08.VII.2013г. Консултация – Табаков

(1) $G = \{(x,y) \mid x,y \in \mathbb{R}\}$

0) $(x_1,y_1)\circ(x_2,y_2) = (x+x_2, y_1e^{-x_2}+y_2e^{x_1}) \in G$

$x_1+x_2 \in \mathbb{R}$ $\quad y_1e^{-x_2}+y_2e^{x_1}\in\mathbb{R}$

$\Uparrow x_1,x_2\in\mathbb{R}$

$G=\{(a,x) \mid a,x\in\mathbb{Q}, a\neq 0\}$

$(a_1,x_1)\circ(a_2,x_2)=(a_1a_2, x_2+a_2x_1)$

$a_1\neq 0, a_2\neq 0 \Rightarrow a_1a_2\neq 0$

$a_1a_2\in\mathbb{Q},\ x_2+a_2x_1\in\mathbb{Q}$

1) асоц.

2) $\exists (a,b)\in G : (a,b)(x,y)=(x,y)(a,b)=(x,y)$

$\left|\begin{array}{l} a+x=x \\ be^{-x}+ye^{a}=y \end{array}\right. \Rightarrow \begin{cases} a=0 \\ \Rightarrow be^{-x}=0 \Rightarrow b=0 \end{cases}$

$(0,0)\in G$

3) $\exists$ обратен в $G$

$(x,y)^{-1}\in G$?

$(x,y)\circ(c,d)=(c,d)\circ(x,y)=(0,0)$

$x+c=0$ $\quad c=-x$ $\quad (-x,-y)$

$ye^{-c}+de^{x}=0$ $\quad d=-y$

0), 1), 2), 3) $\Rightarrow$ G-група относно $\circ$.

б) $H = \{(0,y) \in G \mid y \in \mathbb{R}\} \triangleleft G$ ; $G/H \cong \mathbb{R}$ , $H \cong \mathbb{R}$

$\{\emptyset\} = H \subset G$ $\quad H < G \quad (0,y)^{-1} = (0,-y) \in H$

1) $(0,y_1) \circ (0,y_2) = (0, y_1 + y_2)$

2) $\exists (0,y)^{-1} \in H$

$(0,y) \circ (0,a) = (0,a) \circ (0,y) = (0,0) \Leftrightarrow y + a = 0 \quad a = -y$

1)+2) $\Rightarrow H < G$

$\varphi : H \to \mathbb{R}$

$(H, \circ) \to (\mathbb{R}, +)$

$\varphi((0,y)) = y$

Проверка за хомоморфизъм:

$\varphi[(0,y_1) \circ (0,y_2)] \overset{?}{=} \varphi(0,y_1) + \varphi(0,y_2)$

$(C_3, *)$
$(\mathbb{Z}_3, +)$ $\Big\rangle$ циклични групи

$\varphi(0, y_1 + y_2) = y_1 + y_2$

$y_1 + y_2 = y_1 + y_2$ ✓ $\Rightarrow \varphi$ е ХММ

Биекция, проверка:

Допускаме, че $\left.\begin{array}{l} (0,y_1) \neq (0,y_2) \\ \varphi(0,y_1) = \varphi(0,y_2) \\ y_1 = y_2 \ \lightning \end{array}\right\}$ инекция

$\forall y \in \mathbb{R} \ \exists (0,y) \in H$ – сюрекция

$\Rightarrow \varphi$ е изоморфизъм $H \cong \mathbb{R}$

$G/H \cong \mathbb{R}$

$\psi : G \to \mathbb{R} \qquad \psi(x,y) = x$

$\operatorname{Ker}\psi = \{(x,y) \in G \mid \psi(x,y) = 0\} = H = \{(0,y) \in G \mid y \in \mathbb{R}\}$

$(x_1,y_1) \circ H = (x_2,y_2) \circ H \iff (x_1,y_1)^{-1} \circ (x_2,y_2) \in H$

$(-x_1,-y_1) \circ (x_2,y_2) \in H$

$(\underbrace{-x_1+x_2}_{0}, \underbrace{-y_1 e^{-x_2} + y_2 e^{-x_1}}_{y}) \in H$

$\Updownarrow$

$x_1 = x_2$

$\psi[(x_1,y_1)\circ(x_2,y_2)] = \psi(x_1,y_1) + \psi(x_2,y_2)$

$\Rightarrow \psi$ е ХММ на групи

$\forall x \in \mathbb{R} \ \exists (x,y) \in G \Rightarrow \psi$ е сюрекция

По Th. за ХММ $\Rightarrow G/H \cong \mathbb{R}$, $H \triangleleft G$

(2) $I = (3+2i) \triangleleft \mathbb{Z}[i]$

а) $J = \{a+bi \in \mathbb{Z}[i] \mid 2a-3b \equiv 0 \pmod{13}\}$ $\qquad I = J$

цели Гаусови числа

$\mathbb{Z}[i] = \{a+bi \mid a,b \in \mathbb{Z},\ i$ - имагин. единица$\}$

б) $\mathbb{Z}[i]/I \cong \mathbb{Z}_{13}$

$R,\ a \in R$

$(a) = \{$

Област

$a, b \in D$

$a \neq 0 \qquad ab = 0 \in D$

$b \neq 0$ област за цялост

$\forall$ поле $\Rightarrow$ област за цялост

$\forall$ област $\not\Rightarrow$ поле

а)

$I \subseteq R$

$\forall a,b \in I$

1) $a-b \in I$

2) $ar \in I \qquad \forall r \in R$

$ra \in I$

$M_n(\mathbb{Z})$

$$i \to \begin{pmatrix} & 0 & \\ a_{i1}\ a_{i2} & \dots & a_{in} \\ & 0 & \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1n} \\ - & - & - \\ b_{n1} & \dots & b_{nn} \end{pmatrix} = \begin{pmatrix} & 0 & \\ i_{i1} & \dots & i_{in} \end{pmatrix}$$

$\in I \qquad \in M_n(\mathbb{Z})$

$$\begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \\ b_{n1} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} & a_{1i} & \\ 0 & a_{2i} & 0 \\ & \vdots & \\ & a_{ni} & \end{pmatrix}$$

$(a) = \{ar \mid r \in R\}$

1) $I \subseteq J$

$\underset{\in I}{(3+2i)}\underset{\in \mathbb{Z}[i]}{(a+bi)} = \underbrace{3a-2b}_{A} + \underbrace{(3b+2a)}_{B} i \in I \ \in J \Rightarrow I \subseteq J$

$2A - 3B \equiv 2(3a-2b) - 3(3b+2a) = \cancel{6a} - 4b - 9b \cancel{-6a} = -13b \equiv$

$\equiv 0 \pmod{13}$

2) $J \subseteq I$

$\underbrace{3a-2b}_{A=x} + \underbrace{(3b+2a)}_{B=y} i \in J$

$x + yi \in J \qquad 2x - 3y \equiv 0 (13)$

$\begin{cases} 3a-2b = x \\ 3b+2a = y \end{cases}$ има единствено решение

$a = \frac{x+2b}{3} \qquad b = \frac{y-2a}{3} = \frac{3y-2x-4b}{9}$

$b = \frac{3y-2x}{13} \in \mathbb{Z}$

$\frac{3y-2x}{13} = \frac{y-2a}{3}$

$9y - 6x = 13y - 26a$

$-4y - 6x = -26a$

$\cancel{a = -\frac{1}{2}y + 1}$ $\quad 13a = 2y + 3x$

$a = \frac{2y+3x}{13} \in \mathbb{Z}$

$2x - 3y \equiv 0 \pmod{13}$

$3x + 2y \equiv 0 \pmod{13}$

$\exists! a, b \in \mathbb{Z}$ $3a - 2b = x$
$3b + 2a = y$
$J \ni x + yi = (3+2i)(a+bi) \in I$

$\Rightarrow J \subseteq I$

б) $\boxed{\varphi : \mathbb{Z}[i] \to \mathbb{Z}_{13}}$ - цели числа, остатъци по модул 13

$Ker\,\varphi = \{ a+bi \in \mathbb{Z}[i] \mid \varphi(a+bi) = \bar{0} \} = I = \{ a+bi \in \mathbb{Z}[i] \mid 2a - 3b \equiv 0 (mod\,13) \}$

$a_1 + b_1 i + I = a_2 + b_2 i + I$
$\Updownarrow$
$(a_1 + b_1 i) - (a_2 + b_2 i) \in I$
$\underbrace{(a_1 - a_2)}_{a} + \underbrace{(b_1 - b_2)}_{b} i \in I$

$\Leftrightarrow 2(a_1 - a_2) - 3(b_1 - b_2) \equiv 0 (mod\,13)$
$2a_1 - 3b_1 \equiv 2a_2 - 3b_2 \,(mod\,13)$

$2a - 3b \equiv 0 (mod\,13) \,/.7$
$a + 5b \equiv 0 (mod\,13)$

$\boxed{\varphi(a+bi) = \overline{a+5b}}$

1) $\varphi[(a_1+b_1 i) + (a_2 + b_2 i)] = \varphi(a_1 + b_1 i) + \varphi(a_2 + b_2 i)$
$\varphi(a_1 + a_2 + (b_1 + b_2) i) = \overline{a_1 + 5b_1} + \overline{a_2 + 5b_2}$
$\overline{a_1 + a_2 + 5(b_1 + b_2)} = \overline{a_1 + a_2 + 5(b_1 + b_2)}$ ✓

2) $\varphi[(a_1 + b_1 i)(a_2 + b_2 i)] = \varphi(a_1 + b_1 i) * \varphi(a_2 + b_2 i)$
$\varphi(a_1 a_2 - b_1 b_2 + (a_2 b_1 + a_1 b_2) i) = (\overline{a_1 + 5b_1})(\overline{a_2 + 5b_2})$
$\overline{a_1 a_2 - b_1 b_2 + 5(a_2 b_1 + a_1 b_2)} = \overline{a_1 a_2 + 25(b_1 b_2) + 5(a_1 b_2 + a_2 b_1)}$
$\overline{a_1 a_2 - b_1 b_2 + 5(a_2 b_1 + a_1 b_2)} = \overline{a_1 a_2 - b_1 b_2 + 5(a_2 b_1 + a_1 b_2)}$

$\Rightarrow 1) + 2) \Rightarrow \varphi$ е ХММ върху

$Ker\,\varphi = I$

по Th. за ХММ на пръстени $\Rightarrow \mathbb{Z}[i] / J \cong \mathbb{Z}_{13}$

③ $f = x^5 - 3x^3 + 2x^2 + x - 2 \quad g = x^3 - 3x + 1, \quad F = \mathbb{Q} \quad f = g q_1 + r_1$

$g = r_1 q_2 + r_2 \qquad \deg r_2 < \deg r_1, \; r_2 \neq 0 \qquad \deg r_1 < \deg g, \; r_1 \neq 0$

$r_1 = r_2 q_3 + r_3$

$\vdots$

$r_{k-1} = r_k q_{k+1} + 0 \qquad r_k \to$ старший коеф. 1 – НОД $= (f, g)$

$(f, 0) = f$

$$\begin{array}{l|l} f = x^5 - 3x^3 + 2x^2 + x - 2 & g = x^3 - 3x + 1 \\ -\;x^5 - 3x^3 + x^2 & q_1 = x^2 \\ \hline r_1 = x^2 + x - 2 & \end{array} \qquad F = \mathbb{Q}$$

$\deg r_1 < \deg g$

$$\begin{array}{l|l} g = x^3 - 3x + 1 & r_1 = x^2 + x - 2 \\ -\;x^3 + x^2 - 2x & q_2 = x - 1 \\ \hline -x^2 - x + 1 & \\ -\;-x^2 - x + 2 & \\ \hline r_2 = -1 & \end{array}$$

$\deg r_2 < \deg r_1$

$(f, g) = 1 = -r_2$

$\cancel{(f,g)}\; \exists ?\, u, v \in \mathbb{Q}[x] : uf + vg = (f, g) = 1$

$r_2 = g - q_2 r_1 = g - (x-1) r_1 = g - (x-1)(f - x^2 g) =$

$= -(x-1) f + (1 + x^3 - x^2) g$

$u = x - 1$

$v = -x^3 + x^2 - 1$

4 a) $f(x) = x^n - 1$ $\quad g(x) = x^3 - 3x - 2$ $\quad$ остатък

$f = gq + r$

$\deg r < \deg g$

$r = ax^2 + bx + c$

$g(x) = x^3 - 3x - 2$

решаваме или по Хорнер или Виет

корени на $g$: $x_{1,2} = -1$

$x_3 = 2$

$$2^n - 1 = f(2) = \underbrace{g(2)q(2)}_{0} + r(2) = 4a + 2b + c$$

$$(-1)^n - 1 = f(-1) = \underbrace{g(-1)q(-1)}_{0} + r(-1) = a - b + c$$

$\alpha$ е $k$-кратен корен на $f$

$f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(k-1)}(\alpha) = 0$

$f^{(k)}(\alpha) \neq 0$

$$n(-1)^{n-1} = f'(-1) = \underbrace{g'(-1)q(-1)}_{0} + \underbrace{g(-1)q'(-1)}_{0} + r'(-1) = -2a + b$$

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$$\Sigma = \frac{x_1^2}{3+x_1} + \frac{x_2^2}{3+x_2} + \frac{x_3^2}{3+x_3}$$

$x_1, x_2, x_3$ - корени на $f = x^3 + px + q$

$$\left| \begin{array}{l} x_1 + x_2 + x_3 = 0 \\ x_1x_2 + x_1x_3 + x_2x_3 = p \\ x_1x_2x_3 = -q \end{array} \right.$$

$$\frac{x_1^2}{3+x_1} = \frac{x_1^2 + 9 - 9}{3+x_1} = \frac{(x_1-3)(x_1+3)+9}{3+x_1} =$$

$$= x_1 - 3 + \frac{9}{3+x_1}$$

$$\Sigma = \underbrace{(x_1 + x_2 + x_3)}_{0} \cdot \cancel{(-9)} - 9 + 9\sum_{i=1}^{3} \frac{1}{3+x_i}$$

$\boxed{7.3}$ б) $\displaystyle\sum_{i=1}^{3} \frac{1}{a - x_i} = \frac{f'(a)}{f(a)}$

$f$ - старши коеф. 1

$f(a) \neq 0$ $\quad x_i$ - корен

$$\sum = \frac{x_1^2+2x_2x_3}{x_2+x_3-2x_1} + \frac{x_2^2+2x_1x_3}{x_1+x_3-2x_2} + \frac{x_3^2+2x_1x_2}{x_1+x_2-2x_3} = -\frac{1}{3}(x_1+x_2+x_3) + \frac{2q}{3}\sum_{i=1}^{3}\frac{1}{x_i^2}$$

$x_1 \neq 0$

$$\frac{x_1^2-\frac{2q}{x_1}}{-3x_1} = \frac{x_1^3-2q}{-3x_1^2} =$$

$$f(x_1) = 0 = x_1^3+px_1+q$$

$$x_1^3 = -px_1-q$$

$$= -\frac{x_1}{3} + \frac{2q}{3}\cdot\frac{1}{x_1^2} = \frac{2q}{3}\cdot\frac{p^2}{q^2} =$$

$$= \frac{2p^2}{3q}$$

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2} = \frac{x_1^2x_2^2+x_2^2x_3^2+x_1^2x_3^2}{(x_1x_2x_3)^2} = \frac{}{q^2}$$

$$p^2 = x_1^2x_2^2+x_1^2x_2^2+x_2^2x_3^2+2\underbrace{(x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2)}_{\underbrace{(x_1+x_2+x_3)x_1x_2x_3}_{0}}$$