Mathematical modelling

Additional Lecture Notes

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1 Lyapunov stability

Let us consider the following autonomous system Σ of ordinary differential equations:

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $f : D \to R^n$ is a locally Lipshitz map from a domain $D \subset R^n$ into R^n. Let us assume that the point $\bar{x}$ is an equilibrium point of (1), i.e. $f(\bar{x}) = 0$.

Let us assume that $\bar{x} \neq 0$. Consider the change of the coordinates $y := x - \bar{x}$. Let $x(\cdot)$ be a solution of (1) defined on some open interval I, and let $y(t) := x(t) - \bar{x}$ for each $t \in I$. Then for each $t \in I$ we have that

$$\dot{y}(t) = \dot{x}(t) = f(x(t)) = f(y(t) + \bar{x}) = g(y(y)),$$

where $g(y) := f(y + \bar{x})$. Clearly, $g(0) = f(\bar{x}) = 0$.

In the new variable y the system Σ has equilibrium at the origin. Thus, without loss of generality, we may assume that the function f satisfies $f(0) = 0$, and we shall study the stability of (1) at the origin.

Definition 1.1

The equilibrium point 0 of (1) is

- ♢ stable, if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for each point x_0 with $\|x_0\| < \delta$ the corresponding solution $x(\cdot, x_0)$ of (1) starting from the point x_0 is defined on the interval $[0, +\infty)$ and $\|x(t, x_0)\| \leq \varepsilon$ for each $t \in [0, +\infty)$.
- ♢ asymptotically stable, if it is stable and $\delta > 0$ can be chosen such that

$$\lim_{t \uparrow \infty} x(t, x_0) = 0.$$

Theorem 1.2 *Let the origin be an equilibrium point for (1). Let the function $V : D \to R$ be continuously differentiable on a neighborhood D of the origin and $V(0) = 0$ and $V(x) > 0$ for each $x \in D \setminus \{0\}$. If $\dot{V}_f(x) := V'(x)f(x) \leq 0$, then (1) is stable at the origin. Moreover, if $\dot{V}_f(x) < 0$ for each $x \in D \setminus \{0\}$, then (1) is asymptotically stable at the origin.*

Proof. Let us fix an arbitrary $\varepsilon > 0$. We choose $r \in (0, \varepsilon)$ so that the ball $B_r := \{x \in R^n : \ \|x\| \leq r\} \subset D$. We set $S_r := \{x \in R^n : \ \|x\| = r\}$ and let $\alpha := \min_{x \in B_r} V(x)$. Then $\alpha > 0$ (because $V(x) > 0$ for each $x \in D \setminus \{0\}$). Take $\beta \in (0, \alpha)$ and define the set $\Omega_\beta := \{x \in B_r : \ V(x) \leq \beta\}$. Clearly, $0 \in \Omega_\beta$.

Let us assume that $\Omega_\beta \cap S_r \neq \emptyset$. Then there exists a point $p \in \Omega_\beta \cap S_r$, and hence $V(p) \geq \alpha > \beta \geq V(p)$. The obtained contradiction shows that $\Omega_\beta \cap S_r = \emptyset$, and hence $\Omega_\beta \subset B_r$.

Let x_0 be an arbitrary point of the set Ω_β and let $x(\cdot, x_0)$ be the corresponding solution of (1) starting from x_0. Since

$$V(x(t, x_0)) = V(x_0) + \int_0^t \dot{V}_f(x(s, x_0))\, ds \leq V(x_0)$$

the solution $x(\cdot, x_0)$ of (1) remains in the closed bounded set Ω_β, and hence it is defined on $[0, +\infty)$.

Since the function V is continuous and $V(0) = 0 < \beta$, there exists $\delta > 0$ so that $V(x) < \beta$ for each $x \in B_\delta$, i.e. $B_\delta \subset \Omega_\beta \subset B_r \subset B_\varepsilon$ which shows that the origin is a stable equilibrium point of (1).

Next we prove that the inequality $\dot{V}_f(x) < 0$ for each $x \in D \setminus \{0\}$ implies that (1) is asymptotically stable at the origin. Let x_0 be an arbitrary point of B_δ and let $x(\cdot)$ denote the corresponding solution of (1) starting from x_0. We shall prove that $\lim_{t \uparrow \infty} x(t) = 0$, i.e. for each $\mu > 0$ there exists $T > 0$ such that $\|x(t, x_0)\| < \mu$ for each $t > T$.

Since $V(x(t))$ is monotonically decreasing and bounded from below by zero, $V(x(t)) \to c \geq 0$ as t tends to infinity. Let us assume that $c > 0$. Since $V(0) = 0$ and V is a continuous function, there is $d \in (0, \varepsilon)$ such that $V(x) < c$ for each $x \in B_d$, i.e. $B_d \subset \Omega_c$. Then the relation $V(x(t)) \to c \geq 0$ as t tends to infinity implies that the trajectory $x(t)$ lies outside the ball B_d for each $t \geq 0$. Let

$$-\gamma = \max_{x \in \{y \in B_r : \ \|y\| \geq d\}} V(x).$$

Then it follows that

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}_f(x(s)\, ds \leq V(x_0) - \gamma t.$$

Since the right-hand side of this inequality tends to $-\infty$ as $t \to \infty$, this inequality contradicts the assumption that $c > 0$. Hence $c = 0$, and hence $V(x(t)) \to 0$ as $t \to \infty$.

Let $\mu > 0$ be an arbitrary positive number. By repetition of the previous arguments, there exists $\nu > 0$ such that $\Omega_\nu \subset B_\mu$. Because $V(x(t)) \to 0$ as t tends to infinity, there exists $T > 0$ such that $0 \le V(x(t)) < \nu$ for each $t > T$, i.e. $x(t) \in \Omega_\nu \subset B_\mu$ for each $t > T$. Hence $x(t) \to 0$ as $t \to \infty$ and this completes the proof. $\diamondsuit$

Example 1.3 Consider the first order differential equation

$$\dot{x} = -g(x),$$

where $g : (-\varepsilon, \varepsilon) \to R$ is a locally Lipschitz function such that $g(0) = 0$ and $xg(x) > 0$ for each $x \in (-\varepsilon, \varepsilon)$. It follows from here that $g(x) > 0$ for $x > 0$ and$g(x) < 0$ for $x < 0$.

Let us consider the function $V : (-\varepsilon, \varepsilon) \to R$ which is defined as follows:

$$V(x) = \int_0^x g(y)\, dy.$$

Clearly, V is continuously differentiable, $V(0) = 0$ and $V(x) > 0$ for each $x \neq 0$ (For example, consider the case $x > 0$. Then

$$V(x) = V(0) + \int_0^x g(y)\, dy > 0.$$

The case $x < 0$ can be considered in the same way.) Since

$$\dot{V}_{-g}(x) = -g(x)^2 < 0 \text{ for each } x \in (-\varepsilon, \varepsilon) \setminus \{0\},$$

we may conclude that this system is asymptotically stable at the origin.

Further we denote $\dot{V}_f$ just by $\dot{V}$ when the right-hand side of the equation $\dot{x} = f(x)$ is known.

Example 1.4 Consider the pendulum equation without friction

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -\frac{g}{l}\sin x \end{cases}$$

on the set $\Omega = \{(x, y) : \; x \in (-2\pi, 2\pi)\}$. As a Lyapunov function we consider the total mechanical energy of the pendulum

$$E(x, y) = \frac{1}{2}ml^2y^2 + mgl(1 - \cos x).$$

Direct calculation shows that

$$\dot{E}(x,y) = 12ml^2y(-\frac{g}{l}\sin x) + mgl(\sin x)y = 0.$$

Hence, the pendulum without friction is stable at the origin.

Example 1.5 Consider the pendulum equation with friction

$$\begin{cases} \dot{x} &= & y \\ \dot{y} &= & -\frac{g}{l}\sin x - \frac{k}{m}y \end{cases}$$

on the set $\Omega = \{(x,y):\ x \in (-2\pi, 2\pi)\}$. As a Lyapunov function we consider again the total mechanical energy of the pendulum

$$E(x,y) = \frac{1}{2}ml^2y^2 + mgl(1-\cos x).$$

Direct calculation shows that

$$\dot{E}(x,y) = ml^2y(-\frac{g}{l}\sin x - \frac{k}{m}y) + mgl(\sin x)y = -kl^2y^2 \le 0.$$

Hence, we can conclude that the pendulum with friction is stable at the origin.

Next, let us look for a more general Lyapunov function of the form

$$V(x,y) = \frac{1}{2}(x,y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + mgl(1-\cos x),$$

where $a > 0$ and $ac - b^2 > 0$ (this means that the corresponding quadratic form is positive definite. Clearly,

$$V(x,y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2 + d(1-\cos x).$$

Then

$$\dot{V}(x) = ax\dot{x} + b(\dot{x}y + x\dot{y}) + cy\dot{y} + d\dot{x}\sin x,$$

i.e.

$$\dot{V}(x) = \dot{x}(ax + by + mgl\sin x) + \dot{y}(bx + cy).$$

By simplifying we obtain

$$\dot{V}(x) = y(ax + by + mgl\sin x) + (bx + cy)\left(-\frac{g}{l}\sin x - \frac{k}{m}y\right)$$

and

$$\dot{V}(x) = xy\left(a - b\frac{k}{m}\right) + y^2\left(b - c\frac{k}{m}\right) + y\sin x\left(d - c\frac{g}{l}\right) - x\sin x b\frac{g}{l}.$$

We set $d = \dfrac{k}{m}$, $b = \dfrac{1}{2}\dfrac{k}{m}$, $a = \dfrac{1}{2}\dfrac{k^2}{m^2}$ and $c = 1$. Then we obtain that

$$\dot{V}(x) = -\frac{1}{2}\frac{k}{m}y^2 - \frac{1}{2}\frac{k}{m}\frac{g}{l}x\sin x < 0 \text{ for } (x, y) \neq (0, 0).$$

The term $x \sin x < 0$ for each $|x| < \pi$. Hence, we can conclude that the pendulum with friction is asymptotically stable at the origin.

2 LaSalle's Invariance Principle

Studying the pendulum equation with friction (cf. Example 1.5), we saw that $\dot{E}(x, y) = -(k/m)y^2 \leq 0$. Notice that $\dot{E}$ is negative everywhere except the line $l := \{(x, y) : \ y = 0\}$, where $\dot{E} \equiv 0$. If a trajectory of the considered pendulum stays on l, then $y(t) \equiv 0$. This implies that $\dot{x}(t) = y(t) \equiv 0$. So, that $x(t) \equiv$ constant. Also, $y(t) \equiv 0$ implies that $\dot{y}(t) \equiv 0$, and hence, $\sin x(t) = 0$. Thus on the interval $-\pi < x < \pi$ the equality $\sin x = 0$ is possible only for $x = 0$. Therefore, $E(x(t), y(t))$ degreases to zero as $t \to \infty$.

This arguments shows that if a positive function V is known such that its derivative $\dot{V}$ is negative semidefinite and no trajectory of the system can stay on the set $\{x \in R^n : \dot{V}(x) = 0\}$ except the origin, then the origin is asymptotically stable. This leads to the so called LaSalle invariance principle.

To formulate and prove this principle, we introduce some notions: Let $\varphi(\cdot, x_0)$ be a solution of (1) starting from the point x_0. A point p is said to be a positive limit point of the trajectory $\varphi(\cdot, x_0)$ if there is a sequence $t_n \to \infty$ as $n \to \infty$ such that $\varphi(t_n, x_0) \to p$ as $n \to \infty$. The set of all positive limit points is called positive limit set and is denoted by $\Omega(x_0)$. A set S is said to be (positive, negative) invariant with respect to (1), if $\varphi(t, x_0) \in S$ for each point $x_0 \in S$ and for each real t ($t \geq 0$, $t \leq 0$). We also say that $\varphi(\cdot, x_0)$ approaches the set S as $t \to \infty$, if for each $\varepsilon > 0$ there is $T > 0$ such that $\text{dist}\,(\varphi(t, x_0), S) < \varepsilon$ for each $t > T$. Here $\text{dist}\,(x, S) := \inf(\|x - y\| : \ y \in S)$.

Imagine that a stable limit cycle L exist for a two-dimensional system (1). This stable cycle L is the positive limit set $\Omega(x_0)$ for each point x_0 which is sufficiently closed to L, i.e. the solution $\varphi(t, x_0)$ approaches L as $t \to \infty$.

Notice, that the solution $\varphi(t, x_0)$ does not approach any fixed point of L, i.e. the relation $\varphi(t, x_0) \to L$ as $t \to \infty$ does not imply that $\lim_{t\to\infty} \varphi(t, x_0)$ exists. Also, notice that an equilibrium point and a limit cycle are invariant sets. Also the set $M_c := \{x \in R^n : \ V(x) \leq c\}$ is positive invariant whenever $\dot{V}(x) \leq 0$. Indeed, let x_0 be an arbitrary point of M_c and let $\varphi(\cdot, x_0)$ be the solution of (1) starting from the point x_0. Then we have that

$$V(\varphi(t, x_0)) = V(x_0) + \int_0^t \dot{V}(\varphi(s, x_0))\, ds \leq V(x_0),$$

and hence, the set $M_c := \{x \in R^n : \ V(x) \leq c\}$ is positive invariant.

A fundamental property of the limit sets is given by the following

Lemma 2.1 *Let the solution $\varphi(\cdot, x_0)$ of (1), starting from the point x_0, be bounded. Then the set $\Omega(x_0)$ is nonempty compact connected and invariant with respect to (1). Moreover, $\lim_{t\to\infty} \varphi(t, x_0) = \Omega(x_0)$.*

Proof. Let t_k be an arbitrary sequence tending to $+\infty$ as $k \to +\infty$. Since the sequence $\varphi(t_k, x_0)$ is bounded, there exist a convergent sequence $\{x_{k_j}\}_{j=1}^{\infty}$ of the sequence $\{x_k\}_{k=1}^{\infty}$. Let $\{x_{k_j}\} \to x$ as $j \to \infty$. Then $x \in \Omega(x_0)$. So that the set $\Omega(x_0)$ is nonempty.

Let us choose $b > 0$ so that $\|\varphi(t, x_0)\| \leq b$ for each $t \geq 0$. Let x be an arbitrary point of the set $\Omega(x_0)$. Then there exists a sequence $\{t_k\} \to \infty$ as $k \to \infty$ such that $\varphi(t_k, x_0) \to x$ as $k \to \infty$. From the inequality $\|\varphi(t_k, x_0)\| \leq b$ follows that $\|x\| \leq b$. Because x is an arbitrary point of the set $\Omega(x_0)$, we obtain that $\Omega(x_0)$ is bounded.

Let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence of points belonging to $\Omega(x_0)$ such that $x_k \to x$ as $k \to \infty$. Then for every positive integer k there exists a sequence t_{k_j} tending to $+\infty$ as $j \to \infty$ such that $\varphi(t_{k_j}, x_0) \to x_k$ as $j \to \infty$. Then there exists two positive integers J_1 and J_2 such that $t_{k_j} > k$ for each $j > J_1$ and $\|\varphi(t_{k_j}, x_0) - x_k\| < 1/k$ for each $j > J_2$. Let us set $\tau_k = t_{k_{\bar{j}}}$ for some $\bar{j} > \max(J_1, J_2)$ Then $\tau_k > k$ and $\|\varphi(\tau_k, x_0) - x_k\| < 1/k$ for each positive integer k. Let $\varepsilon > 0$ be arbitrary. Then there exists positive integers N_1 and N_2 such that

$$\|\varphi(\tau_k, x_0) - x_k\| < \frac{\varepsilon}{2} \text{ for each positive integer } k > N_1$$

and

$$\|x_k - x\| < \frac{\varepsilon}{2} \text{ for each positive integer } k > N_2.$$

It follows from here that

$$\|\varphi(\tau_k, x_0) - x\| < \varepsilon \text{ for each positive integer } k > \max(N_1, N_2),$$

and hence $\varphi(\tau_k, x_0) \to x$ as $k \to \infty$. In this way we have proved that the set $\Omega(x_0)$ is closed, and so it is compact (because it is bounded).

Let us assume that the set $\Omega(x_0)$ is not connected. Then there exist two open sets U and V such that

$$\Omega(x_0) \cap U \neq \emptyset,\ \Omega(x_0) \cap V \neq \emptyset,\ U \cap V \neq \emptyset,\ \Omega(x_0) \subset U \cup V.$$

Then there exists two sequences $t_k \to \infty$ and $s_k \to \infty$ as $k \to \infty$ such that

$$\varphi(s_k, x_0) \in U,\ \varphi(t_k, x_0) \in V \text{ and } s_k < t_k < s_{k+1} \text{ for each positive integer } k.$$

Because for each positive integer k the set $\{\varphi(t, x_0) : t \in [s_k, t_k]\}$ is a connected curve going from a point in U to a point in V, there must exist a point $\tau_k \in [s_k, t_k]$ such that $\varphi(\tau_k, x_0) \in R^n \setminus (U \cup V)$. On the other hand, since the sequence $\{\varphi(\tau_k, x_0)\}_{k=1}^{\infty}$ is bounded, there exists a point $\bar{x}$ and a subsequence $\tau_{k_j} \to \infty$ as $j \to \infty$ such that $(\varphi(\tau_{k_j}, x_0) \to \bar{x}$ as $j \to \infty$. Hence the point $\bar{x}$ belongs to $\Omega(x_0)$. But this is impossible, the set $R^n \setminus (U \cup V)$ is closed, and hence the inclusion $\varphi(\tau_{k_j}, x_0) \in R^n \setminus (U \cup V)$ implies that $\bar{x} \in R^n \setminus (U \cup V)$. Then

$$\bar{x} \notin U \cup V \supset \Omega(x_0) \ni \bar{x}.$$

The obtained contradiction shows that the set $\Omega(x_0)$ is connected.

Let x be an arbitrary point of the set $\Omega(x_0)$ and let $\varphi(t, x)$ be the solution of (1) starting from the point x. Since $x \in \Omega(x_0)$, there exists a sequence t_k tending to $+\infty$ as $k \to \infty$ such that $\varphi(t_k, x_0) \to x$ as $j \to \infty$. We set $x_k = \varphi(t_k, x_0)$. By the uniqueness of the solution of (1) one can prove that

$$\varphi(t, x_k) = \varphi(t, \varphi(t_k, x_0)) = \varphi(t + t_k, x_0)$$

Since $t_k \to \infty$ as $k \to \infty$ we have that there exists $K > 0$ such that $t_k + t > 0$ for each positive integer $k > K$. From the continuity of solutions of (1) with respect to the initial point we have that

$$\varphi(t, x) = \lim_{k \to \infty} \varphi(t, x_k) = \lim_{k \to \infty} \varphi(t + t_k, x_0),$$

and hence $x \in \Omega(x_0)$. Hence the set $\Omega(x_0)$ is invariant with respect to the trajectories of (1).

Let us assume that $\varphi(t, x_0)$ does not approach the set $\Omega(x_0)$ as $t \to \infty$. This means that there exists $\varepsilon_0 > 0$ and a sequence $\{t_k\} \to \infty$ as $k \to \infty$ such that dist $(\varphi(t_k, x_0), \Omega(x_0)) \geq \varepsilon_0$ for each positive integer k. Since the sequence $\{\varphi(t_k, x_0)\}_{k=1}^{\infty}$ is bounded, there exists a point $\bar{x}$ and a subsequence $t_{k_j} \to \infty$ as $j \to \infty$ such that $(\varphi(t_{k_j}, x_0) \to \bar{x}$ as $j \to \infty$. Hence the point $\bar{x}$ belongs to $\Omega(x_0)$. But this is impossible, because the distance between $\bar{x}$ and the set $\Omega(x_0)$ is greater than or equal to ε_0. The obtained contradiction shows that $\varphi(t, x_0)$ approaches the set $\Omega(x_0)$ as $t \to \infty$. $\diamondsuit$

Now we can formulate the LaSalle invariance principle:

Theorem 2.2 *Let S be a closed and bounded subset of R^n which is positive invariant with respect to the trajectories of (1). Let $V : S \to R$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ for each point $x \in S$. We set $Z = \{x \in S : \dot{V}(x) = 0\}$ and let M be the largest subset of S which is invariant with respect to the trajectories of (1). Then every solution of (1) starting from a point of S approaches M as $t \to \infty$.*

Proof. Let x_0 be an arbitrary point of S and let $\varphi(\cdot, x_0)$ be the solution of (1) starting from x_0. Because the set S is closed bounded and positive invariant with respect to the trajectories of (1), $\varphi(\cdot, x_0)$ is well defined on $[0, +\infty)$. Since $\dot{V}(x) \leq 0$ for each point $x \in S$, the function $V(\varphi(\cdot, x_0))$ is monotonically decreasing on $[0, +\infty)$. Since V is continuous and the set S is bounded and closed, the set $\{V(\varphi(t, x_0)) : \ t \geq 0\}$ is bounded. Let c denote its exact lower limit, i.e. $c \leq V(\varphi(t, x_0))$ for each $t \geq 0$, and for each positive integer n there exists $t_n \geq 0$ so that $V(\varphi(t_n, x_0)) < c + 1/n$. Hence, $c \leq V(\varphi(t, x_0)) < c + 1/n$ for each $t \geq t/n$ which means that $V(\varphi(t, x_0)) \to c$ as $t \to \infty$. Note that $\Omega(x_0)$ is contained in S (because S is closed and positive invariant with respect to the trajectories of (1). Then for a each point $p \in \Omega(x_0)$ there exists a sequence $\{t_k\} \to \infty$ as $k \to \infty$ such that $\varphi(t_k, x_0) \to p$. The continuity of V implies that $V(p) = V(\varphi(t_k, x_0)) \to c$ as $k \to \infty$. Hence $V(p) = c$ for each $p \in \Omega(x_0)$. Because the set S is invariant with respect to the trajectories of (1), $\dot{V}(p) = 0$ for each $p \in \Omega(x_0)$. Thus

$$\Omega(x_0) \subset M \subset Z \subset S.$$

Since $\varphi(t, x_0)$ is bounded, $\varphi(t, x_0)$ approaches the set $\Omega(x_0)$ as $t \to \infty$. Hence $x(t)$ approaches M as $t \to \infty$. This completes the proof. $\diamondsuit$

Let $V : R^n \to R$ be a continuously differentiable function. It is said that V is radially unbounded if $\lim_{\|x\| \to \infty} V(x) = \infty$. For example, $V_1(x, y) = x^4 + y^4$ is radially unbounded. But the function $V_1(x, y) = (x - y)^4$ is not radially unbounded. If the function V is radially unbounded, then for each real number c the set $S_c := \{x \in R^n : V(x) \leq c\}$ is bounded.

Corollary 2.3 (Barbashin and Krasovskii's theorem) *Let* 0 *be an equilibrium point for (1),* D *be a neighborhood of the origin and* $V : D \to R$ *be a continuously differentiable, positive definite function such that* $\dot{V}(x) \leq 0$ *for each point* $x \in D$*. Let* $Z = \{x \in D : \ \dot{V}(x) = 0\}$*. Assume that the only solution of (1) that can stay forever in* Z *is the trivial solution. Then the origin is asymptotically stable. If, in addition,* V *is radially unbounded, then the origin is globally asymptotically stable.*

Example 2.4 Consider the system

$$\begin{aligned} \dot{x} &= \ y \\ \dot{y} &= \ -g(x) - h(y), \end{aligned}$$

where

$$g(0) = 0, \ \ xg(x) < 0 \text{ for each } 0 \neq x \in (-\varepsilon, \varepsilon)$$

and

$$g(0) = 0, \ \ yh(y) < 0 \text{ for each } 0 \neq y \in (-\varepsilon, \varepsilon)$$

Clearly, the origin is an isolated equilibrium of this system. This system can be considered as a generalized pendulum equation with $h(y)$ as the friction term. Hence, as a Lyapunov function can be taken the following energy-like function:

$$V(x, y) := \int_0^x g(s)ds + \frac{1}{2}y^2$$

We set $D := \{(x, y) : \ x \in (-\varepsilon, \varepsilon), y \in (-\varepsilon, \varepsilon)\}$. Then V is positive definite on D. The derivative of V with respect to the trajectories of this system is

$$\dot{V}(x, y) = yg(x) + y(-g(x) - h(y)) \leq 0.$$

Note that if $\dot{V}(x, y) = 0$ then y must be equal to 0. Hence,

$$Z = \{(x, y) : \ \dot{V}(x, y) = 0\} = \{(x, y) : \ y = 0\}.$$

Assume that $(x(t), y(t))$ is a trajectory belonging to the set Z. Then $y(t) \equiv 0$. From here we obtain that $\dot{x}(t) \equiv 0$, and so, $x(t) \equiv c$, where the constant $c \in (-\varepsilon, \varepsilon)$. On the other hand, $y(t) \equiv 0$ implies that $\dot{y}(t) \equiv 0$, and so, $g(c) = 0$. Since the last equality is possible only for $c = 0$, the origin is asymptotically stable.

Example 2.5 Consider the system

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x^5. \end{aligned}$$

We set $V(x, y) := x^2 + y^2$. Then one can check that

$$\dot{V}(x, y) = -2xy - 2x^4 + 2x^5 y,$$

and hence

$$\dot{V}(\varepsilon, \pm\varepsilon) = \pm\varepsilon^2 - 2\varepsilon^4 \pm \varepsilon^5 = \pm\varepsilon^2(1 \mp \varepsilon^2 + \varepsilon^3).$$

Thus the function V changes its sign if we choose $\varepsilon > 0$ to be sufficiently small. The same conclusion can be made if $V(x, y)$ is chosen to be $V(x, y) = x^2 + \alpha xy + \beta y^2$, where the constants α and β satisfy the inequality: $\alpha^2 - 4\beta < 0$.

If we set $V_\alpha(x, y) = x^6 + \alpha y^2$, then

$$\dot{V}_\alpha(x, y) = -6x^8 + 2(\alpha - 3)x^5 y,$$

and hence for $\alpha = 3$ we obtain that $\dot{V}_3(x, y) = -6x^8 \leq 0$. Note that if $\dot{V}_3(x, y) = 0$ then x must be equal to 0. Hence,

$$Z = \{(x, y) : \dot{V}_3(x, y) = 0\} = \{(x, y) : x = 0\}.$$

Assume that $(x(t), y(t))$ is a trajectory belonging to the set Z. Then $x(t) \equiv 0$. From here we obtain that $\dot{y}(t) \equiv 0$, and so, $y(t) \equiv c$, where c is a constant. On the other hand, $x(t) \equiv 0$ implies that $\dot{x}(t) \equiv 0$, and so, $c = 0$. In this way we obtain that the origin is asymptotically stable, and hence there exists a Lyapunov function.

One can guess that the function $V(x, y) = x^6 + 3y^2 + xy^3$ is a Lyapunov function for this example. To prove this, we use the following

Lemma 2.6 (Young's inequality): *If $a \geq 0$ and $b \geq 0$ then the following inequality holds true:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $p > 1$, $q > 1$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Clearly, the inequality holds true for $b = 0$. Let $b > 0$. We set

$$g(x) := \frac{x^p}{p} + \frac{b^q}{q} - bx.$$

Then one can check that

$$g'(x) = x^{p-1} - b, \ \lim_{x\to 0} g'(x) = -b < 0, \ \lim_{x\to\infty} g'(x) = \infty.$$

Hence, the function $g'(x)$ is increasing and has a global minimum at $b^{1/(p-1)}$. Since $g'(b^{1/(p-1)}) = 0$, the proof is completed.

Let us return to the function $V(x, y) = x^6 + 3y^2 + xy^3$. One can check that

$$\dot{V}(x, y) = (6x^5 + y^3)(-y - x^3) + (6y + 3xy^2)x^5 = -6x^8 - y^4 - x^3y^3 + 3x^6y^2.$$

Applying the Young's inequality for $p = 6$ and $q = 6/5$, we obtain that

$$|x| \ |y|^3 \leq \frac{x^6}{6} + \frac{5}{6}|y|^{18/5} \leq \frac{x^6}{6} + \frac{5}{6}|y|^2$$

for $|y| \leq 1$. Hence

$$V(x, y) \geq x^6 + 3y^2 - |x| \ |y|^3 \geq \frac{x^6}{6}x^6 + \frac{13}{6}y^2,$$

i.e. the function V is positive definite.

Analogously, we apply the Young's inequality for $p = 8/3$ and $q = 8/5$ and obtain that

$$|x|^3 \ |y|^3 \leq \frac{3x^8}{8} + \frac{5}{8}|y|^{24/5} \leq \frac{3x^8}{8} + \frac{5}{8}|y|^4$$

for $|y| \leq 1$.

Also, we apply the Young's inequality for $p = 4/3$ and $q = 4$ and obtain that

$$|x|^6\ |y|^2 \le \frac{3x^8}{4} + \frac{1}{4}y^8 \le \frac{3x^8}{4} + \frac{1}{4}\frac{1}{16}y^4$$

for $|y| \le 1/2$.

Hence

$$\dot{V}(x,y) \le -6x^8 - y^4 + |x|^3|y|^3 + 3x^6y^2 \le -6x^8 - y^4 + \frac{3x^8}{8} + \frac{5}{8}y^4 + \frac{9x^8}{4} + \frac{3}{64}y^4,$$

i.e.

$$\dot{V}(x,y) \le -\frac{27}{8}x^8 - \frac{21}{64}y^4$$

i.e. the function $\dot{V}$ is negative definite. So have proved that V is a Lyapunov function.

3 Barbalat's lemma

This a very useful assertion for studying asymptotic stability:

Definition 3.1 *It is said that $f : R \to R$ is uniformly continuous if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|t_2 - t_1| < \delta$, then $|f(t_2) - f(t_1)| < \varepsilon$.*

Clearly, if $f(\cdot)$ is differentiable and its derivative is bounded, then $f(\cdot)$ is uniformly continuous.

Barbalat's lemma: Let $y : (0, \infty) \to R$ be Riemann integrable and uniformly continuous then

$$\lim_{t\to\infty} y(t) = 0.$$

An equivalent formulation of Barbalat's lemma: Let $f : R \to R$ be differentiable and has a finite limit as $t \to \infty$. If $f'(\cdot)$ is uniformly continuous then $f'(t) \to 0$ as $t \to \infty$.

Further note that the uniform continuity is required to prevent sharp "spikes" that might prevent the limit from existing. For example suppose we add a spike of height 1 and area 2^{-n} at every integer. Then the function is continuous and L^1 (and thus Riemann integrable), but $y(t)$ would not have a limit at infinity.

Proof of Barbalat's lemma. We suppose that $y(t) \not\to 0$ as $t \to \infty$. Then there exists a sequence $\{t_n\}$ in R such that $t_n \to \infty$ as $n \to \infty$ and

$|y(t_n)| \geq \varepsilon$ for all n. By the uniform continuity of y there exists a δ such that, for all n and all $t \in R$

$$|t_n - t| \leq \delta \Rightarrow |y(t_n) - y(t)| \leq \frac{\varepsilon}{2}.$$

So, for all $t \in [t_n, t_n + \delta]$ and for all n we have

$$|y(t)| = |y(t_n) - (y(t_n) - y(t))| \geq |y(t_n)| - |y(t_n) - y(t)| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2},$$

i.e. the sign of $y(t)$ is constant on $[t_n, t_n + \delta]$. Therefore,

$$\left| \int_0^{t_n+\delta} y(t)dt - \int_0^{t_n} y(t)dt \right| = \left| \int_{t_n}^{t_n+\delta} y(t)dt \right| = \int_{t_n}^{t_n+\delta} |y(t)|dt \geq \frac{\varepsilon\delta}{2}.$$

for each n. By the hypothesis, the improprer Riemann integral $\int_0^\infty y(t)dt$ exists, and thus the left hand side of the inequality converges to 0 as $n \to \infty$ yielding a contradiction.

Corollary 3.2 *Let $f : R \to R$ be twice differentiable, has a finite limit as $t \to \infty$ and its second derivative is bounded. Then $f'(t) \to 0$ as $t \to \infty$.*

In general, the fact that derivative tends to zero does not imply that the function has a limit. Take for example the function $f(t) = \sin(\ln(t))$. Then $f(\cdot)$ does not have a limit as $t \to \infty$, while

$$\lim_{t\to\infty} f'(t) = \lim_{t\to\infty} \frac{\cos(\ln(t))}{t} = 0.$$

Also, the converse is not true. Take for example the function $f(t) = e^{-t}\sin(e^{2t})$. Then $f(t) \to 0$ as $t \to \infty$, while

$$\lim_{t\to\infty} f'(t) = \lim_{t\to\infty} -e^{-t}\sin(e^{2t}) + \lim_{t\to\infty} 2e^t\cos(e^{2t})$$

does not exist.

4 A mathematical model of chemostat

The mass balance model in a continuously stirred tank bioreactor is described by the following nonlinear system Σ of ordinary differential equations

$$\frac{d}{dt}s(t) = u(s_{in} - s(t)) - k\ \mu(s(t))\ x(t) \tag{2}$$
$$\frac{d}{dt}x(t) = (\mu(s(t)) - \alpha u)x(t) \tag{3}$$

The state variables s_1, s_2 and x_1, x_2 denote substrate and biomass concentrations, respectively. The input substrate concentrations s_{in}, the dilution rate u, the proportion of dilution rate α (reflecting the process heterogeneity) and the model parameter k are constant.

The function μ is the specific growth rates of the microorganisms. We do not assume to know explicit expressions for the latter, we only impose the following general assumptions on μ:

Assumption A1: The function $\mu(s_i)$ is defined for $s \in [0, +\infty)$, $\mu(0) = 0$, $\mu(s) > 0$ for $s > 0$; $\mu(s)$ is continuously differentiable and bounded for all $s \in [0, +\infty)$.

One can directly verify that the point $(\bar{s}, \bar{x})$ which is determined by the equalities

$$\mu(\bar{s}) = \alpha u,\ \ \bar{x} = \frac{s_{in} - \bar{s}}{\alpha k} \tag{4}$$

is a nontrivial equilibrium point for the system.

Assumption A2: The following inequalities hold true: $0 < \bar{s} < s_{in}$, $\mu(s) < \alpha u$ for each $s \in (0, \bar{s})$ and $\mu(s) > \alpha u$ for each $s \in (\bar{s}, \hat{s}]$ with $\hat{s} > s_{in}$.

Lemma 4.1 (cf. [5]) *Let the Assumptions A1 and A2 be satisfied and (s_0, x_0) be an arbitrary point satisfying the inequalities $x_0 > 0$ and $s_0 > 0$. Then there exists $T > 0$ such that the corresponding solution $(s(t),\ x(t))$ of Σ satisfies the inequality: $s(t) < s_{in}$ for all $t > T$.*

Proof. Notice that $\dot{s}(t) < u(s_{in} - s(t))$ for each $t \geq 0$. Let us assume that $s(t) \geq s_{in}$ for each $t \geq 0$. Then $\dot{s}(t) < u(s_{in} - s(t)) \leq 0$. From here we obtain that $\frac{d}{dt}(s(t) - s_{in}) \leq u(s_{in} - s(t))$ and hence $s(t) \leq s_{in} + e^{-tu}(s_0 - s_{in})$. Thus for each $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that $s_{in} \leq s(t) \leq s_{in} + \varepsilon$ for each $t \geq T_\varepsilon$. The inequality $\mu(s_{in}) > \alpha u$ and the continuity of μ imply the existence of $\varepsilon_0 > 0$ so that $\mu(s) - \alpha u > 0$ for each $s \in (s_{in} - \varepsilon_0, s_{in} + \varepsilon_0)$. Hence

$\mu(s(t)) - \alpha u > 0$ for each $t \geq T_{\varepsilon_0}$, and so $\dot{x}(t) = x(t)(\mu(s(t)) - \alpha u) > 0$ for each $t \geq T_{\varepsilon_0}$. The last inequality implies that $x(t) > x(T_{\varepsilon_0})$ for each $t \geq T_{\varepsilon_0}$. But then $\dot{s}(t) \leq -k\mu(s(t))x(t) < -k\alpha u x(T_{\varepsilon_0}) < 0$ for each $t \geq T_{\varepsilon_0}$. This inequality shows that there exists $\bar{t} > 0$ such that $s(\bar{t}) < s_{in}$. If for some $\hat{t} > \bar{t}$ we have that $s(\hat{t}) = s_{in}$, then from $\dot{s}(\hat{t}) = -k\mu(\hat{t})x(\hat{t}) < 0$ we obtain that $s(\hat{t} + \tau) < s(\hat{t}) = s_{in}$ for each sufficiently small $\tau > 0$ and so, $s(\hat{t} + \tau)$ can not be larger than s_{in}. Thus there exists $T := T_{\varepsilon_0}$ such that $s(t) < s_{in}$ for each $t \geq T$. ♢

Lemma 4.2 (cf. [7]) *Let the Assumption A1 be satisfied and (s_0, x_0) be an arbitrary point satisfying the inequalities $x_0 > 0$ and $s_{in} > s_0 > 0$. Then for any $\varepsilon > 0$, there exists $T > 0$ such that the corresponding solution $(s(t), x(t))$ of Σ satisfies the inequality:*

$$s_{in} - \varepsilon < s(t) + k\ x(t) < \frac{s_{in}}{\alpha} + \varepsilon$$

for all $t > T$.

Proof. We us fix an arbitrary point (s_0, x_0) satisfying the inequalities $x_0 > 0$ and $s_{in} > s_0 > 0$ and denote by $(s(\cdot), x(\cdot))$ the corresponding solution of Σ with initial condition $s(0) = s_0$ and $x(0) = x_0$. Clearly, $s(t) > 0$ and $x(t) > 0$ for each $t > 0$ where the solution is defined. We set

$$p(t) := s(t) + k\ x(t) - \frac{s_{in}}{\alpha}$$

and

$$q(t) := s(t) + k\ x(t) - s_{in}.$$

One can directly check that

$$\begin{aligned} \dot{p}(t) &= u(s_{in} - s(t)) - \alpha k u x(t)) \\ &\leq -\alpha u \left(s(t) + k\ x(t) - \frac{s_{in}}{\alpha} \right) = -\alpha u p(t), \end{aligned}$$

and hence

$$p(t) = s(t) + k\ x(t) - \frac{s_{in}}{\alpha} \leq p(0) \cdot e^{-t\alpha u}.$$

This inequality shows that $p(t)$ is bounded from above. From here and using the fact that the values of $s(t)$ and $x(t)$ are positive, one can easily obtain that the trajectory $(s(t), x(t))$ is well defined and bounded for all $t \geq 0$. Analogously

$$\begin{aligned}\dot{q}(t) &= u(s_{in} - s(t)) - \alpha k u x(t)) \\ &\geq -u\,(s(t) + k\; x(t) - s_{in}) = -\alpha u q(t),\end{aligned}$$

and hence

$$q(t) = s(t) + k\; x(t) - s_{in} \geq q(0) \cdot e^{-tu}.$$

This completes the proof. $\diamondsuit$

Theorem 4.3 *Let the Assumptions A1 and A2 be satisfied. The system Σ is asymptotically stable at the point $(\bar{s}, \bar{x})$ which is determined by (4).*

Proof. Following [2], we set

$$V(s, x) := \int_{\bar{s}}^{s} Q(\xi)\; d\xi + \int_{\bar{x}}^{x} \frac{\zeta - x^*}{\zeta}\; d\zeta,$$

where the smooth function Q will be chosen later. One can directly check that the Lie derivative of V with respect to the trajectories of the system Σ is

$$\begin{aligned}\dot{V}(s, x) = \;& Q(s)[u(s_{in} - s) - k\mu(s)x(s)] + \frac{x - \bar{x}}{x}(\mu(s) - \alpha u)x \\ = \;& [Q(s)u(s_{in} - s) - (\mu(s) - \alpha u)\bar{x}] \\ & + (\mu(s) - \alpha u)x \left[1 - \frac{Q(s)k\mu(s)}{\mu(s) - \alpha u}\right] \\ & .\end{aligned}$$

We set

$$Q(s) := \frac{(\mu(s) - \alpha u)\bar{x}}{u(s_{in} - s)}$$

and obtain

$$\dot{V}(s, x) = (\mu(s) - \alpha u)x \left[1 - \frac{(\mu(s) - \alpha u)\bar{x}}{u(s_{in} - s)} \frac{k\mu(s)}{\mu(s) - \alpha u}\right] \tag{5}$$

$$= (\mu(s) - \alpha u)x \left[1 - \frac{(\mu(s) - \alpha u)\bar{x}}{u(s_{in} - s)} \frac{k\mu(s)}{\mu(s) - \alpha u}\right] \tag{6}$$

$$= (\mu(s) - \alpha u)x \left[1 - \frac{s_{in} - \bar{s}}{s_{in} - s} \frac{\mu(s)}{\alpha u}\right] \tag{7}$$

Since

$$1 - \frac{\mu_1(s)(s_{in} - \bar{x})}{\alpha k(s_{in} - s)} \quad \begin{cases} < 0, & \text{if } 0 < s < \bar{s} \\ > 0, & \text{if } \bar{s} < s \leq s_{in}, \end{cases}$$

we obtain that

$$\dot{V}(s,x) = (\mu(s) - \alpha u)x \left[1 - \frac{s_{in} - \bar{s}}{(s_{in} - s)} \frac{k\mu(s)}{\alpha k u}\right] \leq 0.$$

Hence,

$$Z = \{(s,x) : \ \dot{V}(s,x) = 0\} = \{(s,x) : \ x = 0 \text{ or } s = \bar{s}\}.$$

According to the Assumption A2 we have that $\bar{s} < s_{in}$. Let us choose $\varepsilon > 0$ so small that $3\varepsilon < s_{in} - \bar{s}$. Then there exists $T > 0$ such that for each $t \geq T$ we have that

$$s_{in} - \varepsilon < s(t) + k\ x(t) < \frac{s_{in}}{\alpha} + \varepsilon$$

Let us assume that for some $\bar{t} > T$ we have that $0 < x(\bar{t}) \leq \varepsilon/k$. For the corresponding value of $s(\bar{t})$ we obtain that

$$s(\bar{t}) > s_{in} - \varepsilon - k\ x(\bar{t}) > \bar{s} + 2\varepsilon - \varepsilon = \bar{s} + \varepsilon > \bar{s}$$

Applying again Assumption A2, we obtain that $\dot{x}(\bar{t}) = x(\bar{t})(\mu(\bar{s}(\bar{t}) - \alpha u) > x(\bar{t})(\mu(\bar{s} + \varepsilon) - \alpha u) > 0$. This implies that $x(t) \geq x(\bar{t}) > 0$ for each $t > \bar{t}$ such that $x(t) \leq \varepsilon/k$. Hence the maximal invariant subset of Z is the point $(\bar{s}, \bar{x})$. This completes the proof. $\diamondsuit$

5 Stabilization of the mathematical model of chemostat

Let us consider again the model in a continuously stirred tank bioreactor which is described by the following nonlinear system Σ of ordinary differential equations

$$\frac{d}{dt}s(t) = u(s_{in} - s(t)) - k\ \mu(s(t))\ x(t) \quad (8)$$

$$\frac{d}{dt}x(t) = (\mu(s(t)) - \alpha u)x(t) \quad (9)$$

The state variables s_1, s_2 and x_1, x_2 denote substrate and biomass concentrations, respectively. The input substrate concentrations s_{in}, the dilution rate u, the proportion of dilution rate α (reflecting the process heterogeneity) and the model parameter k are constant. Here we consider u as a control function.

Again, the function μ is the specific growth rates of the microorganisms. We do not assume to know explicit expressions for the latter, we only impose the following general assumptions on μ:

Assumption A1: The function $\mu(s_i)$ is defined for $s \in [0, +\infty)$, $\mu(0) = 0$, $\mu(s) > 0$ for $s > 0$; $\mu(s)$ is continuously differentiable and bounded for all $s \in [0, +\infty)$.

One can directly verify that the point $(\bar{s}, \bar{x})$ which is determined by the equalities

$$\mu(\bar{s}) = \alpha u, \;\; \bar{x} = \frac{s_{in} - \bar{s}}{\alpha k} \tag{10}$$

is a nontrivial equilibrium point for the system.

Assumption A2: The following inequalities hold true: $0 < \bar{s} < s_{in}$, $\mu(s) < \alpha u$ for each $s \in (0, \bar{s})$ and $\mu(s) > \alpha u$ for each $s \in (\bar{s}, \hat{s}]$ with $\hat{s} > s_{in}$.

Assumption A3: The values of the functions $s : [0, \infty) \to R^+$ and $y : [0, \infty) \to R^+$ with $y(t) := \lambda\mu(s(t))x(t)$ are on-line measurable.

We have already discussed assumptions A1 and A2 in the previous section. Assumptions A3 is again very general. Real sensors or numerical estimators can indeed be used to obtain online the quantity s and y. Remark that, for a large part of bioprocesses, the production (or consumption) of gaseous components (O2,CO2...) is monitored and is directly related to the reaction kinetics y.

First, we propose an output feedback controller, that achieves the global asymptotic stabilization of a bioprocess, without any knowledge of its kinetics and with respect to the non-negativity constraint of the input. However, this static controller requires accurate knowledge of the parameters k, s_{in} and λ to achieve asymptotic regulation without error.

Let us denote $s^* \in (0, s_{in})$ the desired set point for substrate concentration. We set $x^* = \dfrac{s_{in} - s^*}{\alpha k}$ and define the following static feedback control law:

$$u(s,x) := \frac{k\mu(s)x}{s_{in} - s^*}.$$

Proposition 5.1 *Let the assumptions A1 and A2 hold true. Then the above written static feedback control law globally stabilizes the model Σ towards the point (s^*, x^*).*

Proof. We substitute the feedback control law $u(s,x)$ in the model Σ and obtain

$$\frac{d}{dt}s(t) = -\frac{k\mu(s(t))x(t)}{s_{in} - s^*}(s(t) - s^*) \quad (11)$$

$$\frac{d}{dt}x(t) = -\frac{\alpha k\mu(s(t))x(t)}{s_{in} - s^*}(x(t) - x^*) \quad (12)$$

From Assumption A1, it is straightforward that the non-negative orthant of the state space is positively invariant by system (16)-(17). Thus, for any positive initial state conditions (that are assumed to be positive throughout the paper), the feedback control law $u(s,x)$ takes only nonnegative values. Integrating the system (16)-(17), one can obtain that

$$\max(x(0), x^*) \geq x(t) \geq \min(x(0), x^*)$$

$$\max(s(0), s^*) \geq s(t) \geq \min(s(0), s^*)$$

Using Assumption A1, we conclude that for any positive initial state conditions and for all non-negative time, the function the feedback control law $u(s,x)$ is bounded below by a positive constant. Considering the closed loop system (16)-(17), it is straightforward to see that (s^*, x^*) is globally exponentially stable.

The static feedback control law $u(s,x)$ proposed above, requires perfect knowledge of the parameters k/λ and s_{in} to perform the stabilization towards the targeted set point without static error. However, identification of these parameters is a difficult task, especially for bioprocesses. To solve this drawback, we propose an adaptive feedback control law. In the sequel, we suppose that the Assumptions A1, A2 and A3 hold true. Moreover, we assume that s^* belongs to the interval $(0, s_{in})$. Let us denote by $z = (s, x, \gamma)$ the new state vector and set $\gamma^* := k/(\lambda(s_{in} - s^*))$. We choose the constants γ_M and γ_m so that the following inequalities hold true $\gamma_m < \gamma^* < \gamma_M$.

Proposition 5.2 *We consider the control system $\bar{\Sigma}$: (8)-(9) and*

$$\dot{\gamma}(t) = -K\lambda\mu(s(t))x(t)(s(t) - s^*)(\gamma(t) - \gamma_m)(\gamma_M - \gamma(t))$$

with initial data $s(0) = s_0 > 0$, $x(0) = x_0 > 0$ and $\gamma(0) = \gamma_0 \in (\gamma_m, \gamma_M)$. Then the following adaptive feedback control law $u(s, x, \gamma) = \lambda\gamma x\mu(s)$ stabilizes asymptotically $\bar{\Sigma}$ towards the point (s^*, x^*, γ^*).

Proof. Substituting the adaptive feedback control law $u(s, x, \gamma)$ in $\bar{\Sigma}$, we obtain the following closed-loop system:

$$\begin{aligned} \frac{d}{dt}s(t) &= \lambda\mu(s(t))x(t)(\gamma(t)(s_{in} - s(t)) - \frac{k}{\lambda}) && (13) \\ \frac{d}{dt}x(t) &= \lambda\mu(s(t))x(t)(\frac{1}{\lambda} - \alpha\gamma(t)x(t)) && (14) \\ \frac{d}{dt}\gamma(t) &= -K\lambda\mu(s(t))x(t)(s(t) - s^*)(\gamma(t) - \gamma_m)(\gamma_M - \gamma(t)) && (15) \end{aligned}$$

One can directly checked that $s(t) > 0$, $x(t) >$ and $\gamma(t) \in (\gamma_m, \gamma_M)$ for each t, where the solution is defined. We make the following time change: $t' := f(t)$ with $f(t) := \lambda \int_0^t \mu(s(\tau))x(\tau)d\tau$. Clearly, $\frac{d}{dt}f(t) = \lambda\mu(s(t))x(t) > 0$, and hence there exists $f^{-1}(t')$. Differentiating the identity $t = f^{-1}(f(t))$, we obtain that

$$1 = \frac{d}{dt'}f^{-1\prime}(f(t)).\frac{d}{dt}f(t), \text{ i.e. } \frac{d}{dt'}f^{-1\prime}(t') = \frac{1}{\lambda\mu(s(f^{-1}(t')))x(f^{-1}(t'))}.$$

Hence

$$\begin{aligned} \frac{d}{dt'}s(t) = \frac{d}{dt'}s(f^{-1}(t')) &= \frac{d}{dt}s(f^{-1}(t'))\ \frac{d}{dt'}f^{-1}(t') = \\ &= \frac{d}{dt}s(f^{-1}(t'))\frac{1}{\lambda\mu(s(f^{-1}(t')))x(f^{-1}(t'))}. \end{aligned}$$

Similarly one can calculate $\frac{d}{dt'}x(t)$. We set $\tilde{s}(t') = s(f^{-1}(t'))$, $\tilde{x}(t') = x(f^{-1}(t'))$ and $\tilde{\gamma}(t') = \gamma(f^{-1}(t'))$. Also, we make the following change of coordinate: $v(t') = s_{in} - \tilde{s}(t')$. Then the system (16)-(18) becomes (denoting with a prime the time derivatives with respect to t' and $v^* = s_{in} - s^*$):

$$\begin{aligned} v'(t') &= \frac{k}{\lambda} - \gamma(t')v(t') = \gamma^*v^* - \gamma(t')v(t') && (16) \\ \tilde{x}'(t') &= \frac{1}{\lambda} - \alpha\gamma(t')x(t') && (17) \\ \tilde{\gamma}'(t) &= K(v(t') - v^*)(\gamma(t') - \gamma_m)(\gamma_M - \gamma(t')) && (18) \end{aligned}$$

If $v(t') \leq 0$, then $v'(t') \geq \gamma^* v^* > 0$. This observation shows that there exists $t'_0 > 0$ with $v'(t'_0) > 0$. Let us consider the subsystem $\tilde{\Sigma}$ of $\bar{\Sigma}$ (with dynamics determined by the differential equations (16) and(18)) on the set $S := \{(v, \gamma) : v > 0, \gamma \in (\gamma_m, \gamma_M)\}$. Clearly, the set S is a bounded set which is positively invariant with respect to the trajectories of $\tilde{\Sigma}$.

We set

$$W(v,\gamma) := \int_{v^*}^{v} \frac{w - v^*}{w} dw + \int_{\gamma^*}^{\gamma} \frac{\xi - \gamma^*}{K(\xi - \gamma_m)(\gamma_M - \xi)} d\xi,$$

We check that $W(v,\gamma)$ is defined, non-negative on the set S and vanishes only for $v = v^*$ and $\gamma = \gamma^*$. Furthermore, one can check that

$$\dot{W}(v,\gamma) = W'_v(v,\gamma)(\gamma^* v^* - \gamma v) + W'_\gamma(v,\gamma)(K(v - v^*)(\gamma - \gamma_m)(\gamma_M - \gamma)) =$$

$$= \frac{v - v^*}{v}(\gamma^* v^* - \gamma v) + \frac{\gamma - \gamma^*}{K(\gamma - \gamma_m)(\gamma_M - \gamma)}(K(v - v^*)(\gamma - \gamma_m)(\gamma_M - \gamma)) =$$

$$= \frac{v - v^*}{v}(\gamma^* v^* - \gamma v) + (\gamma - \gamma^*)(v - v^*) =$$

$$= \frac{v - v^*}{v}(\gamma^* v^* - \gamma v + \gamma v - \gamma^* v) = -\gamma^* \frac{(v - v^*)^2}{v} \leq 0,$$

and $\dot{W}(v,\gamma) = 0$ only for $v = v^*$.

According to Lasalle's invariance principle we obtain that every solution trajectory of $\tilde{\Sigma}$ approaches the largest invariant subset M of the set $\{(v,\gamma) : v = v^*,\ \gamma \in (\gamma_m, \gamma_M)\}$. Now, consider a trajectory starting from (v^*, γ_0) with $\gamma \in (\gamma_m, \gamma_M)$ and $\gamma \neq \gamma^*$. It is clear that this trajectory escapes from and therefore that the largest invariant set in the set S is the fixed point (v^*, γ^*). Then (v^*, γ^*) is a globally attractive fixed point for system $\tilde{\Sigma}$.

A straightforward Jacobian matrix computation at the point (v^*, γ^*) proves that this fixed point is locally stable too. Then, we can conclude that (v^*, γ^*) is a globally asymptotically stable fixed point for system $\tilde{\Sigma}$.

Next we study the behavior of the original system Σ on the set $\Omega = \{(v^*, x, \gamma^*) :\ x > 0\}$. The dynamics is linear, has a single equilibrium for $x^* = \dfrac{1}{\alpha\lambda\gamma^*}$ and is globally asymptotically stable at x^*. To finish the proof, we apply the main of [1] for autonomous system.

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