18. 12. 2013 г.

Упражнение

$r(t) = (x(t), y(t))$

$v(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$

$\dot{v}(t) = (\ddot{x}(t), \ddot{y}(t), \ddot{z}(t))$

m_0 е маса на покой за дад. мат. точка

$$m\ddot{r}(t) = \vec{F} = \begin{pmatrix} \frac{d}{dt}(m\dot{x}) \\ \frac{d}{dt}(m\dot{y}) \\ \frac{d}{dt}(m\dot{z}) \end{pmatrix}$$

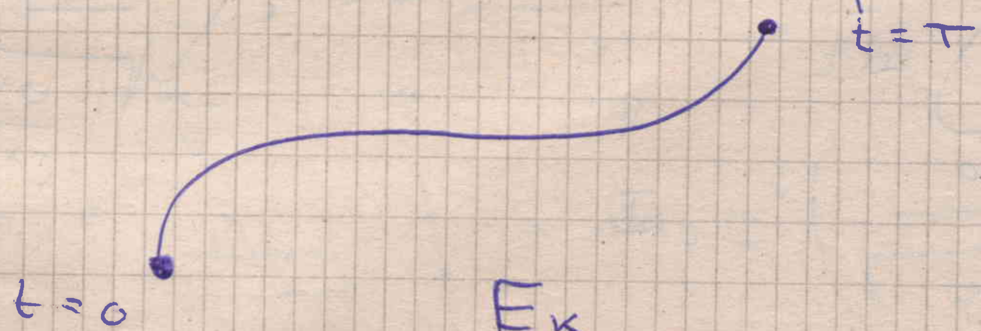
$$\frac{d}{dt}(mv) = \vec{F}$$

скорост на нарастване на импулса

В класическата механика $m = m_0 = const$
(скорости, много по-малки от c)
В рел. мех ($v \sim c$)

$$m(t) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad v^2 = x^2 + y^2 + z^2 \; (\langle v, v \rangle)$$

Нека мат. точка се движи от т. $(0,0,0)$ ($v = 0$) до т. (X, Y, Z) със скорост V.



$E_к$

$$E_k = \int_\Gamma \langle F, dr \rangle = \int_0^T \langle \frac{d}{dt}(mv), v \rangle dt =$$

\# Ако x и y са вектори

$$\frac{d}{dt}\langle x, y \rangle = \langle \frac{d}{dt}x, y \rangle + \langle x, \frac{d}{dt}y \rangle$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$ - релативистки фактор (Айнщайн)

$$\frac{d}{dt}\langle mv, v \rangle = \langle \frac{d}{dt}(mv), v \rangle + \langle mv, \frac{d}{dt}v \rangle$$

$$\Rightarrow \langle \frac{d}{dt}(mv), v \rangle = \frac{d}{dt}\langle mv, v \rangle - \langle mv, \frac{d}{dt}v \rangle$$

$$\Rightarrow E_k = \int_0^T \left(\frac{d}{dt}\langle mv, v \rangle - \langle mv, \frac{d}{dt}v \rangle \right) dt =$$

$$= \underbrace{\int_0^T \frac{d}{dt}\langle mv, v \rangle dt}_{I_1} - c^2 \underbrace{\int_0^T \frac{m_0 (\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z})}{c^2\sqrt{1-\frac{v^2}{c^2}}} dt}_{I_2}$$

$$\frac{d}{dt}\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2\sqrt{1-\frac{v^2}{c^2}}}\left(-\frac{\frac{d}{dt}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)}{c^2}\right) =$$

$$= \frac{-1}{2\sqrt{1-\frac{v^2}{c^2}}}\,\frac{\left(2\dot{x}\ddot{x}+2\dot{y}\ddot{y}+2\dot{z}\ddot{z}\right)}{c^2}$$

$$E_x = \int_0^T \frac{d}{dt}\left[m v^2 + c^2 m_0\sqrt{1-\frac{v^2}{c^2}}\right]dt =$$

$$= \int_0^T \left[\frac{m_0 \cdot v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_0 c^2\sqrt{1-\frac{v^2}{c^2}}\right]dt =$$

$$= \int_0^T \frac{d}{dt}\left[\frac{m_0 v^2 + m_0 c^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}\right]dt =$$

$$= \int_0^T \frac{d}{dt}\left[\frac{m_0 v^2 + m_0 c^2 - v^2 m_0}{\sqrt{1-\frac{v^2}{c^2}}}\right]dt =$$

$$= m_0 c^2 \left.\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right|_0^T = m_0 c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

$$\boxed{= mc^2 - m_0c^2}$$

пълна енергия

$$\Rightarrow \boxed{E_k = mc^2 - m_0c^2}$$ — енергия на покой.

$$mc^2 = E_k + \underbrace{m_0c^2}_{\text{„}E_п\text{"}}$$

„потенциална"

1.5 Намерете дължината на кривите

1) $x = \dfrac{b + a\cos\psi}{2}$ $\quad y = \dfrac{(b + a\cos\psi)\sqrt{3}}{2}$

$z = a\sin\psi$

$$0 \le \psi \le 2\pi$$

$$L = \int_{\Gamma} 1\, d\sigma = \int_0^{2\pi} 1.\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, d\psi$$

$$\dot{x} = -\frac{a}{2}\sin\psi$$

$$\dot{y} = -\frac{a\sqrt{3}}{2}\sin\psi$$

$$\dot{z} = a\cos\psi$$

$$= \int_0^{2\pi} \sqrt{\frac{a^2}{4}\sin^2\psi + \frac{a^2 3}{4}\sin^2\psi + a^2\cos^2\psi} =$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2\psi + a^2\cos^2\psi}\, d\psi =$$

$$= 2\pi \cdot a$$

I род

Зад.

$$\int_L x^{\frac{4}{3}} + y^{\frac{4}{3}}\, dl \quad L: x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

$$a > 0$$

$$x = \cancel{\ } a\cos^3\varphi$$

$$y = a\sin^3\varphi$$

$$L: a^{\frac{2}{3}}\cos^2\varphi + a^{\frac{2}{3}}\sin^2\varphi = a^{\frac{2}{3}}$$

$$\varphi \in [0; 2\pi]$$

$$\dot{x} = -a\,3\cos^2\varphi\sin\varphi$$

$$\dot{y} = 3\sin^2\varphi\, a\cos\varphi$$

$$I = \int_0^{2\pi} a^{\frac{4}{3}}(\cos^4\varphi + \sin^4\varphi)\sqrt{9a^2(\sin^2\varphi\cos^4\varphi + \sin^4\varphi\cos^2\varphi)}$$

$$= \int_0^{2\pi} \frac{a^{\frac{4}{3}}}{2} \left(\cos^4\varphi + \sin^4\varphi\right) \cdot 3a\sqrt{\sin^2\varphi\cos^2\varphi}\, d\varphi =$$

$$= \frac{3a^{\frac{7}{3}}}{2} \int_0^{2\pi} \left(\cos^4\varphi + \sin^4\varphi\right) \sqrt{\sin^2(2\varphi)}\, d\varphi =$$

$$= \frac{3a^{\frac{7}{3}}}{\cancel{2}} \int_0^{2\pi} \left(\cos^4\varphi + \sin^4\varphi\right) \left|\sin\varphi\cos\varphi\right| =$$

$$= 3a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} \underbrace{\left(\cos^4\varphi + \sin^4\varphi\right)\sin\varphi\cos\varphi}_{\phi(\varphi)}\, d\varphi$$

$$+ 3a^{\frac{7}{3}} \int_{\pi}^{\frac{3\pi}{2}} \phi(\varphi)\, d\varphi - 3a^{\frac{7}{3}} \int_{\left[\frac{\pi}{2};\pi\right] \cup \left[\frac{3\pi}{2};2\pi\right]} \phi(\varphi)\, d\varphi$$

$$I_1 = \int_0^{\frac{\pi}{2}} \cos^5\varphi \sin\varphi\, d\varphi + \int_0^{\frac{\pi}{2}} \sin^5\varphi\cos\varphi\, d\varphi \ldots$$

Зад. $\int_L \operatorname{arctg}\frac{y}{x}$ $\quad L: r(\varphi) = \varphi$ спирала

$x^2 + y^2 \leq \frac{\pi^2}{16}$

параметризация на спирала

$$\boxed{\begin{aligned} x &= \varphi\cos\varphi \\ y &= \varphi\sin\varphi \end{aligned}} \qquad 0 \le \varphi \le \frac{\pi}{4}$$

<u>зад.</u>

$$\int_{(1,2)}^{(3,4)} x\,dx + y\,dy$$

$$F(x,y) \qquad \frac{x^2}{2} + \frac{y^2}{2}$$

$$F_x = x$$
$$F_y = y$$

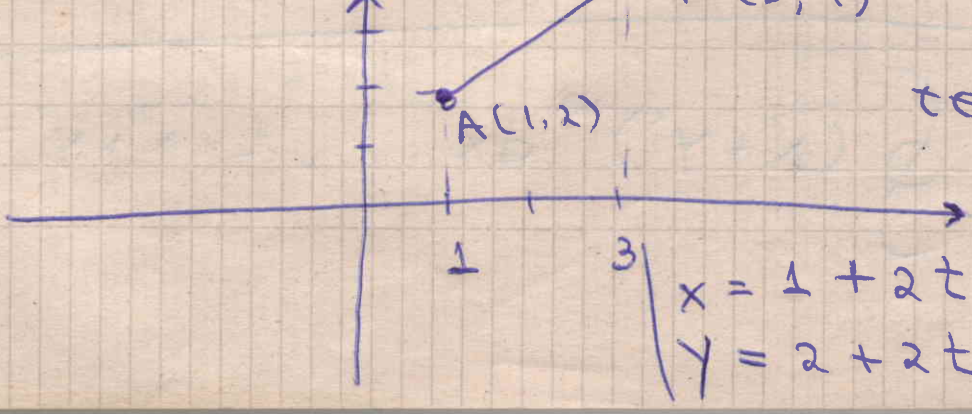
$$F(x,y) = \frac{x^2}{2} + \frac{y^2}{2} \quad \text{потенциал}$$

$$= F(3,4) - F(1,2) = \frac{9}{2} + \frac{16}{2} - \frac{1}{2} - \frac{4}{2} =$$
$$= 10$$

<u>зад.</u>

$$\int_L xy\,dx + y^2\,dy$$

Отс.
L: (1, 2) (3, 4)



$t \in [0, 1]$

$$\left| \begin{aligned} x &= 1 + 2t \\ y &= 2 + 2t \end{aligned} \right.$$

$\dot{x} = 2$
$\dot{y} = 2$

не е потенциално

$$I = 2\int_0^1 (1+2t)^2 \sqrt{2^2+2^2}\, dt =$$

$$= 4\sqrt{2}\int (1+4t+4t^2)\, dt$$

$$I = \int_0^1 (1+2t)(2+2t).2 + (2+2t)^2 .2\, dt =$$

$$= 4\int_0^1 (1+2t)(1+t) + 2(1+t)^2\, dt =$$

$$= 4\int_0^1 (1+t)(1+2t+2+2t)\, dt =$$

$$= 4\int_0^1 (1+t)(2+3$$

Зад. $\oint_L (x^2+y^2)^{10}\, dx \quad , L: x^2+y^2=R^2$

$x = R\cos\varphi$ $\qquad \varphi \in [0; 2\pi]$

$y = R\sin\varphi$

обратно

$x = R\cos(2\pi-\varphi)$

$y = R\sin(2\pi-\varphi)$

$\varphi \in [0; 2\pi]$

края
смение аргумента

$$I = \int_0^{2\pi} R^{20}(-R)\sin\varphi \, d\varphi = 0$$

зад.

$$\int \frac{yz\,dx + xz\,dy + xy\,dz}{1 + x^{\bullet} y^{\bullet} z^{\bullet}}$$

$x = \cos t$ $\qquad 0 \le t \le 2\pi$

$y = \sin t$

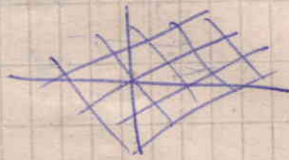
$z = t$

$$\ln(1+xyz)' = \frac{1}{1+xyz} \cdot yz$$

$$xyz = \cos t \sin t \cdot t = \frac{\sin 2t}{2}\, t$$

$$t \in \{0; 2\pi]$$

$$\sin 2t = \frac{-2}{t}$$

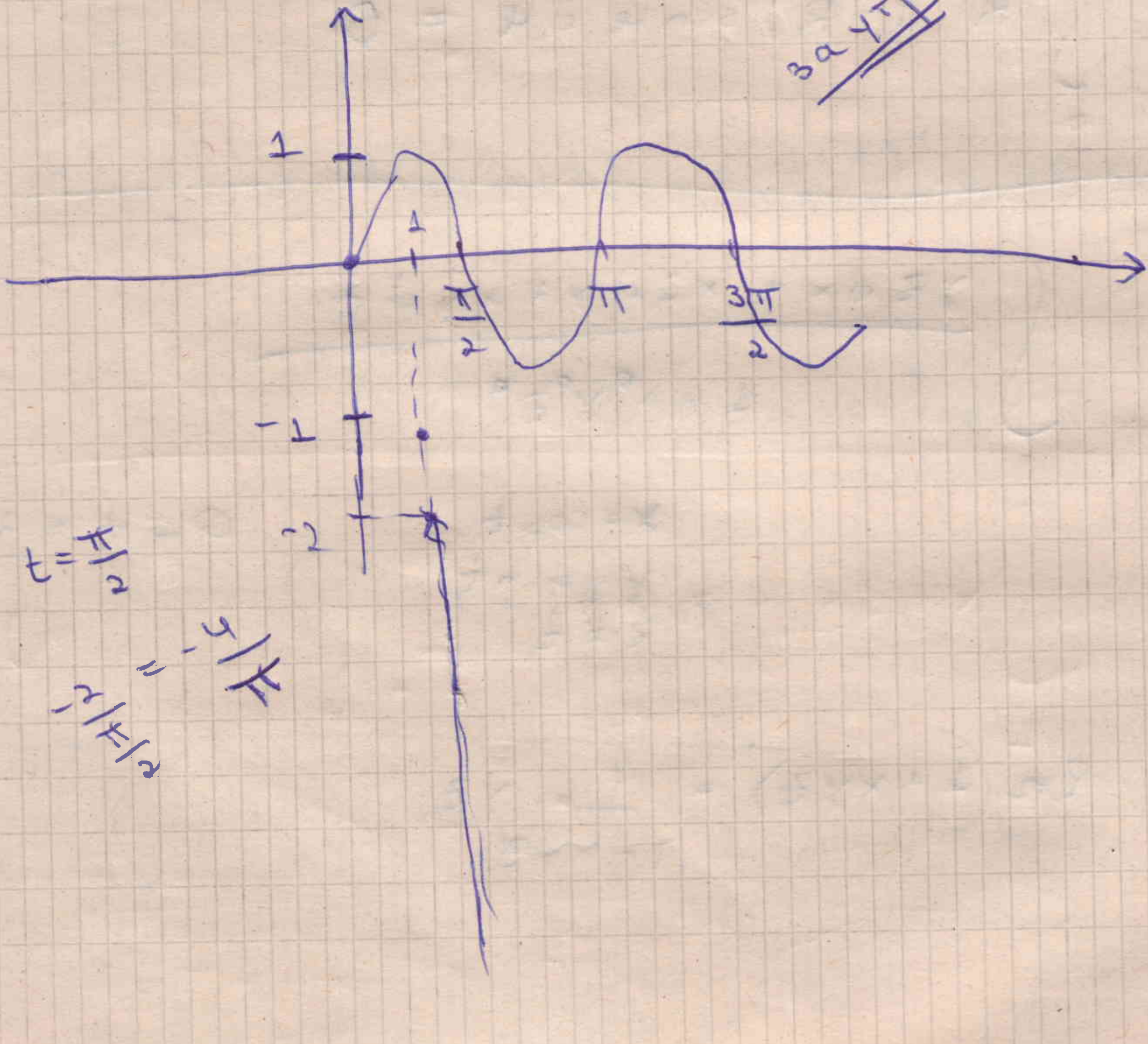


$$\sin 2t \cdot t = -2$$

$$t \sin 2t + 2$$

$$\sin 2t + t \cos 2t \cdot t$$

заутре



$\dot{x} = -\sin t$

$\boxed{\text{зад. } 2.22\text{e)}}$

$$\int_L \frac{yz\,dx + xz\,dy + xy\,dz}{1 + x^2y^2z^2}$$

$x = \cos t \qquad \dot{x} = -\sin t$

$y = \sin t \qquad \dot{y} = \cos t$

$z = t \qquad \dot{z} = 1$

$$\int_0^{2\pi} \frac{-\sin t\, \overset{t}{\cancel{\cos t}} \sin t + t\cos^2 t + \cos t \sin t\, 1}{1 + t^2\sin^2 t\cos^2 t}\,dt$$

$$= \cancel{\int_0^{2\pi} \frac{\cos t\,(-\sin^2 t + t\cos^2 t + \sin^2 t)}{1 + \sin^2 t\cos^2 t}\,dt}$$

$$= \int_0^{2\pi} \frac{t\left(\cos 2t + \frac{\sin 2t}{2}\right)}{1 + t^2\frac{\sin^2 2t}{4}}\,dt$$

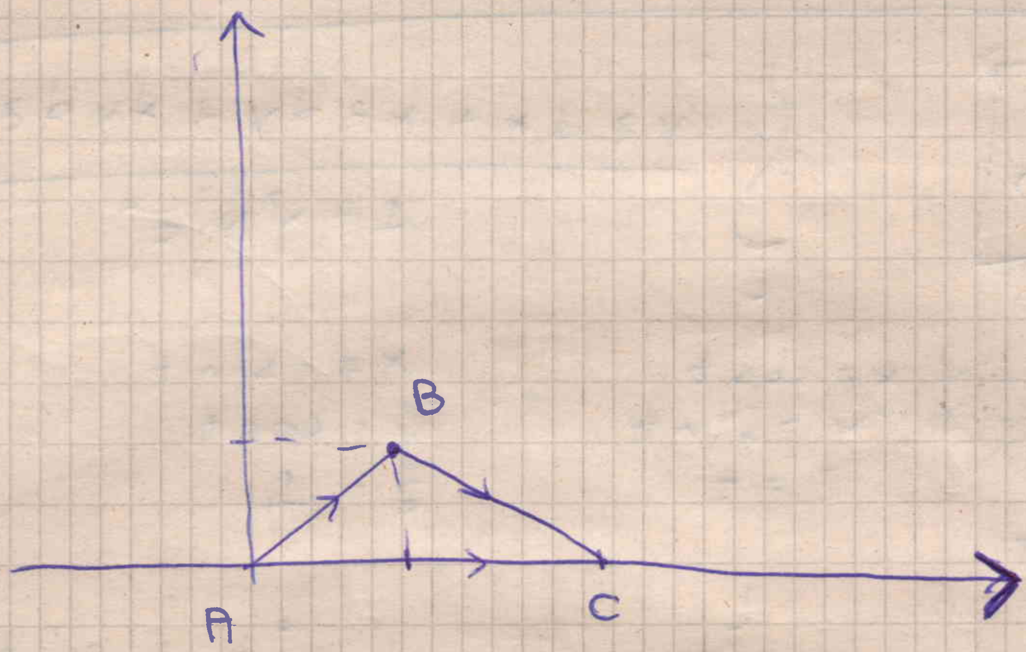
зад.

$$\oint \frac{xy\,dx}{1+x} - \ln\,(1+x)\,dy$$

$C_3 \longrightarrow \triangle ABC$

$A(0,0) \qquad C(2,0)$

$B(1,1)$



Формулата на Грийн

$$I = -\iint_D \frac{\partial}{\partial x}\left(-\ln(1+x)\right) - \frac{\partial}{\partial y}\left(\frac{xy}{1+x}\right) dx\,dy =$$

* D е вътрешността на Δ-ка

$$= -\iint_D -\frac{1}{1+x} - \frac{x}{1+x}\, dx\,dy =$$

$$= \iint_D dx\,dy = S(D) = 1$$

зад.
$$\oint_{C_1} \operatorname{arctg} y\, dx - \frac{xy^2}{1+y^2}\, dy$$

C_1: единичният кръг

$$x^2+y^2 \le 1 \qquad dx\,dy$$

$$I = \iint_D \frac{\partial}{\partial x}\left(\frac{-xy^2}{(1+y^2)}\right) - \frac{\partial}{\partial y}\left(\operatorname{arctg} y\right)$$

$$\Rightarrow \quad I = \iint_{x^2+y^2<1} \frac{-y^2}{1+y^2} - \frac{1}{1+y^2} = -\pi$$

зад.
$$\oint_{C_1} \overset{P}{2(2+x)y}\, dx + \overset{Q}{(2x+x^2)}\, dy$$

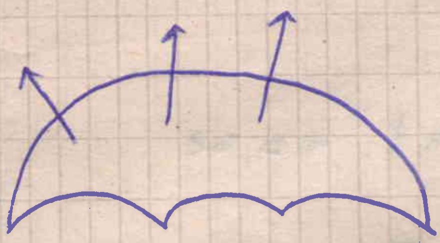
$$2(2+x) \overset{?}{=} 2+2x$$

$$[\text{illegible}] \qquad 0$$

$$\cancel{I_D} = \overbrace{\oint 2(1+x)y\,dx + (2x+x^2)\,dy + \underbrace{\oint_{C_1} 2y\,dx}}$$

$$\oint_{C_1} 2y\,dx \overset{+0dy}{=} \iint_{\substack{D\\ x^2+y^2\le 1}} \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}2y = -2\pi$$

Лекция



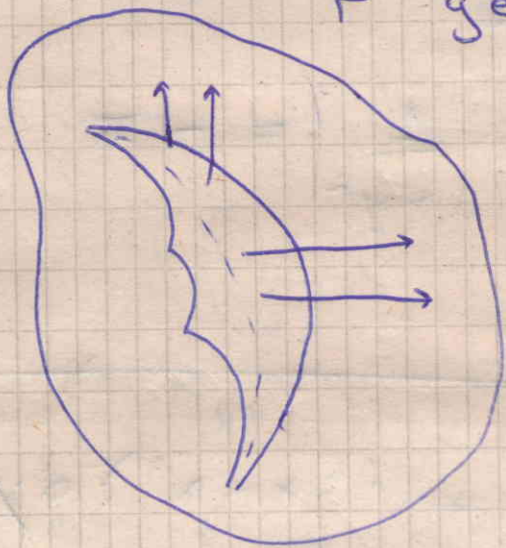
ориентация

Повърхнинен интеграл от II род

(S, n)

F деф. в околност на S

флуид



$$\int_S \langle F, n \rangle \, ds$$

$\varphi : \Omega \longrightarrow \mathbb{R}^3$, φ - инекция

$K \subset \Omega \subset \mathbb{R}^2$, $\varphi \in C^1$, $\operatorname{rg} \varphi' = 2$

област

измерим компакт

$$S = \varphi(K)$$

+ - ориентация, съгласувана с параметризацията

$$n(\varphi(u)) = \frac{\varphi'_{u_1}(u) \times \varphi'_{u_2}(u)}{\|\varphi'_{u_1}(u) \times \varphi'_{u_2}(u)\|}$$

$$\int_S \langle F, n \rangle \, ds$$ F - непрекъснато

$$\| $$

$$\int_K \langle F(\varphi(n)), n(\varphi(n)) \rangle \cdot \sqrt{\det(\varphi'^T(u)\varphi(u)} \; du =$$

от $\circledast$: $\|\varphi'_{u_1}(u) \times \varphi'_{u_2}(u)\|$

$$= \int_K \langle F(\varphi(u)), \frac{\varphi'_{u_1}(u) \times \varphi'_{u_2}(u)}{\|\varphi'_{u_1}(u) \times \varphi'_{u_2}(u)\|} \rangle \cdot \|\varphi'_{u_1}(u) \times \varphi'_{u_2}(u)\|$$

$\varphi'_{u_1}(u) = a$

$\varphi'_{u_2}(u) = b$

$\circledast$

$$\boxed{\sqrt{\langle a, a \rangle \langle b, b \rangle - \langle a, b \rangle^2} = \|a \times b\|}$$

$$\Rightarrow = \int_K \langle F(\varphi(u)) \cdot \varphi'_{u_1}(u) \times \varphi'_{u_2}(u)) \rangle \, du$$

двоен $\int$

$\boxed{\text{поток на полето } F \text{ през ориентираната повърхнина } S}$

$$\int_S \langle F, n \rangle = \int_K \langle F(\varphi(u)), \varphi'_{u_1}(u) \times \varphi'_{u_2}(u) \rangle \, du$$

$$\varphi'_{u_1} = \left(\frac{\partial \varphi_1}{\partial u_1}, \frac{\partial \varphi_2}{\partial u_1}, \frac{\partial \varphi_3}{\partial u_1} \right)$$

$$\varphi'_{u_2} = \left(\frac{\partial \varphi_1}{\partial u_2}, \frac{\partial \varphi_2}{\partial u_2}, \frac{\partial \varphi_3}{\partial u_2} \right)$$

$$= \int_K \Bigg(F_1(\varphi(u)) \cdot \left(\frac{\partial \varphi_2}{\partial u_1} \frac{\partial \varphi_3}{\partial u_2}(u) - \frac{\partial \varphi_3}{\partial u_1} \frac{\partial \varphi_2}{\partial u_2}(u) \right) +$$

$$+ F_2(\varphi(u)) \cdot \left(\frac{\partial \varphi_3}{\partial u_1} \cdot \frac{\partial \varphi_1}{\partial u_2}(u) - \frac{\partial \varphi_3}{\partial u_2} \frac{\partial \varphi_1}{\partial u_1} \right) =$$

$$+ F_3(\varphi(u)) \cdot \left(\frac{\partial \varphi_1}{\partial u_1} \frac{\partial \varphi_2}{\partial u_2} - \frac{\partial \varphi_2}{\partial u_1} \cdot \frac{\partial \varphi_1}{\partial u_2} \right) \Bigg) du$$

$$= \int_S F_1 \, dx_2 \, dx_3 + F_2 \, dx_3 \, dx_1 + F_3 \, dx_1 \, dx_2$$

$$x_i = \varphi_i(u)$$

$$dx_i \, dx_j = \begin{vmatrix} \frac{\partial \varphi_i}{\partial u_1} & \frac{\partial \varphi_i}{\partial u_2} \\ \frac{\partial \varphi_j}{\partial u_1} & \frac{\partial \varphi_j}{\partial u_2} \end{vmatrix} (u) \, du_1 \, du_2$$

Пример:

$$F = (0, 0, F_3)$$

$$\varphi(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix} \qquad (x,y) \in K$$

$$\int_S \langle F, n \rangle = \int_K F_3(x, y, f(x,y)) \cdot dx\,dy$$

:))

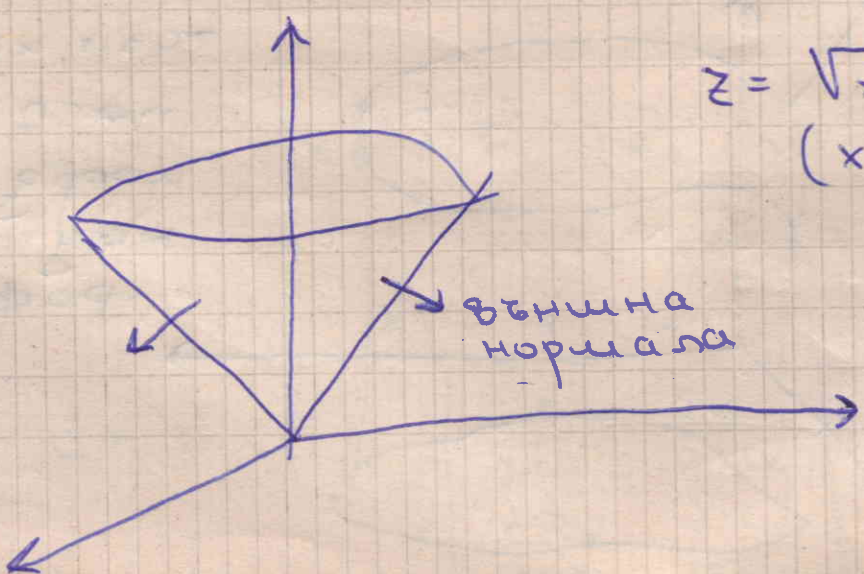
зад. $F(x,y,z) = (0,0,z)$

$$\iint_S z\,dx\,dy$$

S конус

$$\begin{cases} z^2 = x^2 + y^2 \\ 0 \le z \le 1 \end{cases} \cancel{0 \le z \le 1}$$

външна нормала



$z = \sqrt{x^2 + y^2}$

$(x,y) \in K = \{x^2 + y^2 \le 1\}$

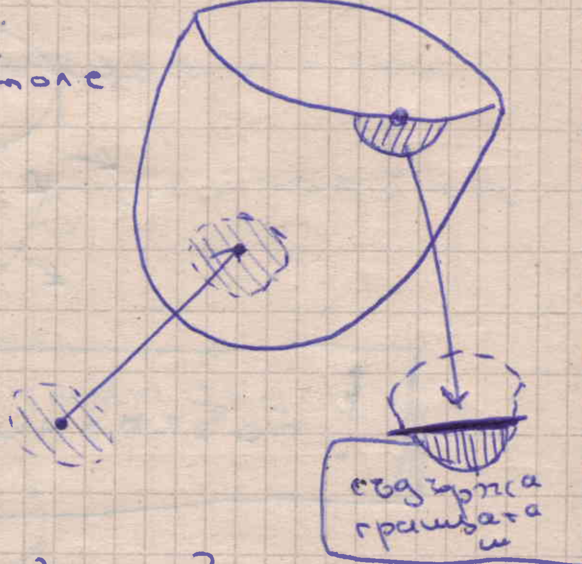
не е съгл. с параметризацията

$$\Rightarrow I = -\iint\limits_{\substack{K \\ x^2+y^2\le 1}} \sqrt{x^2+y^2}\,dx\,dy = -\int_0^{2\pi}\left(\int_0^1 \rho^2\,d\rho\right)d\varphi =$$

$$= -\frac{2\pi\cdot 1}{3} = -\frac{2}{3}\pi$$:))

Деф. Гладка повърхнина S с край ∂S (ориентирана) (S, n) непр. норм. в. поле

$S \subset \mathbb{R}^3$



ако

1) $\forall\, p \in S \setminus \partial S$

съществува

V_p отв. О в $\mathbb{R}^3$,

$p \in V_p$ (ок. т нар)

и $\varphi_p : \{u \in \mathbb{R}^2 : u_1^2 + u_2^2 < 1\} \longrightarrow V_p \cap S$

φ_p е гладко, $\operatorname{rg} \varphi_p' \equiv 2$, биекция

(тогава φ_p^{-1} е гладка)

$$\left(n(\varphi_p(u)) = \frac{\varphi'_{p u_1}(u) \times \varphi'_{p u_2}(u)}{\| \varphi'_{p u_1}(u) \times \varphi'_{p u_2}(u) \|} \right.$$

$u \in \{u_1^2 + u_2^2 < 1\}$

2) $p \in \partial S$

Същ. V_p отворено в $\mathbb{R}^3$, $p \in V_p$

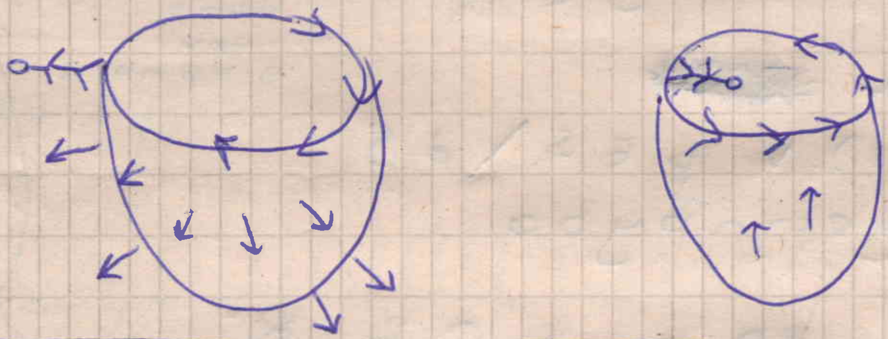
и същ: $\varphi_p : \{u \in \mathbb{R}^2 : u_1^2 + u_2^2 < 1\} \to \mathbb{R}^3$

т.че φ_p е инекция, $\varphi_p \in C^1$, rg $\varphi_p' \equiv 2$.

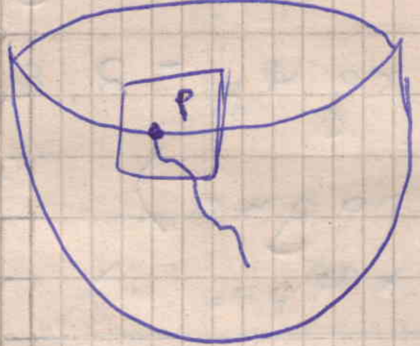
При това $\varphi_p\big|_{\overline{\Omega}} : \underbrace{\left\{u \in \mathbb{R}^2 : \begin{array}{l} u_1^2 + u_2^2 < 1 \\ u_2 \le 0 \end{array}\right\}}_{\overline{\Omega}} \to V_p \cap S$

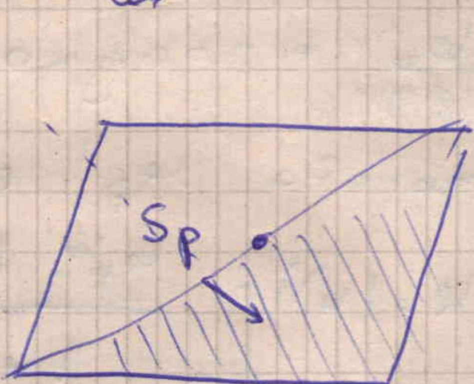
<u>е биекция.</u>

съгласуване с ориентацията



$\boxed{\text{! повърхнината да е отляво}}$





допир. п-во

навътре, навън, допир, към края

$\mathbb{R}^3$

$S_p \ni (p;v)$ сочи навътре, ако $\exists$ гладка крива

$$\alpha : [0,1] \longrightarrow S$$

$$\alpha \in C^1$$

$$\alpha(0) = p$$

$$\dot{\alpha}(0) = v$$

ν_p - единичен в-р в S_p, който е $\perp$ [redacted] към кр. и сочи навътре

$$\nu_p \perp \partial S$$

$\tilde{\tau}(p)$ е единичен в-р. допирателен към ∂S в т. p

и $(\tilde{\tau}(p), \nu(p), n(p))$ е дясна тройка

ориент. на повърхнината

(ортонорм. тройка
$\det = \underline{1}$)

D $\quad$ $n = (0,0,1)$

областта да стои отляво

$S_p \ni (p; v)$ сочи навътре, ако $\exists$ гладка крива

$$\alpha : [0,1] \longrightarrow S$$

$$\alpha \in C^1$$

$$\alpha(0) = p$$

$$\dot{\alpha}(0) = v$$

ν_p - единичен в-р в S_p, който е $\perp$ [redacted] към кр. и сочи навътре

$$\nu_p \perp \partial S$$

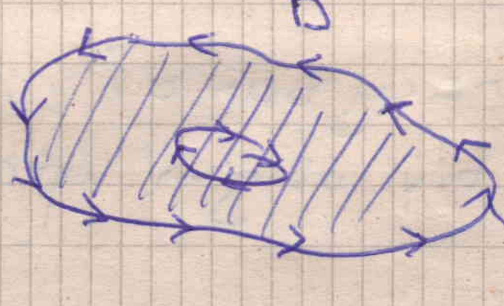
$\tau(p)$ е единичен в-р допирателен към ∂S в т. p

и $(\tau(p), \nu(p), n(p))$ е дясна тройка

ориент. на повърхнината

$\left(\text{ортонорм. тройка} \quad \det = 1 \right)$

D



$n = (0,0,1)$

областта да стои отляво

$\tau : \partial S \to \mathbb{R}^3$

$\forall$ точка от кр.-един. допир. в-р

индуцирана ориентация на края

Деф. Частично гладки ориент. повърхнини

S_i гладки пов. с край ∂S_i

$i = 1, \dots, k$

$S = \bigcup_{i=1}^{k} S_i$ частично гладка пов., ако

$S_i \cap S_j \subset \partial S_i \cap \partial S_j$

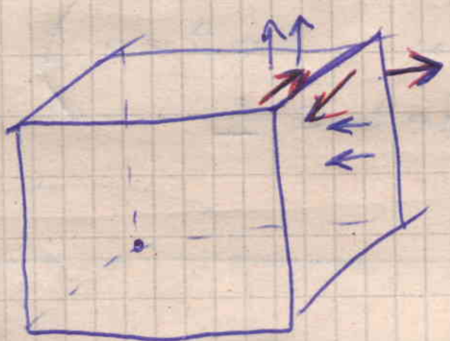
$\forall\, i \neq j$

$n : S \to \mathbb{R}^3$

~~ед. норм. непр. в-но поле~~ е

ориентация на S, ако

1) $n\big|_{S_i}$ ориентация на S_i $\forall\, i = 1, 2, \dots, k$



лошо :(

$\rightarrow$:)

но се прекъсват по края

кога са съгласувани?

ориентация в/у общия край

съгласуване на ориентация

2) $\forall p \in \partial S_i \cap \partial S_j$: $\tilde{\tau}_i(p)$ инд $\parallel$ $-\tilde{\tau}_j(p)$ инд. от $n|_{S_j}$

$$\tilde{\tau}_i(p) = -\tilde{\tau}_j(p)$$

!Th Стокс

(S, n) двукратно гладка (параметризациите имат ″) ориентирана компактна повърхнина с край ∂S

F гладко векторно поле, дефинирано в околност на S.
Тогава

$$\underbrace{\int_{\partial S}}_{\text{с индуцирана ориент.}} \langle F, dr \rangle = \iint_S \langle \underbrace{\operatorname{rot}}_{\text{ротор}} F, n \rangle$$

$$\operatorname{rot} F = \nabla \times F$$

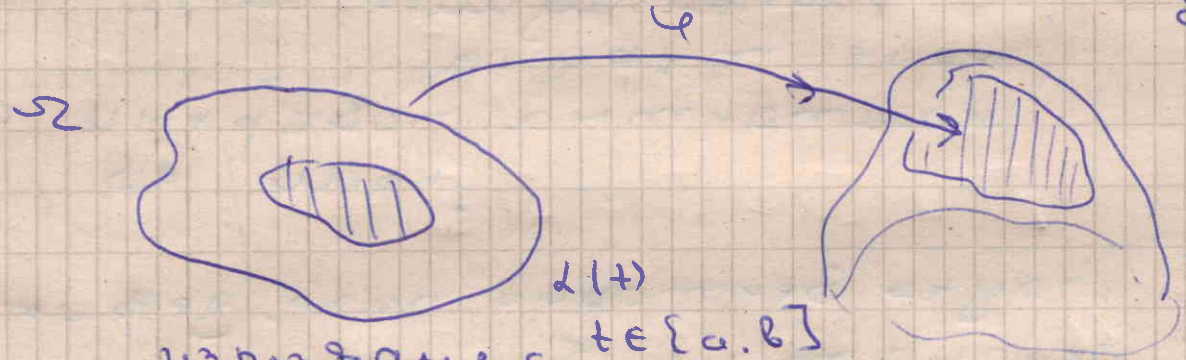
$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

$$F = (F_1, F_2, F_3)$$

$$\Rightarrow \operatorname{rot} F = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) = \iiint (\operatorname{rot} F)_1 dx_2 dx_3 + (\operatorname{rot} F_2) dx_3 dx_1 + (\operatorname{rot} F_3) dx_1 dx_2$$

мери завихрянето на полето t

потока на завихрянето



изрязване с гладка крива

$t \in [a.b]$

пар. на $\partial\Omega$ в прав. посока

$$F = (F_1, F_2, 0)$$

$$\varphi(u_1, u_2) = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$$

$$dx_1 dx_2$$

Ω област в $\mathbb{R}^2$ с граница $\partial\Omega$ частично гл. крива

Формула на Грийн

$$\int_{\partial\Omega} F_1\,dx_1 + F_2\,dx_2 = \iint_{\Omega} \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1\,dx_2$$

частен случай на Стокс

Лема 1 Ω област в $\mathbb{R}^2$ с граница $\partial\Omega$ частично гладка крива

$$\tilde{\Omega} \supset \bar{\Omega} = \Omega \cup \partial\Omega$$

отв.

F гладко в-но поле, деф. в ок. на $\bar{\Omega}$

$\partial\Omega \to$ индуцирана ориентация

$\bar{\Omega}$ - крив. трапец по двете пром.

$$\int_{\partial\Omega} F_1\,dx_1 + F_2\,dx_2 = \iint_{\bar{\Omega}} \left(\frac{\partial F_2}{\partial x_1}(x) - \frac{\partial F_1}{\partial x_2}(x)\right) dx_1\,dx_2$$

$\iint$

$\bar{\Omega} = \{ (x_1, x_2) \in \mathbb{R}^2$

$a \le x_1 \le b$

$g_1(x_1) \le x_2 \le g_2(x_1)$

$g_1 \le g_2$

непр.

част. гладки

$$\iint_{\overline{\Omega}} -\frac{\partial F_1}{\partial x_2}(x)\,dx_1\,dx_2 =$$

$$= \int_a^b \left(\int_{g_1(x_1)}^{g_2(x_1)} \left(-\frac{\partial F_1}{\partial x_2}(x_1, x_2) \right) dx_2 \right) dx_1 =$$

$$= \int_a^b \left(- F_1(x_1, x_2) \Big|_{g_1(x_1)}^{g_2(x_1)} \right) dx_1 =$$

$$= \int_a^b \left(-F_1(x_1, g_2(x_1)) + F_1(x_1, g_1(x_1)) \right) dx_1$$

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

$\Gamma_1 : \gamma_1(t) = (t, g_1(t))$

t от a до b

$\Gamma_2 : \gamma_2(t) = (b, t)$

t от $g_1(b)$ до $g_2(b)$

$\Gamma_3 : \gamma_3(t) = (t, g_2(t))$

t от b до a

$\Gamma_4 : \gamma_4(t) = (a, t)$

t от $g_2(a)$ до $g_1(a)$

$$-\int_a^b F_1(x_1, g_2(x_1))\, dx_1$$

$x_1 \to t$

$$= \underbrace{\int_b^a F_1(t, g_2(t))\, dt}_{\int_{\Gamma_3} F_1\, dx_1} +$$

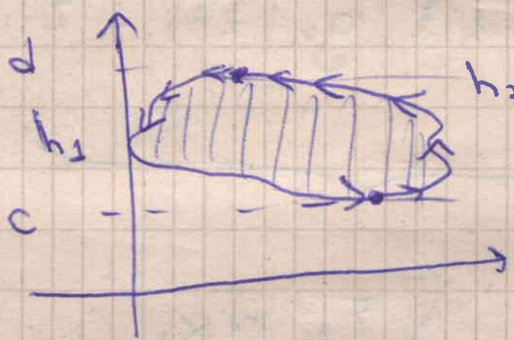
полето $(F_1, 0)$

$$- \underbrace{\int_a^b F_1(t, g_1(t))\, dt}_{\int_{\Gamma_1} F_1\, dx_1} + \underbrace{0}_{\int_{\Gamma_2} F_1\, dx_1} + \underbrace{0}_{\int_{\Gamma_4} F_1\, dx_1}$$

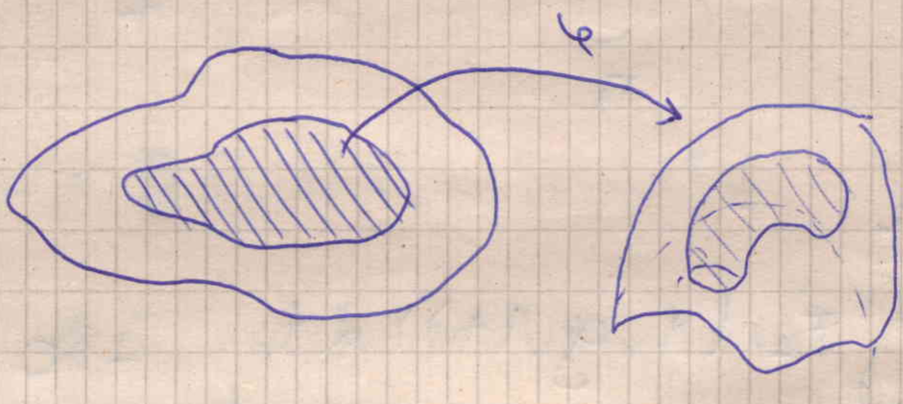
$$= \int_{\partial\Omega} F_1\, dx_1$$

$$\Rightarrow \iint\limits_{\Omega} \left(- \frac{\partial F_1}{\partial x_2}(x_1, x_2) \right) dx_1\, dx_2 = \int\limits_{\partial \Omega} F_1\, dx_1$$

$$\iint\limits_{\Omega} \frac{\partial F_2}{\partial x_1}(x_1, x_2)\, dx_1\, dx_2 = \int\limits_{\partial \Omega} F_2\, dx_2$$



Лема 2



$\Omega \subset \mathbb{R}^2$ област с граница $\partial \Omega$
част. гладка крива

$\tilde{\Omega} \supset \overline{\Omega} = \Omega \cup \partial \Omega$
отв. O

$\varphi : \overline{\Omega} \to \mathbb{R}^3$

$\varphi \in C^2$, $\operatorname{rg} \varphi' = 2$ и е инекция

$S = \varphi(\overline{\Omega})$

Формулата на Грийн е в сила за $\forall$ гладко в-но поле, деф. в ок. на $\overline{\Omega}$.
F гладко векторно поле, деф. в ок. на S.

$$\Rightarrow \int_{\partial S} \langle F, dr \rangle = \iint_{S} \langle \operatorname{rot} F, n \rangle$$

$$\int_{\partial S} \langle F, dr \rangle$$

$\alpha(t)$, $t \in [a, b]$
пар.-ия $\partial\Omega$ в прав. посока
$\beta(t) = \varphi(\alpha(t))$, $t \in [a; b]$
пар. на ∂S в прав. посока

$$\int_{\partial S} \langle F, dr \rangle = \int_a^b \langle F(\beta(t)), \dot{\beta}(t) \rangle dt =$$

$$= \int_a^b$$

$$\beta(t) = \begin{pmatrix} \varphi_1(\alpha_1(t), \alpha_2(t)) \\ \varphi_2(\alpha_1(t), \alpha_2(t)) \\ \varphi_3(\alpha_1(t), \alpha_2(t)) \end{pmatrix}$$

$$\dot{\beta}(t) = \left(\frac{\partial \varphi_i}{\partial u_1} (\alpha(t)) \dot{\alpha}_1(t) \right) + \frac{\partial \varphi_i}{\partial u_2} (\alpha(t) \cdot \dot{\alpha}_2(t)$$

$i = 1, 2, 3$

$$= \int_a^b \Big(F_1(\alpha(t)) \Big(\frac{\partial \varphi_1}{\partial u_1} (\alpha(t)) \cdot \dot{\alpha}_1(t) + \frac{\partial \varphi_1}{\partial u_2} \alpha(t) \cdot \dot{\alpha}_2(t) \Big)$$

$$+ F_2(\alpha(t)) \cdot \Big(\frac{\partial \varphi_2}{\partial u_1} \alpha(t) \cdot \dot{\alpha}_1(t) + \frac{\partial \varphi_2}{\partial u_2} (\alpha(t)) \cdot \dot{\alpha}_2(t) \Big)$$

$$+ F_3(\alpha(t)) \Big(\frac{\partial \varphi_2}{\partial u_1} (\alpha(t)) \cdot \dot{\alpha}_1(t) + \frac{\partial \varphi_3}{\partial u_2} (\alpha(t) \cdot \dot{\alpha}_2(t)) \Big) dt$$

$$= \int_a^b \Big(< F(\varphi(\alpha(t))) , \frac{\partial \varphi}{\partial u_1} (\alpha(t)) > \dot{\alpha}_1(t) +$$

$$+ < F(\varphi(\alpha(t)), \frac{\partial \varphi}{\partial u_2} (\alpha(t)) > \dot{\alpha}_2(t) \Big) dt$$

$$P(u_1, u_2) = < F(\varphi(u_1, u_2)), \frac{\partial \varphi}{\partial u_1} (u_1, u_2) >$$

$$Q(u_1, u_2) = < F(\varphi(u)), \frac{\partial \varphi}{\partial u_2} (u) >$$

$$\int\limits_{\partial S} \langle F, dr \rangle = \int\limits_{\partial \Omega} P\,du_1 + Q\,du_2 =$$

$$\underset{\underline{\underline{\text{Грийн}}}}{=} \iint\limits_{\Omega} \left(\frac{\partial Q}{\partial u_1} - \frac{\partial P}{\partial u_2} \right) du_1 du_2 \quad *$$

$$\frac{\partial Q}{\partial u_1}(u) = \frac{\partial}{\partial u_1} \left(\sum_{i=1}^{3} F_i(\varphi(u)) \cdot \frac{\partial \varphi_i}{\partial u_2}(u) \right)$$

$$= \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \frac{\partial F_i}{\partial x_j}(\varphi(u)) \cdot \frac{\partial \varphi_j}{\partial u_1}(u) \right) \cdot \frac{\partial \varphi_i}{\partial u_2}(u) +$$

$$+ \sum_{i=1}^{3} F_i(\varphi(u)) \cdot \frac{\partial^2 \varphi_i}{\partial u_2 \partial u_1}(u)$$

$$\frac{\partial P}{\partial u_2}(u) = \frac{\partial}{\partial u_2} \left(\sum_{i=1}^{3} F_1(\varphi(u)) \cdot \frac{\partial \varphi_i}{\partial u_1}(u) \right) =$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial F_i}{\partial x_j}(\varphi(u)) \cdot \frac{\partial \varphi_j(u)}{\partial u_2} \Bigg) \frac{\partial \varphi_i}{\partial u_1}(u)$$

$$+ \sum_{i=1}^{3} F_i(\varphi(u)) \cdot \frac{\partial^2 \varphi_i}{\partial u_1 \partial u_2}(u) \Bigg)$$

изваж-
дане
и
Шварц за
равенство на смесените

частни производни

⊛

$$= \iint\limits_{\Omega} \sum_{i,j=1}^{3} \frac{\partial F_i}{\partial x_j}(\varphi(u)) \cdot \underbrace{\left(\frac{\partial \varphi_j}{\partial u_1}(u) \cdot \frac{\partial \varphi_i}{\partial u_2} - \frac{\partial \varphi_j}{\partial u_2}(u) \frac{\partial \varphi_i}{\partial u_1}(u) \right)}_{\frac{\det(\varphi_j, \varphi_i)}{\det(u_1, u_2)}} du_1 du_2$$

$$\frac{\det(\varphi_j, \varphi_i)}{\det(u_1, u_2)} = \cancel{\quad}$$

$$= \begin{cases} 0 \text{ ако } i = j \\ - \cancel{\det} \dfrac{D(\varphi_i, \varphi_j)}{D(u_1, u_2)}, & i \neq j \end{cases}$$

$$= \iint\limits_{\Omega} \left(\frac{\partial F_3}{\partial x_2}(\varphi(u)) - \frac{\partial F_2}{\partial x_3}(\varphi(u)) \right) \begin{vmatrix} \frac{\partial \varphi_2}{\partial u_1} & \frac{\partial \varphi_2}{\partial u_2} \\ \frac{\partial \varphi_3}{\partial u_1} & \frac{\partial \varphi_3}{\partial u_2} \end{vmatrix}(u)$$

$$+ \left(\frac{\partial F_1}{\partial x_3}(\varphi(u)) - \frac{\partial F_3}{\partial x_1}(\varphi(u)) \right) \begin{vmatrix} \frac{\partial \varphi_3}{\partial u_1} & \frac{\partial \varphi_3}{\partial u_2} \\ \frac{\partial \varphi_1}{\partial u_1} & \frac{\partial \varphi_1}{\partial u_2} \end{vmatrix}(u)$$

$$+ \left(\frac{\partial F_2}{\partial x_1}(\varphi(u)) - \frac{\partial F_1}{\partial x_2}(\varphi(u)) \right) \begin{vmatrix} \frac{\partial \varphi_1}{\partial u_1} & \frac{\partial \varphi_1}{\partial u_2} \\ \frac{\partial \varphi_2}{\partial u_1} & \frac{\partial \varphi_2}{\partial u_2} \end{vmatrix}(u) \, du_1 du_2$$

$$= \iint_S \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) dx_2\, dx_3 +$$

$$+ \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) dx_3\, dx_1 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1\, dx_2 =$$

$$= \int_S \langle \operatorname{rot} F, n \rangle$$

?

08.01.2013г.

Упражнение

Повърхнинен интеграл от II тип

Деф. Нека (P, Q, R) е гладко поле и Φ - повърхнина е гладка и ориентируема. Повърхнинен интеграл от II род означаваме с

$$\iint_\Phi P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

I

$$\iint_{\Phi} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy =$$

$$= \iint_{\Phi} (P\cos x + Q\cos y + R\cos z)\,d\sigma,$$

където в-рът $(\cos x, \cos y, \cos z)$

$$\underbrace{(\cos X, \cos Y, \cos Z)}_{\substack{\text{нормален} \\ \text{в-р към} \\ \text{повърхнината}}} = \frac{(A, B, C)}{\sqrt{A^2+B^2+C^2}}$$

Тъй като има 2 начина да ориентираме повърхнината, векторът от директорните косинуси може евентуално да бъде обратен по знак. Ето защо при пресмятане на интегралите от този род е нужно да се уточни накъде сочи нормалният вектор.

Ако параметризираме повърхнината Φ

$$r(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in D,$$

то

$$I = \iint\limits_D \frac{P(u,v).A(u,v)+Q(u,v)B(u,v)+R(u,v)C(u,v)}{\cancel{\sqrt{A^2+B^2+C^2}}}\,\cancel{\sqrt{EG-F^2}}\,du\,dv$$

$$= \iint\limits_D P(u,v)\cdot A(u,v)+Q(u,v)B(u,v)+R(u,v)C(u,v)\,du\,dv =$$

$$= \iint\limits_D \left\langle \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \right\rangle du\,dv$$

$$(A,B,C) = \pm\, r_u \times r_v$$

Смяната на ориентацията на повърхнината сменя знака на интеграла.

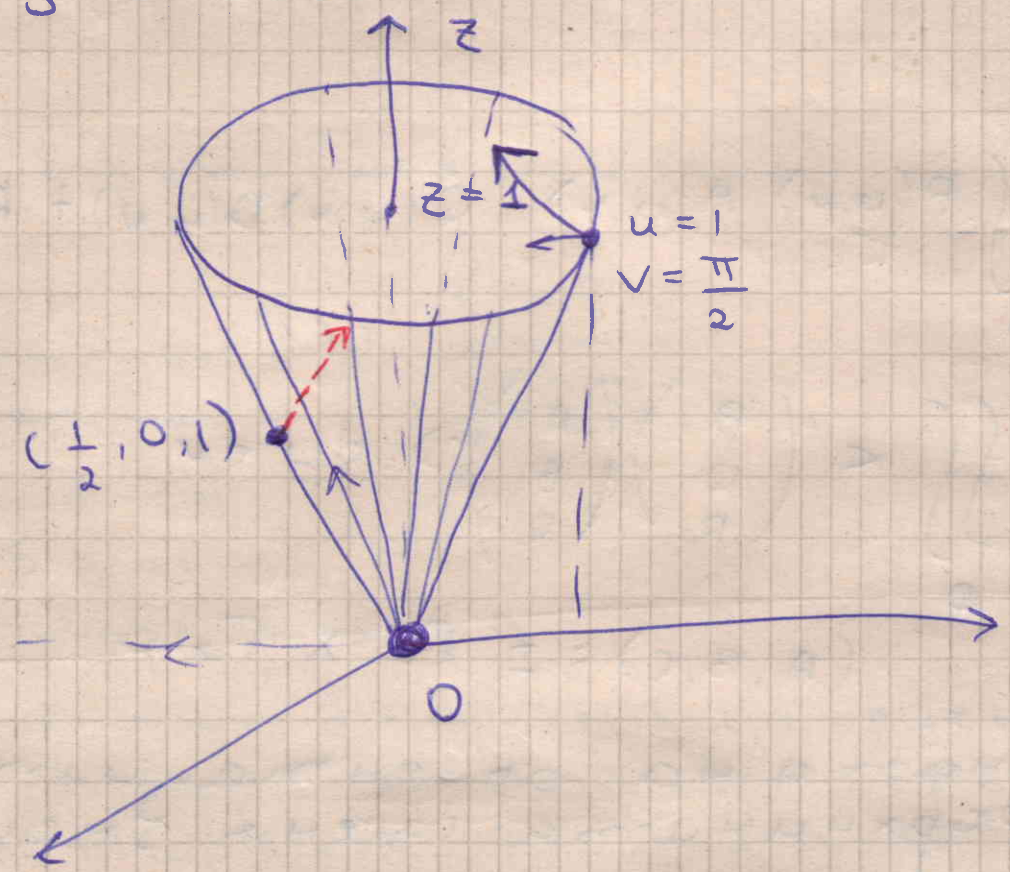
<u>зад.</u> Да се пресметне

$$\iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy,$$

S е външната страна на конуса

$x^2+y^2=z^2 \quad z \in [0;1]$

Заб. Под външна страна на повърхнина разбираме, че тя е ориентирана навън.
(в случая от центъра на коорд. с-ма навън)



$x = u\cos v$
$y = u\sin v$
$z = u$ $\quad(\pm)$

$$D \begin{cases} u \in [0, 1] \\ v \in [0; 2\pi] \end{cases}$$

$$r_u = (\cos v, \sin v, 1)$$

лич обв.

$$\begin{cases} r_u & \cos v \quad \sin v \quad 1 \\ r_v = (-u\sin v,\ u\cos v,\ 0) \end{cases}$$

$$r_u \times r_v = \left(-u\cos v,\ -u\sin v,\ u\right)$$

Проверяваме каква ориентация сме получили като проверим накъде сочи векторът (A, B, C) в някоя фиксирана точка от повърхнината.

Например в т. $\left(\frac{1}{2}, 0, \frac{1}{2}\right) \in$ конуса

и отговаря на $u = \frac{1}{2}$ $v = 0$

$$(A, B, C)\Big|_{\substack{u=\frac{1}{2}\\ v=0}} = \frac{1}{2}\left(-1, 0, 1\right)$$

Получената ориентировка не е външната и => сменяме знака на $r_u \times r_v$.

$$(A, B, C) = (u\cos v, u\sin v, -u)$$

Тогава

$$I = \iint_D \underbrace{u\cos v}_{X} \cdot \underbrace{u\cos v}_{A} + \underbrace{u\sin v}_{Y} \cdot \underbrace{u\sin v}_{} + \underbrace{u}_{Z} \cdot \underbrace{u}_{C} \, du\,dv$$

$$= \int_0^{2\pi}\int_0^{1} 0\,du\,dv = 0 \quad :))$$

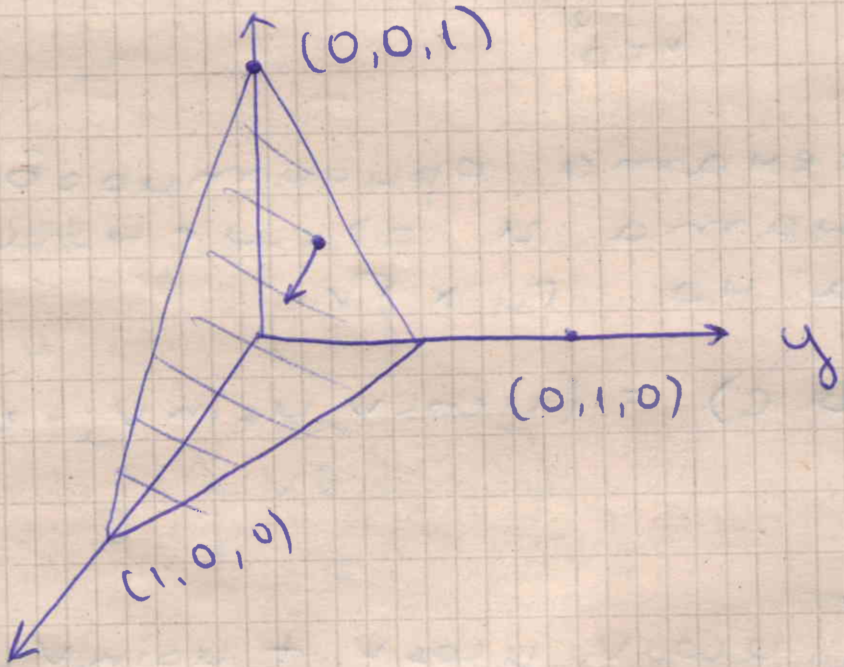
Зад. Да се пресметне

$$\iint_{\Phi} yz\,dy\,dz + xz\,dx\,dz + xy\,dx\,dy,$$

където Φ е триъгълникът

$A(1,0,0)$
$B(0,1,0)$
$C(0,0,1)$

Взета е неговата вътр. страна (нормалният в-р сочи към коорд. начало)



$x+y+z-1=0$

$\perp$ ~~норм.~~ в-р $(1,1,1)$, $(-1,-1,-1)$

Параметризиране Δ-ка

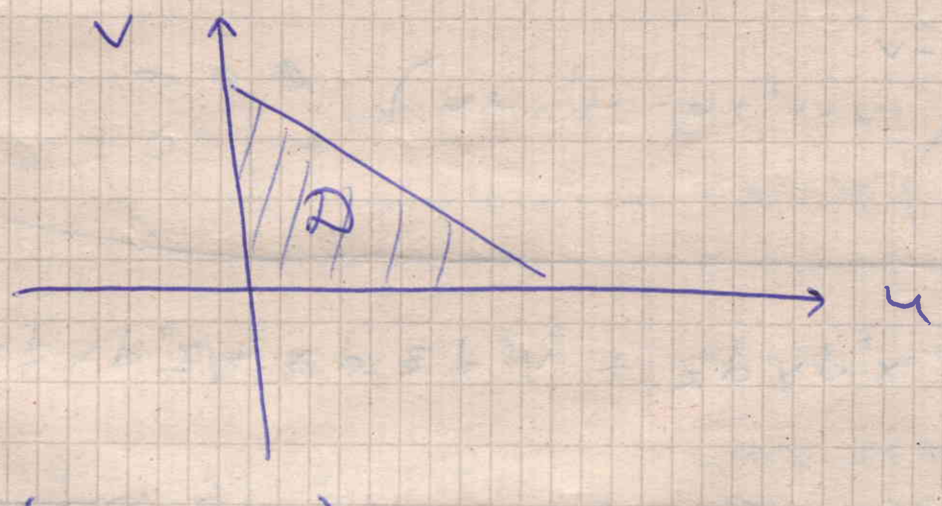
т. $A(1,0,0)$

$\overrightarrow{AB}(-1,1,0)$

$\overrightarrow{AC}(-1,0,1)$

$$r(u,v) = \begin{pmatrix} 1 + u(-1) + v.(-1) \\ 0 + u.1 + v.0 \\ 0 + u.0 + v.1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1-u-v \\ u \\ v \end{pmatrix} \quad \begin{array}{|l} \underline{\underline{u+v \leq 1}} \\ 0 \leq u \leq 1 \\ 0 \leq v \leq 1 \end{array} \quad \mathcal{D}$$



$r_u = (-1,1,0)$

$r_v = (-1,0,1)$

$$r_u \times r_v = \left(\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} \right) =$$

$$= (1, 1, 1) ;$$

$$(A, B, C) = (-1, -1, -1)$$

$$\cancel{I = \iint_D xz \ldots}$$

$$I = -\iint_D \cancel{(\ldots)}\, uv(-1) + (1-u-v)v(-1) + (1-u-v)u(-1)\, du\, dv =$$

$$= -\iint_D -\cancel{uv} - \cancel{v} + \cancel{uv} + v^2 + \cancel{u} + u^2 \cancel{+v}\, du\, dv$$

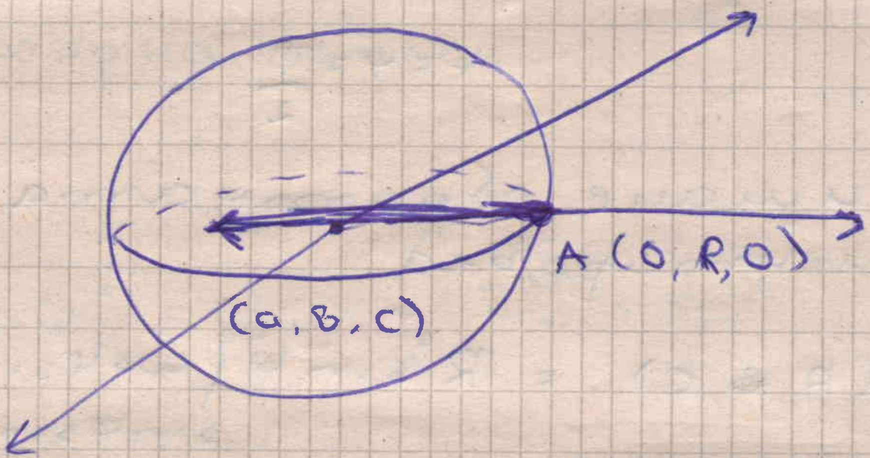
$$= -\int_0^1 \int_0^{1-v} \left(v - v^2 + u - u^2 - uv \right) \ldots$$

__зад.__ $$\iint_S x^2\, dy\, dz + y^2\, dz\, dx + z^2\, dx\, dy$$

S е външната страна на сферата

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2$$

$$x = R\cos\varphi\sin\psi + a$$
$$y = R\sin\varphi\sin\psi + b$$
$$z = R\cos\psi + c \qquad \varphi\in[0;2\pi]$$
$$\psi \in [0;\pi]$$



$$r_\varphi = \left(-R\sin\varphi\sin\psi;\ R\cos\varphi\sin\psi,\ -\cancel{R}\cdot 0\right)$$
$$r_\psi = R\left(\cos\varphi\cos\psi,\ \sin\varphi\cos\psi,\ -\sin\psi\right)$$

$$r_\varphi\times r_\psi = \left(R^2\left(\begin{vmatrix}\cos\varphi\sin\psi & 0\\ \sin\varphi\cos\psi & -\sin\psi\end{vmatrix},\right.\right.$$
$$\left.\left.\begin{vmatrix}0 & -\cancel{R}\sin\varphi\sin\psi\\ -\sin\psi & \cos\varphi\cos\psi\end{vmatrix},\ \begin{vmatrix}-\sin\varphi\sin\psi & \cos\varphi\sin\psi\\ \cos\varphi\cos\psi & \sin\varphi\cos\psi\end{vmatrix}\right)\right)$$

$$= R^2\left(-\cos\varphi\sin^2\psi,\ -\sin\varphi\sin^2\psi,\ -\sin\psi\cos\psi\right) =$$

$$= -R^2 \sin\psi \left(\cos\varphi\sin\psi, \sin\varphi\sin\psi, \cos\psi\right) =$$

$\neq$.

т. с коорд. $(a, b, c) + R(0, 1, 0)$

отг. на $\varphi = \psi = \frac{\pi}{2}$

$$\Rightarrow (A, B, C)\Big|_{\varphi=\psi=\frac{\pi}{2}} = -R^2 \cdot 1\,(0, 1, 0)$$

$\Rightarrow$ вземаме обратната ориентировка

$$(A, B, C) := R^2 \sin\psi \left(\cos\varphi\sin\psi, \sin\varphi\sin\psi, \cos\psi\right)$$

$$I = R^2 \iint_D (a + R\cos\varphi\sin\psi)^2 \sin\psi\cos\varphi + (b + R\sin\varphi\sin\psi)^2 \sin^2\psi\sin\varphi + (R\cos\psi + c)^2 \cos\psi\sin\psi \, d\varphi \, d\psi$$

Нека (F_1, F_2, F_3) е гладко поле.

$$\cancel{\nabla F = \left(\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}, \frac{\partial F_3}{\partial z}\right)}$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Ако $Q: \mathbb{R}^3 \to \mathbb{R}$

$$\nabla Q = \operatorname{grad} Q = \left(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z} \right)$$

$$\Delta Q = \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2}$$

оператор на Лаплас

ИЗПИТ

$$\operatorname{div}(\nabla Q) = \operatorname{div}\left(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z} \right) =$$

$$= \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} = \Delta Q$$

Да се пресметне

$\nabla \|x\|$ $\quad \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$$\nabla \|x\| = \left(\frac{2x_1}{2\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{2x_2}{2\sqrt{}}, \frac{2x_3}{2\sqrt{}} \right) =$$

$$= \frac{x}{\|x\|}$$

$$\nabla e^{\|x\|} = \nabla e^{\sqrt{x_1^2+x_2^2+x_3^2}} =$$

$$= \left(\frac{e^{\|x\|} \cdot 2x_1}{2\sqrt{}}, \frac{e^{\|x\|} \cdot 2x_2}{2\sqrt{}}, \dots \right) =$$

$$= x \frac{e^{\|x\|}}{\|x\|}$$

скалар

вектор

Формула на Гаус-Остроградски

Нека S е затворена, частично гладка повърхнина с външна ориентация. Нека полето $F = (P, Q, R)$ е гладко в/у S.
Тогава

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy =$$

$$= \iiint_G \underbrace{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}_{\operatorname{div} F} \,dx\,dy\,dz$$

$G \subseteq \mathbb{R}^3$ е областта,

което S загражда

Пример: $S: x^2+y^2+z^2=R^2$ външна

$$\iint_S x^3 dy\,dz + y^3 dz\,dx + z^3 dx\,dy =$$

сф. смяна

$$= \iiint_{x^2+y^2+z^2\le R^2} 3\left(x^2+y^2+z^2\right) dx\,dy\,dz =$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} 3.1.\, r^2 \sin\psi \; dr\,d\psi\,d\varphi =$$

$$= \cancel{3}\int_0^{2\pi}\int_0^{\pi} \sin\psi \,\frac{R^3}{\cancel{3}} = 2\pi.\left(-\cos\psi\right)\Big|_0^{\pi} R^3 =$$

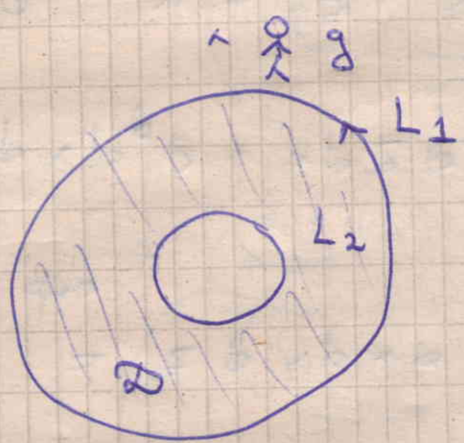
$$= 2\pi\,(1+1)\,R^3 =$$

$$= \underline{4\pi R^3}$$

Тв. Нека $V \subseteq \mathbb{R}^3$ е измерима област (П. Ж). Тогава обема на $V = \iiint_V dx\,dy\,dz$

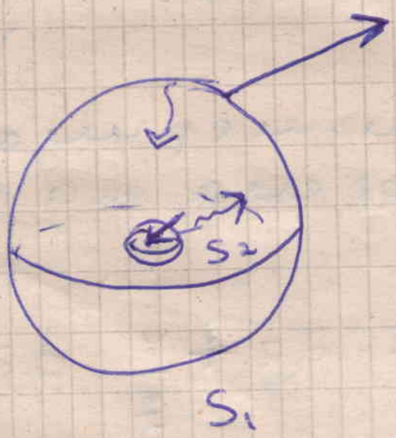
$$V = \iint_{\Phi} \frac{x}{3}\, dy\,dz + \frac{y}{3}\, dz\,dx + \frac{z}{3}\, dx\,dy$$

където Φ е контурната пов. на V (ограждаща), ориентирана външно



$\langle L_1 \cup L_2 \rangle = D$, ако L_2 се обикаля по часовниковата.

! отляво



синхронично ориентирани

за $\langle S_1, S_2 \rangle$ говорим, ако

S_1 е външно ор.

S_2 е вътр-ор.

или обратното

зад. от Изпит !

Да се пресметне

$$\iint\limits_S \frac{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}{\sqrt{x^2+y^2+z^2}^{\,3}}, \text{ където}$$

S е затворена гладка повърхнина

а) S не загражда 0-та

б) S загражда 0

В а) прилагаме директно Г-О

$$\frac{\partial}{\partial x}\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\sqrt{}^{\,3} - x \cdot \frac{1}{3}\cdot\frac{2x}{\sqrt{x^2+y^2+z^2}}}{\left(x^2+y^2+z^2\right)^3} =$$

$$= \frac{\sqrt{x^2+y^2+z^2}}{\left(x^2+y^2+z^2\right)^3}\cdot\left(x^2+y^2+z^2 - 3x^2\right) =$$

$$= \left(x^2+y^2+z^2\right)^{-\frac{5}{2}}\cdot\left(y^2+z^2-2x^2\right)$$

Аналог.

$$\frac{\partial}{\partial y}\left(\frac{y}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) = (x^2+y^2+z^2)^{-\frac{5}{2}}\left(x^2+z^2-2y^2\right)$$

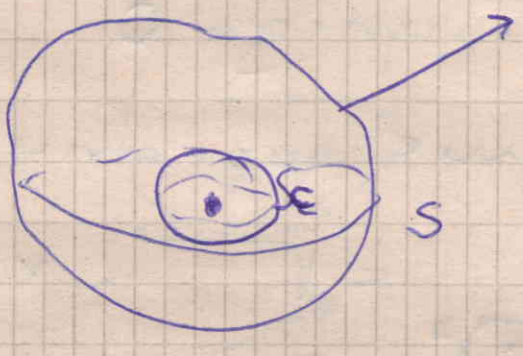
$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{}}\right) = (x^2+y^2+z^2)^{-\frac{5}{2}}\left(y^2+x^2-2z^2\right)$$

$$I = \iiint_G (x^2+y^2+z^2)^{-\frac{5}{2}}\left(\cancel{x^2}+\cancel{y^2}-\cancel{2z^2}+\cancel{x^2}+\cancel{z^2}-\cancel{2y^2}+\cancel{y^2}+\cancel{z^2}-\cancel{2x^2}\right) + 0\, dx\,dy\,dz$$

$= 0$:))

G е вътр. на S

б)



Полето не е гладко
Ограждаме O със сфера с радиус ε

Нека S_ε е сфера с радиус $R=\varepsilon$ с вътрешна ориентация и

$\varepsilon < \min d(O;S)$

разоми

$$\iint\limits_{S\cup S_\varepsilon} (F, n)\, d\sigma = \iint\limits_{S} (F, n)\, d\sigma +$$

$$+ \iint\limits_{S_\varepsilon} (F, n)\, d\sigma =$$

$$= \iiint\limits_{G=\langle S_1\cup S_\varepsilon\rangle} \underbrace{\operatorname{div} F\, dx\, dy\, dz}_{0} \Rightarrow \iint\limits_{S} (F, n)\, d\sigma =$$

$$= -\iint\limits_{S_\varepsilon^-} (F, n)\, d\sigma =$$

$$= \iint\limits_{S_\varepsilon^+} (F, n)\, d\sigma$$

$$x = \varepsilon \cos\varphi \sin\psi \qquad \varphi \in [0, 2\pi]$$
$$y = \varepsilon \sin\varphi \sin\psi$$
$$z = \varepsilon \cos\psi$$

$$r_\varphi = \varepsilon(-\sin\varphi \sin\psi,\ \cos\varphi \sin\psi,\ 0)$$

$$r_\psi = \varepsilon(\cos\varphi\cos\psi,\ \sin\varphi\cos\psi,\ -\sin\psi)$$

$$r_\varphi \times r_\psi = -\varepsilon\sin\psi\,(x, y, z)$$

$$(A, B, C) = \varepsilon \sin\psi\,(x, y, z)$$

$$I = \int_0^{2\pi}\int_0^{\pi} \frac{x\varepsilon\sin\psi x + y\varepsilon\sin\psi y + z\varepsilon\sin\psi z}{\varepsilon^3}\, d\varphi\, d\psi$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\cancel{\varepsilon}}{\cancel{\varepsilon^3}} \sin\varphi . \cancel{\varepsilon^2} \, d\varphi \, d\psi =$$

$$= 2\pi . 2 = 4\pi$$

Дом.: $\oint\limits_{\Gamma} \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$, където

а) Г не загр. 0

б) Г загр. 0

част.

Г-гладка проста крива

Лекция

Th Стокс

(S, n) ориентирана повърхнина с
край ∂S
компактна
двукратно гладка

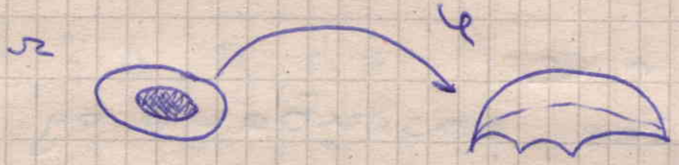
F гладко поле, деф. в ок-т на S

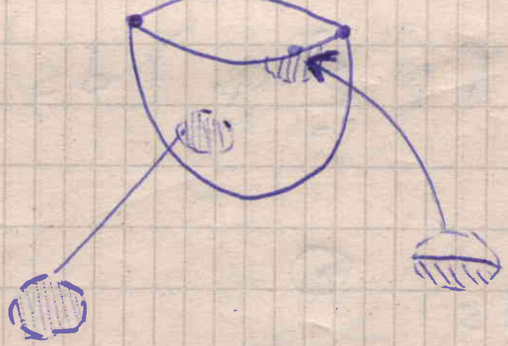
$$\Rightarrow \int\limits_{\partial S} \langle F, dr \rangle = \iint\limits_{S} \langle \operatorname{rot} F, n \rangle \, dS$$

↑
индуцирана ориентация

Л1 : ф-лата на Грийн

Л2 : ф-ла Стокс





Деф. Разбиване на единицата в/у повърхнина ~~Sf~~ (S, n)

$\{g_i\}_{i=1}^{n}$ кр. мн. от гл. ф-ции

1) $g_i : S \to \cancel{\mathbb{R}}\ [0,1]$
рестрикция на гладко изобр., дефинирано в ок-т на S.

2) $\sum_{i=1}^{n} g_i(p) = 1 \quad \forall p \in S$

3) За $\forall\, i = 1, 2, \dots, n \ \exists$
лок. параметризация
$\varphi_i : U_i \to S$

и $K_i \subset U_i$, K_i компакт
такива че

$$\varphi_i(K_i) \supset \{p \in S : g_i(p) \neq 0\}$$

$\{g_i\}_{i=1}^n$ разбиване ед. в/у S

$$f : S \to \mathbb{R}$$

$$\iint_S f\,ds = \iint_S f\left(\sum_{i=1}^n g_i\right) ds =$$

$$= \sum_{i=1}^n \iint_{\substack{S \\ \shortparallel \\ \varphi_i(K_i)}} f g_i\,ds \quad (*)$$

можем да разбиваме по ни.начинк-
едно и също число

$$\{g_i\}_{i=1}^n \longrightarrow \{\varphi_i\}_{i=1}^n$$
$$K_i$$

$$\{\tilde{g}_i\}_{i=1}^{\tilde{n}} \longrightarrow \{\tilde{\varphi}_i\}_{j=1}^{\tilde{n}}$$

2 разбивания на 1

$$\circledast = \sum_{i=1}^{n} \iint\limits_{\varphi_i(U_i)} f g_i \left(\sum_{j=1}^{\tilde{n}} \tilde{g}_j \right) ds =$$

линейност

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{\tilde{n}} \iint\limits_{\varphi_i(U_i) \cap \tilde{\varphi}_j(\tilde{U}_j)} f g_i \tilde{g}_j \, ds \right) =$$

$$= \sum_{j=1}^{\tilde{n}} \left(\sum_{i=1}^{n} \iint\limits_{\tilde{\varphi}_j(\tilde{U}_j)} f g_i \tilde{g}_j \, ds \right) =$$

$$= \sum_{j=1}^{\tilde{n}} \iint\limits_{\tilde{\varphi}_j(\tilde{U}_j)} f \tilde{g}_j \overbrace{\left(\sum_{i=1}^{n} g_i \right)}^{=1} ds = \sum_{j=1}^{\tilde{n}} \iint\limits_{\tilde{\varphi}_j(\tilde{U}_j)} f \tilde{g}_j \, ds$$

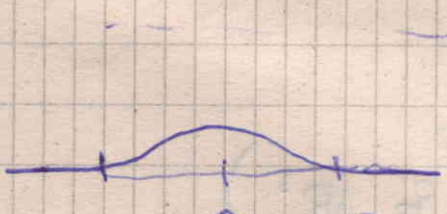
$= \circledast$:))

деф. е коректна

Th (S, n)

S компактна гладка повърхнина с край ∂S (може $\partial S = \varnothing$)

$\Rightarrow$ В/у S $\exists$ разбиване на единицата



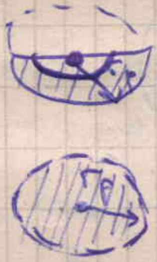
$$\mu(t) = \begin{cases} 0 & |t| \geq r \\ e^{-\frac{1}{r^2 - t^2}}, & |t| < r \end{cases}$$

$\mu \in C^\infty(\mathbb{R})$

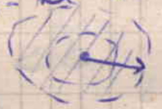
$p \in S$

$\varphi_p : U_p \longrightarrow S$

U_p



$\tilde{U}_p$ достатъчно-малък радиус)



r_p

$\varphi_p(\tilde{U}_p) = V_p$

$\downarrow$

рад. $\dfrac{r_p}{2}$

$q \in S$

$h_p(q) = \mu\left(\underline{\|\varphi_p^{-1}}(q) - \varphi_p^{-1}(p)\|\right)$

$\operatorname{supp} h_p = \overline{\tilde{U}_p}$ зат. обвивка

$=: K_p$

$p \in V_p$ отв. в S

$(V_p = S \cap W_p$

W_p отв. в $\mathbb{R}^3)$

$\bigcup\limits_{p \in S} W_p = \bigcup\limits_{p \in S} V_p = S$ компакт

$\Rightarrow \exists$ крайно подпокритие

$$\bigcup_{i=1}^{n} W_{p_i} \supset S$$

$$V_{p_i} = S \cap W_{p_i}$$

$$\downarrow$$

$$h_{p_i} \quad , \quad K_{p_i}$$

гладки

$$g_i(p) = \frac{h_{p_i}}{\sum_{j=1} h_{p_j}(p)}$$

$\{V_{p_i}\}_{i=1}^{n}$ е покритие на S

$p \in V_{p_j}$ за някое j

$\Rightarrow h_{p_j}(p) > 0$

$$\sum_{i=1}^{n} g_i(p) = 1 \quad \forall p \in S$$

g_i гладки колкото φ_i

$g_i \geq 0$

$\operatorname{supp} g_i \subset \varphi_i(K_i)$

$K_i \subset U_i$

K_i компакт

Д-во Th Стокс:

(S, n) - компактна
двукратно гладка

$\Rightarrow \exists$ разбиване на 1-та в/у S

$$\{g_i\}_{i=1}^{n}$$

$$\iint\limits_S \langle \mathrm{rot}\, F, n \rangle \, ds =$$

$$= \iint\limits_S \langle \mathrm{rot}\, F, n \rangle \sum_{i=1}^{n} g_i \, ds =$$

$$= \sum_{i=1}^{n} \iint\limits_{\varphi_i(k_i)} \langle \mathrm{rot}\, F, n \rangle g_i \, ds \quad (*)(*)$$

$K_i \longrightarrow$  крив трапеци :)

$g \,\mathrm{rot}\, F$

$$\mathrm{rot}(g.F) = g.\mathrm{rot}\, F + \mathrm{grad}\, g \times F$$

$$\begin{matrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{matrix}$$

Ikoopg.

$$[\mathrm{rot}\,(gF)]_1 = \frac{\partial}{\partial x_2}(gF_3) - \frac{\partial}{\partial x_3}(gF_2) =$$

$$= \frac{\partial g}{\partial x_2}\cdot F_3 + g\,\frac{\partial F_3}{\partial x_2} - \frac{\partial g}{\partial x_3}F_2 - g\cdot\frac{\partial F_2}{\partial x_3} =$$

$$= g\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) + \frac{\partial g}{\partial x_2}F_3 - \frac{\partial g}{\partial x_3}F_2 =$$

$$= g\,(\mathrm{rot}\,F) + [\mathrm{grad}\,g \times F]_1$$

$$\Rightarrow ** = \sum_{i=1}^{n} \iint_{\varphi_i(K_i)} \langle \mathrm{rot}\,(g_i F) - (\mathrm{grad}\,g_i)\times F, n\rangle\, ds =$$

$$= \sum_{i=1}^{n} \iint_{\varphi_i(K_i)} \langle \mathrm{rot}\,g_i F, n\rangle\, ds -$$

$= 0$:))

$$- \iint_{\varphi_i(K_i)} \langle \overbrace{(\mathrm{grad}(\underbrace{\textstyle\sum_{i=1}^{n} g_i}_{1}))}^{0} \times F, n\rangle\, ds =$$

$$= \sum_{i=1}^{n} \int_{\varphi_i(\partial K_i)\cap \partial S} \langle g_i F, dr \rangle =$$

$\neq 0$ накрая

извън него $g_i = 0$

$$= \sum_{i=1}^{n} \int_{\partial S} \langle g_i F, dr \rangle = \int_{\partial S} \langle F . \left(\sum_{i=1}^{n} g_i \right) dr \rangle$$

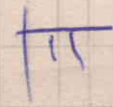
$$= \int_{\partial S} \langle F, dr \rangle$$

Заб. 1

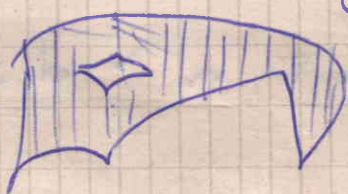
лок. парам.

пак кр. тр. $\to$ о. к.

$\xrightarrow{\varphi}$



∂S е частично гладка крива



Заб 2 (S^n, частично гладка ориент. повърхнина

$$\iint_S \langle \operatorname{rot} F, n \rangle ds =$$

$$= \int_{\partial S} \langle F, dr \rangle$$

инд. ориент.

$$S = \bigcup_{i=1}^{n} S_i$$

секат само по гр. ориент. в/у гр. индуцират противоп. ориент.

$$\iint_S \langle \operatorname{rot} F, n \rangle \, ds =$$

$$= \sum_{i=1}^{k} \iint_{S_i} \langle \operatorname{rot} F, n \rangle \, ds = \sum_{i=1}^{k} \int_{\partial S_i} \langle F, dr \rangle =$$

инд.

$+ - 1 = 0$

против. ориент.

$$= \int_{\partial S} \langle F, dr \rangle$$

$$\int_{\partial S} \omega = \int_S d\omega$$

3 ориг

$$\int_a^b F'(x)\,dx = F(b) - F(a) \quad \text{Л-Н}$$

$$\int_{\partial S} \omega = \int_S d\omega$$

F гладко поле в Ω област в $\mathbb{R}^2$

F потенциално $\underset{def}{\Leftrightarrow}$ $F = \operatorname{grad} u$

$\Leftrightarrow$ независимост от пътя на

$$\int_\Gamma \langle F, dr \rangle \quad \Leftrightarrow$$

! $\int_L \langle F, dr \rangle = 0$ $\forall$ L проста
затв. наг.
без самопресичане
$L \subset \Omega$
Ω едносвързан.

НУ за потенциалност $\Rightarrow$

$$\frac{\partial F_2}{\partial x_1} = \frac{\partial F_1}{\partial x_2} \quad \text{в } \Omega$$

ТВ. Ω едносвързана област в $\mathbb{R}^2$

F-гладко в-но поле в Ω

$$\frac{\partial F_2}{\partial x_1} = \frac{\partial F_1}{\partial x_2} \quad \text{в } \Omega$$

$\Rightarrow F$ е потенциално

Ω едносвързана

$\forall \Gamma$ проста затв. непр. крива в Ω

$\Omega_\Gamma \subset \Omega$

областта, заградена от Γ

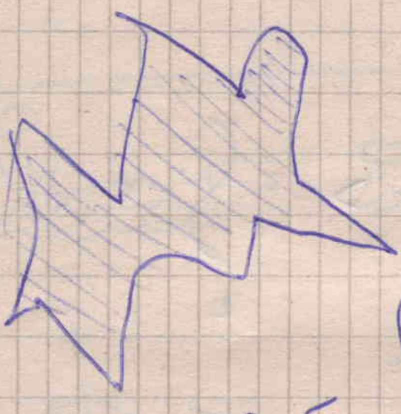
D-во: Ф-ла Грийн

$L \subset \Omega$, проста затв. нач. линия

$\Omega_L \subset \Omega$, $\partial \Omega_L = L$

В ок-т на $\overline{\Omega_L} = \Omega_L \cup L$

Грийн за Ω_L



$$0 = \iint_{\overline{\Omega_L}} \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dx_1 dx_2 =$$

(от усл. за пот.)

$$= \int_L \langle F, dr \rangle$$

затв. начупена без самопресичане

$\Rightarrow$ полето е потенциално

$\Omega \subset \mathbb{R}^3$ област
F гладко в-но поле в Ω

НУ за потенц.

$$\left|\begin{array}{l} \dfrac{\partial F_2}{\partial x_1} = \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_3}{\partial x_2} = \dfrac{\partial F_2}{\partial x_3} \\ \dfrac{\partial F_1}{\partial x_3} = \dfrac{\partial F_3}{\partial x_1} \end{array}\right.$$

$$\mathrm{rot}\, F = 0$$

Стокс

$$0 \overset{?}{=} \iint_{S_L} \langle \mathrm{rot}\, F, n \rangle \, ds = \cancel{\int_{\partial S} \langle F, dr \rangle} = \int_L \langle F, dr \rangle$$

L-проста затв. начупена

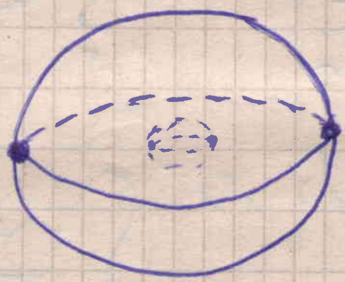
$L \subset \Omega$

? S_L - шито край

$\partial S_L \in L$

, повърхнинно-едносвързани

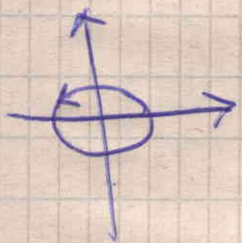
Трябва да поискаме
за $\forall$ L-прост. затв. крив. в Ω
$\exists$ ориент. пов. $S_L \subset \Omega$
$\partial S_L = L$



повърхнинно-едносвързана

не е пов. едносвързано

Пример: $F(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$



$\Omega = \mathbb{R}^2 \setminus \{0\}$

$\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ в Ω

$\displaystyle\int_{C_r} \langle F, dr \rangle = 2\pi$

$\Gamma \subset \Omega \; (\theta \notin \Gamma)$

Γ проста затворена частично гладка крива

Ω_Γ област, крайна, $\partial \Omega_\Gamma = \Gamma$

1) $\theta \notin \Omega_\Gamma$ $\longrightarrow$ $\int_\Gamma \langle F, dr \rangle$

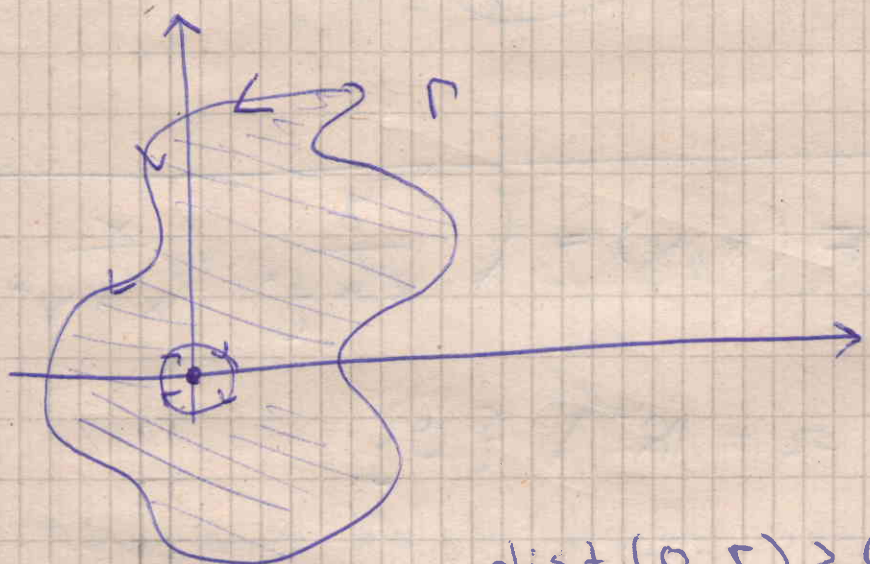
2) $\theta \in \Omega_\Gamma$

$$= \int_\Gamma \frac{-y\,dx + x\,dy}{x^2+y^2} =$$

1)

$$\underset{\text{Грийн}}{=} \iint_{\Omega_\Gamma} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = 0$$

2)



$\operatorname{dist}(0, \Gamma) > 0$

$\overline{B}_\varepsilon(0) \subset \Omega_\Gamma \subset \Omega$

$\tilde{\Omega}_\Gamma = \Omega_\Gamma \setminus \overline{B}_\varepsilon(0)$

$$0 = \iint_{\tilde{\Omega}_r} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \int_{\partial \tilde{\Omega}_r} \langle F, dr \rangle =$$

$$\partial \tilde{\Omega}_r = \Gamma \cup \overline{C_\varepsilon}(0)$$

Γ е с правилна ориентация

$C_\varepsilon(0)$ е с обратна

$$\Rightarrow \; = \int_\Gamma \langle F, dr \rangle + \int_{\overline{C_\varepsilon}} \langle F, dr \rangle =$$

$$= \int_\Gamma \langle F, dr \rangle - \int_{C_\varepsilon} \langle F, dr \rangle$$

$$\int_\Gamma \langle F, dr \rangle = \int_{C_\varepsilon} \langle F, dr \rangle = 2\pi$$

Пропуснато от

11.12.2013г.

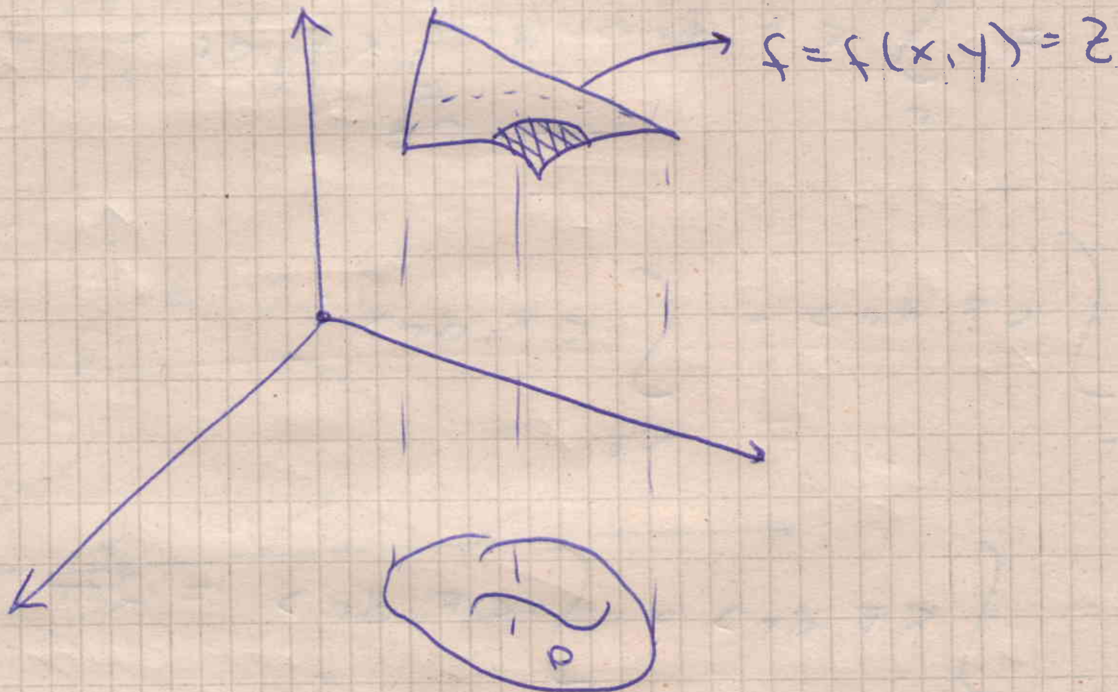
Упражнение

Повърхнини и повърхнинни интеграли

1/ Повърхнини

$f: \mathbb{R}^2 \to \mathbb{R}^3$ $(\mathbb{R}^2)$ е повърхнина

$f = f(x,y)$



Графика на ф-я на 2 променливи

$$(x,y) \to (x,y,f(x,y))$$

$$(*) \quad \underset{\mathbb{R}^2}{f(u,v)} = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \\ f_3(u,v) \end{pmatrix} \in \mathbb{R}^3$$

∀ повърхнина зависи от 2 параметъра

Параметрично представяне на повърхнина:

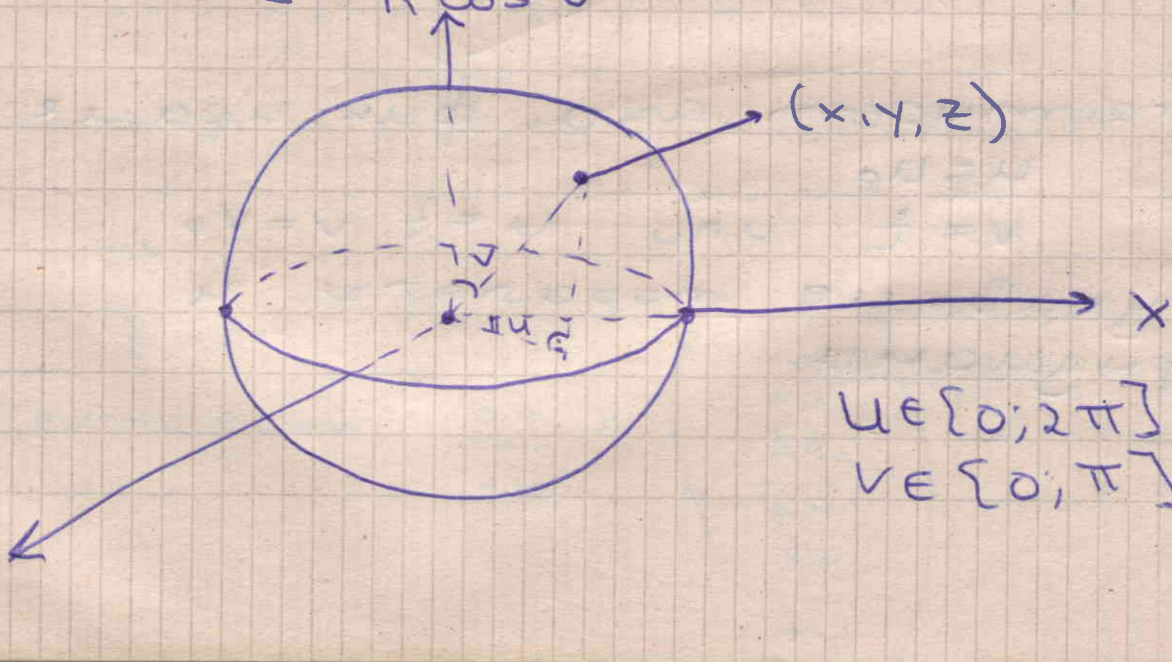
$$f'(u,v) = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix} = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right)$$

гладка повърхнина наричаме

$f_1, f_2, f_3 \in C_1$ и $\operatorname{rg} f' = 2$

Примери: 1) Сфера

$$x = R \cos u \sin v$$
$$y = R \sin u \sin v$$
$$z = R \cos v$$

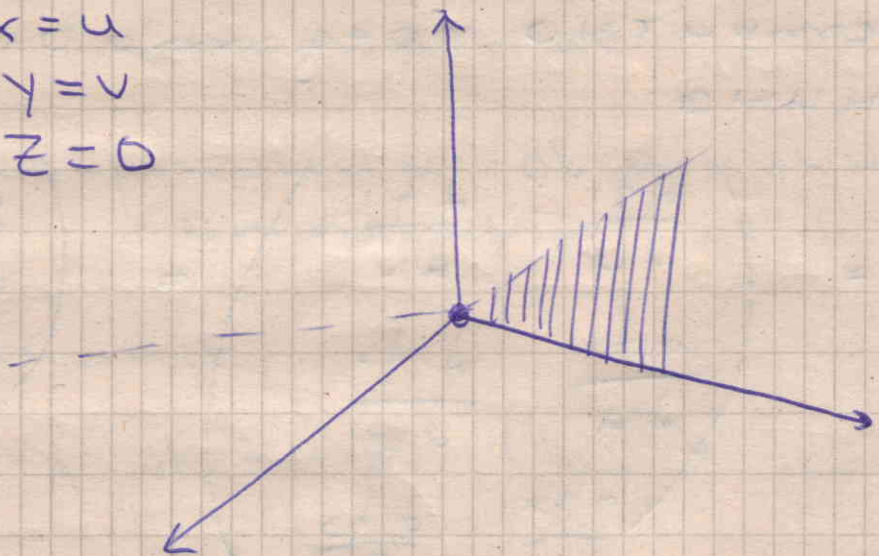


$u \in [0; 2\pi]$

$v \in [0; \pi]$

2) Проста равнина

$$\begin{cases} x = u \\ y = v \\ z = 0 \end{cases}$$



линии на ниво: криви в/у повърхнината, получени при ограничаване до крива в D

$$\begin{aligned} u &= u(t) \\ v &= v(t) \end{aligned} \longrightarrow \begin{pmatrix} f_1(u(t), v(t)) \\ f_2(u(t), v(t)) \\ f_3(u(t), v(t)) \end{pmatrix}$$

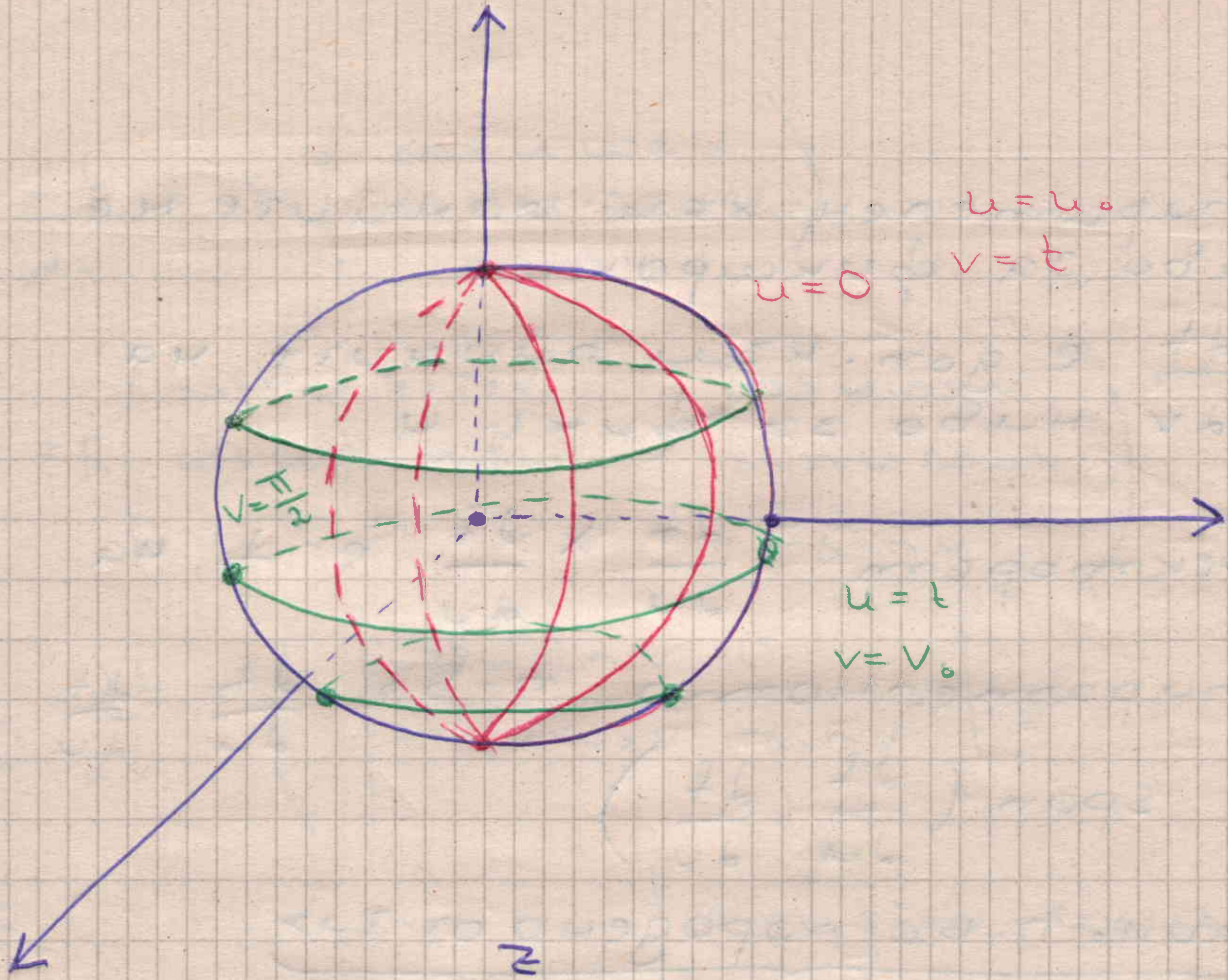
Крива в $\mathbb{R}^3$

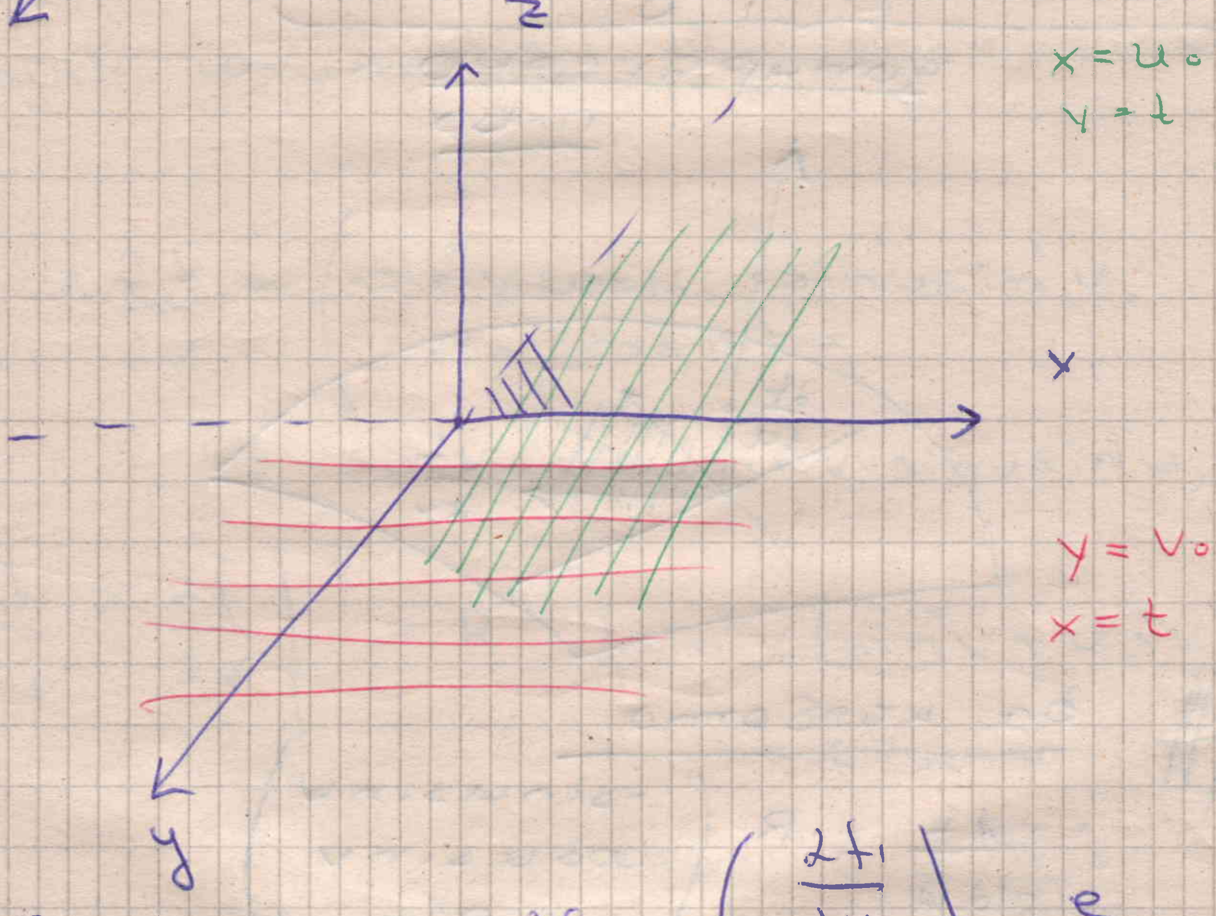
В частност, ако фиксираме

$u = u_0$

$v = t$ или $u = t$ $v = v_0$,

получаваме паралели и меридиани.





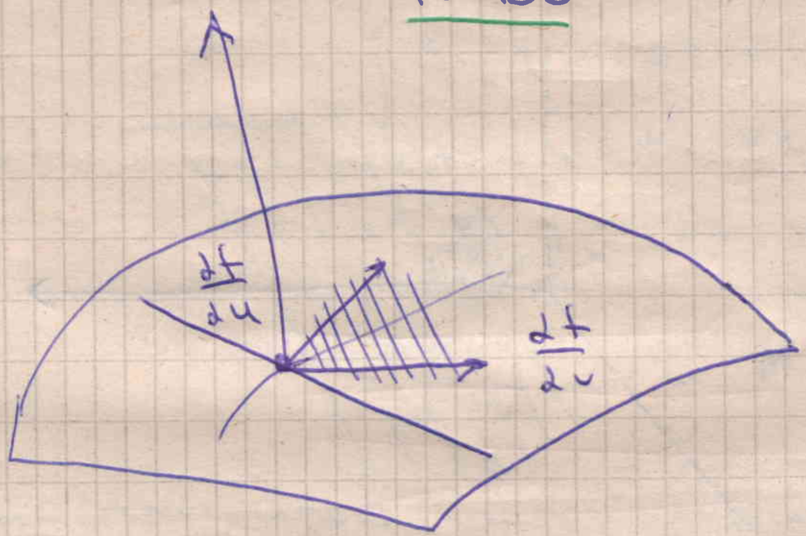
! Векторът $\frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{pmatrix}$ e допирателен към линиите на ниво за фиксирано v

$\frac{\partial f}{\partial v}$ е доп. към линиите на ниво за фикс. u

Векторът $\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$ е $\perp$ на допирателното п-во $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$

$$\underbrace{\text{span}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)}_{\text{лин. п-во, породено от тях}}$$

допирателно п-во



За кълбото:

$$\frac{\partial f}{\partial u} = R \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial v} = R\begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

$$f' = R\begin{pmatrix} -\sin u \sin v & \cos u \cos v \\ \cos u \sin v & \sin u \cos v \\ 0 & -\sin v \end{pmatrix}$$

$$\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} = \left(\begin{vmatrix} \cos u \sin v & 0 \\ \sin u \cos v & -\sin v \end{vmatrix}, \right.$$

$$\left. \begin{vmatrix} 0 & -\sin u \sin v \\ -\sin v & \cos u \cos v \end{vmatrix}, \begin{vmatrix} -\sin u \sin v & \cos u \sin v \\ \cos u \cos v & \sin u \cos v \end{vmatrix} \right)$$

$$\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} = \left(-\sin^2 v \cos u, -\sin u \sin^2 v, -\sin v \cos v \right) =$$

$$= -\sin v \left(\cos u \sin v, \sin v \sin u, \cos v \right)$$

$$\left| \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right|^2 = R^4 \left(\sin^2 v \left(\sin^2 v + \cos^2 v \right) \right) =$$

$$= R^4 \sin^2 v$$

без 0 и $\frac{\pi}{2}$

$$\left| \frac{\partial f}{\partial u} \right|^2 = R^2 \sin^2 v$$

$$\left| \frac{\partial f}{\partial v} \right|^2 = R^2$$

$\boxed{\text{Стандартни означения}}$

$$\left(\frac{\partial f}{\partial u}\right)^2 = \left\langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u} \right\rangle = E$$

$$\left(\frac{\partial f}{\partial v}\right)^2 = \left\langle \frac{\partial f}{\partial v}, \frac{\partial f}{\partial v} \right\rangle = G$$

$$\left\langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right\rangle = F$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \underbrace{\begin{pmatrix} \langle f_u, f_u\rangle & \langle f_u, f_v\rangle \\ \langle f_u, f_v\rangle & \langle f_v, f_v\rangle \end{pmatrix}}_{\text{Грам}}$$

A, B, C - трите координати на вектора $\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$

$$\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} = (A, B, C)$$

В случая, когато повърхнината е зададена като графика на ф-я на 2 променливи:

$$z = f(x, y)$$

Имаме стандартна парамет-ризация

$$\begin{cases} x = x \\ y = y \\ z = f(x, y) \end{cases}$$

$$\begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix} \qquad r' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix}$$

$$r_x = (1, 0, f_x)$$
$$r_y = (0\ 1\ f_y)$$

$$E = \langle r_x, r_x \rangle = 1 + f_x^2$$
$$G = \langle r_y, r_y \rangle = 1 + f_y^2$$
$$F = \langle r_x, r_y \rangle = f_x \cdot f_y$$

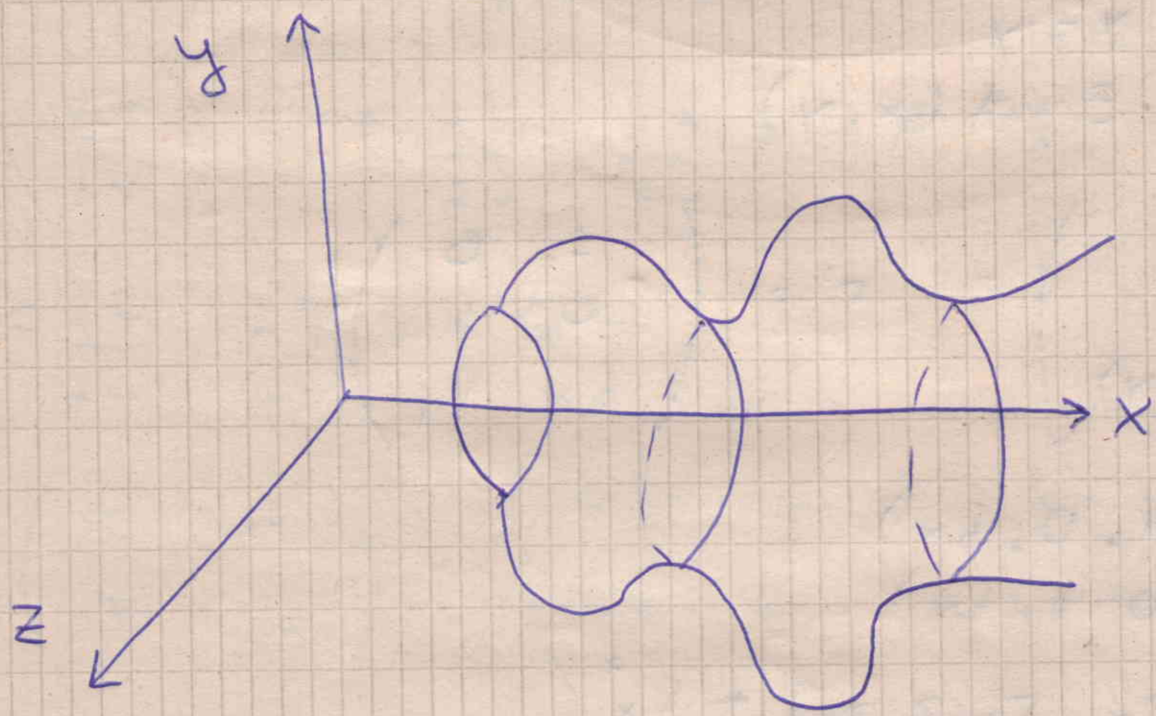
$$(\langle r_x \times r_y \rangle)^2 = (\underset{A}{-f_x}, \underset{B}{-f_y}, \underset{C}{1})$$

$$|r_x \times r_y|^2 = 1 + f_x^2 + f_y^2$$

$$\begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2 = (1 + f_x^2)(1 + f_y^2) - f_x^2 f_y^2 = \underline{1 + f_x^2 + f_y^2}$$

$\cancel{EF}$ $\boxed{EG - F^2 = A^2 + B^2 + C^2}$

Ротационна повърхнина:



имаме крива $\begin{array}{l} x = x(u) \\ y = y(u) \end{array}$ в $\mathbb{R}^2$

Завъртаме кривата на 360° около оста $\overrightarrow{Ox}$, описвайки ротац. пов. Тогава тази рот. пов. има параметризация

$$\boxed{\begin{array}{l} x = x(u) \quad \text{— оста, около която въртим} \\ y = y(u)\cos v \\ z = y(u)\sin v \end{array}}$$

$$r = \begin{pmatrix} x(u) \\ y(u)\cos v \\ y(u)\sin v \end{pmatrix}$$

$$r' = \begin{pmatrix} x'_u & 0 \\ y'_u \cos v & -y(u)\sin v \\ y'_u \sin v & y(u)\cos v \end{pmatrix}$$

$$r_u = (x' \quad y'\cos v \quad y'\sin v)$$
$$r_v = (0, \ -y(u)\sin v \quad y\cos v)$$

$$E = \langle r_u, r_u \rangle = x'^2 + y'^2$$

$$G = \langle r_v, r_v \rangle = y^2$$

$$F = \langle r_u, r_v \rangle = -y'y\cos v\sin u +$$
$$+ y'y\sin v\cos v = 0$$

$$EG - F^2 = EG = (x'^2 + y'^2)y^2$$

$$r_u \times r_v = (yy', \ -yx'\cos v \quad -x'y\sin v)$$

Параметризиране на повърхнини в случая, когато са зададени чрез уравнения:

$$F(x, y, z) = 0$$

Примери: ① $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c^2}$

$$\begin{cases} x = a\cos u \cdot t \\ y = b\sin u\, t \\ z = c^2 t^2 \end{cases} \quad \text{или} \quad \begin{array}{l} x = a\cos u \sqrt{t} \\ y = b\sin u \sqrt{t} \\ z = c^2 t \\ t \geq 0 \\ u \in [0; 2\pi] \end{array}$$

② Конус

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\begin{cases} x = a\cos u\, t \\ y = b\sin u\, t \\ z = ct \end{cases}$$

③ Цилиндрични повърхнини

→ парамет. $f(x, y)$ кривата

z

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = u \end{cases} \quad u \in [a; b]$$

Пример: Да пар. ок. пов. на цилиндър с радиус R и вис. h

$$x^2 + y^2 = R^2$$
$$z \in [0; H]$$

$$\begin{cases} x = R\cos t \\ y = R\sin t \\ z = q \end{cases} \quad \begin{array}{l} t \in [0; 2\pi) \\ q \in [0; H] \end{array}$$

Повърхнинен интеграл от I род

Нека $\mathcal{F}$ е гладка повърхнина с параметризация

$$r(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$ и

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ е непр. в/у $\mathcal{F}$

Пов. инт. от I род

$$\iint\limits_{\mathcal{F}} f \, d\sigma = \iint\limits_{D} f(x(u,v), y(u,v) z(u,v)) \sqrt{EG-F^2} \, du \, dv,$$

където $D \subseteq \mathbb{R}^2$ е областта на параметризация на $\mathcal{F}$.

В частност, ако $f \equiv 1$

$$\iint\limits_{\mathcal{F}} d\sigma = S(\mathcal{F})$$

Примери: Сферата $\longrightarrow S = 4\pi R^2$

$$r(u,v) = \begin{pmatrix} R\cos u \sin v \\ R \sin u \sin v \\ R\cos v \end{pmatrix} \quad \begin{matrix} u \in [0; 2\pi] \\ v \in [0; \pi] \end{matrix}$$

$r_u = R(-\sin u \sin v, \cos u \sin v, 0)$
$r_v = R(\cos u \cos v, \sin u \cos v, -\sin u)$

$E = \langle r_u, r_u \rangle = R^2 \sin^2 v$
$F = R^2$

$G = \langle r_u, r_v \rangle = 0$

$EF = R^4 \sin^2 v$

$\sqrt{EF - G^2} = R^2 \sin v$

$$S(\text{Сф.}) = \iint\limits_{[0,2\pi)\times[0,\pi]} 1 . R^2 \sin v = R^2 \int_0^{2\pi}\int_0^{\pi} (\sin v \, dv) du$$

$$R^2 \int_0^{2\pi} -\cos v \Big|_0^{\pi} dv = R^2 \int_0^{2\pi} (-(-1-1)) = $$
$$= 4\pi R^2$$

<u>Пример 2</u> : ф като графика на ф-ята $f(x,y)$ $(x,y) \in D$

$$r = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$$

$r_x = (1, 0, f_x)$
$r_y = (0, 1, f_y)$

$E = \langle r_x, r_x \rangle = 1 + f_x^2$

$F = 1 + f_y^2 = \langle r_y, r_y \rangle$

$G = \langle r_x, r_y \rangle = f_x f_y$

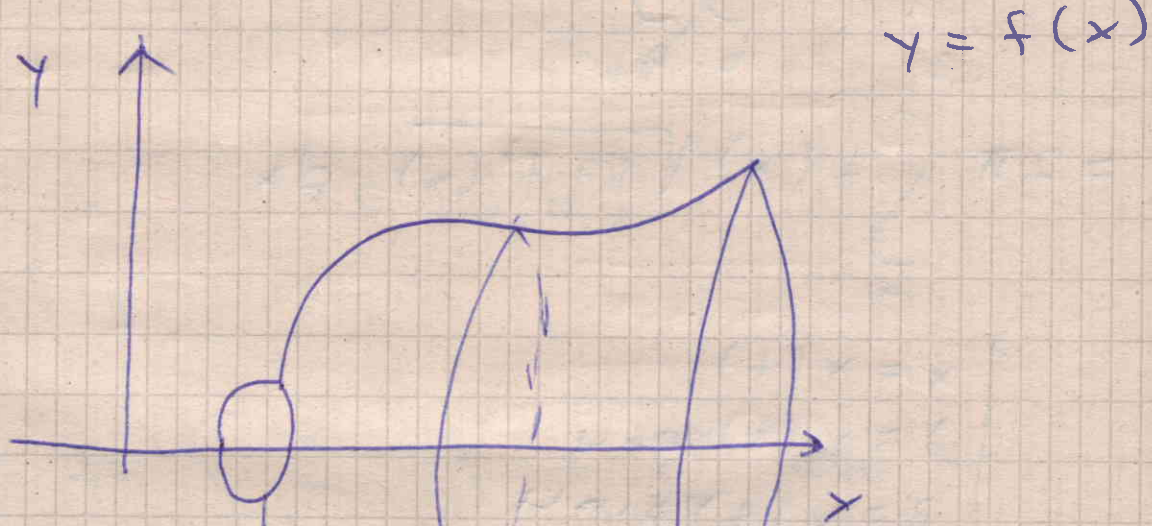
$$\sqrt{EF - G^2} = \sqrt{1 + f_x^2 + f_y^2}$$

$$S(\text{гр.}) = \iint\limits_D \sqrt{EF - G^2}\, dx\, dy = \iint\limits_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

~~Rot~~ **Пример 3**: Ротационна повърхнина

$y = f(x) \quad x \in [a, b]$

и образ. рот. пов. с ос $\overrightarrow{Ox}$



$$r = \begin{pmatrix} x \\ f(x)\cos u \\ f(x)\sin u \end{pmatrix}$$

$x \in [a, b]$
$u \in [0, 2\pi]$

$r_x = (1, f' \cos u, f' \sin u)$
$r_u = (0, -f \sin u, f \cos u)$

$E = \langle r_x, r_x \rangle = 1 + f'^2$
$G = \langle r_x, r_u \rangle = 0$
$F = \langle r_u, r_u \rangle = f^2$

$$\sqrt{EF - G^2} = \sqrt{f^2(1 + f'^2)} \qquad f > 0$$

$$\Rightarrow S(\Phi) = \int_0^{2\pi} \int_a^b |f| \sqrt{1 + f'^2}\, dx\, du =$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + f'(x)}\, dx$$

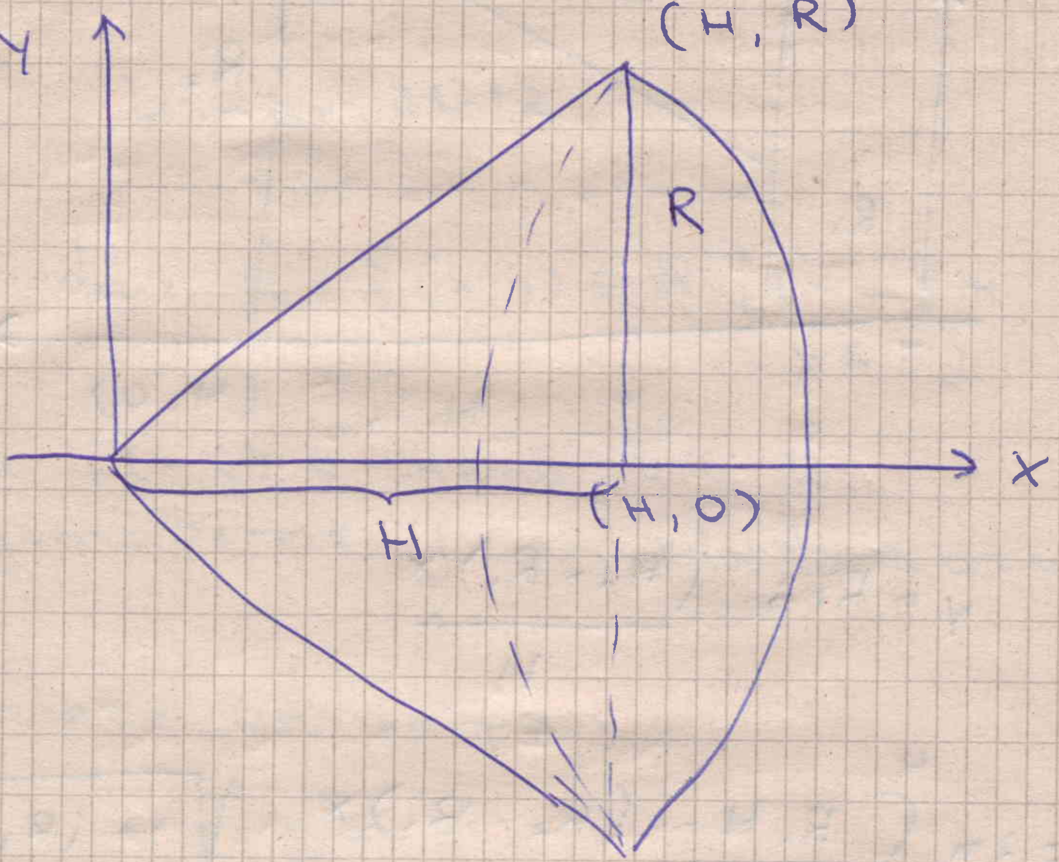
$$\left| \begin{array}{l} x = x(t) \\ y = y(t) \cos u \\ z = y(t) \sin u \end{array} \right.$$

$r_x = (x', y' \cos u, y' \sin u)$
$r_u = (0, -y \sin u, y \cos u)$

$E = x'^2 + y'^2$
$G = 0$
$F = y^2$

$$\boxed{S = 2\pi \int_{a}^{b} y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt}$$

Пр.



$$y = \underbrace{\frac{R}{H} x}_{f(x)} \qquad x \in [0, H]$$

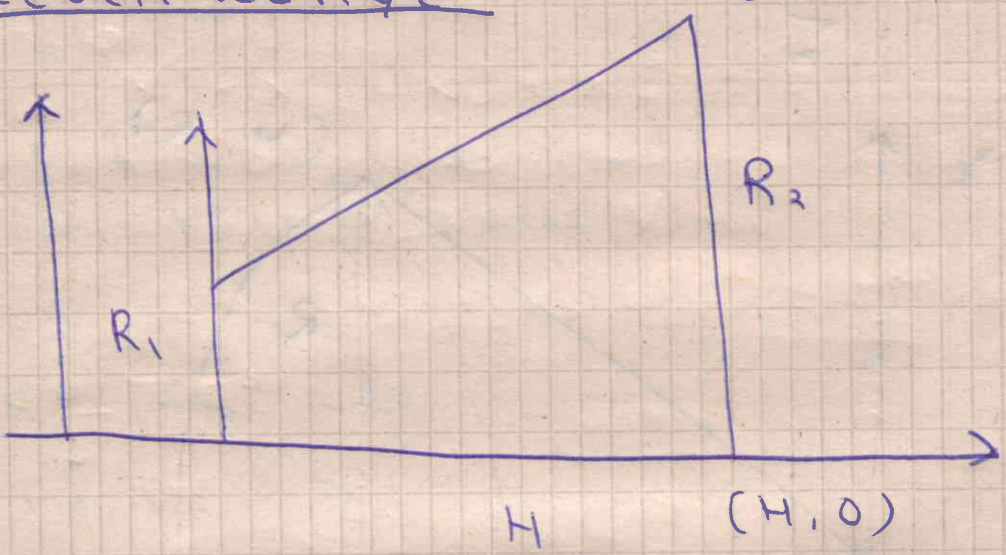
$$S_{ок.п.} = 2\pi \int_0^H \frac{R}{H} x \sqrt{1 + \frac{R^2}{H^2}}\, dx =$$

$$= 2\pi \frac{R}{H^2} \int_0^H x\, dx = \frac{2\pi R\sqrt{R^2+H^2}}{H^2} \left.\frac{x^2}{2}\right|_0^H =$$

$$= \underline{\underline{\pi R\sqrt{R^2+H^2}}} \qquad S_к = \pi R\sqrt{R^2+H^2} + \pi R^2$$

Пресечен конус

$(0, R_2)$



$(H, 0)$

$$y = R_1 + \frac{(R_2 - R_1)x}{H}$$

$$I = 2\pi \int_0^H \frac{R_1 H - (R_2 - R_1)x}{H} \sqrt{1 + \frac{(R_2 - R_1)^2}{H^2}}\, dx =$$

$$= \frac{2\pi \sqrt{H^2 + (R_2 - R_1)^2}}{H^2} \left(\int_0^H R_1 H\, dx + \int_0^H (R_2 - R_1) x\, dx \right)$$

$$= \frac{2\pi}{H^2} \left(H^2 R_1 + \frac{(R_2 - R_1)}{2} H^2 \right) =$$

$$= 2\pi R_1 + \pi (R_2 - R_1) =$$

$$= \pi (R_1 + R_2) \sqrt{H^2 + (R_2 - R_1)^2}$$

15.01.2014г.

Упражнение

Изпит 2012

② Разгл. фиг. $K = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4;\ y \le x\}$

Нека $f(x,y)$ е непр. в/у K
Представете $\iint\limits_K f(x,y)\,dx\,dy$ като

повторен интеграл

$x = \rho\cos\varphi$
$y = \rho\sin\varphi$

$\varphi \in [0; 2\pi]$

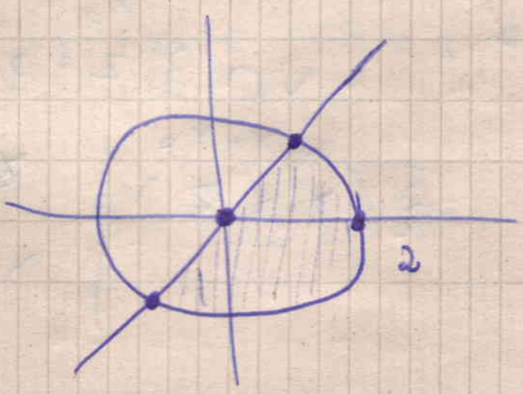
$\rho^2 \le 4$

$0 \le \rho \le 2$

$\rho\sin\varphi \le \rho\cos\varphi$

$0 \le \cot g\varphi$

$tg\varphi \le 0$



$\varphi \in [\pi; 2\pi]$

$$\int_{\pi}^{2\pi}\int_{0}^{2} f(x,y)\,dx\,dy$$

4) Пресм. обема на четиримерното кълбо

$$x_1^2+x_2^2+x_3^2+x_4^2 \le r^2$$ с радиус

r, като използвате Кавалери и факта, че V на

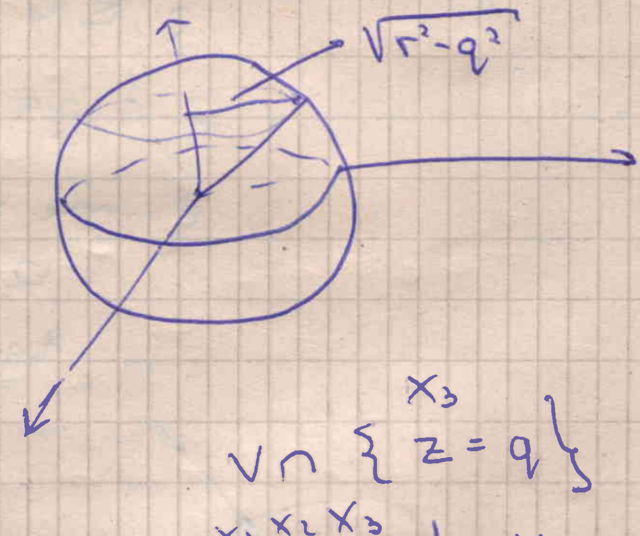
$$x_1^2+x_2^2+x_3^2 \le r^2 \text{ е}$$

$$\frac{4}{3}\pi r^3$$

$$x_4^2 \le r^2-x_1^2-x_2^2-x_3^2$$

$$-\sqrt{r^2-x_1^2-x_2^2-x_3^2} \le x_4 \le \sqrt{r^2-x_1^2-x_2^2-x_3^2}$$

В $\mathbb{R}^3$ $x_1^2+x_2^2+x_3^2=r^2$: V



$$V \cap \{\overset{x_3}{z}=q\} =$$

$$\{(\overset{x_1}{x},\overset{x_2}{y},\overset{x_3}{z}) \mid \cancel{x_3}=q,\ x_1^2+x_2^2=r^2-q^2\}$$

$\cap$

кр

$$V = \bigcup_{q=-r}^{r} C_q .$$ Съгласно Кавалиери

$$\iiint\limits_{x_1^2+x_2^2+x_3^2} 1\,dx_1\,dx_2\,dx_3 = \int_{-r}^{r} \iint\limits_{x_1^2+x_2^2 \le \sqrt{r^2-x_3^2}} 1\,dx_1\,dx_2\,dx_3 \;;$$

В $\mathbb{R}^4$:

$$V: x_1^2 + x_2^2 + x_3^2 + x_4^2 \le r^2$$

$$V \cap \{x_4 = q\} = \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; \begin{array}{l} x_4 = q \\ x_1^2 + x_2^2 + x_3^2 \le \sqrt{r^2 - q^2}^{\,2} \end{array} \right\}$$

$$\parallel$$

$$K_q$$

$$V = \bigcup_{q=-r}^{r} K_q$$

$$V = \int_{-r}^{r} \iiint\limits_{x_1^2+x_2^2+x_3^2 \le \sqrt{r^3-x_4^2}} dx_1\,dx_2\,dx_3 \; dx_4 =$$

$$= \int_{-r}^{r} v\,d(K_{x_4})\,dx_4 =$$

$$= \int_{-r}^{r} \frac{4}{3}\pi \sqrt{r^2 - x_4^2}^{\,3}\, dx_4$$

$$\sqrt{r^2 - x_4^2} \qquad \underline{\underline{x_4 = r\cos\varphi}}$$

$$dx_4 = -r\sin\varphi\, d\varphi$$

$$x_4 = -r$$

$$\varphi = \pi$$

$$x_4 = r$$

$$\varphi = 0$$

$$\Rightarrow \int_{\pi}^{0} \frac{4}{3}\pi \sqrt{\overset{r^2 - r^2\cos^2\varphi}{r^2\sin^2\varphi}}^{\,3} \cdot (-r\sin\varphi\, d\varphi) =$$

$$= \frac{4}{3}\pi r^4 \int_{0}^{\pi} \sin^3\varphi \sin\varphi\, d\varphi$$

$$I = \frac{4}{3}\pi r^4 \int_{0}^{\pi} \sin^4\varphi\, d\varphi$$

$$\sin^4\varphi = (\sin^2\varphi)^2 =$$

$$= \frac{4}{3}\pi r^4 \int_{0}^{\pi} \qquad = \ldots$$

$$\cos 2\varphi = 1 - 2\sin^2\varphi$$

За

Обе
Вы

5. П

е

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Пр

За

Обемът V_n на n-мерно кълбо с $r=1$ е вярно

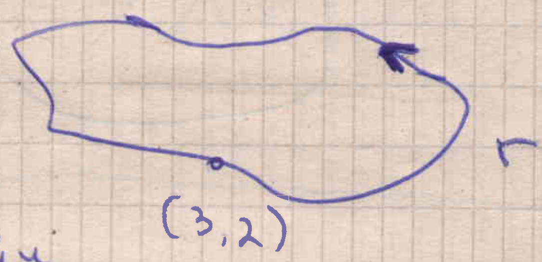
$$\lim_{n\to\infty} V_n = 0 \quad ?$$

(5.) Пресметнете крив. инт. от 2 род

$$\oint_\Gamma \frac{y-2}{(x-3)^2+(y-2)^2}\,dx - \frac{(x-3)}{(x-3)^2+(y-2)^2}\,dy$$, където Γ

е проста частично гладка и $(3,2)\notin\Gamma$

1 сл. $(3,2)\notin \operatorname{int}\langle\Gamma\rangle$



Прилагаме Грийн

$$\iint_D \left(\frac{\partial}{\partial x}Q - \frac{\partial P}{\partial y}\right)dx\,dy =$$

$$= -\iint \frac{1\left((x-3)^2+(y-2)^2\right) + 2(x-3)}{}$$

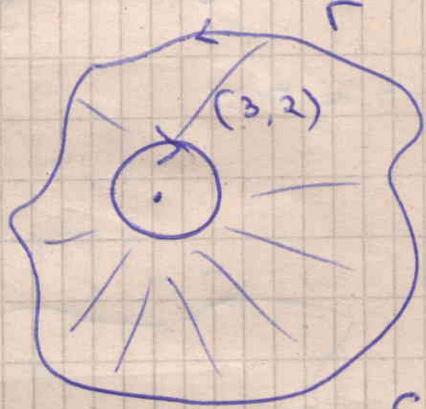
$$\frac{d}{dx}\frac{(x-3)}{\underbrace{(x-3)^2+(y-2)^2}_{=Q}} = \frac{(y-2)^2-(x-3)^2}{Q^2}$$

$$\frac{d}{dy}\frac{y-2}{(x-3)^2+(y-2)^2} = \frac{(x-3)^2-(y-2)^2}{Q^2}$$

$$\frac{1}{2}\ln\left((x-3)^2+(y-2)^2\right)$$

$$\Rightarrow I = 0$$

II сл. $(3,2) \in D$



Ограждаме $(3,2)$ с окр. с радиус $r=\varepsilon$

$\varepsilon < \min((3,2), \Gamma)$ > 0

и нека тази окр. е C_ε.
Имаме, че

$$\oint_{\Gamma \cup C_\varepsilon^-} \langle F, dr \rangle = \iint_{int(\Gamma \cup C_\varepsilon^-)} 0 \, dx\, dy$$

$$\Rightarrow \oint_{\Gamma} \langle F, dr \rangle + \int_{C_\varepsilon^-} \langle F, dr \rangle = 0 \Rightarrow$$

$$\oint_{\Gamma} \langle F, dr \rangle = \oint_{C_\varepsilon^+} \langle F, dr \rangle$$

$$x = 3 + \varepsilon \cos\varphi \qquad \varphi \in [0; 2\pi]$$
$$y = 2 + \varepsilon \sin\varphi$$

$$\dot{x} = -\varepsilon \sin\varphi$$
$$\dot{y} = \varepsilon \cos\varphi$$

$$I = \int_0^{2\pi} \left(\frac{\varepsilon \sin\varphi \cdot (-\varepsilon \sin\varphi)}{\varepsilon^2} + \right.$$

$$\left. + \frac{\varepsilon \cos\varphi}{\varepsilon^2} \cdot \varepsilon \cos\varphi \right) d\varphi =$$

$$= -2\pi$$

---

⑦ Разгледайте хомогенна мат. нишка, разположена по половин арка на циклоидата

$$a(t) = (t - \sin t, 1 - \cos t) \quad 0 \le t \le \pi$$

Намерете центъра на масите на тази нишка.

$$\dot{x} = 1 - \cos t$$
$$\dot{y} = \sin t$$

$$\int_0^{\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt =$$

$$= \int_0^{\pi} \sqrt{1 - 2\sin t \cos t + 1}\, dt =$$

$$M = \int_{\Gamma} \rho\, dl =$$

$$= \int_0^{\pi} \rho \sqrt{\dot{x}^2 + \dot{y}^2}\, dt =$$

$$= \int_0^{\pi} \rho \sqrt{2 - \sin 2t}\, dt$$

$$x_G = \frac{1}{M} \int_{\Gamma} \rho x\, dl$$

$$x_G = \frac{1}{M} \int_0^{2\pi} (t - \overset{\sin}{\cancel{\cos}}\, t) \rho\, 2 \sin \frac{t}{2}\, dt$$

$$y_G = \frac{1}{M} \rho \int_0^{2\pi} (1 - \cos t)\, 2 \sin \frac{t}{2}\, dt$$

(8) Нека F е гладко поле в $\mathbb{R}^3$. Да озн. с S_ε сферата с ц-р v и радиус ε, ориент. с външна нормала. Док., че

$$\operatorname{div} F(x) = \lim_{\varepsilon \to 0} \frac{\iint\limits_{S_\varepsilon} \langle F, n \rangle ds}{\frac{4}{3}\pi\varepsilon^3}$$

$$\operatorname{div} F(x) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \qquad F = (F_1, F_2, F_3)$$

Г-О

$$\iint\limits_{S_\varepsilon} \langle F, n \rangle ds = \iiint\limits_{\cancel{x_1^2+x_2^2+x_3^2 \le \varepsilon^2}\ (x_1-x^1)^2+(x_2-x^2)^2+(x_3-x^3)^2 \le \varepsilon^2} \operatorname{div} F(x_1, x_2, x_3)\, dx_1 dx_2 dx_3$$

$\Rightarrow \quad x = (x^1, x^2, x^3)$ център на кълбото

непр. => Th ср. стойност

$$\frac{\iint\limits_{S_\varepsilon} \langle F, n \rangle ds}{\frac{4}{3}\pi\varepsilon^3} \overset{!}{=} \frac{\iiint\limits_{(x-x^1)^2+\dots+ < \varepsilon^2} \operatorname{div} F(x_1, x_2, x_3)\, dx_1 dx_2 dx_3}{\frac{4}{3}\pi\varepsilon^3} =$$

$$= \frac{\operatorname{div} F\left(\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}\right) \cancel{\frac{4}{3}\pi\varepsilon^3}}{\cancel{\frac{4}{3}\pi\varepsilon^3}} \xrightarrow[\varepsilon \to 0]{} \operatorname{div} F\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

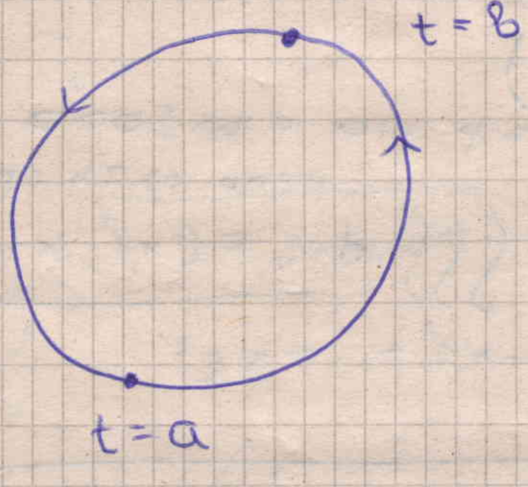
$$\xi_1^2 + \xi_2^2 + \xi_3^2 < \varepsilon^2$$

$\boxed{2012}$ / 6 $\quad F = \left(\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z} \right)$

$$\Gamma: x(t), y(t), z(t) \quad t \in [a, b]$$

$$\int_\Gamma \langle F, dr \rangle = \int_a^b \underbrace{\frac{\partial u}{\partial x} \dot{x} + \frac{\partial u}{\partial y} \dot{y} + \frac{\partial u}{\partial z} \dot{z}}\, dt$$

$$= \int_a^b \frac{d}{dt} u(x(t), y(t), z(t))\, dt = u(b) - u(a)$$



Намерете потенциала

$$F(x, y) = \left(e^x (x + \ln y + 1), \frac{e^x}{y} \right)$$

В каква област е деф. F?

едносвързана ли е $\quad y > 0$

$$\cancel{\frac{dF}{dx}} = e^x \cdot x + e^x \ln y + e^x = \frac{du}{dx}$$

$$u = \int \left(e^x \cdot x + e^x \ln y + e^x \right) dx + A(y)$$

$$= \int e^x \cdot x \, dx + \ln y \cdot e^x + e^x + A(y)$$

$$\int e^x \cdot x \, dx = \int x \, de^x = e^x \cdot x - e^x$$

$$\Rightarrow e^x \cdot x + \ln y \cdot e^x + A(y)$$

$$\frac{du}{dy} = \frac{e^x}{y} = \frac{e^x}{y} + A'(y) \qquad \underline{\underline{A(y) = C}}$$

$$\Rightarrow u = e^x \cdot x + \ln y \cdot e^x + C$$

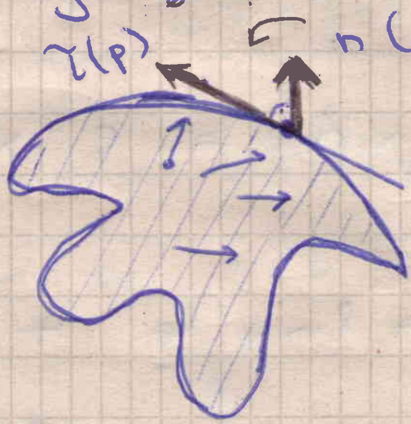
Лекция

<u>Формула на Грийн с дивергенция:</u>

$\Omega \subset \mathbb{R}^2$
област
$\partial\Omega$ - частично гладка

$\tilde{\Omega} \supset \bar{\Omega} = \Omega \cup \partial\Omega$
отв.

F гладко в-но поле в $\tilde{\Omega}$



$$\int_{\partial\Omega} \langle F, n \rangle \, ds$$

$n(p)$, $p \in \partial\Omega$
външната нормала
към Ω

$\|n(p)\| = 1$, сочи навътре

$n(p) \perp \tilde{\tau}(p)$

$$\int_{\partial\Omega} \langle F, n \rangle \, ds$$

част. гладка крива

можем да параметризираме

$$\partial\Omega : \alpha(s); \; s \in [0, S]$$

параметризация с ест. параметър

$(\|\alpha'(s)\| = 1)$ в

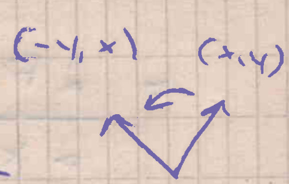
правилна посока

$\alpha'(s) = \tau(\alpha(s))$ единичен в-р по допир. сочи в правилна посока

$\tau(p) = (\tau_1(p), \tau_2(p))$ $\quad n(p) = (n_1(p), n_2(p))$

$$\| \quad = (-n_2(p), n_1(p))$$

ротация на $\frac{\pi}{2}$



$$\int_{\partial\Omega} \langle F, n \rangle \, ds = \int_0^S \Big(F_1(\alpha(s)) \cdot n_1(\alpha(s)) + F_2(\alpha(s)) \cdot n_2(\alpha(s)) \Big) \cdot 1 \, ds$$

естествен параметър $\| \, \| = 1$

$$= \int_0^S F_1(\alpha(s)) . \tilde{\tau}_2(s) - F_2(\alpha(s)) . \tilde{\tau}_1(\alpha(s)) \, ds =$$

Грийн

$$= \int_0^s \langle(F_2, F_1), \tilde{\tau}\rangle \, ds = \int_{\partial\Omega} -F_2\, dx_1 + F_1\, dx_2 =$$

$$= \iint_\Omega \underbrace{\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)}_{\text{дивергенция}} dx_1 \, dx_2 \quad :)$$

Деф. Дивергенция: F гладко в-но поле в област $\Omega \subset \mathbb{R}^n$

$$\operatorname{div} F(x) := \langle \nabla, F\rangle(x) =$$

$$= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \dots + \frac{\partial F_n}{\partial x_n}$$

разходимост на полето

Формула на Гаус-Остроградски

Ω област в $\mathbb{R}^3$

$\partial\Omega$ частично гладка повърхнина

$$= \int_0^S F_1(\alpha(s)) \cdot \tilde{\iota}_2(s) - F_2(\alpha(s)) \cdot \tilde{\iota}_1(\alpha(s))\, ds$$

$$= \int_0^s \langle (F_2, F_1), \tilde{\iota} \rangle\, ds = \int_{\partial\Omega} -F_2\, dx_1 + F_1\, dx_2 \overset{\text{Грийн}}{=}$$

$$= \iint_\Omega \underbrace{\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right)}_{\text{дивергенция}} dx_1\, dx_2 \quad :)$$

Деф. Дивергенция: F гладко в-но поле в област $\Omega \subset \mathbb{R}^n$

$$\operatorname{div} F(x) := \langle \nabla, F \rangle (x) =$$

$$= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \ldots + \frac{\partial F_n}{\partial x_n}$$

разходимост на полето

Формула на Гаус-Остроградски

Ω област в $\mathbb{R}^3$

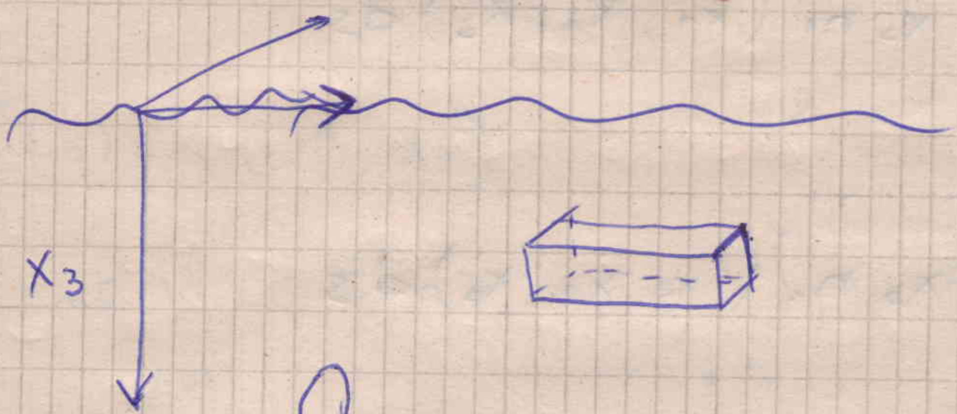
$\partial\Omega$ частично гладка повърхнина

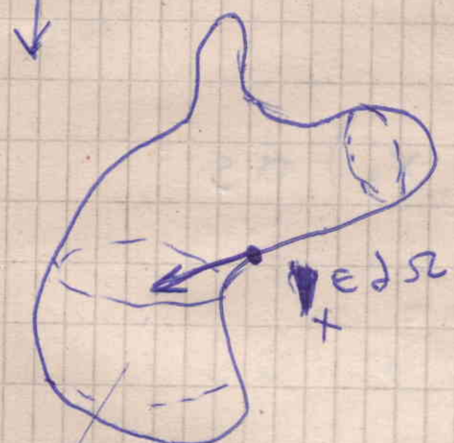
$\partial\Omega$ е ориентирано с външната (за Ω) нормала n.
F - гладко в-но поле в околност на $\bar{\Omega} = \Omega \cup \partial\Omega$

Тогава $$\iint_{\partial\Omega} \langle F, n \rangle \, ds = \iiint_{\Omega} \operatorname{div} F \, dx_1 dx_2 dx_3$$

(цилиндрично тяло по трите)

Закон на Архимед





Ω област
$\partial\Omega$ част. гладка пов.

x_1, x_2 - по повърхността
x_3 - дълбочина

ρ-относително тегло на течността

$$p(x_1, x_2, x_3) = -\rho\, x_3\, n(x_1, x_2, x_3)$$

$$P = \iint_{\partial\Omega} p(x_1, x_2, x_3)\, ds =$$

$$= -\rho \iint_{\partial\Omega} x_3\, n(x_1, x_2, x_3)\, ds$$

$$P_1 = -\rho \iint_{\partial\Omega} x_3\, n_1(x_1, x_2, x_3)\, ds$$

$$P_2 = -\rho \iint_{\partial\Omega} x_3\, n_2(x_1, x_2, x_3)\, ds$$

$$P_3 = -\rho \iint_{\partial\Omega} x_3\, n_3(x_1, x_2, x_3)\, ds$$

Взимаме полето

$$\tilde{F}(x_1, x_2, x_3) = (x_3, 0, 0)$$

$< \tilde{F}, n > =$

$$\Rightarrow p_1 = -\rho \iint_{\Omega} < \tilde{F}, n > ds = \quad \underline{\underline{\Gamma - O}}$$

$$= -\rho \iiint_{\Omega} \underbrace{\operatorname{div} \tilde{F}}_{0} \, dx_1 \, dx_2 \, dx_3 = 0$$

за p_2 $\quad \tilde{\tilde{F}}(x) = (0, x_3, 0)$

$$p_2 = -\rho \iint_{\partial\Omega} < \tilde{\tilde{F}}, n > ds = -\rho \iiint_{\Omega} \underbrace{\operatorname{div} \tilde{\tilde{F}}}_{= 0} dx_1 dx_2 dx_3$$

$\bar{F}(x) = (0, 0, x_3)$

$$p_3 = -\rho \iint_{\partial\Omega} < \bar{F}, n > = -\rho \iiint_{\partial\Omega} \operatorname{div} \bar{F} = -\rho \iiint_{\Omega} 1 \, dx_1 \, dx_2 \, dx_3 =$$

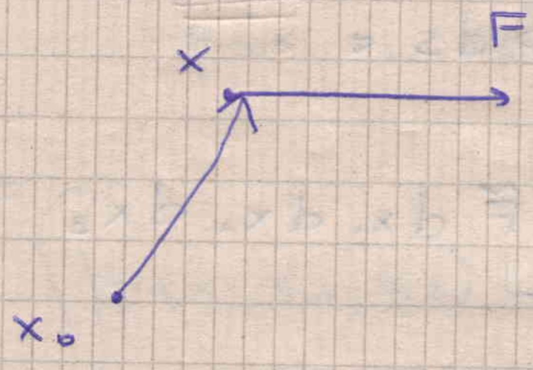
$$\underline{\underline{-\rho V_{\Omega}}}$$

$\Rightarrow P = (0, 0, -\rho V_{\Omega})$

насочена нагоре

головина - тегло изместена течност

зад. Лодка



$M = (x - x_0) \times F$ въртящ момент

? т. x_0, спрямо която $M = 0$
приложна точка на силата

$x_0 = (x_0^1, x_0^2, x_0^3)$

Общият момент

$$M = \iint_{\partial\Omega} (x - x_0) \times P(x)\, ds$$

елемент. налягане

$$M_{x_0} = \iint_{\partial\Omega} (x - x_0) \times P(x)\, ds$$

$$x - x_0 = (x_1 - x_0^1,\ x_2 - x_0^2,\ x_3 - x_0^3)$$

$$p(x) = (-\rho x_3 n_1(x),\ -\rho x_3 n_2(x),\ -\rho x_3 n_3(x))$$

$$M_{x_0}^1 = -\rho \iint_{\partial\Omega} (x_2 - x_0^2)\, x_3 n_3(x) - (x_3 - x_0^3)\, x_3 n_2(x)\, ds$$

$$= -\rho \iiint_{\partial\Omega} \langle (0,\ -x_3(x_3 - x_0^3),\ x_3(x_2 - x_0^2)),\ n \rangle\, ds$$

$$= -\rho \iiint_{\partial\Omega} (0 + 0 + x_2 - x_0^2)\, dx_1\, dx_2\, dx_3 = 0$$

$$\iiint_{\Omega} x_2\, dx_1\, dx_2\, dx_3 = \iiint_{\Omega} x_0^2\, dx_1\, dx_2\, dx_3$$

$$x_0^2 \cancel{\iiint_{\Omega} 1\, dx_1\, dx_2\, d} x_3 \cdot V_\Omega = \iiint_{\Omega} x_2\, dx_1\, dx_2\, dx_3$$

$$x_0^2 = \frac{1}{V_\Omega} \iiint_{\Omega} x_2\, dx_1\, dx_2\, dx_3$$

$$M_{x_0}^2 = -\rho \iiint_{\partial\Omega} (x_3 - x_0^3)\, x_3 n_1(x) - (x_1 - x_0^1) \cdot x_3 n_3(x)\, ds$$

$$M^2_{x_0} = \rho \iint_{\partial\Omega} \langle x_3(x_3 - x_3^0),\ 0,\ -x_3(x_1 - x_0^1)\rangle, n\rangle \, ds$$

$$\overset{\text{Г-О}}{=} -\rho \iiint_{\Omega} \left(0,\ 0,\ -(x_1 - x_0^1)\right) dx_1\, dx_2\, dx_3$$

$$\Rightarrow \iiint_{\Omega} x_1\, dx_1\, dx_2\, dx_3 = \iiint_{\Omega} x_0^1\, dx_1\, dx_2\, dx_3$$

ц. на масите на геом. тяло с равномерна плътност

$$\begin{cases} x_0^1 = \dfrac{1}{V_\Omega} \cdot \displaystyle\iiint_{\Omega} x_1\, dx_1\, dx_2\, dx_3 \\ x_0^2 = \dfrac{1}{V_\Omega} \cdot \displaystyle\iiint_{\Omega} x_2\, dx_1\, dx_2\, dx_3 \end{cases}$$

$$M^3_{x_0} = -\rho \iint_{\partial\Omega} \{(x_1 - x_0^1)x_3 n_2(x) - (x_2 - x_0^2)x_3 n_1(x)\}\, ds =$$

$$= -\rho \iint_{\partial\Omega} \langle -x_3(x_2 - x_0^2),\ x_3(x_1 - x_0^1),\ 0\rangle, n\rangle \, ds =$$

$$\overset{\text{Г-О}}{=} -\rho \iiint_{\Omega} (0, 0, 0)\, dx_1\, dx_2\, dx_3 = 0$$

ОС

правата през т. на мaите на геом. тяло

Лема $\bar{\Omega} = \Omega \cup \partial\Omega$

цилиндрично тяло и по трите променливи

$$\iint_{\partial\Omega} \langle F, n \rangle \, ds = \iiint_{\Omega} \operatorname{div} F \, dx_1 \, dx_2 \, dx_3$$

$$\iiint_{\Omega} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) dx_1 \, dx_2 \, dx_3$$

$$\iiint_{\Omega} \frac{\partial F_3}{\partial x_3} \, dx_1 \, dx_2 \, dx_3 \overset{?}{=} \iint_{\partial\Omega} \langle \tilde{F}, n \rangle \, ds$$

$\bar{\Omega}$ цилиндрично тяло по x_3

$$\tilde{F} = (0, 0, F_3)$$

$$\bar{\Omega} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in D, \; g_1(x_1, x_2) \le x_3 \le g_2(x_1, x_2) \right\}$$

D измерим компакт в $\mathbb{R}^2$

$g_1, g_2 : D \to \mathbb{R}$ $\quad g_1(x_1, x_2) \le g_2(x_1, x_2)$ в D

непр.
част. гладки

$$\iiint_{\Omega} \frac{\partial F_3}{\partial x_3}\, dx_1\, dx_2\, dx_3 =$$

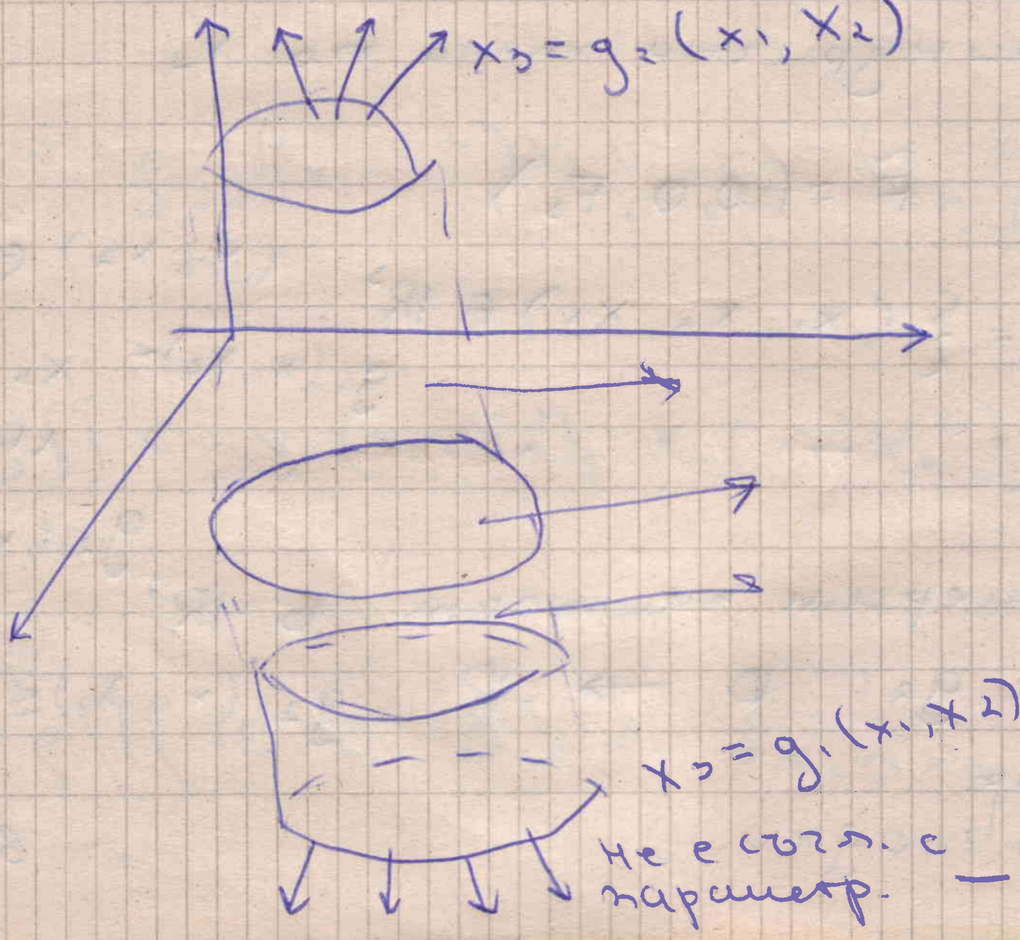
$$= \iint_{D} \int_{g_1(x_1, x_2)}^{g_2(x_1, x_2)} \left(\frac{\partial F_3}{\partial x_3}(x_1, x_2, x_3)\, dx_3 \right) dx_1\, dx_2$$

$F_3(x_1, x_2, \cdot)$

$$= \iint_{D} F_3(x_1, x_2, g_2(x_1, x_2)) - F_3(x_1, x_2, g_1(x_1, x_2))\, dx_1\, dx_2$$

*

Контур на цил. тяло



n - нормала, сочи нагоре
стандартна н.

$$\partial\Omega = \{ \underbrace{(x_1, x_2, g_2(x_1, x_2))}_{(S_1, n)} : \underline{(x_1, x_2) \in D} \} \cup$$

$$\cup \{ \underbrace{(x_1, x_2, g_1(x_1, x_2))}_{(S_2, n)} : (x_1, x_2) \in D \} \cup$$

$$\cup \{ (x_1, x_2, x_3) : (x_1, x_2) \in \partial D, x_3 \in [g_1(x_1, x_2), g_2(x_1, x_2)] \}$$

$$\iint_D F_3 (x_1, x_2, g_2(x_1, x_2))\, dx_1 dx_2 =$$

$$= \iint_{S_1} \langle \tilde{F}, n \rangle$$

$$\tilde{F} = (0, 0, F_3)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ g_2(x_1, x_2) \end{pmatrix} \quad n(x) = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial g_2}{\partial x_1} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial g_2}{\partial x_2} \end{pmatrix}$$

нормала, съгласувана с параметризацията

$$\begin{vmatrix} 1 & 0 & \frac{\partial g_2}{\partial x_1} \\ 0 & 1 & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

$$\left(-\frac{\partial g_2}{\partial x_1}, -\frac{\partial g_2}{\partial x_2}, 1 \right)$$

$$\iint_S \langle \tilde{F}, n \rangle = \iint_D \langle \underset{(0,0,F_3)}{\tilde{F}_1}, \left(-\frac{\partial g_2}{\partial x_1}, -\frac{\partial g_2}{\partial x_2}, 1 \right) \rangle dx_1 dx_2$$

$$= \iint_D F_3 (x_1, x_2, g_2(x_1, x_2)) \, dx_1 dx_2$$

за долната капачка —

$$\circledast$$

$$= \iint_{S_1} \langle \tilde{F}, n \rangle ds + \iint_{S_2} \langle \tilde{F}, n \rangle ds +$$

$$+ \iint_{S_3} \underset{0}{\underset{\parallel}{\langle \tilde{F}, n \rangle}} ds = \iint_{\partial \Omega} \langle \tilde{F}, n \rangle ds$$

∂D парамтр.

$\alpha(t) \quad t \in [a, b]$

∂D

$$\left(\begin{matrix} \alpha_1(t) \\ \alpha_2(t) \\ x_3 \end{matrix} \right)$$

$$\begin{pmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \dot{\alpha}_1 & \dot{\alpha}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left((\dot{\alpha}_2, -\dot{\alpha}_1, 0)(0, 0, F_3) \right)$$

$$= 0 \quad :)$$

$$\Rightarrow I = \iint_{\partial\Omega} \langle \tilde{F}, n \rangle \, ds \qquad \tilde{F} = (0, 0, F_3)$$

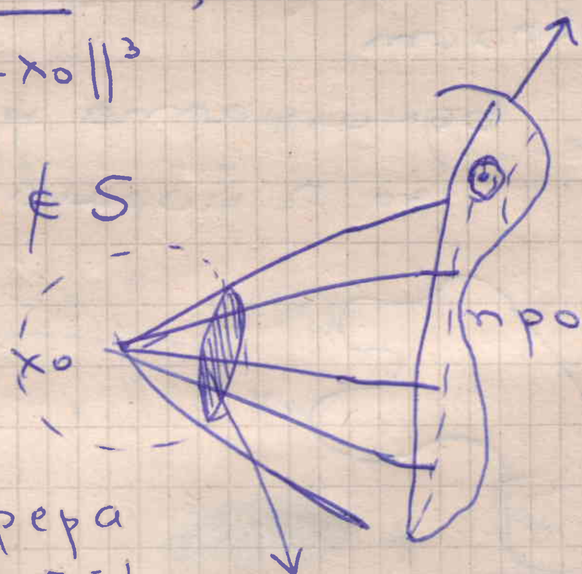
$$= \iint_{\partial\Omega} F_3 \, dx_1 \, dx_2 \qquad :))$$

Интеграл на Гаус:

(s, n)

$$\iint_S \left\langle \frac{x - x_0}{\|x - x_0\|^3}, n \right\rangle ds$$

$x_0 \notin S$



мн. малки парченца -> като допир.

п-во

което в/х единичната сфера ($\|x-x_0\|^2$

$$\iint_S \frac{1}{\|x-x_0\|^2} \cdot \left\langle n, \frac{x-x_0}{\|x-x_0\|} \right\rangle$$

cos на ъгъла

ъгъла, под който се вижда S

$S \equiv \partial\Omega$

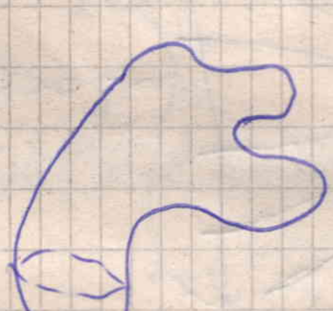
$\Omega \subset \mathbb{R}^3$

област

външната нормала

за Ω нормала

x_0



1 сл. $x_0 \notin \Omega$

$$\iint_{\partial\Omega} \left\langle \frac{x-x_0}{\|x-x_0\|^3}, n \right\rangle ds =$$

$$= \iiint_{\Omega} \operatorname{div} \frac{x-x_0}{\|x-x_0\|^3} \, dx_1 \, dx_2 \, dx_3$$

$$\frac{x-x_0}{\|x-x_0\|^3}$$

$$F(x_1, x_2, x_3) = \frac{x-x_0}{\|x-x_0\|^3} =$$

$$= \left(\frac{x_1 - x_0^1}{\|x-x_0\|^3}, \frac{x_2 - x_0^2}{\|x-x_0\|^3}, \frac{x_3 - x_0^3}{\|x-x_0\|^3} \right)$$

$$\|x-x_0\|^3 = \left((x_1 - x_{01})^2 + (x_2 - x_{02})^2 + (x_3 - x_{03})^2\right)^{\frac{3}{2}}$$

$$\frac{\partial F_1}{\partial x_1} = \left(\frac{x_1 - x_0^1}{\|x-x_0\|^3} \right)'_{x_1} =$$

$$= \frac{1}{\|x-x_0\|^3} + \frac{-\frac{3}{2}(x_1 - x_0^1) \, 1 \, . \, 2(x_1 - x_0^1)}{\|x-x_0\|^5}$$

$$= \frac{1}{\|x-x_0\|^3} - \frac{3(x_1-x_0^1)^2}{\|x-x_0\|^5}$$

$$\cancel{\frac{\partial F_2}{\partial x_2}} =$$

$$\frac{\partial F_1}{\partial x_1} = \frac{1}{\|x-x_0\|^3} - \frac{3(x_1-x_0^1)^2}{\|x-x_0\|^5}$$

$$\frac{\partial F_2}{\partial x_2} = \frac{1}{\|x-x_0\|^3} - \frac{3(x_2-x_0^2)^2}{\|x-x_0\|^5}$$

$$\frac{\partial F_3}{\partial x_3} = \frac{1}{\|x-x_0\|^3} - \frac{3(x_3-x_0^3)^2}{\|x-x_0\|^5}$$

$$\operatorname{div} F(x) = \frac{3}{\|x-x_0\|^3} - \frac{3}{\|x-x_0\|^5}\underbrace{\left[(x_1-x_0^1)^2 + (x_2-x_0^2)^2 + (x_3-x_0^3)^2\right]}_{=\|x-x_0\|^2}$$

$$= 0$$

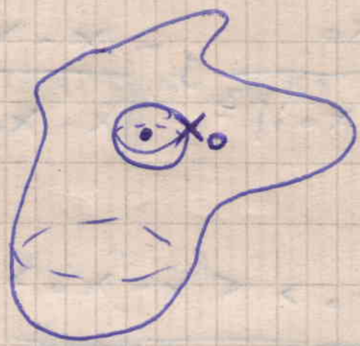
$S = \partial\Omega$

Ω - област в $\mathbb{R}^3$

n външната за Ω нормала

1) $x_0 \notin \overline{\Omega}$

$$\iint_S \left\langle \frac{x-x_0}{\|x-x_0\|^3}, n \right\rangle ds = \iiint_\Omega \operatorname{div} F = 0$$



2) $x_0 \in \Omega$

$\varepsilon > 0 \qquad \overline{B_\varepsilon}(x_0) \subset \Omega$

$$\tilde{\Omega} = \Omega \setminus \overline{B_\varepsilon}(x_0)$$

$$\iint\limits_{\partial\tilde{\Omega}} \langle F, n \rangle = \iiint\limits_{\tilde{\Omega}} \operatorname{div} F = 0$$

сфера

$$\partial_{n}\tilde{\Omega} = \partial\Omega \cup S_\varepsilon(x_0)$$

$(\partial\Omega, n)$

$$x \in S_\varepsilon(x_0) \qquad n(x) = \frac{x_0 - x}{\|x_0 - x\|} = -\frac{x - x_0}{\|x - x_0\|}$$

външната за $\tilde{\Omega}$ нормала

$$\iint\limits_{\partial\tilde{\Omega}} \langle F, n \rangle = \iint\limits_{\partial\Omega} \langle F, n \rangle + \iint\limits_{S_\varepsilon(x_0)} \langle F, n \rangle = 0$$

$$\iint\limits_{\partial S} \langle F, n \rangle = -\iint\limits_{S_\varepsilon(x_0)} \langle F, n \rangle \, ds =$$

$$= - \iint\limits_{S_\varepsilon(x_0)} \left< \frac{x - x_0}{\|x - x_0\|^3}, \frac{x - x_0}{\|x - x_0\|} \right> ds =$$

$$= \iint\limits_{S_\varepsilon(x_0)} \frac{\langle x - x_0 \rangle \langle x - x_0 \rangle}{\|x - x_0\|^4} ds =$$

$$= \iint\limits_{S_\varepsilon(x_0)} \frac{ds}{\|x - x_0\|^2} = \frac{1}{\varepsilon^2} \iint\limits_{S_\varepsilon(x_0)} ds =$$

$$= \frac{1}{\cancel{\varepsilon^2}} 4\pi \cancel{\varepsilon^2} = \underline{\underline{4\pi}}$$ мярото на единична сфера.

22.01.2014г.

Упражнение

Писмен изпит

$\boxed{5.1.}$

$\boxed{5.4a}$

тетраедър

5.1) $K_n(a) = \left| \begin{array}{l} x_1 + x_2 + \ldots + x_n \le a \\ x_i \ge 0 \end{array} \right| \; a > 0$

$$= I_n$$

$$V = \underbrace{\iiint\cdots\int}_{n} dx_1\, dx_2\, dx_3 \dots dx_n$$

$$\begin{cases} x_1 + x_2 + \dots + x_n = a \\ x_i \geq 0 \end{cases}$$

$$0 \leq x_n \leq a - x_1 - \dots - x_{n-1}$$

$$I_{n-1} = \underbrace{\iint\cdots\int}_{n-1} dx_1\, dx_2 \dots dx_{n-1}$$

$$V = \int_0^a \left(I_{n-1}\, dx_1\, dx_2 \dots dx_{n-1} \right) =$$

$$= \int_0^a$$

$$\begin{cases} x_1 + x_2 + x_3 \leq a \\ x_i \geq 0 \end{cases}$$

$$I_3$$

$$0 \leq x_3 \leq a - x_1 - x_2$$

$$I_n = \int_0^a$$

$$\Rightarrow I_3 = \int_0^a \left(\iiint_{x_1 + x_2 \leq a - x_3} 1 \right) dx_1\, dx_2\, dx_3 =$$

$$= \int_0^a \left(\int_0^a \int_0^{a - x_1 - x_3} dx_1\, dx_2 \right) dx_3 =$$

$$= \int_0^a \int_0^a (a - x_2 - x_3)\, \cancel{dx_1}\, dx_2\, dx_3 =$$

$$= \int_0^a \cancel{\int_0^a \left(a - \frac{x_2^2}{2} - x_3\right) dx_2\, dx_3} =$$

$$= \int_0^a \cancel{\left(a - \frac{a^2}{2} - x_3\right)} dx_3 =$$

$$= \cancel{a\frac{a^2}{2}} \quad a - a^2$$

$$\cancel{I = a - a^2 - a^3 - \dots}$$

$$\left|\begin{array}{l} x_1 + x_2 + \dots + x_n \le a \\ x_i \ge 0 \end{array}\right.$$

$$0 \le x_n \le a - \cancel{x_n} - x_{n-1} - \dots - x_1$$

$$I_n = \int_0^a \left(\int_{D_{n-1}} dx_1\, dx_2 \dots dx_{n-1} \right) dx_n$$

$$I_{n-1} = \int_{D_{n-1}} 1\, dx_1\, dx_2 \dots dx_{n-1}$$

$$D_{n-1}: \left| \begin{array}{c} x_1 + x_2 + \dots + x_{n-1} \leq a \\ x_i \geq 0 \end{array} \right.$$

$$D'_{n-1}: \left| \begin{array}{c} x_1 + x_2 + \dots + x_{n-1} \leq a - x_n \\ x_i \geq 0 \end{array} \right.$$

$$I_n = \int_0^a \left(\int_{x_1 + x_2 + \dots + x_{n-1} \leq a - x_n} dx_1\, dx_2 \dots dx_{n-1} \right) =$$

$$\cancel{I_{n-1}(a-x_n) = \int_0^a}$$

$$\cancel{x_1 + x_2 + \dots + x_{n-1}}$$

$$I_n = \int_0^a I_{n-1}(a-x_n)\, dx_1\, dx_2 \dots dx_{n-1} =$$

$$\cancel{x_n = 0} \quad \cancel{x_n = a}$$

$$= \int_0^a I_{n-1}(a-x_n)\, dx_1\, dx_2 \dots dx_{n-1}$$

$$I_n = \int_0^a I_{n-1}(a)\, dx_1\, dx_2 \dots dx_{n-1}$$

5.4 a) 1

$$I_{n,1} = \int_{K_n} (x_1 + x_2 + \dots + x_n)\, dx_1 \dots dx_n$$

$$K_n : \{ 0 \leq x_i \leq 1 ,\ i = 1, 2, \dots, n$$

$$I_{n,1} = \int_0^1 \left(\int (x_1 + x_2 + \dots + x_{n-1})\, dx_1\, dx_2 \dots \cancel{dx_{n-1}} \dots \cancel{dx_1} \dots \right) dx_n =$$

$$= \int_0^1 dx_n \int_0^1 x\, dx_{n-1} \dots \int_0^1 x_1\, dx_n =$$

$$= \sum \sum_{i=1}^{n} \int_0^1 x_i\, dx_i =$$

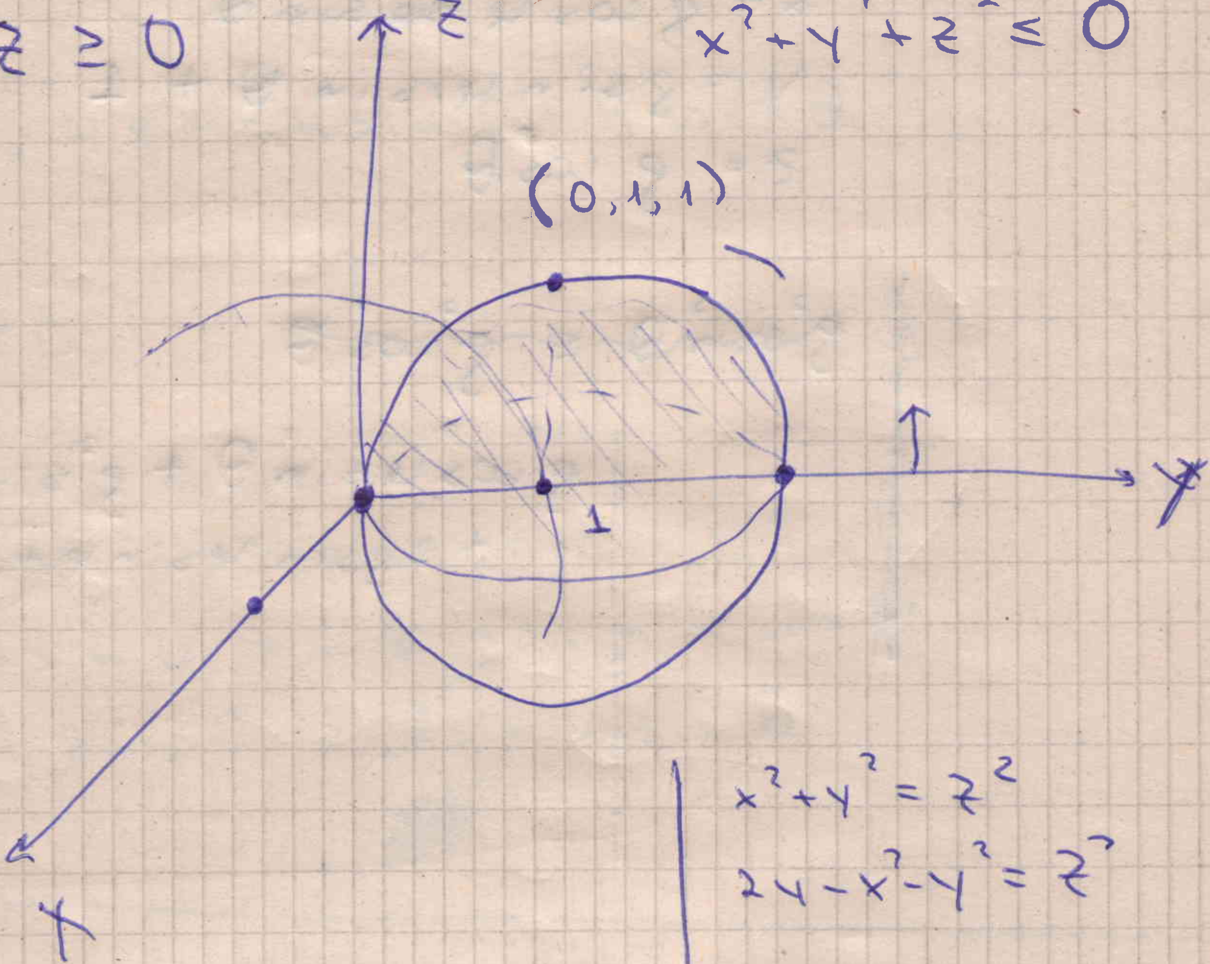
$$\sum \frac{x_i^2}{2} \Big|_0^1 = \frac{n}{2}$$

Задача 2

$$\begin{cases} x^2+y^2 \le z^2 \le 2y - x^2 - y^2 \\ z \ge 0 \end{cases}$$

$$\begin{cases} x^2+(y-1)^2+z^2 \le 1 \\ z \ge 0 \end{cases}$$

$$x^2+y^2+z^2 \le 0$$



$$\begin{cases} x^2+y^2 = z^2 \\ 2y-x^2-y^2 = z^2 \end{cases}$$

$$x^2+y^2 = 2y-x^2-y^2$$

$$2x^2+2y^2 = 2y$$

$$\cancel{2y^2}$$

$$2y^2-2y+2x^2=0$$

$$x = \rho\cos\varphi\,\cancel{\sin\theta}\ \cos\theta$$

$$y = 1+\rho\cos\varphi\sin\theta$$

$$z = \rho\sin\varphi\sin\theta$$

$$\begin{cases} x^2 + y^2 \le z^2 \\ x^2 + (y-1)^2 + z^2 \le 1 \\ z \ge 0 \end{cases}$$

$$x = \rho \cos\varphi \sin\theta$$
$$y = \rho \sin\varphi \sin\theta + 1$$
$$z = \rho \cos\theta$$

$$\cancel{\rho^2 \sin^2\theta \le \rho^2 \cos^2\theta}$$

$$\rho^2 \cos^2\varphi \sin^2\theta + \rho^2 \sin^2\varphi \sin^2\theta + 2\rho \sin\varphi \sin\varphi + 1 \le \rho^2 \cos^2\theta$$

$$\begin{cases} \rho^2 \sin^2\theta + 2\rho \sin\varphi \sin\theta + 1 \le \rho^2 \cos^2\theta \\ \rho \le 1 \\ \cos\theta\, \rho \ge 0 \end{cases}$$

$$\rho^2 \cos 2\theta$$

$$\begin{cases} \rho^2 \cos 2\theta - 2\rho \sin\varphi \sin\theta - 1 \geq 0 \\ 0 \leq \rho \leq 1 \\ \theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \end{cases}$$

$$D = 4\rho^2 \sin^2\varphi \sin^2\theta + 4\cos^2 2\theta$$

$f(0) \leq 0$

$f(1) \geq 1$

$$\cos 2\theta \geq 2 \sin\varphi \sin\theta - 1$$

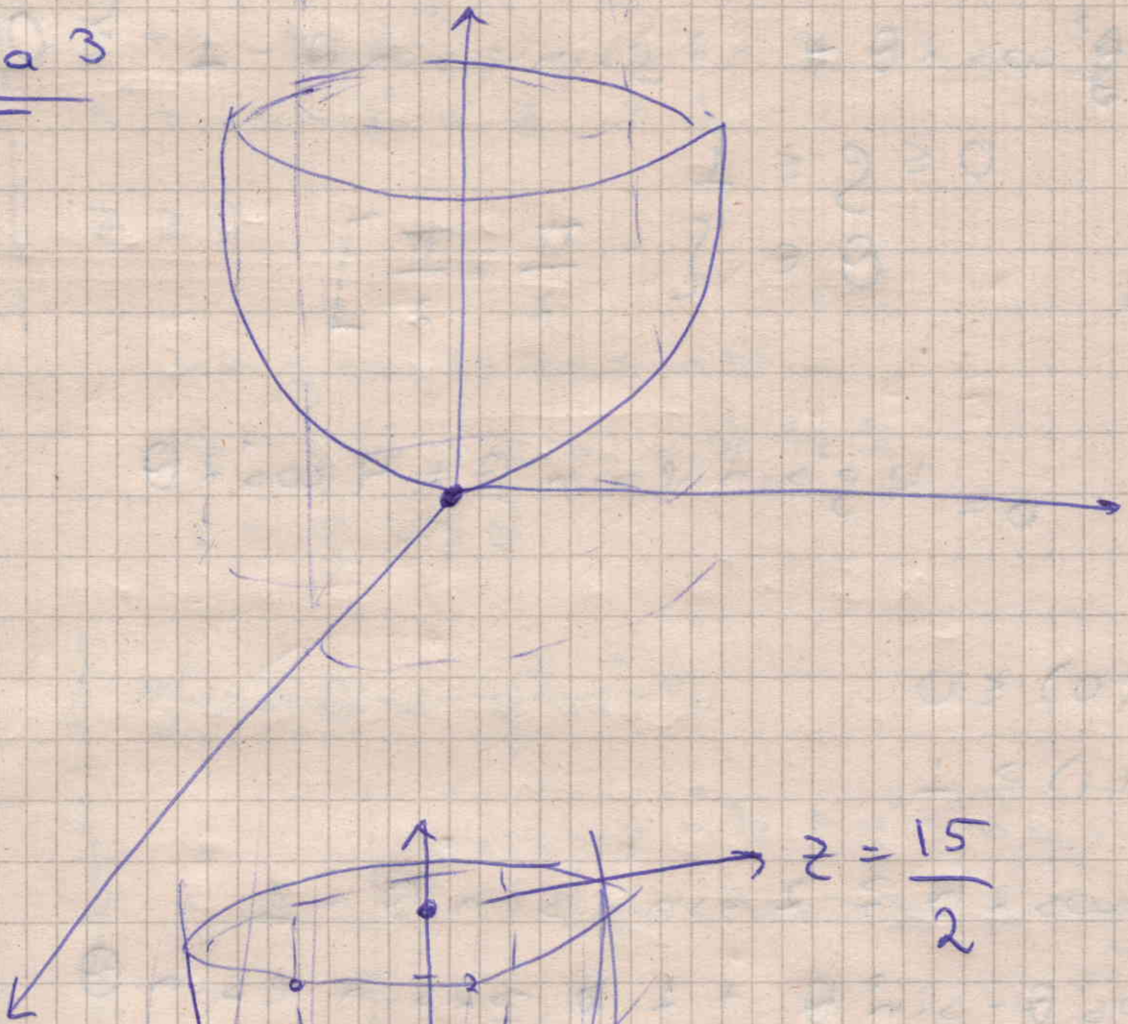
$$\cos^2\theta - \sin^2\theta + 1 \geq 2 \sin\varphi \sin\theta$$

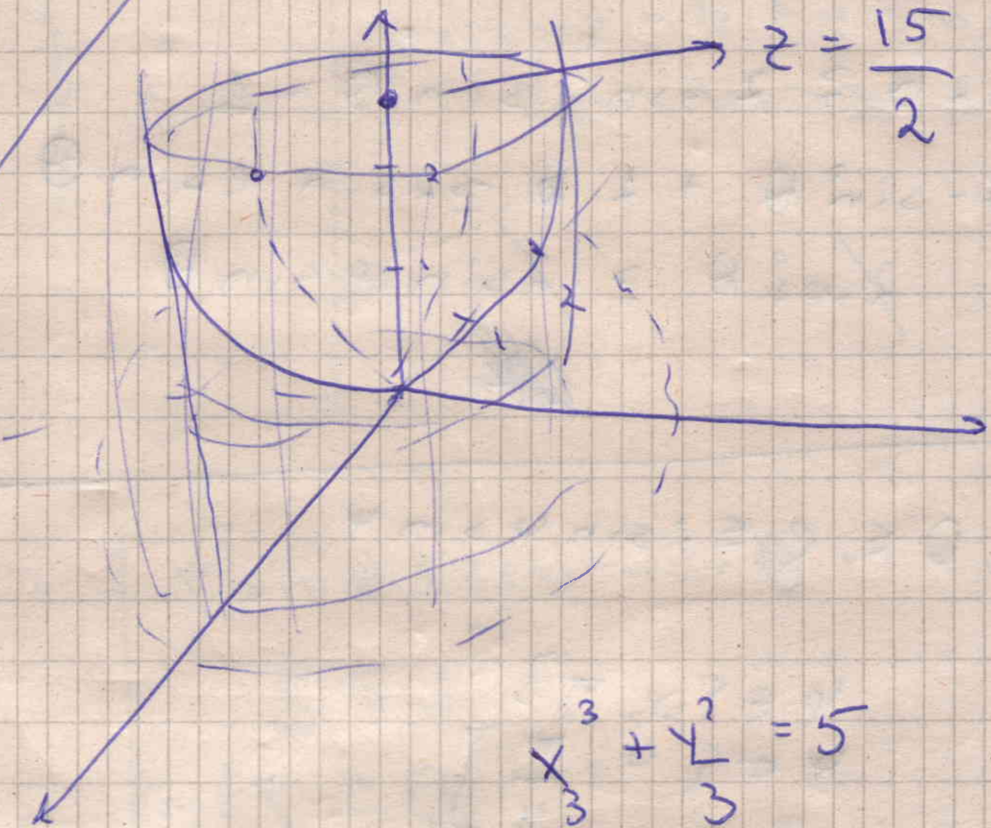
$$\cancel{2}\cos^2\theta \geq \cancel{2}\sin\varphi \sin\theta$$

$$0 \leq \rho \leq 2 \sin\varphi \sin\psi$$

$$\varphi \in [0; \pi]$$
$$\psi \in \left[0; \frac{\pi}{4}\right]$$

Задача 3





$$z = \frac{15}{2}$$

$$\frac{x^3}{3} + \frac{y^2}{3} = 5$$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = \dfrac{\rho^2}{2} \end{cases}$$

$\varphi \in \{0; 2\pi]$

$\rho^2 \leq 15$

$\rho \in \{0, \sqrt{15}]$

$z = \dfrac{\rho^2}{2}$

$$S(\Pi) = \iint 1\, d\sigma =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{15}} \sqrt{EF - G^2}\, d\rho\, d\varphi$$

$G_x = G_y = 0$ симетрична

$$G_z = \frac{1}{M} \int_0^{2\pi}\int_0^{\sqrt{15}} \rho^2/2 \sqrt{EG - F^2}$$

$$r_\rho = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ \rho \end{pmatrix}$$

$E = \rho^2$

$G = \rho^2$

$F = 0$

$$r_\varphi = \begin{pmatrix} -\rho\sin\varphi \\ \rho\cos\varphi \\ 0 \end{pmatrix}$$

$$\begin{matrix} 1 & 0 & \frac{2-x}{\sqrt{4x-x^2-y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{4x-x^2-y^2}} \end{matrix}$$

$$= \begin{pmatrix} \frac{-(2-x)}{\sqrt{4x-x^2-y^2}} \\ \frac{y}{\sqrt{4x-x^2-y^2}} \\ 1 \end{pmatrix}$$

$$\frac{(2-x)^2}{4x-x^2-y^2} + \frac{y^2}{4x-x^2-y^2} + 1 =$$

$$= \frac{4 - \cancel{4x} + \cancel{x^2} + \cancel{4x} - \cancel{x^2} - \cancel{y^2} + \cancel{y^2}}{4x-x^2-y^2} =$$

$$= \frac{4}{4x-x^2-y^2}$$

$$\Rightarrow n(\varphi(x,y)) = \frac{\sqrt{4x-x^2-y^2}}{2} \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

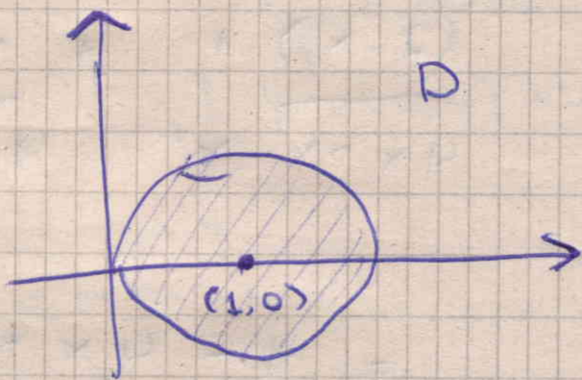
$$\Rightarrow n(\varphi(x,y)) = \begin{pmatrix} \frac{x-2}{2} \\ \frac{y}{2} \\ \frac{\sqrt{4x-x^2-y^2}}{2} \end{pmatrix}$$

единично нормално векторно поле

$$\Gamma = \partial S$$

$$D = \{x^2+y^2 \le 2x\}$$

$(x,y) \in D$



$$\int_\Gamma (y^2+z^2)\,dx + (x^2+z^2)\,dy + (x^2+y^2)\,dz =$$

$$= \iint_S \langle \operatorname{rot} F, n \rangle \, ds =$$

$$F(x,y,z) = \begin{pmatrix} y^2 + z^2 \\ x^2 + z^2 \\ x^2 + y^2 \end{pmatrix}$$

$$= \iint_D \left\langle \operatorname{rot} F, \frac{\varphi'_x \times \varphi'_y}{\cancel{\|\varphi'_x \times \varphi'_y\|}} \right\rangle \cancel{\|\varphi'_x \times \varphi'_y\|} \, dx\,dy$$

$$\operatorname{rot} F \quad \begin{matrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{matrix}$$

$$\operatorname{rot} F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= (2y - 2z, 2z - 2x, 2x - 2y)$$

$$\varphi'_x \times \varphi'_y = \begin{pmatrix} \frac{x-2}{\sqrt{4x - x^2 - y^2}} \\ \frac{y}{\sqrt{4x - x^2 - y^2}} \\ 1 \end{pmatrix}$$

$$= \iint_D \frac{2(y-z)(x-2)}{\sqrt{4x - x^2 - y^2}} + \frac{2(z-x)y}{\sqrt{4x - x^2 - y^2}} +$$

$$+ \quad 2(x-y)\Big) \, dx\,dy =$$

$$= 2 \iint_D \frac{\qquad\qquad}{\qquad}$$

$$(y-z)(x-2) + (z-x)y =$$

$$= \cancel{xy} - 2y - xz + 2z + zy - \cancel{xy}$$

$$= 2(z-y) + z(y-x)$$

$$! \quad z = \sqrt{4x-x^2-y^2}$$

$$= \iint_D \Bigg(\frac{2\left(y-\sqrt{4x-x^2-y^2}\right)(x-2)}{\sqrt{4x-x^2-y^2}} +$$

$$+ \frac{2\left(\sqrt{4x-x^2-y^2} - x\right)y}{\sqrt{\qquad}} +$$

$$+ 2(x-y)\Bigg) \, dx\,dy =$$

$$= \left(y - \sqrt{4x-x^2-y^2}\right)(x-2) + \left(\sqrt{4x-x^2-y^2} - x\right)$$

$= \cancel{xy} - 2y - x\sqrt{\quad} + 2\sqrt{\quad}$

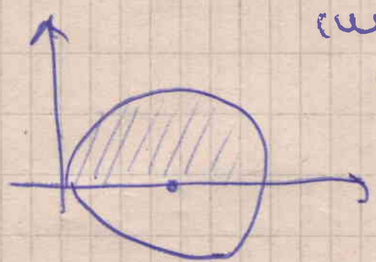
$+ y\sqrt{\quad} - \cancel{xy} =$

$= \sqrt{4x-x^2-y^2}\,(y - x + 2) - 2y$

$= 2\iint\limits_D \left(\cancel{y} - \cancel{x} + 2 - \frac{2y}{\sqrt{4x-x^2-y^2}} + \cancel{x} - \cancel{y} \right) =$

$\qquad\qquad \text{,, } 0$

$= 4\iint\limits_D 1\,dx\,dy - 4\iint\limits_D \frac{y}{\sqrt{4x-x^2-y^2}}\,dx\,dy$



симетрично
спрямо
абсцисата
$y, -y$

$= 4\pi$

Лема U рел. отв. в $D \iff \forall x \in U$

$\exists\, \varepsilon > 0$

$B_\varepsilon(x) \cap D \subset U$

02.10.2013г.

Упражнение

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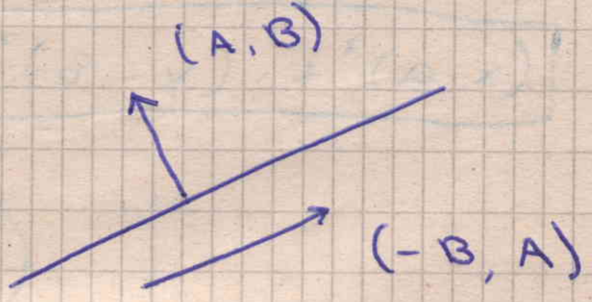
2 контролни
2 домашни
2 конт. упр.
Ръководство по мат. анализ II част

АГ в равнината
Криви и области в $\mathbb{R}^2$
Конични сечения

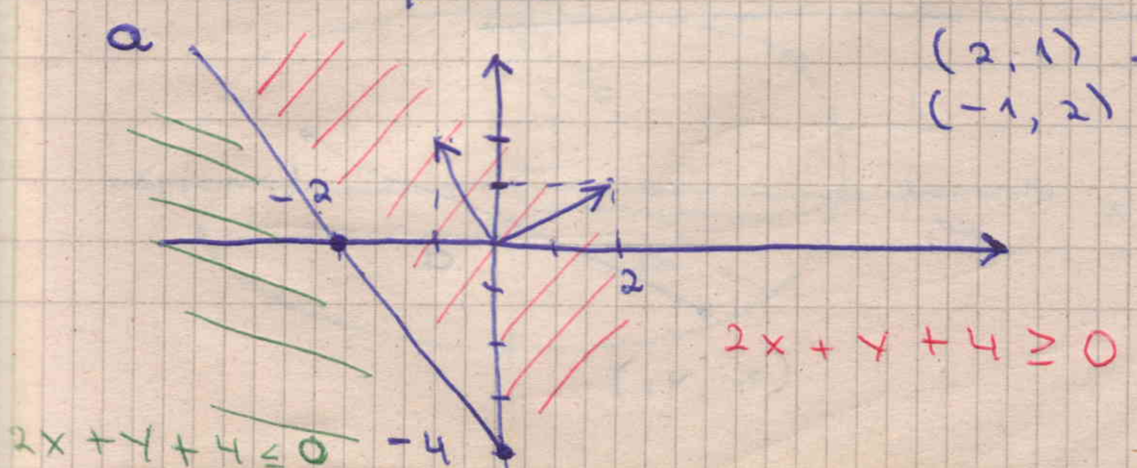
I.

$\boxed{Ax + By + C = 0}$ (A, B) е нормален



$2x + y + 4 = 0$

$(2, 1) \perp a$
$(-1, 2) \parallel a$

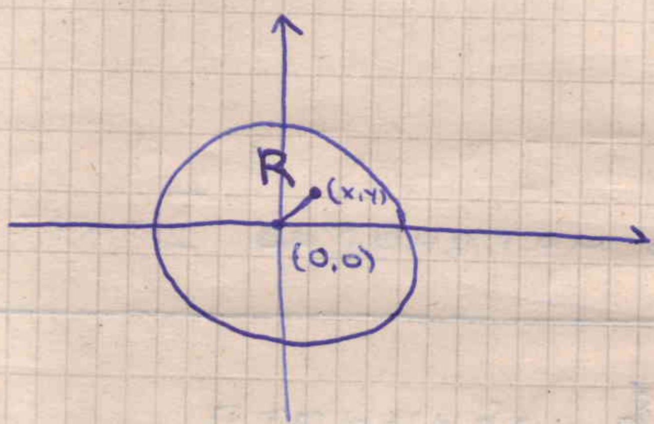


>0 - точки отгоре

<0 - отдолу

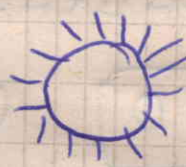
II Окръжност

$$x^2+y^2=R^2$$

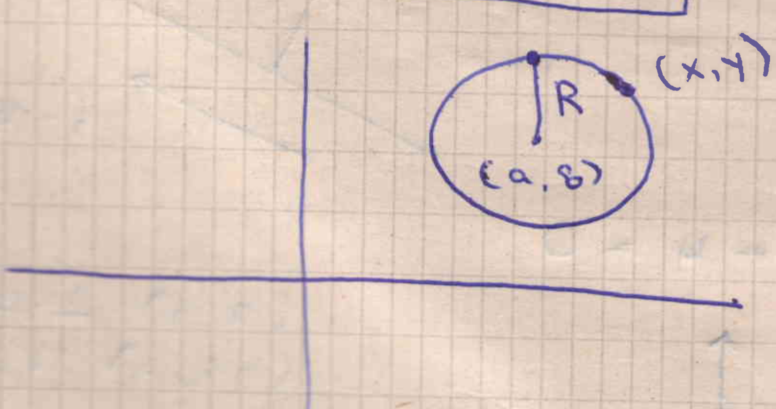


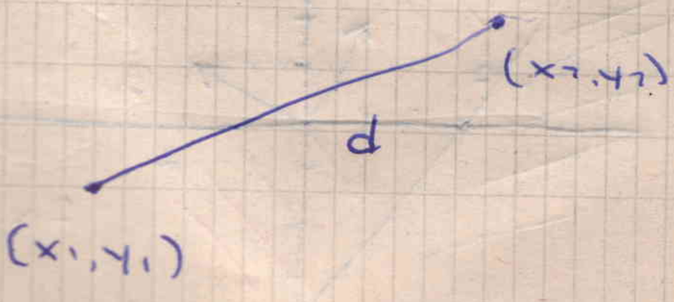
$$x^2+y^2\le R^2$$

$$x^2+y^2\ge R^2$$



$$\boxed{(x-a)^2+(y-b)^2=R^2}$$





$$d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

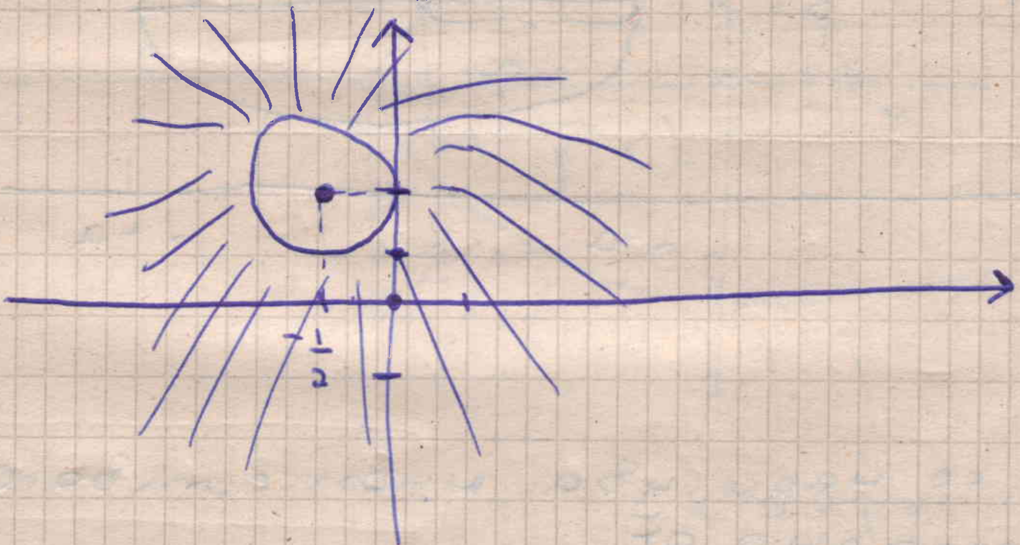
$$(x-a)^2 + (y-b)^2 \le R^2$$ 

$$(x-a)^2 + (y-b)^2 \ge R^2$$ 

зад. $x^2 - 2y + y^2 + x > -1$

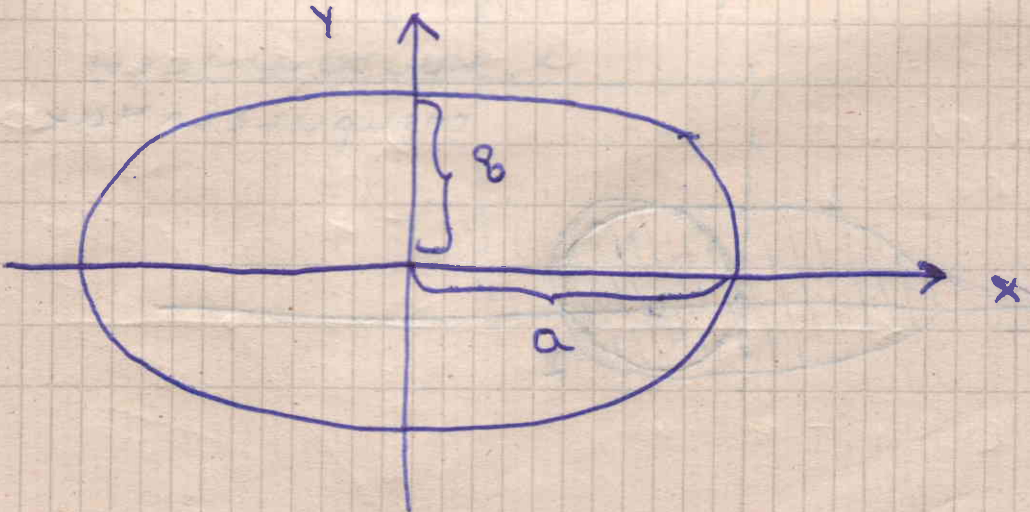
$$x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} + y^2 - 2y + 1 - \frac{5}{4} > -1 + \frac{5}{4}$$

$$\left(x+\frac{1}{2}\right)^2 + (y-1)^2 > \frac{1}{4} = \left(\frac{1}{2}\right)^2$$



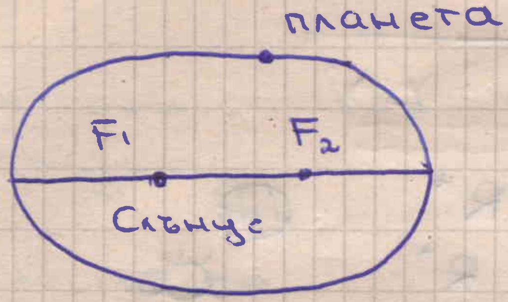
III Елипса

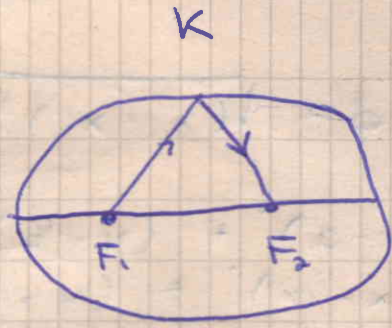
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



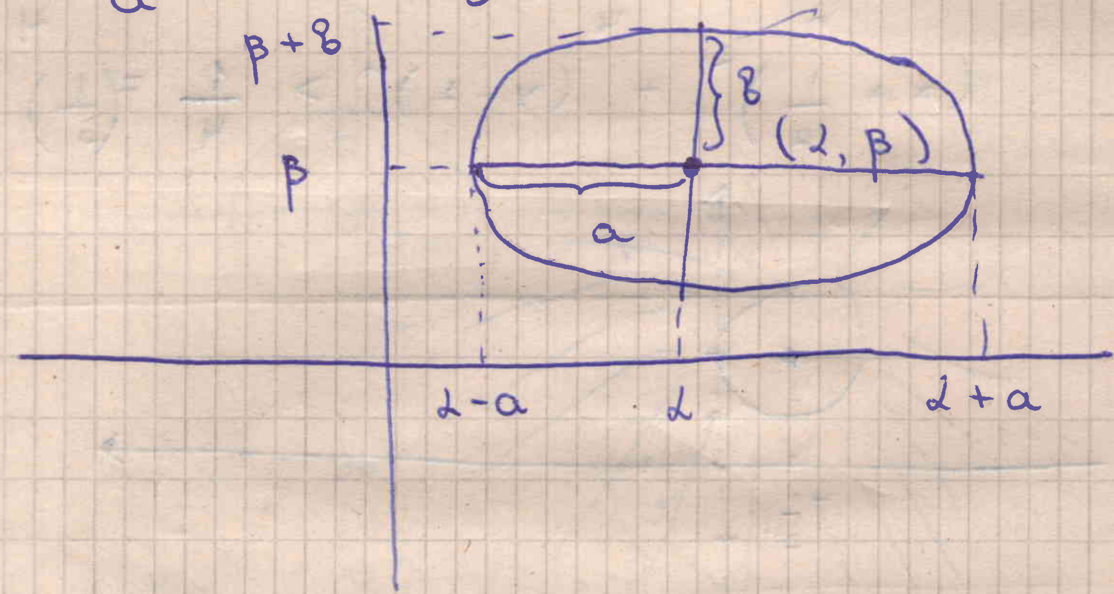
$a > b$

$$KF_1 + KF_2 = const$$





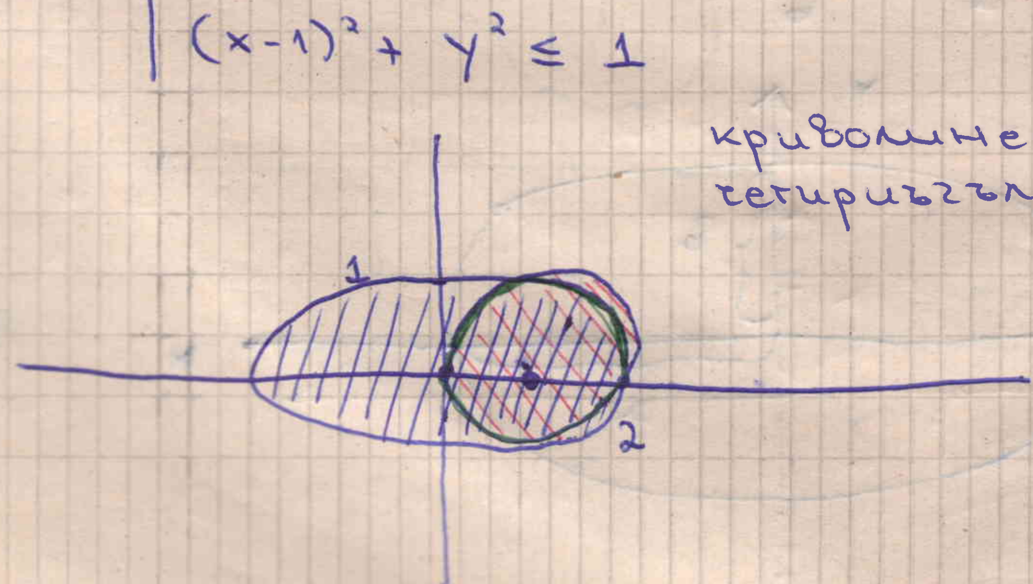
$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1$$



Зад. Да се нарисува множеството, определено от

$$\begin{cases} \frac{x^4}{4} + y^2 \le 1 \\ (x-1)^2 + y^2 \le 1 \end{cases}$$

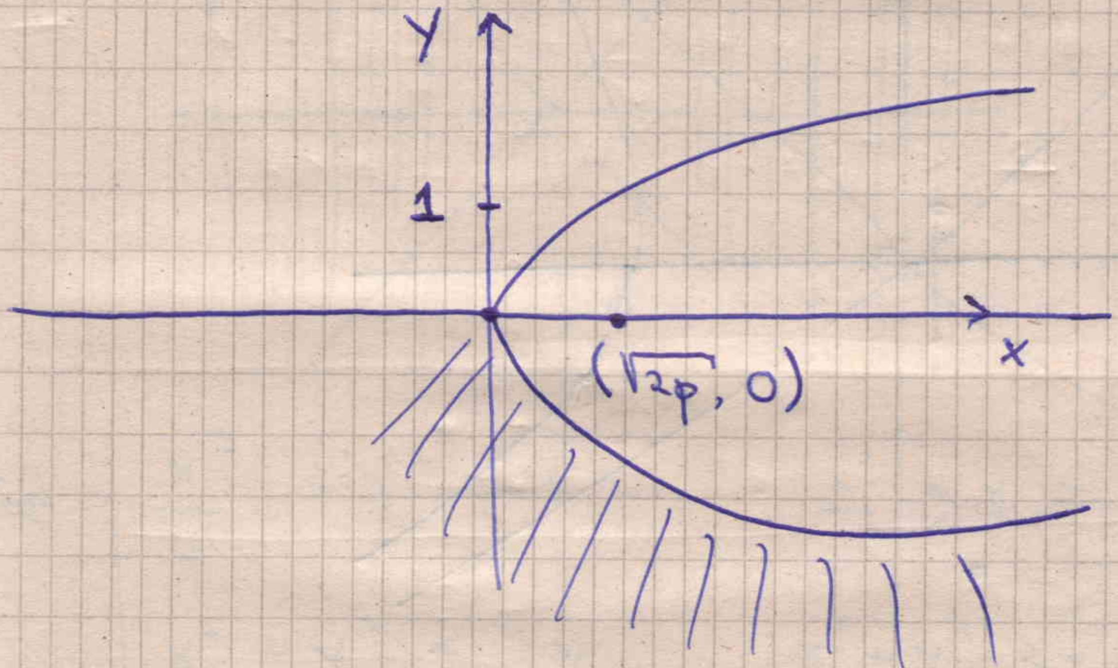
криволинеен четириъгълник



IV Парабола

Общо у-е на параб. е

$$x^2 = 2py$$



вътре

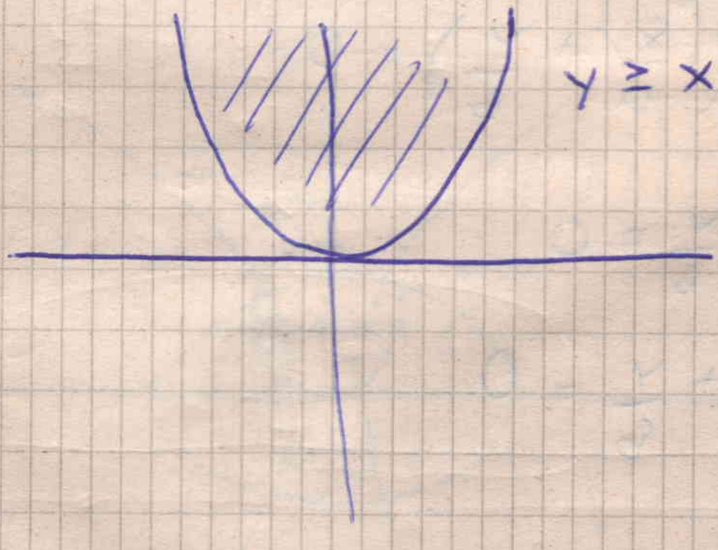
↑

$x^2 \geq 2py$

$\leq$

$x^2 \leq 2py$

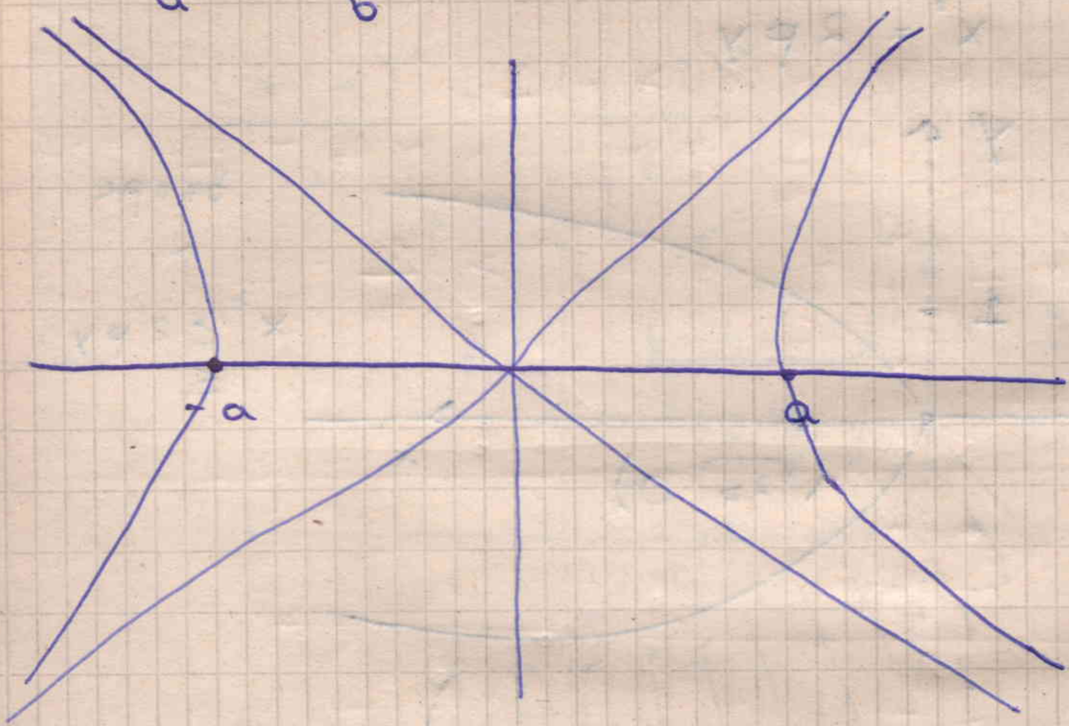
$y = x^2$



$y \geq x^2 + C$

IV Хипербола

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$ → асимптотични прави

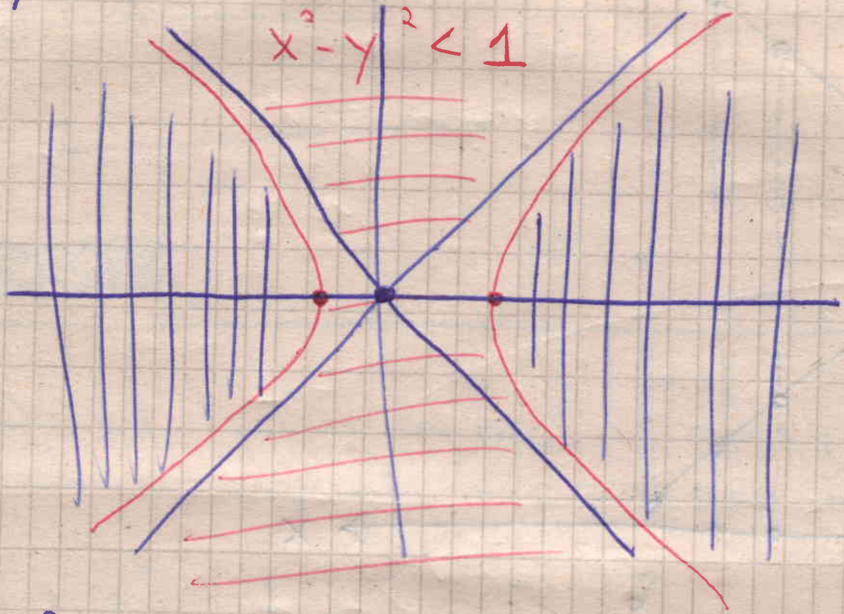
$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

$$\left|\begin{array}{l} \frac{x}{a} - \frac{y}{b} = 0 \\ \frac{x}{a} + \frac{y}{b} = 0 \end{array}\right.$$

$x^2 - y^2 = 1$

$x - y = 0$

$x + y = 0$



$x^2 - y^2 \geq 1$

зад.

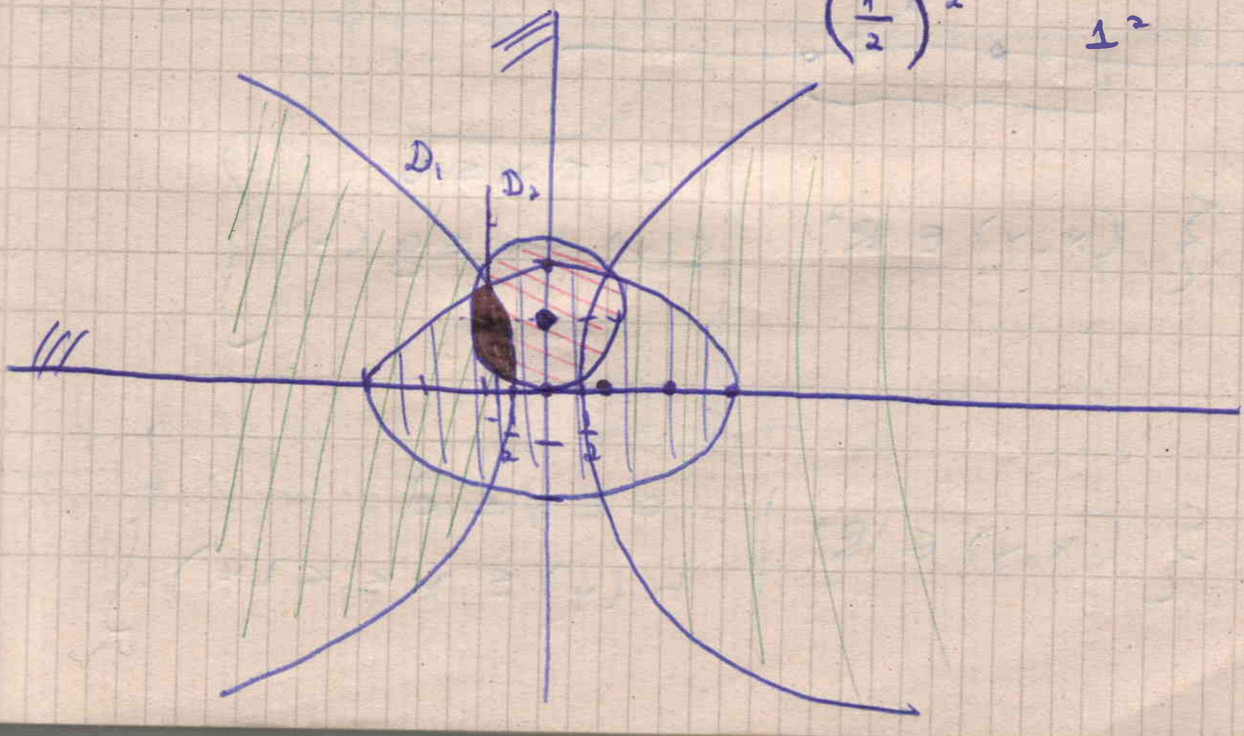
$$\begin{cases} \dfrac{x^2}{4} + y^2 \leq 1 \\ x^2 + y^2 < 2y \\ 4x^2 - y^2 \geq 1 \\ x < 0 \\ y > 0 \end{cases}$$

$\dfrac{x^2}{2^2} + y^2 \leq 1$

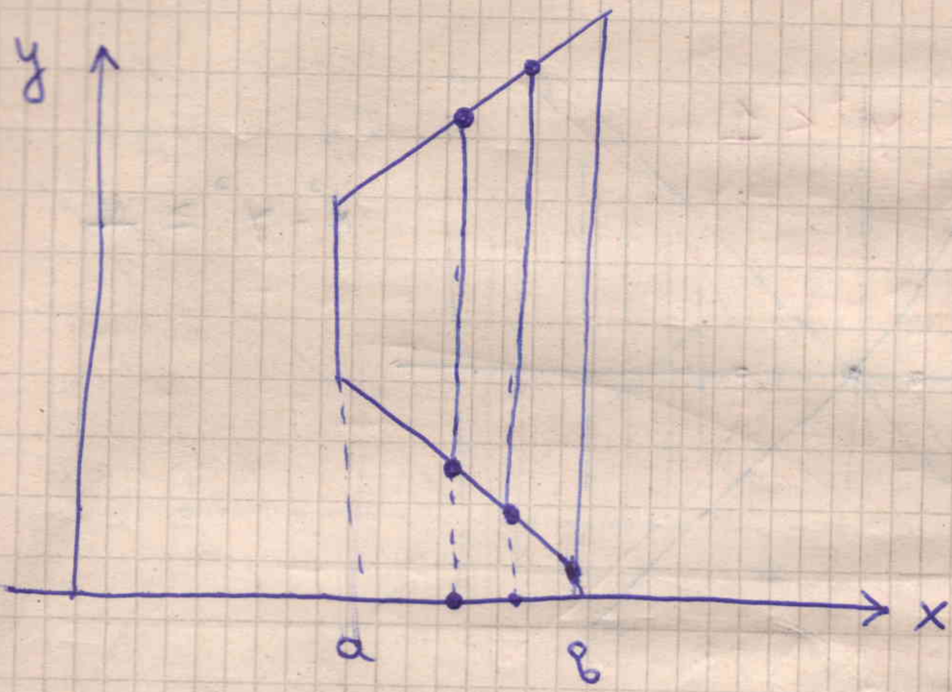
$x^2 + y^2 - 2y + 1 < 1$

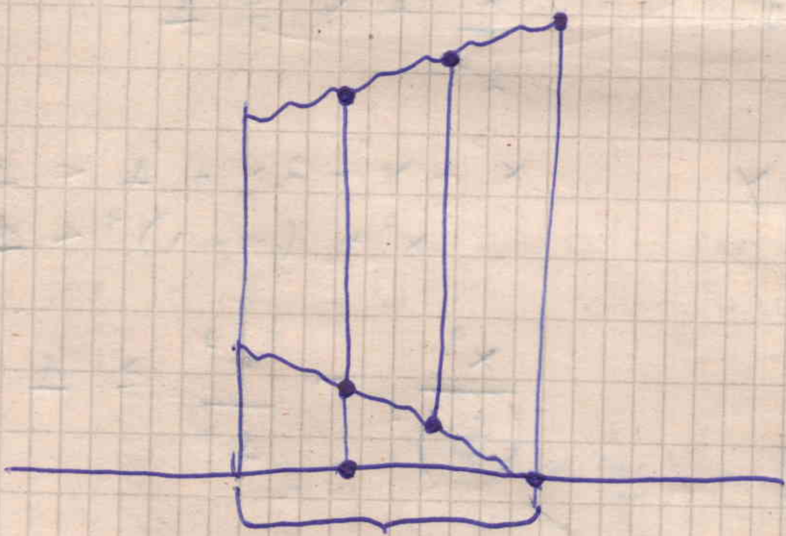
$x^2 + (y-1)^2 < 1$

$\dfrac{x^2}{\left(\frac{1}{2}\right)^2} - \dfrac{y^2}{1^2} \geq 1$



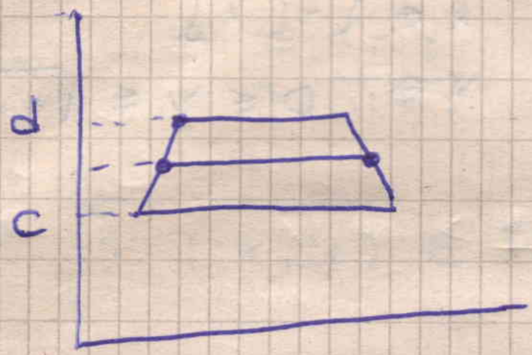
Деф. Криволинеен трапец





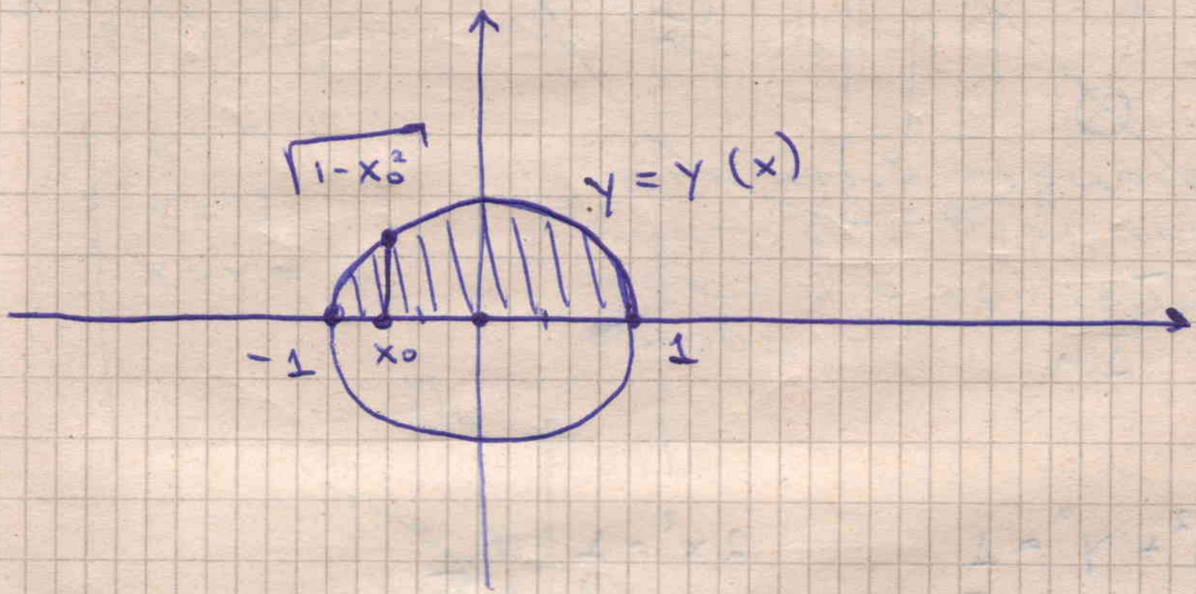
$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} a \le x \le b \\ f(x) \le y \le g(x) \end{array} \right\}$$

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} c \le y \le d \\ h(y) \le x \le k(y) \end{array} \right\}$$



зад. Да се нарисува и представи като крив. трапец множеството

$$\begin{cases} x^2 + y^2 \le 1 \\ y \ge 0 \end{cases}$$

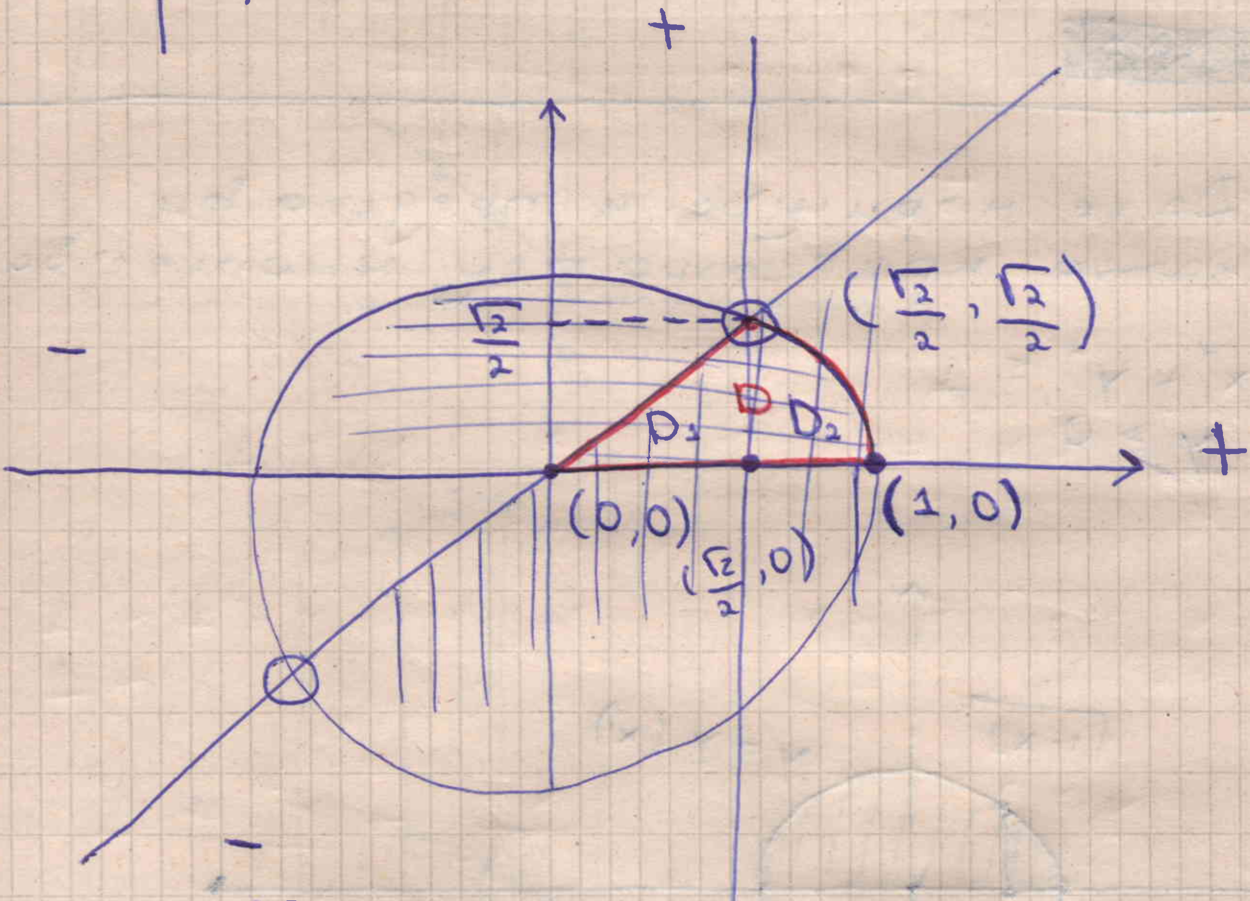


$$y^2 \le 1 - x^2 \qquad x \in [-1, 1]$$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x \in [-1,1] \\ \cancel{x \in} \quad 0 \le y \le \sqrt{1-x^2} \end{array} \right.$$

δ) $\begin{cases} x^2 + y^2 \le 1 \\ y \ge 0 \\ y - x \le 0 \end{cases}$



$\underline{\text{I}}$ H.

$y \le x$

$\begin{cases} x^2 + y^2 = 1 \\ y - x = 0 \end{cases}$ $\qquad 2x^2 = 1$

$x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\begin{cases} y = x \\ x = \pm \frac{\sqrt{2}}{2} \end{cases}$

$$x^2 \le 1 - y^2$$

$$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$$

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \le y \le \frac{\sqrt{2}}{2} \\ y \le x \le \sqrt{1-y^2} \end{array} \right\}$$

II н. $D = D_1 \cup D_2$

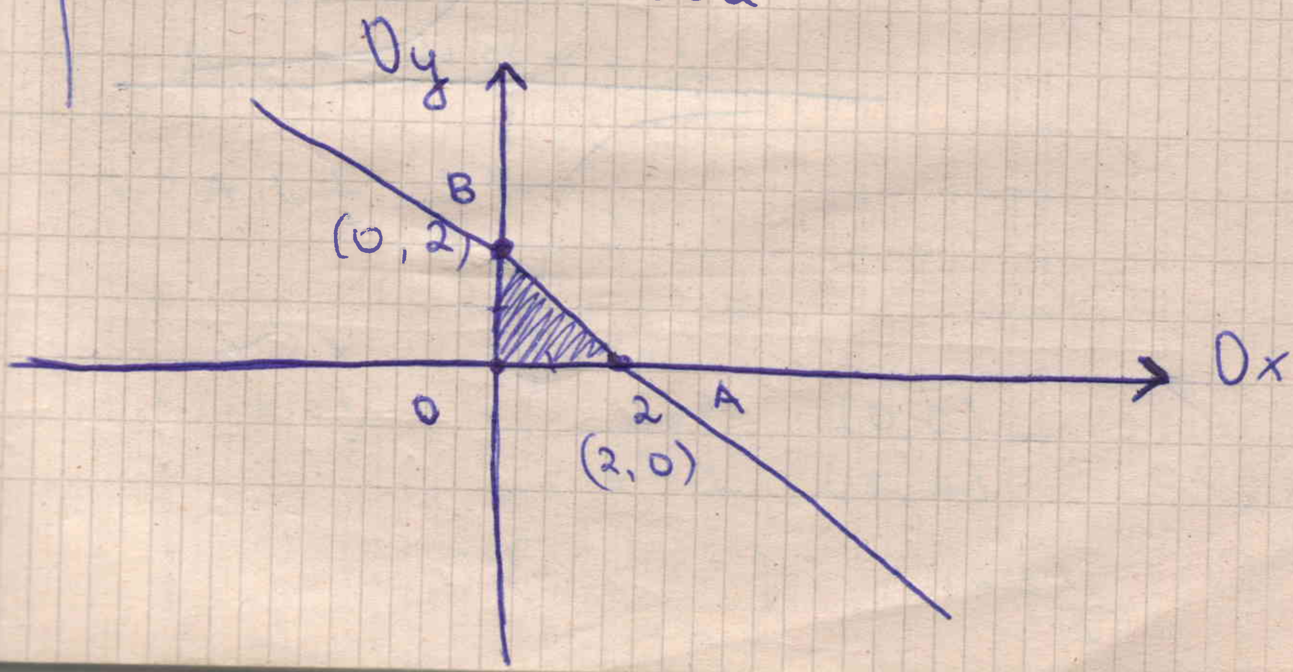
$$D_1 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \le x \le \frac{\sqrt{2}}{2} \\ 0 \le y \le x \end{array} \right\}$$

$$D_2 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} \frac{\sqrt{2}}{2} \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{array} \right\}$$

Зад. Представете множеството заградено от кривите

$$\begin{cases} x = 0 \\ y = 0 \\ x + y = 2 \end{cases}$$

като крив. трапец, както с вертикални, така и с хоризонтални основи



$$T = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x \in [0,2] \\ 0 \le y \le 2-x \end{array} \right\}$$

$$\left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} y \in [0,2] \\ 0 \le x \le 2-y \end{array} \right\}$$

! Канонизация

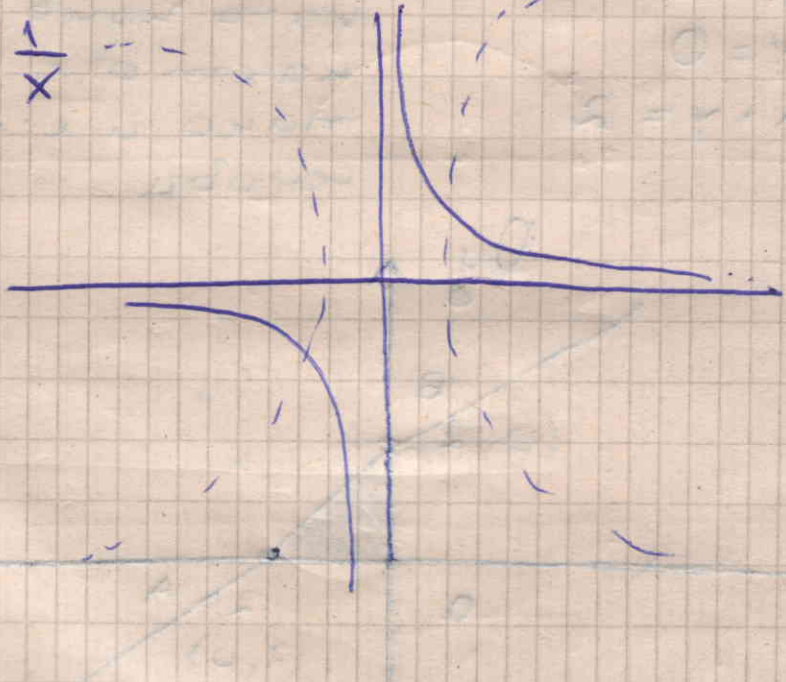
$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1x + b_2y + c = 0$$

Ако една крива от II степен не е изродена, то тя е или елипса, или хипербола, или парабола, или двойка прави.

$$x^2 + y^2 = -1$$

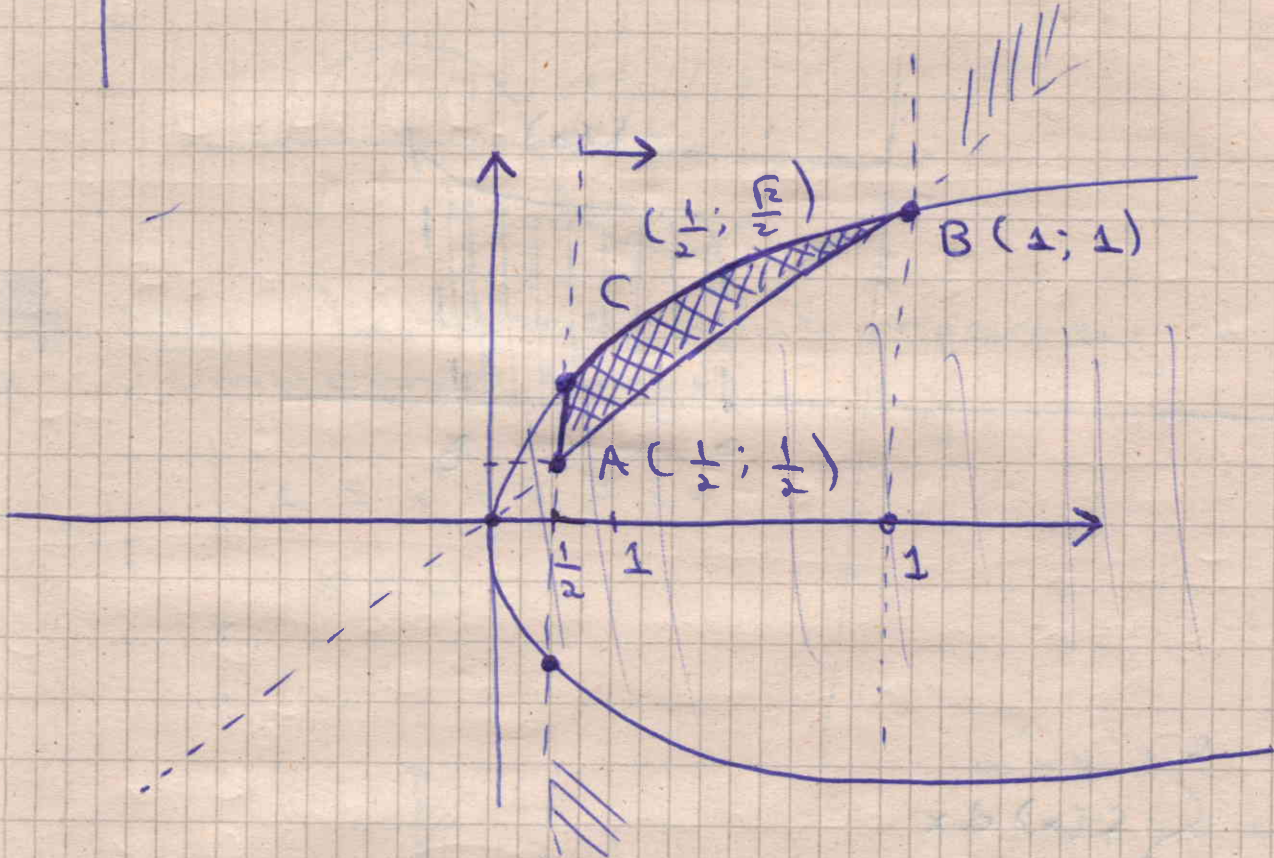
$$xy = 1$$

$$y = \frac{1}{x}$$



зад. Представете като крив. трапец

$$\begin{cases} y^2 \le x \\ \frac{1}{2} \le x \le y \end{cases}$$



A : $x = y \cap x = \frac{1}{2}$

$\Rightarrow A\left(\frac{1}{2}; \frac{1}{2}\right)$

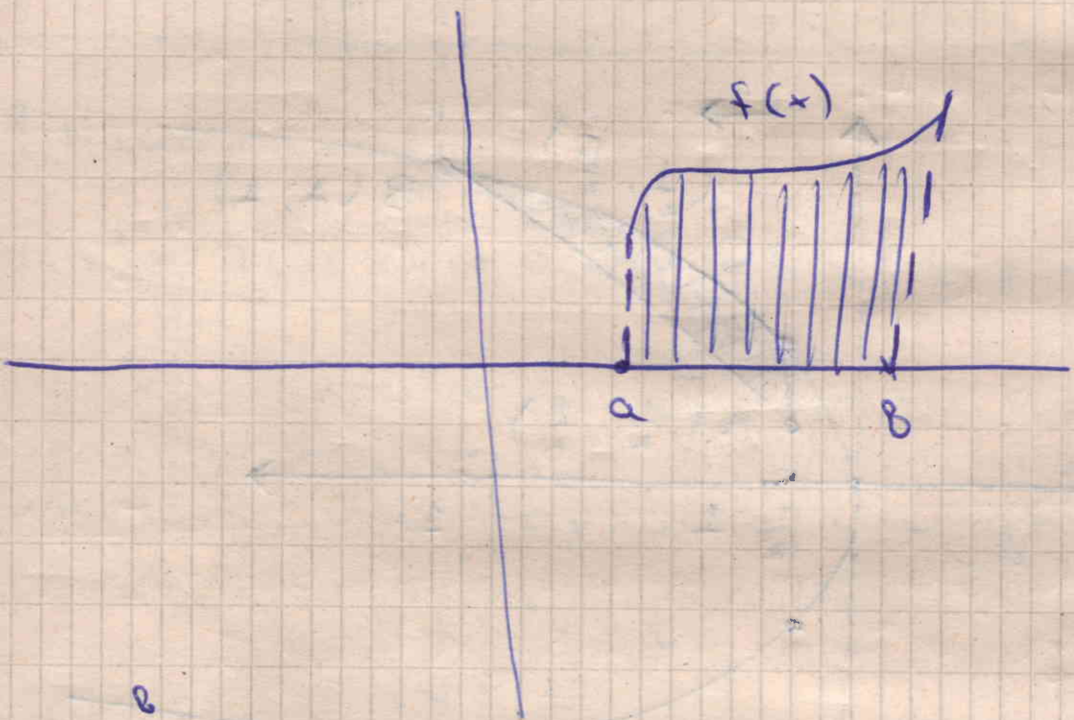
B : $\begin{cases} y^2 = x \\ x = y \end{cases}$ $\quad x = x^2$

$x = 0 \quad (0, 0)$

$x = 1 \quad (1, 1)$ $\qquad C\left(\frac{1}{2}; \frac{\sqrt{2}}{2}\right)$

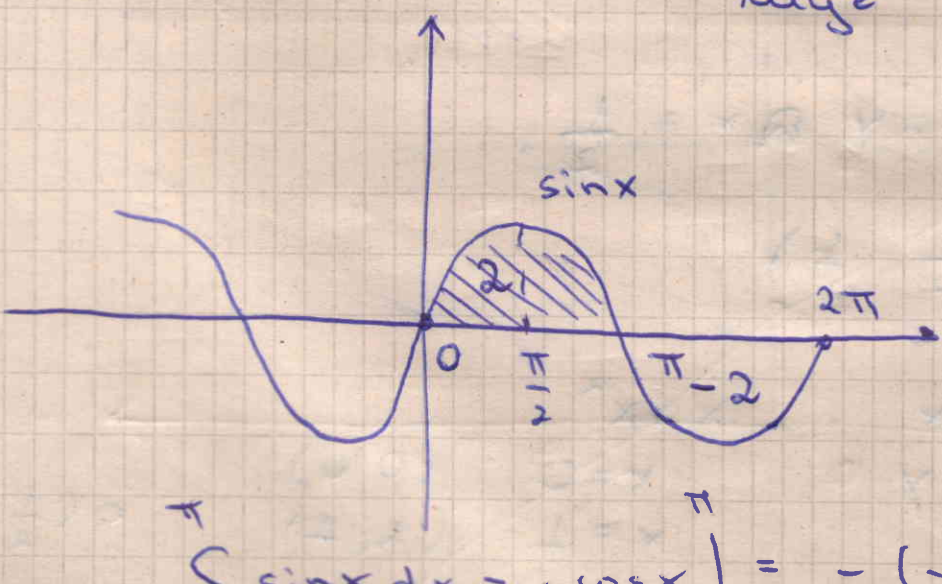
C : $\begin{cases} x = \frac{1}{2} \\ y^2 = x \end{cases}$ $\quad y = \pm\frac{\sqrt{2}}{2}$

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} \frac{1}{2} \le x \le 1 \\ x \le y \le \sqrt{x} \end{array} \right\}$$



$$\int_a^в f(x)\,dx$$

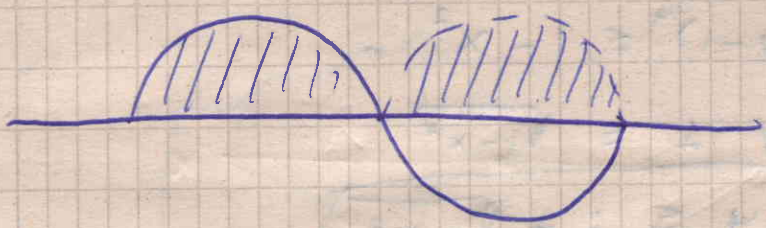
ориентирано лице



$$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = -(-1-1) = 2$$

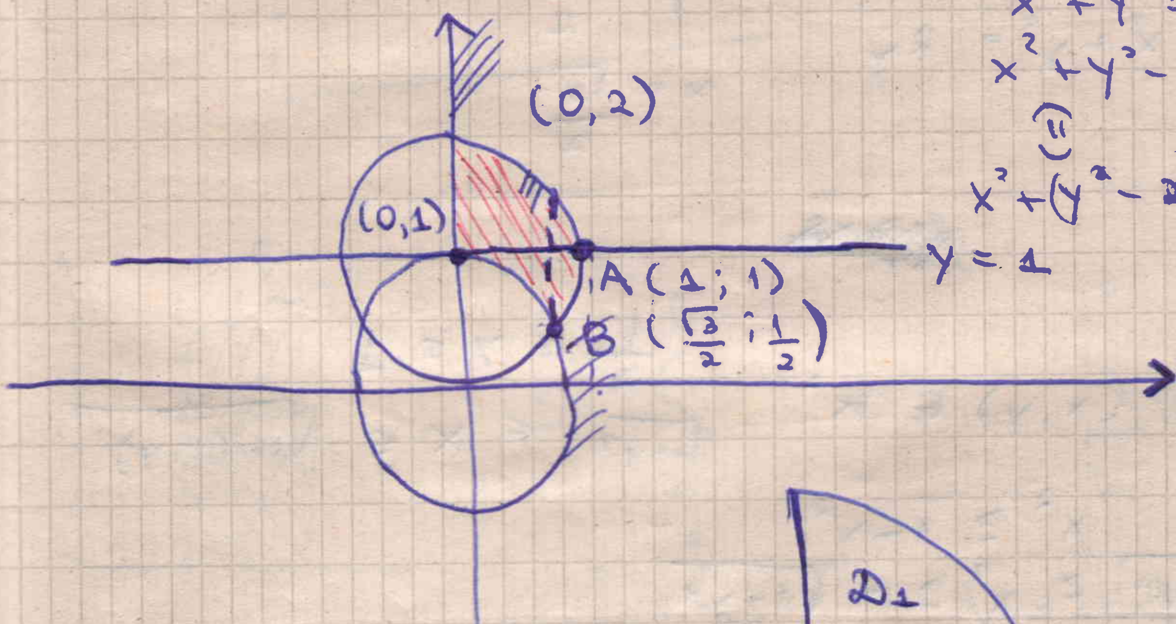
$$\int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big|_{\pi}^{2\pi} = -(1-(-1)) = -2$$

$$\int_{0}^{2\pi} |\sin x| \, dx \text{ - лице}$$



зад. Пр. като крив. трапец и пресметнете лицето му

$$D: \begin{cases} 1 \le x^2 + y^2 \le 2y \\ x \ge 0 \end{cases}$$

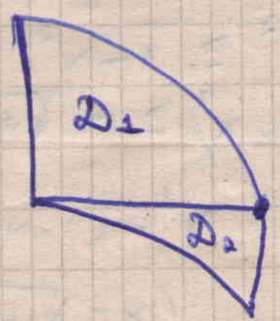


$x^2 + y^2 \ge 1$

$x^2 + y^2 - 2y \le 0$

$\Updownarrow$

$x^2 + (y - 1)^2 \le 1$



$$A: \begin{cases} y = 1 \\ x^2 + y^2 = 2y \end{cases} \Rightarrow \begin{matrix} y = 1 \\ x^2 + 1 = 2 \end{matrix} \quad x = 1$$

$D_1: \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} 0 \le x \le 1 \\ 1 \le y \le \end{matrix} \right.$

$\boxed{x^2 + (y-1)^2 \le 1}$ → трябва да изразим y

$$(y-1)^2 \le 1 - x^2$$

$$-\sqrt{1-x^2} \le y - 1 \le \sqrt{1-x^2}$$

$$1 - \sqrt{1-x^2} \le y \le 1 + \sqrt{1-x^2}$$

$$1 \le y \le 1 + \sqrt{1-x^2}$$

~~D~~:

$B: \left| \begin{matrix} x^2 + y^2 = 1 \\ x^2 + y^2 = 2y \end{matrix} \right.$ $\quad y = \frac{1}{2}$ $\quad x = \frac{\sqrt{3}}{2}$

$D_2: \left\{ (x,y) \in \mathbb{R}^2 \quad \begin{matrix} \frac{1}{2} \le y \le 1 \\ \sqrt{1-y^2} \le x \le \sqrt{1-(1-y)^2} \end{matrix} \right.$

$$x^2 \ge 1 - y^2$$

$$\sqrt{1-y^2} \ge x^2 \ge -\sqrt{1-y^2}$$

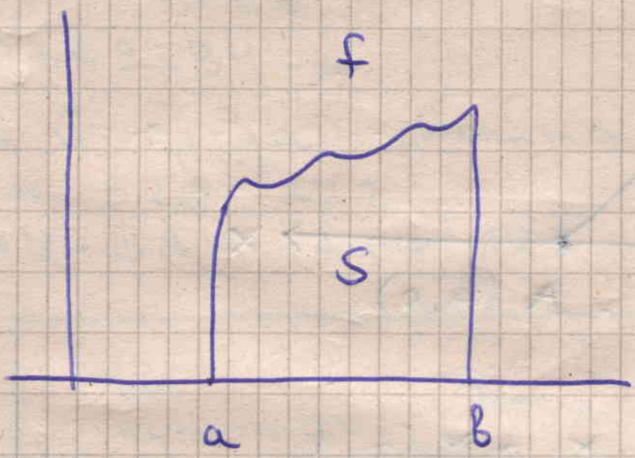
$$x^2 - (1-y^2) \ge 0$$

$$\left(x - \sqrt{1-y^2}\right)\left(x + \sqrt{1-y^2}\right) \ge 0$$

$$x^2 = 1 - (y-1)^2$$

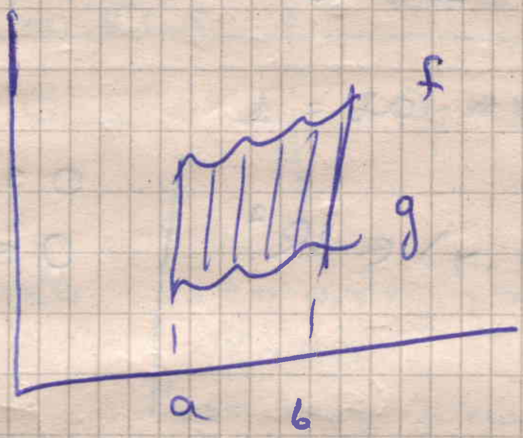
$$x \leq \sqrt{1-(y-1)^2}$$

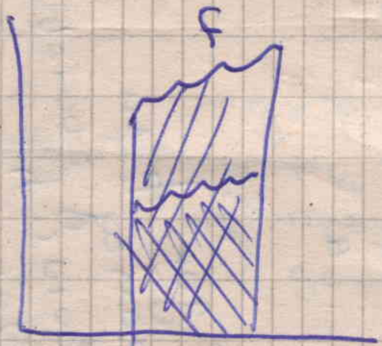
$< \cancel{x}$



$$S = \int_a^b f(x)\,dx$$

$S =$





$$S = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx =$$

$$= \int_a^b \left[f(x) - g(x)\right] dx$$

$$D_1: \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ 1 \le y \le \end{array} \right.$$

$$\boxed{x^2 + (y-1)^2 \le 1} \to \text{трябва да изразим } y$$

$$(y-1)^2 \le 1 - x^2$$

$$-\sqrt{1-x^2} \le y - 1 \le \sqrt{1-x^2}$$

$$1 - \sqrt{1-x^2} \le y \le 1 + \sqrt{1-x^2}$$

$$1 \le y \le 1 + \sqrt{1-x^2}$$

~~D2~~:

$$B: \left| \begin{array}{l} x^2 + y^2 = 1 \\ x^2 + y^2 = 2y \end{array} \right. \qquad y = \frac{1}{2} \qquad x = \frac{\sqrt{3}}{2}$$

$$D_2: \left\{ (x,y) \in \mathbb{R}^2 \quad \begin{array}{l} \frac{1}{2} \le y \le 1 \\ \sqrt{1-y^2} \le x \le \sqrt{1-(1-y)^2} \end{array} \right.$$

$$x^2 \ge 1 - y^2$$

$$\sqrt{1-y^2} \ge x^2 \ge -\sqrt{1-y^2}$$

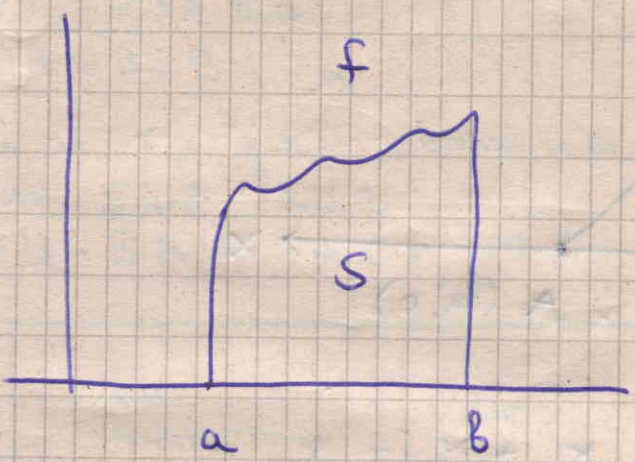
$$x^2 - (1-y^2) \ge 0$$

$$(x - \sqrt{1-y^2})(x + \sqrt{1-y^2}) \ge 0$$

$x^2 = 1 - (y-1)^2$

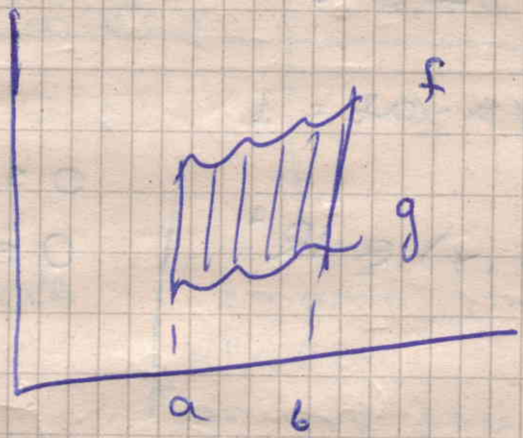
$x \leq \sqrt{1-(y-1)^2}$

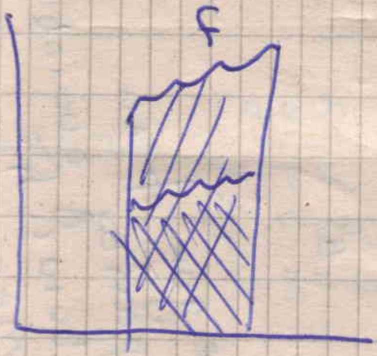
$< x$



$$S = \int_a^b f(x)\,dx$$

$S =$



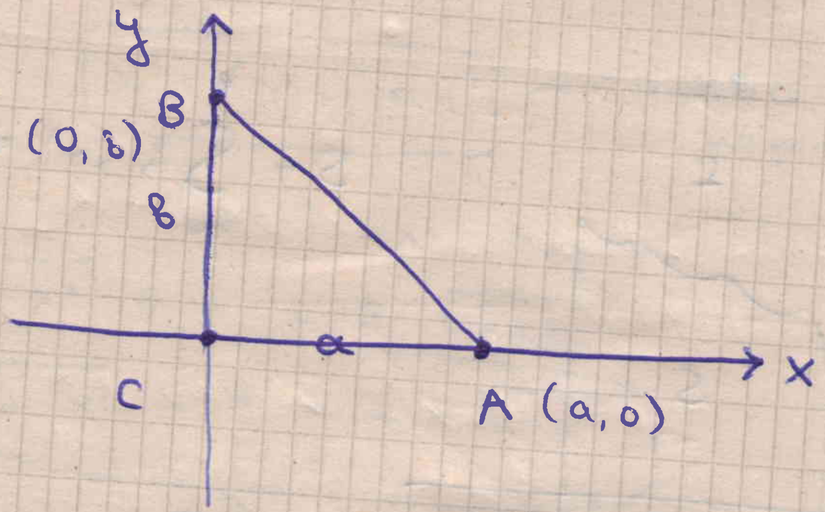


$$S = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx =$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

Даден е $\triangle ABC$ с катети a и b

$S = ?$



$$\frac{x}{a} + \frac{y}{b} = 1$$

$$xb + ya - ab = 1$$

$$\Delta = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \leq \\ 0 \leq \end{array} \right.$$

$$S = \int_0^a \frac{ab - bx}{a}\, dx =$$

$$= b \int_0^a dx - \frac{b}{a} \int_0^a x\, dx =$$

$$= bx \Big|_0^a - \frac{b}{} \, x^2 \Big| =$$

Th на Лебег (Критерий за интегр.)

$f: \Delta \to \mathbb{R}$

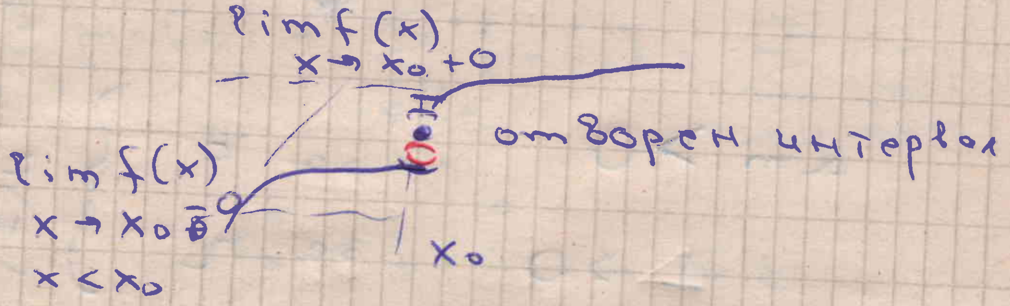
f-ограничена

Δ паралелотоп в $\mathbb{R}^n$

Твърди, че f е интегруема по Риман, тогава и само тогава, когато множеството от точките на прекъсване на f е пренебрежимо по Лебег

$$R_f = \{x \in \Delta : f \text{ е прекъсната в } x\}$$

f монотонно растяща



точка на прекъсване $\longleftrightarrow$ отв. интервал

и те не се секат

изброимо много такива

мон. → изброимо мн. точки на прекъсв

f инт. $\iff$ $\forall \varepsilon > 0$ $\forall \eta > 0$

II форма
критерий
интегруемост

$\exists$ П подразделение

$$\sum_{M_i - m_i \geq \eta} \mu_n(\Delta_i) < \varepsilon$$

f интегруема ~~на~~ на f $\Rightarrow$ R_f е пренебрежимо

$\varepsilon > 0$

$$\varepsilon > \sum_{m=1}^{\infty} \alpha_m, \quad \alpha_m > 0 \quad \forall m \in \mathbb{N}$$

$\alpha_m > 0$ за ε

$\frac{1}{m} > 0$ за η

$\Rightarrow$ II ф-ма

$\exists$ $П_m$ подразделение на Δ, т.че

$\sum \mu_n(\Delta_i) <$

Π_m' - елементите на Π_m, в които
...илацията на f е по-голяма
... равна на $\frac{1}{m}$. (които участват в Σ)

...т - множеството от стените
на ел. на Π_m
...но 0 от изродени паралелотопи ($\partial \Delta i \ldots$)

...) $\cup \left(\bigcup_{m=1}^{\infty} P_m \right)$ - изброима фамилия от паралелотопи в $\mathbb{R}^n$

1) $$\sum_{\Delta \in \Pi} \mu_n(\Delta) = \sum_{\Delta \in \bigcup_{m=1}^{\infty} P_m} \mu_n(\Delta) + \sum_{m=1}^{\infty} \left(\sum_{\Delta \in \Pi_m'} \mu_n(\Delta) \right) \le$$

$$\le 0 + \sum_{m=1}^{\infty} \alpha_m < \varepsilon$$

...$_f \subset \bigcup_{\Delta \in \Pi} \Delta$

$x \notin \bigcup_{\Delta \in \Pi} \Delta$ ...

$\Rightarrow$ за $\forall m \in \mathbb{N}$ във вътр.

$x \in \overset{\circ}{\Delta}{}_{i_x}^m$ не може да е в стена

$\Delta_{i_x}^m \in \Pi_m$ m - фикс.

и осцилацията

$$M_{i_x}^m - m_{i_x}^m < \frac{1}{m}$$

$$y \in \Delta_{i_x}^m$$

$$|f(x) - f(y)| \le M_{i_x}^m - m_{i_x}^m < \frac{1}{m}$$

$\forall m \in \mathbb{N}$ $\exists$ $U_m \equiv \overset{\circ}{\Delta}{}_{i_x}^m$ отв. мн. / з-о т. x

$\forall y \in U_m$:

$$|f(x) - f(y)| < \frac{1}{m}$$

$\Rightarrow$ непрекъснатост в т. x

обратно

R_f е пренебр. $\Rightarrow$ f е интегр.

$\left.\begin{matrix} \varepsilon > 0 \\ \eta > 0 \end{matrix}\right\}$ произволни фикс.

и $\sum_{i=1}^{\infty} \mu_n(\Delta_i) < \varepsilon$

$\bigcup \overset{\circ}{\Delta}_i$ отворено

$$\Delta \setminus \left(\bigcup_{i=1}^{\infty} \overset{\circ}{\Delta}_i \right) = C \text{ компакт}$$

отворено

Кантор 1 лекция

$f : \Delta \to R$

$C \subset \Delta$

компакт

f е непр. във $\forall$ т. на C

$C \cap R_f = \emptyset$

$\varepsilon > 0$

$\delta > 0 \quad \forall x' \in C \quad \forall x'' \in \Delta$

$\|x'' - x'\| < \delta :$

$|f(x'') - f(x')| < \eta$

Ако $\square_j \in \Pi$ и $\square_j \cap C \neq \emptyset$, то осцилацията на f в/у $\square_j$

$$M_j - m_j < \eta$$

$$x_j \in \square_j \cap C$$

$x \in \square_j$, $y \in \square_j$ произволни

$$x, x_j \in \square_j \Rightarrow \|x - x_j\| < \delta$$

$$x_j \in C$$

$$|f(x) - f(x_j)| < \frac{\eta}{4}$$

за y

$$y, x_j \in \square_j$$

$$|f(y) - f(x_j)| < \frac{\eta}{4}$$

$$|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| < \frac{\eta}{2}$$

Осцилацията на f в/у $\square_j$

$$\sum_{M_j - m_j \geq \eta} \mu_n(\square_j)$$

$$\sum_{\square_j \subset \bigcup_{i=1}^{\infty} \mathring{\Delta}_i} \mu_n(\square_j)$$

$$C = \Delta \setminus \left(\bigcup_{i=1}^{\infty} \mathring{\Delta}_i\right)$$

$$\sum_{M_j - m_j \geq \eta} \mu_n(\square_j) \leq \sum_{\square_j \subset \bigcup_{i=1}^{\infty} \mathring{\Delta}_i} \mu_n(\square_j)$$

$$K = \bigcup_{\square_j \subset \bigcup_{i=1}^{\infty} \mathring{\Delta}_i} \square_j$$ компакт

Непрекривају се

$$\subset \bigcup_{i=1}^{\infty} \mathring{\Delta}_i \Rightarrow$$

$$K \subset \bigcup_{i=1}^{n} \mathring{\Delta}_i$$

$\parallel$

Лема

$$\Rightarrow \sum_{M^j - m^j \geq \eta} \mu_n(\square_j) \leq \sum_{\square_j \subset \bigcup_{i=1}^{\infty} \Delta_i} \mu_n(\square_j) \leq \sum_{i=1}^{\infty} \mu_n(\Delta_i)$$

$$\leq \sum_{i=1}^{\infty} \mu_n(\Delta_i) < \varepsilon \quad :))$$

<u>Следствие 2</u>: $f, g : \Delta \to \mathbb{R}$ интегруеми

$\Delta \subset \mathbb{R}^n$ пар.

$\Rightarrow f+g,\ f-g,\ f \cdot g,\ \dfrac{f}{g}$ (ако

$|g(x)| \geq \varepsilon_0 > 0 \quad \forall x \in \Delta$)
равномерно далеч от 0

са интегруеми по Риман

$\dfrac{f}{g} : \Delta \to \mathbb{R}$ и е ограничена, защото

$|f(x)| \leq M$

мн. от т. на прекъсване

$R_{f/g} \subset R_f \cup R_g$

$\Updownarrow$

$R^n \setminus R_{f/g} \supset (R^n \setminus R_f) \cap (R^n \setminus R_g)$

$\frac{f}{g}$ е непр. в x $\Leftarrow$ f непр. в т. x g непр. в т. x

и Th Лебег

Задача: $F: \mathbb{R}^2 \to \mathbb{R}$ непр.

$f, g: \Delta \to \mathbb{R}$ интегруеми $\Delta \subset R^n$ пар.

$\Rightarrow F(f,g)$ е инт. в Δ.

$\{(f(x), g(x)) : x \in \Delta\}$ - ограничено подмн. на $\mathbb{R}^2$

F непр.

$\Rightarrow F(f,g)$ огр.

$R_{F(f,g)} \subset R_f \cup R_g$