,q}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \cap L^{\mathrm{loc}}_1(\mathbb{R}^n) \quad \text{if } s > \sigma_p = n\Big(\frac{1}{p} - 1\Big)_+. \tag{E.28}$$

This refers to the shaded area in Figure E.3 aside. If $s < \sigma_p$, then (E.28) is not true (which means that there are singular distributions belonging to $A^s_{p,q}(\mathbb{R}^n)$). The case $s = \sigma_p$ is somewhat tricky. One may consult [Tri01, Theorem 11.2, pp. 168/169] and the references given there. In addition, Exercise 3.18 illuminates the situation.

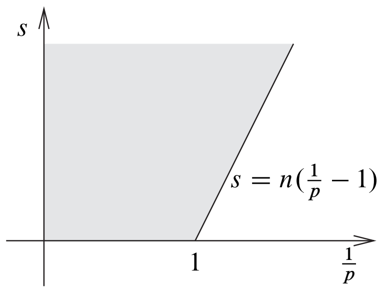



Figure E.3

**Sobolev spaces**. Let $1 < p < \infty$ and $s \in \mathbb{R}$. It is usual nowadays to call

$$H^s_p(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n), \quad \|f \,|H^s_p(\mathbb{R}^n)\| = \|I_{-s} f \,| L_p(\mathbb{R}^n)\|, \tag{E.29}$$

*Sobolev spaces* (fractional Sobolev spaces, Bessel potential spaces). If $1 < p < \infty$ and $s = k \in \mathbb{N}_0$, then they contain *classical Sobolev spaces*

$$W^k_p(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \quad \|f \,| W^k_p(\mathbb{R}^n)\| = \sum_{|\alpha| \le k} \|\mathrm{D}^\alpha f \,| L_p(\mathbb{R}^n)\| \tag{E.30}$$

as special cases. Furthermore,

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{E.31}$$

is called the *Paley–Littlewood property* of the Sobolev spaces. Obviously, the norms in (E.29), (E.30) and in connection with (E.31) are equivalent to each other (for the indicated parameters). We dealt with spaces of this type in Sections 3.1, 3.2 where we had been interested in the case $p = 2$ especially. Then these spaces are Hilbert spaces,

$$H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n) = F^s_{2,2}(\mathbb{R}^n) = B^s_{2,2}(\mathbb{R}^n), \quad s \in \mathbb{R}. \tag{E.32}$$

One may also consult Note 3.6.1.

**Hölder–Zygmund spaces**. The spaces

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n), \quad s \in \mathbb{R}, \tag{E.33}$$

are usually called *Hölder–Zygmund spaces*, sometimes restricted to $s > 0$, where one has

$$\mathcal{C}^s(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n), \quad s > 0. \tag{E.34}$$

Here $C(\mathbb{R}^n)$ has the same meaning as in Definition A.1. Let again

$$(\Delta_h f)(x) = f(x+h) - f(x), \quad x \in \mathbb{R}^n, \ h \in \mathbb{R}^n, \tag{E.35}$$

be the usual differences in $\mathbb{R}^n$. Let $\Delta_h = \Delta^1_h$ and for $m \in \mathbb{N}$, $m \geq 2$,

$$(\Delta^m_h f)(x) = \Delta^1_h(\Delta^{m-1}_h f)(x), \quad x \in \mathbb{R}^n, \ h \in \mathbb{R}^n, \tag{E.36}$$

be the iterated differences. Let

$$0 < s = \ell + \sigma, \quad \ell \in \mathbb{N}_0, \ s > \ell. \tag{E.37}$$

Assume $\sigma < m \in \mathbb{N}$. Then $\mathcal{C}^s(\mathbb{R}^n)$ is the collection of all $f \in C(\mathbb{R}^n)$ such that

$$\|f|\mathcal{C}^s(\mathbb{R}^n)\|_{\ell,m} = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup |h|^{-\sigma}|\Delta^m_h(\mathrm{D}^\alpha f)(x)| < \infty, \tag{E.38}$$

where the second supremum is taken over all $x \in \mathbb{R}^n$, all $h \in \mathbb{R}^n$ with $0 < |h| < 1$ and all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq \ell$ (equivalent norms). In particular, if $0 < s < 1$, then one obtains the usual *Hölder norm*

$$\begin{aligned} \|f|\mathcal{C}^s(\mathbb{R}^n)\|_{0,1} &= \|f|C^s(\mathbb{R}^n)\| \\ &= \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{\substack{x \in \mathbb{R}^n, \\ 0<|h|<1}} |h|^{-s}|f(x+h) - f(x)|. \end{aligned} \tag{E.39}$$

If $s = 1$, this leads to the so-called *Zygmund class* $\mathcal{C}^1(\mathbb{R}^n)$ which can be normed by

$$\|f|\mathcal{C}^1(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{\substack{x \in \mathbb{R}^n, \\ 0<|h|<1}} |h|^{-1}|\Delta^2_h f(x)|. \tag{E.40}$$

We refer also to (3.44) and Exercises 3.20, 3.21.

**Besov spaces.** The spaces $B^s_{p,q}(\mathbb{R}^n)$ according to Definition E.1 are called *Besov spaces*. Let

$$0 < p \le \infty, \quad 0 < q \le \infty \quad \text{and} \quad s > \sigma_p = n\Big(\frac{1}{p} - 1\Big)_+. \tag{E.41}$$

Then (E.28) can be strengthened by

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_{\max(p,1)}(\mathbb{R}^n). \tag{E.42}$$

Furthermore, let $\Delta^m_h$ be as in (E.36) with $s < m \in \mathbb{N}$. Then

$$\|f\,|L_p(\mathbb{R}^n)\| + \Big(\int\limits_{|h|\le 1} |h|^{-sq} \|\Delta^m_h f\,|L_p(\mathbb{R}^n)\|^q \,\frac{\mathrm{d}h}{|h|^n}\Big)^{1/q} \tag{E.43}$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$. This covers in particular the classical Besov spaces

$$B^s_{p,q}(\mathbb{R}^n), \quad s > 0, \quad 1 \le p \le \infty,\ 1 \le q \le \infty. \tag{E.44}$$

We also refer to Note 3.6.1.

**Remark E.10.** We de not give specific references. All may be found in the books mentioned in Note 3.6.3, especially in [Tri83], [Tri92b], [Tri06]. This appendix is not a brief survey. We wanted to make clear that some assertions for the special spaces on (E.1), (E.2) proved in this book are naturally embedded in the larger framework of the recent theory of function spaces.

# Selected solutions

**Exercise 1.5.** $\inf\{x_2 : (x_1, x_2) \in \Omega\} > 0$.

**Exercise 1.18.** (a) Let $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n_+$, $x^0_\blacktriangledown = (x_1^0, \ldots, x_{n-1}^0, -x_n^0)$,

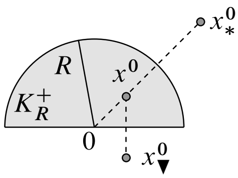
$$g_{\mathbb{R}^n_+}(x^0, x) = \begin{cases} -\dfrac{1}{2\pi}(\ln|x - x^0| - \ln|x - x^0_\blacktriangledown|), & n = 2, \\ \dfrac{1}{(n-2)|\omega_n|}\left(\dfrac{1}{|x - x^0|^{n-2}} - \dfrac{1}{|x - x^0_\blacktriangledown|^{n-2}}\right), & n \geq 3, \end{cases}$$

$$u(x^0) = \frac{2x_n^0}{|\omega_n|} \int\limits_{\sigma_n = 0} \frac{u(\sigma)}{|\sigma - x^0|^n}\, d\sigma, \quad x^0 \in \mathbb{R}^n_+.$$

(b) Let $x^0 = (x_1^0, \ldots, x_n^0) \in K_R^+$,

$$x^0_* = x^0 \frac{R^2}{|x^0|^2},$$

$$x^0_\blacktriangledown = (x_1^0, \ldots, x_{n-1}^0, -x_n^0),$$




$$g(x^0, x) = g_{K_R}(x^0, x) - g_{K_R}(x^0_\blacktriangledown, x)$$

$$= g_{\mathbb{R}^n_+}(x^0, x) - \frac{R}{|x^0|} g_{\mathbb{R}^n_+}(x^0_*, x).$$

Let $n = 3$, $x^0 \in K_R^+ \subset \mathbb{R}^3_+$,

$$u(x^0) = \frac{R^2 - |x^0|^2}{4\pi R} \int\limits_{\substack{|\sigma| = 1, \\ \sigma_3 > 0}} u(\sigma)\left[\frac{1}{|\sigma - x^0|^3} - \frac{1}{|\sigma - x^0_\blacktriangledown|^3}\right] d\sigma$$

$$+ \frac{x_3^0}{2\pi} \int\limits_{\substack{\sigma_1^2 + \sigma_2^2 \leq 1, \\ \sigma_3 = 0}} u(\sigma)\left[\frac{1}{|\sigma - x^0|^3} - \frac{R^3}{|x^0|^3} \frac{1}{|\sigma - x^0_*|^3}\right] d\sigma.$$

**Exercise 1.24.** (b) $\Delta u = 0$, then Theorem 1.23 (iii) gives

$$\sup_{(x_1, x_2) \in K_R} u(x_1, x_2) = R^2, \qquad \inf_{(x_1, x_2) \in K_R} u(x_1, x_2) = -R^2.$$

(c) $f(0,0,0) = -1$, $f(x_1,x_2,x_3) \equiv 0$ if $(x_1,x_2,x_3) \in \partial K_1$, thus Theorem 1.23 (iii) fails and $f$ cannot be harmonic in $K_1$.

**Exercise 1.41.** (b) $u(x_1,x_2) = \dfrac{4C}{R^4}(x_1^3 x_2 - x_1 x_2^3)$.

**Exercise 1.42.** (a) Let $\varphi \in C(\partial K_R^+)$, then for $x \in K_R^+$,

$$u(x) = \frac{R^2-|x|^2}{4\pi R} \int\limits_{\substack{|\sigma|=1,\\ \sigma_3>0}} \varphi(\sigma)\left[\frac{1}{|\sigma-x|^3} - \frac{1}{|\sigma-x_\blacktriangledown|^3}\right]\mathrm{d}\sigma$$
$$+ \frac{x_3}{2\pi} \int\limits_{\substack{\sigma_1^2+\sigma_2^2\le 1,\\ \sigma_3=0}} \varphi(\sigma)\left[\frac{1}{|\sigma-x|^3} - \frac{R^3}{|x|^3}\frac{1}{|\sigma-x_*|^3}\right]\mathrm{d}\sigma$$

with $x_\blacktriangledown = (x_1,x_2,-x_3)$, $x_* = x\,\dfrac{R^2}{|x|^2}$, and

$$u(x) = \varphi(x), \quad x \in \partial K_R^+.$$

(b) $u(x_1,x_2) = x_1^2 - x_2^2$.

**Exercise 1.49.** $u(x_1,x_2) = \dfrac{1}{4}(x_1^2+x_2^2) + \dfrac{7}{4} - \dfrac{1}{8}\left(1+\dfrac{3}{e^2}\right)\log(x_1^2+x_2^2)$; log taken with respect to base $e$.

**Exercise 2.13.** (b) For the *regular* case one may choose a smooth function $f$ such that $T_f^\alpha = T_{\mathrm{D}^\alpha f}$; in the *singular* case, say, with $n=1$, $\Omega = (-1,1)$, $\alpha = 1$, let

$$f(x) = \begin{cases} x^\nu, & x>0,\\ 0, & x\le 0,\end{cases}$$

with $-1<\nu<0$. Then $f \in L_1^{\mathrm{loc}}(\Omega)$, but there is no $g \in L_1^{\mathrm{loc}}(\Omega)$ with $T_g = T_f^1$ since $f' \notin L_1^{\mathrm{loc}}(\Omega)$.

**Exercise 2.17.** (b) $\dfrac{\mathrm{d}g}{\mathrm{d}t} = \chi(t) - \chi(-t)$, $\quad \dfrac{\mathrm{d}^2 g}{\mathrm{d}t^2} = 2\delta$.

**Exercise 2.39.** $\mathrm{D}^\alpha(\mathcal{F}^{-1}\varphi)(\xi) = i^{|\alpha|}\,\mathcal{F}^{-1}(x^\alpha\varphi(x))(\xi)$,

$$\xi^\alpha(\mathcal{F}^{-1}\varphi)(\xi) = i^{|\alpha|}\,\mathcal{F}^{-1}(\mathrm{D}^\alpha\varphi)(\xi).$$

**Exercise 2.51.** $k > \dfrac{n}{p'}$, e.g., $k = 1 + \max\left\{m \in \mathbb{N}_0 : m \le \dfrac{n}{p'}\right\}$.

**Exercise 2.52.** (b) Recall that $\varphi(x) = e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$.

**Exercise 2.61.** $\mathcal{F}^{-1}(\mathrm{D}^\alpha T) = (-i)^{|\alpha|}\, x^\alpha (\mathcal{F}^{-1}T)$,
$\mathcal{F}^{-1}(x^\alpha T) = (-i)^{|\alpha|}\, \mathrm{D}^\alpha (\mathcal{F}^{-1}T)$.

**Exercise 2.68.** (a) $\mathcal{F}(e^{-a|x|})(\xi) = \dfrac{1}{\sqrt{2\pi}} \dfrac{2a}{a^2+\xi^2}$.

(b) $\mathcal{F}(\operatorname{sgn}(x)e^{-|x|})(\xi) = \dfrac{1}{\sqrt{2\pi}} \dfrac{-2i\xi}{1+\xi^2}$.

(c) $$\mathcal{F}(\chi_{[-a,a]}(x))(\xi) = \frac{2a}{\sqrt{2\pi}} \begin{cases} \dfrac{\sin(a\xi)}{a\xi}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

(d) $$\mathcal{F}((1-|x|)_+)(\xi) = \frac{1}{\sqrt{2\pi}} \begin{cases} \dfrac{2(1-\cos\xi)}{\xi^2}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

**Exercise 2.70.** (b) $\mathcal{F}^{-1}(f * g)(\xi) = (2\pi)^{n/2} (\mathcal{F}^{-1}f)(\xi)\,(\mathcal{F}^{-1}g)(\xi)$.

(c) $h_a = \dfrac{1}{\sqrt{2\pi}} \mathcal{F}(e^{-a|\xi|})$, $g_a = \dfrac{1}{\sqrt{2\pi}} \mathcal{F}(\chi_{[-a,a]})$, thus (2.168) and Exercise 2.68 (a), (c) imply,

$$\begin{aligned} h_a * h_b &= \mathcal{F}(\mathcal{F}^{-1}(h_a * h_b)) \\ &= \sqrt{2\pi}\, \mathcal{F}\left(\frac{1}{\sqrt{2\pi}}(e^{-a|\xi|}) \frac{1}{\sqrt{2\pi}}(e^{-b|\xi|})\right) = h_{a+b}, \\ g_a * g_b &= \sqrt{2\pi}\, \mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\, \chi_{[-a,a]}\, \frac{1}{\sqrt{2\pi}}\, \chi_{[-b,b]}\right) = g_{\min(a,b)}. \end{aligned}$$

**Exercise 3.16.** (a) $w_s\, \mathcal{F}(e^{-|\xi|}) \sim \langle x\rangle^{\frac{s}{2}-1} \in L_2(\mathbb{R})$ if, and only if, $s < \frac{3}{2}$.

(b) $w_s\, \mathcal{F}\chi_{[-a,a]} \sim \langle x\rangle^{\frac{s-1}{2}} \in L_2(\mathbb{R})$ if, and only if, $s < \frac{1}{2}$.

(c) $w_s(x)\mathcal{F}(\chi_A(\xi))(x) = w_s(x) \prod_{j=1}^n \mathcal{F}\chi_{[-a,a]}(x_j)$, thus

$$w_s \mathcal{F}(\chi_A) \in L_2(\mathbb{R}^n) \quad \text{if, and only if,} \quad s < \frac{1}{2}.$$

(d) $\mathcal{F} f \sim (\mathcal{F} \chi_{[-a,a]})^r$, hence

$$w_s \, \mathcal{F} f \in L_2(\mathbb{R}^n) \quad \text{if, and only if,} \quad s < r - \frac{1}{2}.$$

**Exercise 3.18.** (a) $w_s \, \mathcal{F} \delta \sim \langle x \rangle^{s/2} \in L_2(\mathbb{R}^n)$ if, and only if, $s < -\frac{n}{2}$.

(b), (c) $f \notin L_1^{\text{loc}}(\mathbb{R})$ for $\sigma \geq 0$, and

$$\| w_s \, \mathcal{F} f | L_2(\mathbb{R}) \|^2 \sim \sum_{k=0}^{\infty} 2^{2k(s+\sigma)} < \infty$$

if $s + \sigma < 0$, i.e., $\sigma < |s|$.

(d) Choose $\sigma$ with $0 < \sigma < |s|$, let $f$ be given by (3.37), then

$$g(x) = f(x_1)\psi(x_2, \ldots, x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

with $\psi \in \mathcal{D}(\mathbb{R}^{n-1})$, $\operatorname{supp} \psi \subset [0,1]^{n-1}$, is an example.

**Exercise 3.20.** (b) $f(x) = \min(|x|, 1) \in \operatorname{Lip}(\mathbb{R}^n) \setminus C^1(\mathbb{R}^n)$.

**Exercise 3.29** (b) First estimate $\int_{|h|>1} \cdots$ by $c \, \| f_\gamma | L_2(\mathbb{R}^n) \|$, using $s > 0$, Exercise 3.19 (a) and $f_\gamma \in L_2(\mathbb{R}^n)$.

Secondly, for small $h$, $0 < |h| < 1$, by similar arguments

$$\left( \int_{|x|<2m|h|} |\Delta_h^m f_\gamma(x)|^2 \mathrm{d}x \right)^{1/2} \leq c \, |h|^{\gamma + \frac{n}{2}},$$

for $c$ independent of $h$.

Finally, for $2m|h| < |x| < m + 2$ all differences of $f_\gamma$ are smooth (since $\operatorname{supp}(\Delta_h^m f_\gamma) \subset K_{m+2}$) and can be estimated by their derivatives leading to $|\Delta_h^m f_\gamma(x)| \leq c'|h|^m |x|^{\gamma - m}$; thus

$$\left( \int_{2m|h|<|x|<m+2} |\Delta_h^m f_\gamma(x)|^2 \mathrm{d}x \right)^{1/2} \leq c'' \, |h|^{\gamma + \frac{n}{2}},$$

since $m > \gamma + \frac{n}{2}$, i.e., $\| \Delta_h^m f_\gamma | L_2(\mathbb{R}^n) \| \leq C |h|^{\gamma + \frac{n}{2}}$; in view of $s < \gamma + \frac{n}{2}$ this completes the argument.

**Exercise 3.33.** Let $s = 1$, $\ell = 0$, $n = p = 2$, then there are functions (e.g., $h_\varkappa$ and $g$ as in Exercise 3.6) in $W_2^s(\mathbb{R}^n) = W_2^1(\mathbb{R}^n)$ which are not bounded.

**Exercise 3.36.** (b) Let $f \equiv 1$, then for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\| f - \varphi | C^t(\mathbb{R}^n) \| \geq \| f - \varphi | C(\mathbb{R}^n) \| \geq 1.$$

**Exercise 4.7.** [EE87, Section V.3.1, pp. 222/223].

**Exercise 4.8.** (b) Let $u$ be such that $\|u|L_p(\mathbb{R})\| = 1$, $\|u'|L_p(\mathbb{R})\| \le c$, e.g., $u(x) = c_p(\omega^{1/p}(x-2) - \omega^{1/p}(x+2))$ where $\omega$ is given by (1.58) (with $n = 1$) and $c_p$ appropriately chosen; put

$$u_k(x) = k^{-1/p} u(k^{-1}x),$$

then $\|u_k|L_p(\mathbb{R})\| = \|u|L_p(\mathbb{R})\| = 1$, but

$$\|u_k'|L_p(\mathbb{R})\| = k^{-1}\|u'|L_p(\mathbb{R})\| \le ck^{-1} \to 0 \text{ for } k \to \infty.$$

**Exercise 4.19.** Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \ge 0$, be such that $\|\varphi|W_p^k(\mathbb{R}^n)\| = 1$, hence $\|\varphi|L_p(\mathbb{R}^n)\| > 0$ (take, e.g., $\varphi(x) = c_p\omega(4x)$ with $\omega$ from (1.58) and appropriate $c_p > 0$); let $\varphi_m = \varphi(\cdot - m)$, $m \in \mathbb{Z}^n$, then $\|\varphi_m|W_p^k(\mathbb{R}^n)\| = 1$, $m \in \mathbb{Z}^n$, but

$$\begin{aligned}\|\varphi_m - \varphi_r|W_p^\ell(\mathbb{R}^n)\| &\ge \|\varphi_m - \varphi_r|L_p(\mathbb{R}^n)\| \\ &= 2^{1/p}\|\varphi|L_p(\mathbb{R}^n)\| > 0,\end{aligned}$$

since $\operatorname{supp}\varphi_m \cap \operatorname{supp}\varphi_r = \emptyset$, $m, r \in \mathbb{Z}^n$.

**Exercise 5.8.** If the rotation $\boldsymbol{H}$ and its transpose $\boldsymbol{H}^\top = \boldsymbol{H}^{-1}$ are such that

$$\boldsymbol{H}^\top \boldsymbol{A}\boldsymbol{H} = \begin{pmatrix} d_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & d_n \end{pmatrix}, \quad \boldsymbol{A} = (a_{jk})_{j,k=1}^n,$$

and $d_i > 0$ are the eigenvalues of $\boldsymbol{A}$, then $\xi = \boldsymbol{H}\eta$ and $\eta = \boldsymbol{H}^\top\xi$.

**Exercise 5.20.** $f^\varepsilon(x', x_n) = \psi(x_n)x_n f(x', 0)$ with $\psi \in \mathcal{D}([0, 2\varepsilon])$, $\psi(y) = 1$ for $0 \le y \le \varepsilon$.

**Exercise 5.58.** $g(x^0, x) > 0$ by the same arguments as for Corollary 1.28, but $g(x^0, x) < -\frac{1}{2\pi}\ln|x - x^0|$ only when $\max_{y\in\partial\Omega}|x^0 - y| < 1$, recall also Exercise 1.13; in general we obtain

$$g(x^0, x) \le \frac{1}{2\pi}\left(\max_{y\in\partial\Omega}\ln|y - x^0| - \ln|x - x^0|\right)$$

and

$$g(x^0, x) \ge \frac{1}{2\pi}\left(\min_{y\in\partial\Omega}\ln|y - x^0| - \ln|x - x^0|\right)_+$$

which improves $g(x^0, x) > 0$ if $|x - x^0| < \operatorname{dist}(x^0, \partial\Omega)$.

**Exercise 6.7.** An orthonormal basis $\{h_n\}_{n=1}^{\infty} \subset \mathrm{dom}(A)$ in $\ker(A - \lambda\,\mathrm{id})$ is a Weyl sequence for $\lambda$.

**Exercise 6.9.** [Tri92a, Lemma 4.1.6, Theorem 4.1.6/2, Lemma 4.2.2].

**Exercise 6.14.** [CS90, Section 2.1], [EE87, Proposition II.2.3, Corollary II.2.4].

**Exercise 6.15.** [EE87, Proposition II.1.3], [CS90, Section 1.3].

**Exercise 6.17.** [CS90, Section 2.1].

**Exercise 6.20.** [CS90, Lemma 2.5.2], [EE87, Lemma II.2.9].

**Exercise 7.5.** [ET96, Proposition 3.2.2].

**Exercise 7.9.** For the upper estimates in (7.66) proceed as in Step 1 of the proof of Theorem 7.8, using that obviously

$$h_k(\widetilde{\mathrm{id}}\colon \ell_2^s \hookrightarrow \ell_2^t) \sim h_k(\widetilde{\mathrm{id}}\colon \ell_2^{s-t} \hookrightarrow \ell_2), \quad k \in \mathbb{N}.$$

For the converse apply (7.67), the multiplicativity in Theorem 6.12 (iii), and Theorem 7.8.

**Exercise 7.19.** For $n \le 3$, assume that $2 \le p \le \infty$, and

$$b_1 \in L_{r_1}(\Omega), \quad b_2 \in L_p(\Omega) \quad \text{with } \frac{1}{r_1} = \frac{1}{2} - \frac{1}{p},$$

replacing (7.114). The remaining assumptions (on $\Omega$ and $A$) are the same. Then $B$, given by (7.115), is compact with (7.116), (7.117).

**Exercise 7.22.** For $n = 1$, take $\varphi_j(x) = 2^{\frac{j+1}{2}}\psi(2^j(x^2 - \lambda))$, with $\psi \in \mathcal{D}(\mathbb{R})$ such that

$$\mathrm{supp}\,\psi \subset (1,2) \subset \mathbb{R}, \quad \int_1^2 \psi^2(y)\mathrm{d}y = 1,$$

(e.g. $\psi(y) = \sqrt{\omega(2y-3)}$ and $\omega$ given by (1.58) with $n = 1$); it then follows that $\|\varphi_j|L_2(\mathbb{R})\| = 1$, $\mathrm{supp}\,\varphi_j \cap \mathrm{supp}\,\varphi_k = \emptyset$ for $j \neq k$, and

$$\int_{\mathbb{R}} |x^2 - \lambda|^2 |\varphi_j(x)|^2 \mathrm{d}x \le 2^{2(1-j)} \|\varphi_j|L_2(\mathbb{R})\|^2 \to 0 \quad \text{if } j \to \infty,$$

similarly for $n \in \mathbb{N}$.

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