01.IV.2014г. КА-лекция от средата на ~~втория~~ първия час

Интеграл върху крива

(2) Функция в/у крива

$\gamma : z = \gamma(t) . [a,b] \to \mathbb{C}$

$f : \gamma^* \to \mathbb{C}$

$z \in \gamma^* \quad f(z) := f(\gamma(t)),\ t \in [a,b]$

(3) Деф1: Ако $f : [a,b] \to \mathbb{C}$

$$\int_a^b f := \int_a^b \mathrm{Re} f + i \int_a^b \mathrm{Im} f$$

Деф: Нека $\gamma : [a,b] \to \mathbb{C}$ е гладка крива и $f : \gamma^* \to \mathbb{C}$ е непрекъсната

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\, \gamma'(t)\, dt$$

Ако γ е частично гладка и ~~$\gamma = \bigcup_{k=1} \gamma_k$~~ $\gamma = \bigcup_{k=1}^{n} \gamma_k$

$$\int_\gamma f := \sum_{k=1}^{n} \int_{\gamma_k} f$$

Свойства на $\int_\gamma f$

Деф: Нека ~~γ_1, γ_2~~ $\gamma_1 : z = \gamma_1(t) : [a,b] \to \mathbb{C}$

и $\gamma_2 : z = \gamma_2(\tau) : [c,d] \to \mathbb{C}$ са гладки криви.

Казваме, че двете криви са еквивалентни $\gamma_1 \sim \gamma_2$, ако

$\exists$ ф-я $\tau = \lambda(t)$, т.ч. $\lambda(a) = c$, $\lambda(b) = d$,

$\gamma_2(\lambda(t)) = \gamma_1(t)$, $\forall t \in [a,b]$ и $\exists$ непр. $\lambda'(t) > 0$, $t \in [a,b]$

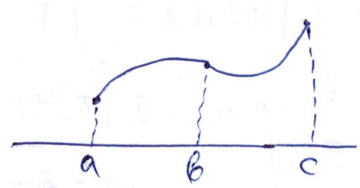
(1) Ако $\gamma_1 \sim \gamma_2$ са гладки криви, то $\int_{\gamma_1} f = \int_{\gamma_2} f$

(2) $\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$, $\forall \alpha, \beta \in \mathbb{C}$

(3) Ако $\gamma_1 : z = \gamma_1(t) ; [a,b] \to \mathbb{C}$

$\gamma_2 : z = \gamma_2(t) : [b,c] \to \mathbb{C}$

$\gamma_1(b) = \gamma_2(b)$



$\gamma_1 \cup \gamma_2$

$$\int_{\gamma_1 \cup \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

По-обща дефиниция:

$$\int_{\gamma_1\cup\gamma_2\cup\ldots\gamma_n} f := \sum_{k=1}^{n} \int_{\gamma_k} f$$

4) $$\int_{\gamma^-} f = -\int_{\gamma} f$$

5) Оценка на $\left|\int_\gamma f\right|$

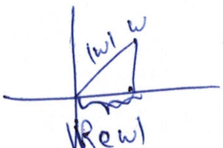
$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|, \qquad \int_\gamma |f(z)|\,|dz| := \int_a^b |f(\gamma(t))|.|\gamma'(t)|\,dt$$

Д-во: Нека $\lambda = \int_\gamma f(z)\,dz$. Имаме, че $\lambda = |\lambda| e^{i\varphi} \Rightarrow |\lambda| = \lambda . e^{-i\varphi}$

Тогава $$\left|\int_\gamma f(z)dz\right| = \mathrm{Re}\left\{\left|\int_\gamma f(z)dz\right|\right\} = \mathrm{Re}\left\{e^{-i\varphi}\int_\gamma f(z)dz\right\} = \mathrm{Re}\left\{\int_\gamma e^{-i\varphi} f(z)dz\right\} =$$

$$= \mathrm{Re}\left\{\int_a^b e^{-i\varphi} f(\gamma(t))\gamma'(t)\,dt\right\} = \int_a^b \mathrm{Re}\left\{e^{-i\varphi} f(\gamma(t))\gamma'(t)\right\}dt$$

(*) $\mathrm{Re}\, w \le |\mathrm{Re}\, w| \le |w|$



$$\overset{(*)}{\le} \int_a^b |e^{-i\varphi} f(\gamma(t))\,\gamma'(t)|\,dt = \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt$$

В частност, ако $|f(z)| \le M,\ z\in\gamma$ то $\left|\int_\gamma f(z)dz\right| \le M\, l(\gamma)$

Примери: 1) $\int_\gamma (z-a)^n dz,\ n\in\mathbb{Z},\ \gamma := \{|z-a| = R\} = \begin{cases} 2\pi i, & n=-1 \\ 0, & n\ne -1\end{cases}$

$\gamma: z = a + Re^{it},\ t\in[0,2\pi]$

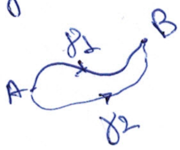
$$\int_\gamma f(z)\,dz = \int_0^{2\pi} R^n e^{int}. R.i.e^{it}\,dt = iR^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt =$$

$$= \begin{cases} n=-1, & 2\pi i \\ n\ne -1, & \frac{R^{n+1}}{n+1}\int_0^{2\pi} de^{i(n+1)t} = \frac{R^{n+1}}{n+1}\left(e^{i2(n+1)\pi} - e^0\right) = 0 \end{cases}$$

2) a

(2) $\int_\gamma z^n \, dz$, $n \geq 0$, $n \in \mathbb{Z}$, $\gamma : [a,b] \to \mathbb{C}$ - гладка, $\gamma(a) = A$, $\gamma(b) = B$.

$$\int_\gamma z^n \, dz = \int_a^b [\gamma(t)]^n \gamma'(t) \, dt = \frac{1}{n+1} \int_a^b d(\gamma(t))^{n+1} = \frac{B^{n+1} - A^{n+1}}{n+1}$$



$\int_{\gamma_1} z^n = \int_{\gamma_2} z^n$

В частност, ако кривата е затворена

$\int_\gamma z^n dz = 0$