<u>06.11.2013г.</u>

Упражнение

Смени в троен интеграл

Нека $\exists\, I = \iiint\limits_D f(x,y,z)\,dx\,dy\,dz$

Нека

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$ е неособена смяна

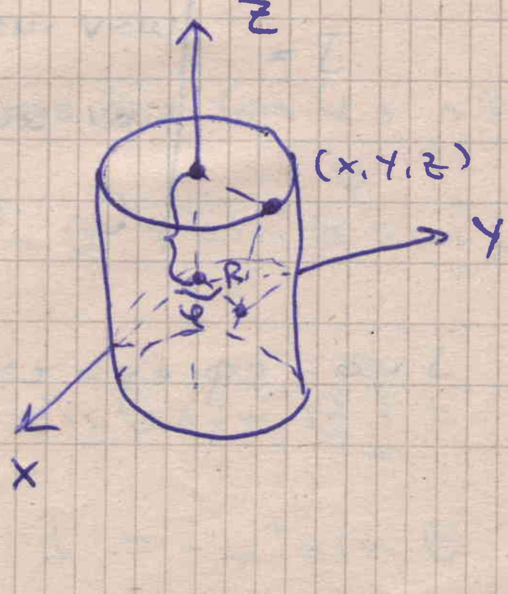
$$\left(J = \frac{D(x,y,z)}{D(u,v,w)} \neq 0 \text{ почти навсякъде} \right)$$

Нека тази смяна изобразява D еднозначно в D'. Тогава

$$I = \iiint\limits_{D'} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J(u,v,w)|\,du\,dv\,dw$$

<u>Цилиндрична смяна:</u>

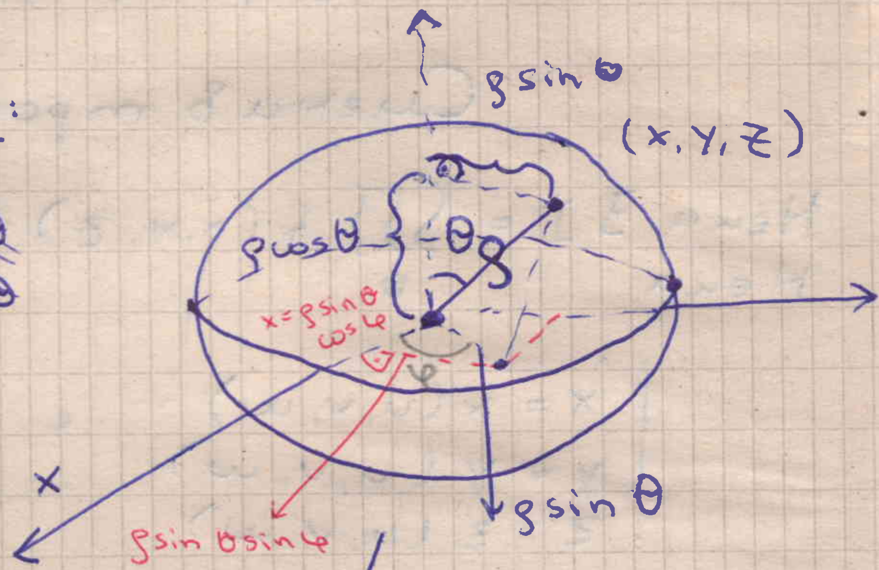
$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = w \end{cases}$$



Цилиндрична смяна най-често правим, когато тялото D е цилиндрично.

<u>Сферична смяна:</u>

$$\cancel{\begin{cases} x = u\cos\varphi\cos\theta \\ y = u\sin\varphi\cos\theta \\ z = u\sin\theta \end{cases}}$$



$0 \leq \varphi \leq 2\pi$

$0 \leq \theta \leq \pi$

$$\begin{cases} x = u\cos\varphi\sin\theta \\ y = u\sin\varphi\sin\theta \\ z = u\cos\theta \end{cases}$$

φ - полярен ъгъл

J на цилиндрична смяна

$$J = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u$$

J на сферична смяна

$$J = \begin{vmatrix} \cos\varphi \sin\theta & -u\sin\varphi\sin\theta & u\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & u\cos\varphi\sin\theta & u\sin\varphi\cos\theta \\ \cos\theta & 0 & -u\sin\theta \end{vmatrix} =$$

$$= u^2 \begin{vmatrix} \cos\varphi\sin\theta & -u\sin\varphi\sin\theta & u\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & u\cos\varphi\sin\theta & - \\ \cos\theta & 0 & \end{vmatrix} =$$

$$= u^2 \cdot \left\{ \cos\theta (-1)^{3+1} \begin{vmatrix} -\sin\varphi\sin\theta & \cos\varphi\cos\theta \\ \cos\varphi\sin\theta & \sin\varphi\cos\theta \end{vmatrix} + \right.$$

$$+ \; -\sin\theta \cdot (-1)^{3+3} \begin{vmatrix} \cos\varphi\sin\theta & -\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta \end{vmatrix} =$$

$$= u^2 \left[+\cos\theta \left(-\sin^2\varphi\sin\theta\cos\theta - \cos^2\varphi\sin\theta\cos\theta \right) - \sin\theta \left(\cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta \right) \right] =$$

$$= u^2 \left[-\cos^2\theta \sin\theta \, \cancel{\cos\theta} - \sin\theta\sin^2\theta \right] =$$

$$= \cancel{u^2 \{\sin\theta\}} - u^2\sin\theta \cdot 1 = -u^2 \sin\theta$$

Обикновено правим сферична смяна, когато едното от ограниченията е свързано с кълбо, а често и когато имаме параболоид хиперболоид, седло, т.е. когато имаме тяло, описано с у-я от втора степен, в което и трите променливи са на квадрат.

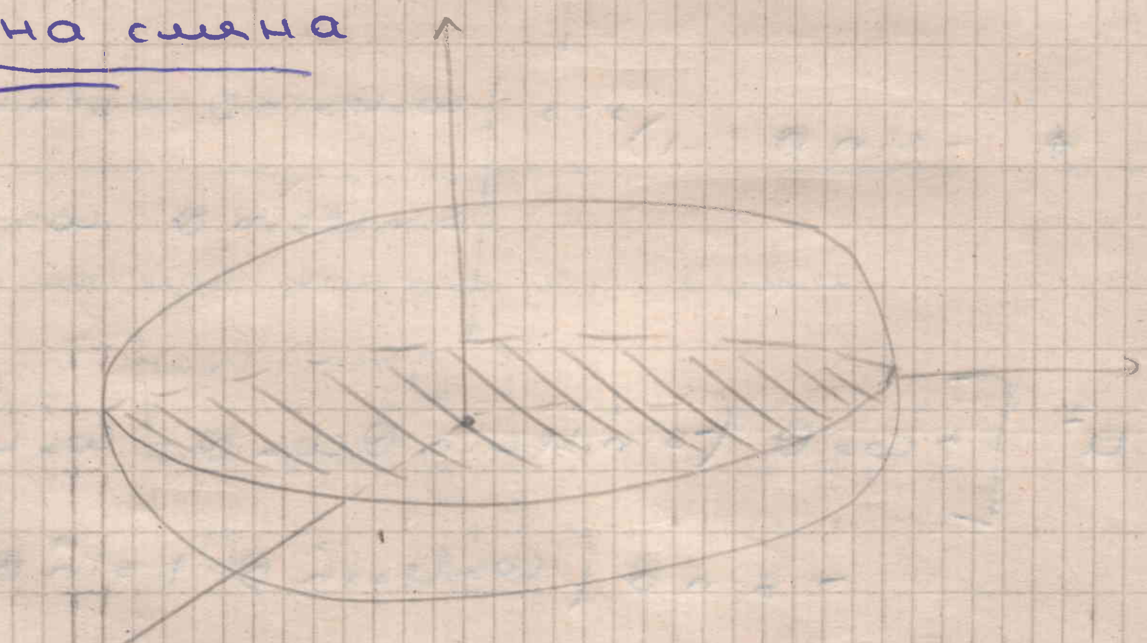
зад. Да се пресметне обема на елипсоида

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\boxed{V=\iiint_E 1\,dx\,dy\,dz}$$

$$a, b, c > 0$$

сферична смяна



I н.

$$E=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{array}{l} -a\le x\le a \\ -b\sqrt{1-\frac{x^2}{a^2}}\le y\le b\sqrt{1-\frac{x^2}{a^2}} \\ c\left(-\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)\le z\le c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \end{array}\right.$$

$$V_E = \left(\int_{-a}^{a} \left(\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \left(\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\right) dy\right) dz$$

$$x = a\rho\cos\varphi\sin\theta$$
$$y = b\rho\sin\varphi\sin\theta$$
$$z = c\rho\cos\theta$$

$$J = abc\rho^2\sin\theta$$

$$\rho^2\sin^2\theta\cos^2\varphi + \rho^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\theta =$$

$$= \rho^2\sin^2\theta \cdot 1 + \rho^2\cos^2\theta = \rho^2 = 1$$

B

Обобщени сферични (елиптични)

у-ето на елипсата е

$$\boxed{\rho = 1}, \text{ а}$$

вътрешността ѝ $\rho \leq 1$

$$0 \leq \varphi \leq 2\pi$$
$$0 \leq \theta \leq \pi$$

$$V = \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{1} 1.\rho^2 d\rho\right) d\theta\right) d\varphi = \quad abc.\sin\theta$$

$$= abc\int_0^{2\pi}\left(\int_0^{\pi} \sin\theta . \frac{1}{3} d\theta\right) d\varphi = \frac{abc}{3}\int_0^{2\pi} -\cos\theta\Big|_0^{\pi} =$$

$$= \frac{2}{3}abc\int_0^{2\pi} d\varphi =$$

$= \frac{4}{3} abc.\pi$

$a = b = c = R$

$V = \frac{4}{3}\pi R^3$

II н. : чрез принципа на Кавалиери

$E = \bigcup_{z=-c}^{c} E_{z_0}$

обединение на елипси, където

E_{z_0} на ниво $z = z_0$ има уравнение

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z_0^2}{c^2} \quad /: 1 - \frac{z_0^2}{c^2}$

Ние знаем, че лицето на елипса с радиуси A и B е $\boxed{\pi AB}$

$$\frac{x^2}{\left(a\left(\sqrt{1-\frac{z_0^2}{c^2}}\right)\right)^2} + \frac{y^2}{\left(b\sqrt{1-\frac{z_0^2}{c^2}}\right)^2} = 1$$

$$S_{E_{z_0}} = \pi ab\left(\sqrt{1-\frac{z_0^2}{c^2}}\right)^2 = \pi ab\left(1-\frac{z_0^2}{c^2}\right)$$

Според принципа на Кавалиери

$$V_E = \int_{-c}^{c} \Big(\underbrace{\iint_{E_z}}\, dx\,dy \Big)\, dz =$$

$$= \int_{-c}^{c} \pi a b \left(1 - \frac{z^2}{c^2}\right) dz =$$

$$= 2\pi a b c - \frac{\pi a b}{c^2} \frac{z^3}{3} \Big|_{-c}^{c} =$$

$$= 2\pi a b c - \frac{\pi a b}{3c^2}\left(c^3 - (-c)^3\right) =$$

$$= 2\pi a b c - \frac{\pi a b 2 c^3}{3c^2} =$$

$$= \frac{4}{3}\pi a b c$$

зад. 4.6 Пресметнете обема на тялото K, опр. с неравенствата

$$K: \left| \begin{array}{l} 1 \le x^2 + y^2 \le 2x \\ x \le \sqrt{3}\, y \\ y^4 z^2 \le x^4 \end{array} \right.$$

$$V = \iiint_K dx\,dy\,dz$$

Правим цил. смяна:

$x = u\cos v$
$y = u\sin v$
$z = w$

$$K_1 \longrightarrow K' : \left| \begin{array}{l} 1 \le u^2 \le 2u\cos v \\ u\cos v \le \sqrt{3}\, u\sin v \\ u^4\sin^4 v \cdot w^2 \le u^4\cos^4 v \\ \qquad w^2 \le \cot g^4 v \end{array} \right.$$

$$\Leftrightarrow \left| \begin{array}{l} u^2 \ge 1 \\ u \le 2\cos v \\ 2\cos v \ge 1 \\ \cot g\, v \le \sqrt{3} \\ -\dfrac{\cos^2 v}{\sin^2 v} \le w \le \dfrac{\cos^2 v}{\sin^2 v} \end{array} \right.$$

$$\left| \begin{array}{l} \frac{\pi}{6} \le v \le \frac{\pi}{3} \\ 1 \le u \le 2\cos v \\ -\cot g^2 v \le w \le \cot g^2 v \end{array} \right.$$

$$\iiint = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\int_{1}^{2\cos v} \left(\int_{-\cot g^2 v}^{\cot g^2 v} u \, dw \right) du \right) dv =$$

$$= \int_{\pi/6}^{\pi/3}\left(\int_{1}^{2\cos v} u \cdot 2\operatorname{ctg}^2 v \, du\right) dv =$$

$$= \int_{\pi/6}^{\pi/3} 2\operatorname{ctg}^2 v \left(\int_{1}^{2\cos v} u \, du\right) dv =$$

$$= \int_{\pi/6}^{\pi/3} \operatorname{ctg}^2 v \, u^2 \Big|_{1}^{2\cos v} dv =$$

$$= \int_{\pi/6}^{\pi/3} \operatorname{ctg}^2 v \,(4\cos^2 v - 1)\, dv =$$

$$= \int_{\pi/6}^{\pi/3} \frac{\cos^4 v}{\sin^2 v} - \int_{\pi/6}^{\pi/3} \frac{\cos^2 v}{\sin^2 v}\, dv$$

$$\operatorname{tg}^2 v = t$$

Да се сметне!!

Отг.: $\left(-3\operatorname{ctg}\varphi - 5\varphi + \sin 2\varphi\right)\Big|_{\pi/6}^{\pi/3}$

$$= 2\sqrt{3} - \frac{5\pi}{6}$$

4.11 | 397 Пресметнете

$$\iiint\limits_{K} yz \, dxdydz \text{ , където}$$

$$K: \begin{cases} x^2+y^2 \le z^2 \le 2y - x^2 - y^2 \\ z \ge 0 \end{cases}$$

$$\begin{cases} x = \rho \cos\varphi \sin\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\theta \end{cases} \qquad K \to K' \qquad \theta \in [0; \pi]$$

$$\rho \cos\theta \ge 0 \qquad \theta \in \left[0; \frac{\pi}{2}\right]$$

$$\sin^2\theta \le \cos^2\theta \iff \sin\theta \le \cos\theta$$

$$\iff tg\theta \le 1$$

$$\iff \theta \in \left[0; \frac{\pi}{4}\right]$$

$$\theta \in \left[0; \frac{\pi}{4}\right]$$

$$\begin{cases} \sin^2\theta \le \cos^2\theta \iff \sin\theta \le \cos\theta \iff tg\theta \le 1 \iff \\ \rho\cos^2\theta \le 2\sin\varphi\sin\theta - \rho\sin^2\theta \\ \theta \in \left[0; \frac{\pi}{2}\right] \end{cases}$$

$$(=)\left|\begin{array}{l} \theta \in [0; \frac{\pi}{4}] \\ 0 \le \rho \le 2\sin\varphi\sin\theta \\ \varphi \in [0; \pi] \end{array}\right.$$ цилиндрично

$$I = \int_0^{\pi}\left(\int_0^{\frac{\pi}{4}}\left(\int_0^{2\sin\varphi\sin\theta} \rho\sin\varphi\sin\theta\,\rho\cos\theta\,\rho^2\sin\varphi\,d\rho\right)d\theta\right)d\varphi =$$

$$= \int_0^{\pi}\left(\int_0^{\frac{\pi}{4}}\sin^2\theta\cos\theta\sin\varphi\left(\int_0^{2\sin\varphi\sin\theta}\rho^4\,d\rho\right)d\theta\right)d\varphi$$

$$= \frac{2^5}{5}\int_0^{\pi}\left(\int_0^{\pi/4}\sin^2\theta\cos\theta\sin\varphi\sin^5\varphi\sin^5\theta\,d\theta\right)d\varphi =$$

$$= \frac{32}{5}\int_0^{\pi}\sin^6\varphi\left(\int_0^{\frac{\pi}{4}}\sin^7\theta\cos\theta\,d\theta\right)d\varphi =$$

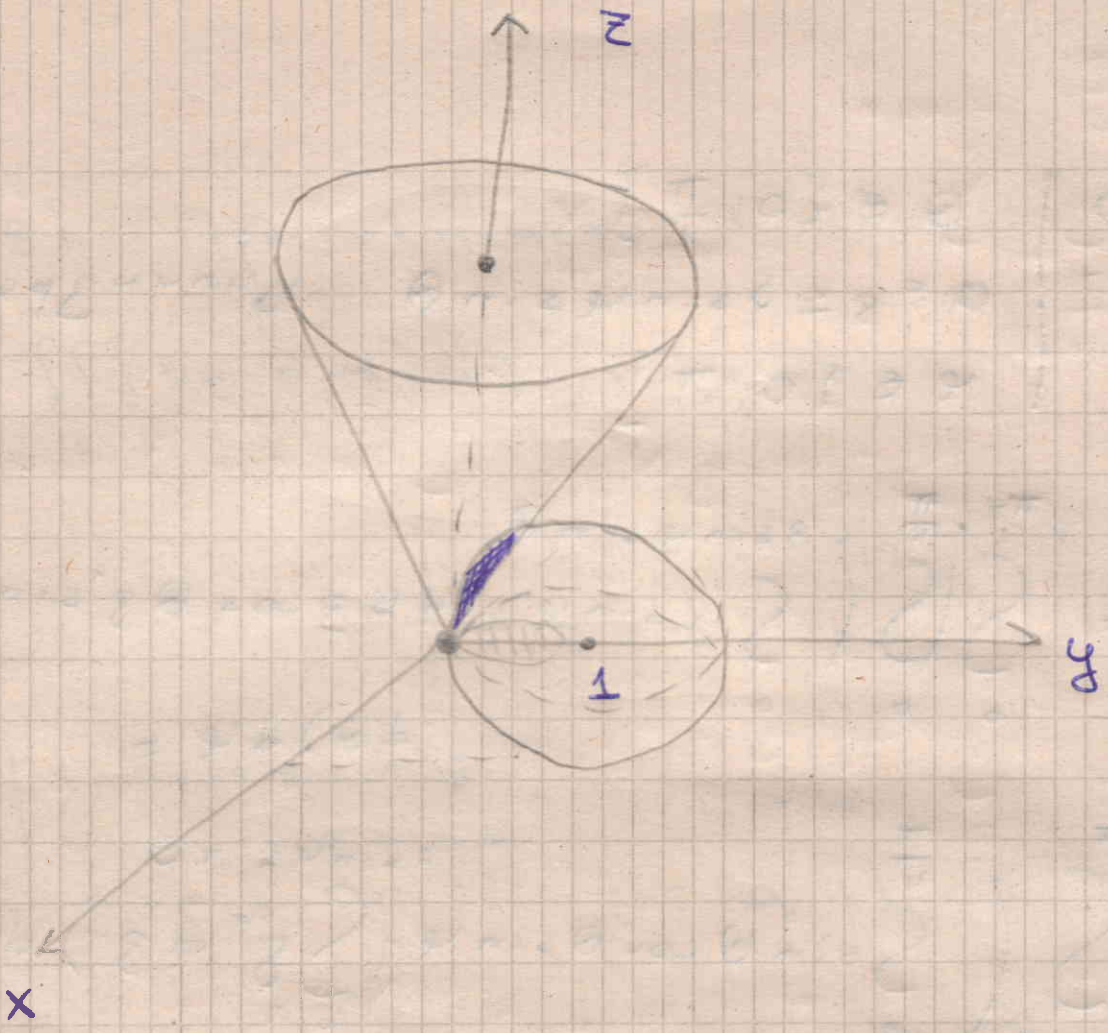
$$= \frac{32}{5}\int_0^{\pi}\sin^6\varphi\,d\varphi\left(\int_0^{\frac{\pi}{4}}\sin^7\theta\,d\sin\theta\right)d\varphi =$$

$tg^2\varphi = t$

$$\cancel{= -\frac{32}{5}\int_0^{\pi}(1-\cos^2\varphi)^2\,d\cos\varphi\cdot\frac{\sin^8\theta}{8}\Big|_{\theta=0}^{\frac{\pi}{4}} =}$$

$$\cancel{= -\frac{32}{5}\cdot\frac{1}{8}\left(\frac{1}{\sqrt{2}}\right)^8\cdot\int_1^{-1}(1-t^2)^2\,dt.}$$

Отг. $\frac{\pi}{64}$



$$x^2 + (y-1)^2 + z^2 \leq 1$$

кълбо

Физична интерпретация на $\iiint$

1) Обем: $V_K = \iiint\limits_K dx\,dy\,dz$

2) Маса на тяло K с плътност

$\rho(x,y,z)$

$$\iiint\limits_K \rho(x,y,z)\,dx\,dy\,dz$$

3) Център на тежестта на тяло K

$$x_G = \frac{1}{M} \iiint_K x \rho \, dx\, dy\, dz$$

$$y \; z_G = \frac{1}{M} \iiint_K y \rho \, dx\, dy\, dz$$

$$z_G = \frac{1}{M} \iiint_K z \rho \, dx\, dy\, dz$$

$$\vec{r}_G = \frac{1}{M} \iiint_K \vec{r} \rho(x,y,z) \, dx\, dy\, dz =$$

$$= \frac{1}{M} \begin{pmatrix} \iiint_K x \rho \, dx\, dy\, dz \\ \iiint_K y \rho \, dx\, dy\, dz \\ \iiint_K z \rho \, dx\, dy\, dz \end{pmatrix}$$

<u>Зад.</u> Намерете масата на неогр. тяло

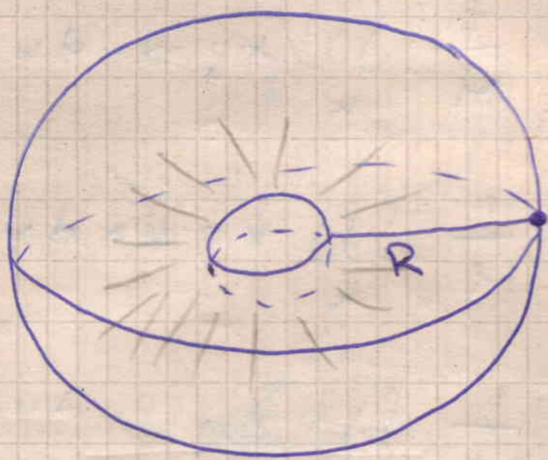
! $K: x^2 + y^2 + z^2 \geq 1$, ако плътността му е

$$\rho(x,y,z) = \mu_0 e^{-k\sqrt{x^2+y^2+z^2}}$$

Ако $\exists \; M(K)$, то

$$M(K) = \lim_{R \to \infty} M(K_R) \quad \text{напр.}$$

$$K_R : \left| \; R^2 \geq x^2 + y^2 + z^2 \geq 1 \right.$$



$$M(K_R) = \iiint_{K_R} \mu_0 e^{-k\sqrt{x^2+y^2+z^2}} \, dx\,dy\,dz$$

$$x = \rho \cos\varphi \sin\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\theta$$

$$K_R : \longrightarrow K_R' \;\Big|\; R^2 \geq \rho^2 \geq 1$$

$$\Leftrightarrow \;\Big|\; R \geq \rho \geq 1$$

$$M(K_R) = M(K_R') = \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_1^{R} \mu_0 e^{-k\sqrt{\rho^2}} \rho^2 \sin\theta \, d\rho\right) d\theta\right) d\varphi =$$

$$= \mu_0 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_1^{R^2} e^{-k\rho} \rho^2 \, d\rho =$$

$$= \mu_0 \cdot 2\pi \cdot 2 \int_1^{R^2} \rho^2 e^{-k\rho} \, d\rho$$

$$\overset{\shortparallel}{J}$$

$$J = \int_1^{R^2} \rho^2 e^{-k\rho}\, d\rho = -k \int_1^{R^2} \rho^2\, d e^{-k\rho} =$$

$$= -k\rho^2 e^{-k\rho}\Big|_1^{R^2} + k\int_1^{R^2} 2\rho e^{-k\rho}\, d\rho =$$

$$= -k\overset{R}{\rho}{}^2 e^{-kR^2} + ke^{-k} + 2k(-k)\int_1^{R^2} d e^{-k\rho} =$$

$$= -kRe^{-kR^2} + ke^{-k} - 2k^2\rho e^{-k\rho}\Big|_1^{R^2} +$$

$$+ 2k^2\int_1^{R^2} e^{-k\rho}\, d\rho = -kR^2 e^{-kR^2} + ke^{-k} -$$

$$- 2k^2R^2 e^{-kR^2} + 2k^2 e^{-k} -$$

$$- 2k^3 e^{-k\rho}\Big|_1^{R^2} =$$

$$= e^{-kR^2} R^2 (k + 2k^2) + ke^{-k} + 2k^2 e^{-k} -$$

$$- 2k^3 e^{-kR^2} + 2k^3 e^{-k}$$

$$M(k) = \lim_{R\to\infty} (4\pi\mu_0 J(R)) =$$

$$= 4\mu_0\pi\left(ke^{-k} + 2k^2 e^{-k} + 2k^3 e^{-k}\right)$$

зад. Пресметнете $F'(t)$, ако

$$F(t) = \iiint\limits_{x^2+y^2+z^2 \le t^2} f(x^2+y^2+z^2)\, dx\, dy\, dz$$

В $F(t) \longrightarrow$ сф. смяна

$$F(t) = \int_0^{2\pi}\int_0^{\pi}\int_0^{t} f(\rho^2)\, \rho^2 \overset{\sin}{\cancel{\cos}}\,\theta \; d\varphi\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi} 1\, d\varphi \int_0^{\pi} \overset{\sin}{\cancel{\cos}}\,\theta\, d\theta \int_0^{t} f(\rho^2)\, \rho^2\, d\rho =$$

$$= 4\pi \int_0^{t} f(\rho^2)\, \rho^2\, d\rho$$

$$F'(t) = 4\pi f(t^2)\, t^2 \qquad \text{Л-Н}$$

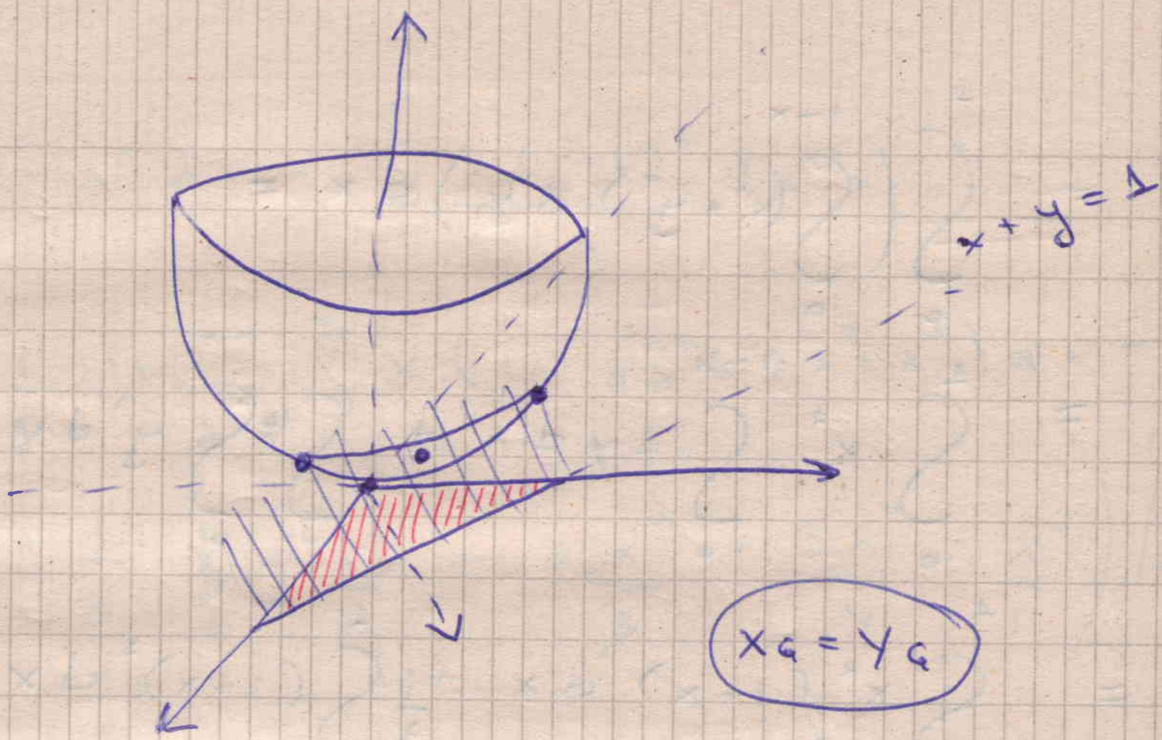
зад. Намерете центъра на тежестта на тялото, оградено от

$$z = x^2 + y^2$$

$$x + y = 1 \qquad y = 1 - x$$

$$\left.\begin{array}{l} x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{array}\right\} \quad \text{~~[illegible]~~}$$

с $\rho(x, y, z) = 1$



$\boxed{x_G = y_G}$

D e $\triangle ABC$ $\quad A(0,0)$

$B(1,0)$

$C(0,1)$

$$K = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} (x,y) \in D \\ 0 \le z \le x^2+y^2 \end{array} \right\} =$$

$$= K = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le x^2+y^2 \end{array} \right\}$$

$$m(K) = \iiint_K 1 \, dx\,dy\,dz = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{x^2+y^2} dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} (x^2+y^2)\, dy \right) dx =$$

$$= \int_0^1 x^2 \int_0^{1-x} dy\, dx + \int_0^1 \int_0^{1-x} y^2\, dy\, dx =$$

$$= \int_0^1 x^2 (1-x)\, dx + \frac{1}{3} \int_0^1 (1-x)^3\, dx =$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} \int_0^1 (1-x)^3\, d(1-x) =$$

$$= \frac{1}{12} - \frac{1}{3}\, \frac{(1-x)^4}{4} \Bigg|_{x=0}^{1} =$$

$$= \frac{1}{12} - \frac{1}{12}(0-1) = \frac{1}{6}$$

$$x_c = \frac{1}{M} \iiint_K x\, dx\, dy\, dz = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} x\, dz\, dy\, dx =$$

$$= \int_0^1 \int_0^{1-x} x(x^2+y^2)\, dy\, dx =$$

$$= \int_0^1 x^3 \int_0^{1-x} dy\, dx + \int_0^{1-x} x \int y^2\, dy\, dx =$$

$$= \int_0^1 x^3(1-x)\,dx + \int_0^1 \frac{x(1-x)^3}{3}\,dx =$$

$$= \frac{1}{4} - \frac{1}{5} + \frac{1}{3}\int_0^1 x\left(1 - 3x + 3x^2 - x^3\right)dx =$$

$$= \frac{1}{20} + \frac{1}{3}\cancel{\int_0^1}\left(\frac{x^2}{2} - \cancel{3}\frac{x^3}{\cancel{3}} + \frac{3x^4}{4} - \frac{x^5}{5}\right)\Bigg|_0^1 =$$

$$= \frac{1}{20} + \frac{1}{3}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) =$$

$$= \frac{1}{20} + \frac{1}{3}\left(-\frac{1}{2} + \frac{3}{4} - \frac{1}{5}\right) =$$

$$= \frac{1}{20} + \frac{1}{3}\left(\frac{-10+15-4}{20}\right) =$$

$$= \frac{1}{20} + \frac{1}{3}\cdot\frac{1}{20} = \frac{4}{3\cdot 20} = \frac{1}{15}$$

$$\cancel{x_G = y_G} \qquad x_G = \frac{1}{M}\cdot\frac{1}{15} = \frac{2}{5}$$

$$z_G = \frac{1}{\frac{1}{6}}\int_0^1\int_0^{1-x}\int_0^{x^2+y^2} z\,dz\,dy\,dx =$$

$$= 6\int_0^1\int_0^{1-x}\frac{(x^2+y^2)^2}{2}\,dy\,dx = \dots$$

$$z_G = \frac{7}{30} \qquad G\ (= r_G\left(\frac{2}{5}, \frac{2}{5}, \frac{7}{30}\right)$$

Дом.: В цилиндрични:

$$\iiint_K \frac{1}{y^2}\,dx\,dy\,dz$$

$$K: \left| \begin{array}{l} 2x^2+3y^2 \le z \le 5y \\ 2 \le x^2+y^2 \end{array} \right.$$

Отг.: $\frac{25}{\sqrt{6}} \operatorname{arctg}\sqrt{\frac{2}{3}} + 10\sqrt{2}\ln(\sqrt{2}-1) + 4 + \frac{\pi}{2}$

04.11

Лекция

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{(n-1)!!}{n!!} \begin{cases} \frac{\pi}{2}, & \text{ако } n \text{ четно} \\ 1, & \text{ако } n \text{ нечет.} \end{cases}$$

Зорич - учебник

Дифеоморфизъм

$\varphi: U \to V$

$U, V \subset \mathbb{R}^n$

отворени

φ биекция

φ, φ^{-1} са гладки - частни произв. $\exists$ непрек.

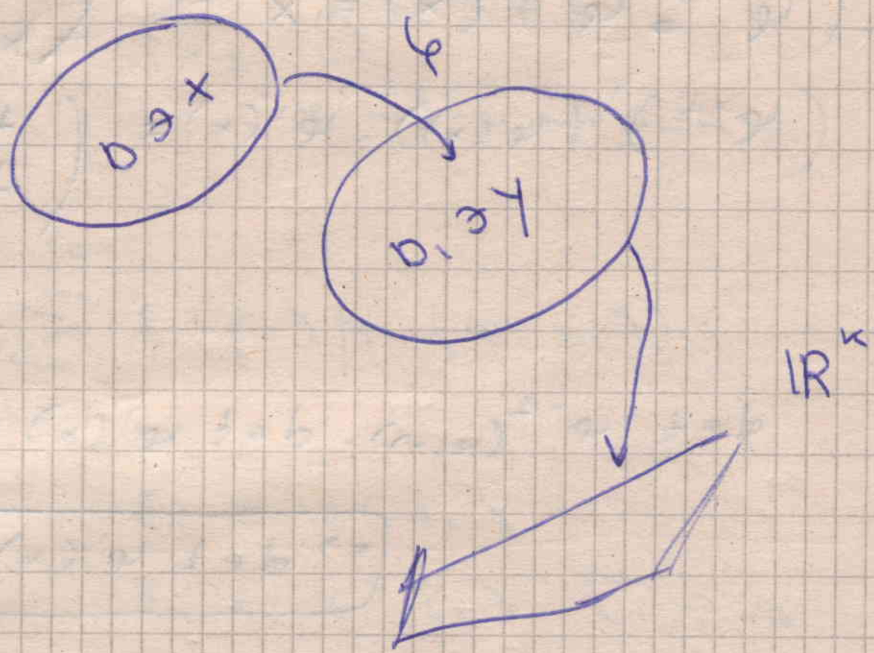
$\varphi, \varphi^{-1} \in C^1$

$\varphi'(x)$

$\varphi : D \to D_1$

$\psi : D_1 \to \mathbb{R}^k$

$D \subset R_n, \quad D_1 \subset \mathbb{R}^m$



$$\varphi(x) = \varphi(x_1, \dots, x_n) = \begin{pmatrix} \varphi_1(x_1, \dots, x_n) \\ \vdots \\ \varphi_m(x_1, \dots, x_n) \end{pmatrix}$$

$$\varphi'(x) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \dots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial \varphi_m}{\partial x_1} & - & - & \frac{\partial \varphi_m}{\partial x_n} \end{pmatrix}(x)$$

$m \times n$

$$d\varphi(x)(h) = \varphi'(x).h$$

$$\boxed{(\psi\circ\varphi)'(x) = \psi'(\varphi(x)).\varphi'(x)}$$

матрици

ψ' $k\times m$ матрица

φ' $m\times n$

$$(\varphi^{-1}\circ\varphi)(x) = x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$(\varphi^{-1})'(\varphi(x)).\varphi'(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 1 \end{pmatrix}$$

$$\det \varphi^{-1}(\varphi(x)).\det\varphi'(x) = 1$$

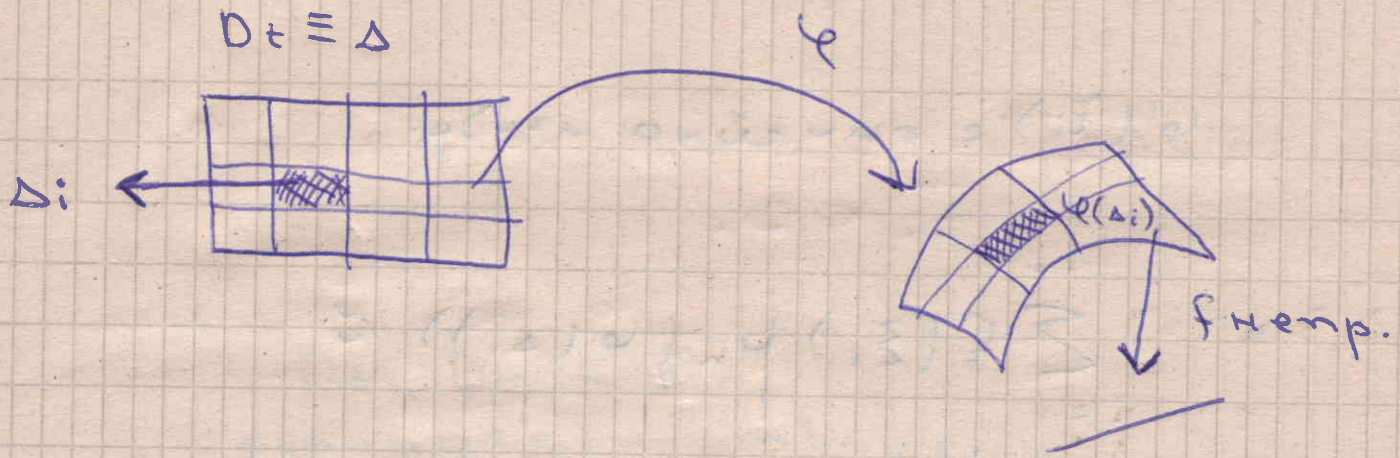
$$\boxed{\Rightarrow \det\varphi'(x) \neq 0}$$

Ако

$$\int_{D_x} f(x)\,dx = \int_{D_t} \,?\, \dots (t)\,dt$$

$\varphi : D_t \longrightarrow D_x$

дифеом.



$\Pi = \{\Delta_i\}_{i=1}^{i_0}$ подр. на Δ

$$\int_{D_x} f(x)\,dx = \sum_{i=1}^{i_0} \int_{\varphi(\Delta_i)} f(x)\,dx =$$

$\varphi(\Delta_i)$ - измерими

$$= \sum_{i=1}^{i_0} f(\xi_i)\,\mu_n(\varphi(\Delta_i))$$

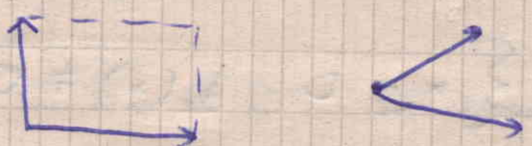
$\xi_i \in \varphi(\Delta_i)$

$\xi_i = \varphi(\tilde{\tau}_i) \quad \tilde{\tau}_i \in \Delta_i$

$f(\varphi(\tilde{\tau}_i))$

Ако φ е линейно

$$\mu_n(\varphi(\Delta_i)) = |\det \varphi'| \cdot \mu_n(\Delta_i)$$



$(\varphi' \equiv \varphi)$

$\varphi'(\tau_i)$ е линейно изобр.

$$\Rightarrow \sum_{i=1}^{i_0} f(\xi_i)\, \mu_n(\varphi(\Delta_i)) \approx$$

$$\sum_{i=1}^{i_0} f(\varphi(\tau_i)) . |\det \varphi'(\tau_i)|\, \mu_n(\Delta_i) =$$

Риманова сума

$$= \sigma_{f\circ\varphi . |\det\varphi'|}(\Pi) \xrightarrow[d(\Pi)\to 0]{}$$

$$\longrightarrow \int_{D_t} (f\circ\varphi)(t) . |\det \varphi'(t)|\, dt$$

$$\boxed{\Rightarrow \int_{D_x} f(x)\,dx = \int_{D_t} f(\varphi(t)) . |\det \varphi'(t)|\, dt}$$

$f : D \to \mathbb{R}$ $\quad D \subset \mathbb{R}^n$

Деф. носител на f

$\operatorname{supp} f = \{ x \in D : f(x) \neq 0 \}$

support

Th за смяна на променливите

$D_t, D_x \subset \mathbb{R}^n$
ограничени, отворени

$\varphi : D_t \to D_x$ дифеоморфизъм

$f : D_x \to \mathbb{R}$ интегруема (по Риман)

и $\operatorname{supp} f \subset D_x$

Тогава

$f \circ \varphi \cdot |\det \varphi'|$ е интегруема в D_t и

$$\int_{D_x} f(x)\,dx = \int_{D_t} f(\varphi(t))\,|\det \varphi'(t)|\,dt$$

Лема 1: D_t, D_x - отворени в $\mathbb{R}^n$
$\varphi : D_t \to D_x$

(a) Ако $E_t \subset D_t$ е пренебрежимо (по Лебег), то образът му
$E_x = \varphi(E_t) \subset D_x$ е пренебрежим (по Лебег)

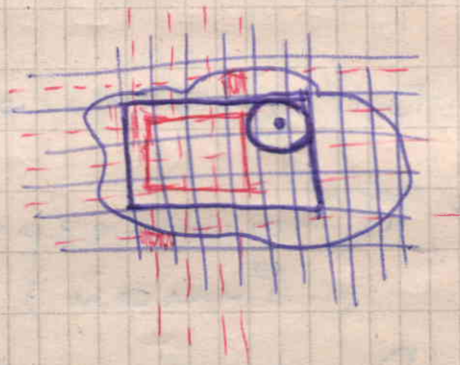
(б) Ако $E_t \subset \overline{E_t} \subset D_t$ е измеримо (по П-Ж), то
зат. обвивка

$E_x = \varphi(E_t)$ е измеримо (по П-Ж) и

$\overline{E_x} \subset D_x$

Д-во: (а) D_t отворено в $\mathbb{R}^n$ - $\cup$ изброимо мн. паралел.

$\bigcup_{i=1}^{\infty} \Delta_i$ в $\mathbb{R}^n$



? ще покрием ли $\forall$ точки

диаг. кубче $a < \frac{r}{\sqrt{n}}$

r - радиус

=> кубчето изцяло в O

=> $\forall$ точки покрити

$\bigcup_{i=1}^{\infty} \varphi(E_t \cap \Delta_i) =$ ~~[illegible]~~

$\varphi\left(\bigcup_{i=1}^{\infty} \Delta_i \cap E_t\right) = \varphi(E_t)$

Б.О.О. $\Delta \subset D_t$, $E_t \subset \Delta$

пренебр.

$\varepsilon > 0$

$\{\Delta_i\}_{i=1}^{\infty}$ $\bigcup_{i=1}^{\infty} \Delta_i \supset E_t$,

Б.О.О. $\Delta_i \subset \Delta$

пар

$\sum_{i=1}^{\infty} \mu_n(\Delta_i) < \varepsilon$

$\exists \; M>0, \quad \dfrac{\partial \varphi_i}{\partial x_j}(x) \le M$ за $\forall \; x \in \Delta$

$\forall \; i = 1, \dots, n$

$j = 1, \dots, n$

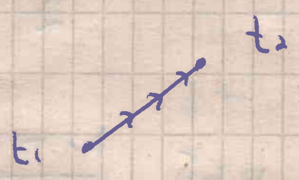
Δ_i - кубчета

! Th ср. ст-ти $n \ge 2$

$\varphi_i(t_1) - \varphi_i(t_2)$

$a(\lambda) = \varphi_i(t_1 + \lambda(t_2 - t_1)) \qquad a(0) = \varphi_i(t_1)$

$a(1) = \varphi_i(t_2)$



$$a(1) - a(0) = a'(\theta) \cdot (1 - 0)$$

$$0 < \theta < 1$$

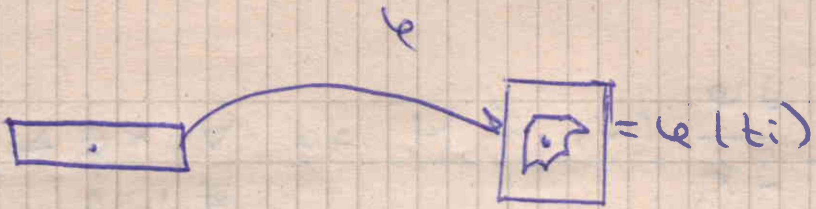
$$a'(\lambda) = \langle \operatorname{grad} \varphi_i(t_1 + \lambda(t_2 - t_1)), \; t_2 - t_1 \rangle$$

$$\varphi_i(t_1) - \varphi_i(t_2) = \left| \langle \operatorname{grad} \varphi_i(t_1 + \theta(t_2 - t_1)), \; t_2 - t_1 \rangle \right| \le$$

НКБШ

$$\| \operatorname{grad} \varphi_i(t_1 + \theta(t_2 - t_1)) \| \, \| t_2 - t_1 \| \le$$

$$\sqrt{n M^2} \, \| t_2 - t_1 \| = M \sqrt{n} \, \| t_2 - t_1 \|$$



Б.О.О. Δ_i са кубчета с център t_i и ръб a_i

$$x_i = \varphi(t_i)$$

$$t \in \Delta_i \Rightarrow \|t - t_i\| \leq \overset{\text{ръб}}{\frac{a_i}{2}}\sqrt{n} \Rightarrow$$

$$|\varphi_j(t) - \varphi_j(t_i)| \leq M\sqrt{n}\,\|t - t_i\| \leq$$

$$\leq M\sqrt{n}\,\frac{a_i}{2}\sqrt{n} = \frac{M n a_i}{2}$$

$\square_i$ кубче с център x_i и ръб $\dfrac{M n a_i}{2}$,

то покрива $\square_i \supset \underline{\varphi(\Delta_i)}$

$$\mu_n(\square_i) = (M n a_i)^n = M^n . n^n . \mu_n(\Delta_i)$$

$$\bigcup_{i=1}^{\infty} \square_i \supset \bigcup_{i=1}^{\infty} \varphi(\Delta_i) = \varphi\Big(\bigcup_{i=1}^{\infty} \Delta_i\Big) \supset \varphi(E_t)$$

$$\sum_{i=1}^{\infty} \mu_n(\square_i) = M^n . n^n \sum_{i=1}^{\infty} \mu_n(\Delta_i) < M^n . n^n . \varepsilon$$

б) $E_t \subset \overline{E_t} \subset O_t$

$E_x = \varphi(E_t)$

$\overline{E_x} = \overline{\varphi(E_t)} = \varphi(\overline{E_t}) \subset D_x$

$\partial E_x = \varphi(\partial E_t)$ φ-дифеоморфизъм и хомоморфизъм

E_t измеримо П-Ж $\Rightarrow \partial E_t$ пренебрежим

$\underset{(a)}{\Rightarrow}$ $\varphi(\partial E_t)$ пренебрежим (Лебег)

$\parallel$

∂E_x

E_t е измеримо $\Rightarrow$ огр. $\Rightarrow \overline{E_t}$ компакт

Вайерщрас $\Rightarrow \varphi(\overline{E_t}) = \overline{E_x}$ компакт

$\Rightarrow$ измеримо по П-Ж $\overline{E_t} \subset D_x$ ||

$\operatorname{supp}\left[(f\circ\varphi)\underbrace{|\det\varphi'|}_{>0}\right] = \operatorname{supp}(f\circ\varphi) =$

$= \varphi^{-1}(\operatorname{supp} f) \subset D_t$

компакт

1) $R(f\circ\varphi)|\det\varphi'|.\chi_{D_t} \equiv R(f\circ\varphi)|\det\varphi'|$

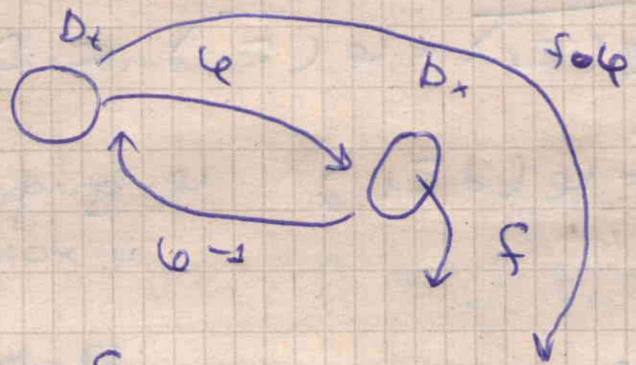
2) $f\circ\varphi.\underbrace{|\det\varphi'|}$ е ограничена

ограничено в/у $\operatorname{supp}(f\circ\varphi)$ компакт

f-огр.

Следствие Лема 1

$$R_{(f\circ\varphi)|\det\varphi'|} \equiv R_{f\circ\varphi} \equiv \varphi^{-1}(R_f)$$



$$(f\circ\varphi)\circ\varphi^{-1} \equiv f$$

композ. на непр. $\to$ непр.

по Лебег

$\varphi^{-1}(R_f)$ е пренебр. а) л. 1

Лема 2: а) I_t, I_x - крайни компактни интервали

$\varphi: I_t \to I_x$ дифеоморфизъм

f интегруема в I_x

$$\Rightarrow \int_{I_x} f(x)\,dx = \int_{I_t} f(\varphi(t))\,|\varphi'(t)|\,dt$$

б) Th е вярна за $n=1$.

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt$$

Интегруемост на $(f\circ\varphi)|\varphi'|$ - Следствие Лема 1

$\Pi_x = \{x_0, x_1, \dots, x_n\}$

$\Pi_t = \{t_0, t_1, \dots, t_m\}$

$\sigma_f(\Pi_x, \xi)$ $\qquad \varphi(t_i) = x_i \;,\; \tau_i \in \Delta t_i$

$= \sum_{i=1}^{n} f(\xi_i)\,|x_i - x_{i-1}| = \sum_{i=1}^{n} f(\xi_i)\,|\varphi(t_i) - \varphi(t_{i-1})| =$

$= \sum f(\varphi(\tau_i))\,|\varphi'(\tau_i)|\,|t_i - t_{i-1}| =$

Избираме $\xi_i = \varphi(\tau_i)$

$= \sigma_{|f\circ\varphi|\varphi'|}(\Pi_t, \tau)$

от φ - диф. морф.

$d(\Pi_x) \to 0 \iff d(\Pi_t) \xrightarrow{\text{равн. непр. } \varphi} 0$

при $d(\Pi_x) \to 0$

$\Rightarrow$

$L \to \int_{I_x} f(x)\,dx \quad \& \quad R \to \int_{I_t} f\circ\varphi(|\varphi'(t)|)\,dt$

$L = R$:))

$\int_{D_x} f(x)\,dx = \int_{D_x} f\circ\varphi(t)\,\det \varphi'(t)\,dt$

$\operatorname{supp} f \subset D_x$ отворено в $\mathbb{R}^1 =$

$$= \bigcup_{i=1}^{\infty} I_i$$

$I_i \cap I_j = \emptyset$

I_i отв. инт.

$\operatorname{supp} f \subset \bigcup_{i=1}^{i_0} I_i$ непресичащи се

След свиване I_i са компактни
$i=1,\dots i_0$ интервали

$$\Rightarrow \int_{D_x} f(x)\,dx = \sum_{i=1}^{i_0} \int_{I_i} f(x)\,dx =$$

$$= \sum_{i=1}^{i_0} \int_{(I_i)} f(\varphi(t))|.\,|\varphi'(t)|\,dt =$$

$$= \int_{D_t} \cancel{\quad} f\circ\varphi(t)\,\varphi'(t)\,dt$$

Вариант на Лема 2 I_t, I_x - кр. затв.
интервали
$\varphi: I_t \to I_x$ дифеом.; f - огр.

$$\Rightarrow \overline{\int_{I_x}} f(x)\,dx = \overline{\int_{I_t}} f(\varphi(t))|\varphi'(t)|\,dt$$

и $\underline{\int}_{I_x} f(x)\,dx = \underline{\int}_{I_t} f(\varphi(t))\,|\varphi'(t)|\,dt$

1 сл.

$f \geq 0$, $f \in M$ в/у I_x

$\Pi_x = \{x_0, \dots, x_{i_0}\}$

$\Pi_t = \{t_0, t_1, \dots, t_{i_0}\}$ $\quad x_i = \varphi(t_i)$

$$S_f(\Pi_x) = \sum_{i=1}^{i_0} \left(\sup_{x \in \underbrace{\Delta x_i}_{\substack{i\text{-тия} \\ \text{интервал от } \Pi_x}}} f(x)\right) |x_i - x_{i-1}| =$$

$$= \sum_{i=1}^{i_0} \left(\sup_{t \in \Delta t_i} f(\varphi(t))\right) |\varphi(t_i) - \varphi(t_{i-1})| \leq$$

$$\leq \sum_{i=1}^{i_0} \left(\sup_{t \in \Delta t_i} f(\varphi(t))\right) \cdot \sup_{t \in \Delta t_i} |\varphi'(t)| \, |t_i - t_{i-1}| =$$

$$\varepsilon = \max \{\omega(\varphi'; \Delta t_i) : i = 1, 2, \dots, i_0\}$$

$$= \sum_{i=1}^{i_0} \sup_{t \in \Delta t_i} (f(\varphi(t))) \cdot \left(\sup_{t \in \Delta t_i} |\varphi'(t)|\right) \cdot |\Delta t_i| \leq$$

$$\leq \sum_{i=1}^{i_0} \sup_{t \in \Delta t_i} (f(\varphi(t)))(|\varphi'(t) + \varepsilon|) \cdot |\Delta t_i|$$

$$= \sum_{i=1}^{i_0} \sup_{t\in\Delta t_i} \left[f(\varphi(t))\, |\varphi'(t)| \right] \cdot |\Delta t_i| + \varepsilon M \cdot |I_t|$$

$$= S_{f\circ\varphi\cdot\varphi'} (\Pi_t) + \varepsilon M ..$$

при $d(\Pi_x) \longrightarrow 0$

$\Rightarrow \varepsilon(\Pi_x) \to 0$

$$\Rightarrow \boxed{\overline{\int_{I_x}} f(x)dx \leq \overline{\int_{I_t}} (f\circ\varphi)(t)\, \varphi'(t)\, dt}$$

при $d(\Pi_x) \to 0 \Leftrightarrow d(\Pi_t) \to 0$

прилагаме φ^{-1} и получаваме обратното неравенство → :))

$$f(x) = \max\{f(x), 0\} - \max\{-f, 0\}$$

Деф. D_t, D_x - отв. в $\mathbb{R}^n$

$\varphi : D_t \to D_x$ дифеоморфизъм

φ се нарича прост, ако $\varphi_i(t)$

менин само k-тата координата

$$\begin{cases} x = \cancel{\varphi(x)}\ \varphi(t) \\ x_1 = \varphi_1(t_1, .. t_n) = t_1 \\ \vdots \\ x_{k-1} = \varphi_k(t_1, .. t_n) = t_{k-1} \\ x_k = \varphi_k(t_1, .. t_n) \\ x_{k+1} = \varphi_{k+1}(t_1, .., t_n)\ t_{k+1} \end{cases}$$

$\varphi_i(t_1, \dots, t_n) = t_i \quad \forall i \neq k$

Лема 3: Th е вярна за прост дифеоморфизъм φ.

$k = n \qquad x_{n-1} = t_{n-1} \quad x_n = \varphi_n(t_1, \dots, t_n)$

$$(\tilde{x}, x_n) := (\underbrace{x_1, \dots, x_{n-1}}_{\tilde{x}}, x_n)$$

$\tilde{x}$ – n-1 мерен вектор

$$(\tilde{t}, t_n) := (\underbrace{t_1, \dots t_{n-1}}_{\tilde{t}}, t_n)$$

$$\varphi'(t) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ - & - & & - \\ & & & \frac{\partial \varphi_n}{\partial x_n} \end{pmatrix} = \frac{\partial \varphi_n}{\partial x_n}$$

$$D_{\tilde{x}_0} = \{ (\tilde{x}, x_n) \in D_x : \tilde{x} \equiv \tilde{x}_0 \}$$

$$D_{\tilde{t}} = \{ (\tilde{t}, t_n) \in D_t : \tilde{t} \equiv \tilde{t}_0 \}$$

$\Delta_x \supset D_x$ ($\Delta_{\tilde{x}} \times \Delta_{x_n}$ „торбин n-1)

~~Δt~~ $\Delta_t \supset D_t$

$$\int_{D_x} f(x)\,dx = \int_{\Delta_x} f(x) \chi_{D_x}(x)\,dx =$$

Th Фубини

$$= \int_{\Delta \tilde{x}} \left(\overline{\int_{\Delta x_n}} f(\tilde{x}, x_n) \chi_{D_x}(\tilde{x}, x_n)\, dx_n \right) d\tilde{x} =$$

$$= \int_{\Delta \tilde{x}} \left(\overline{\int_{D_{\tilde{x}}}} f(\tilde{x}, x_n)\, dx_n \right) d\tilde{x} =$$

$$= \int_{\Delta \tilde{t}} \left(\left(\overline{\int_{D_{\tilde{t}}}} f(\tilde{t}, \varphi_n(\tilde{t}, t_n)) \right) \cdot \left| \frac{\partial \varphi_n}{\partial t_n}(\tilde{t}, t_n) \right| \right.$$

$$\left. \cdot \ dt_n \right) d\tilde{t} =$$

$$= \int_{\Delta \tilde{t}} \left(\left(\overline{\int_{\Delta t_n}} f(\tilde{t}, \varphi_n(\tilde{t}, t_n)) \right) \left| \frac{\partial \varphi_n}{\partial t_n}(\tilde{t}, t_n) \right| \cdot \chi_{D_{\tilde{t}}}(\tilde{t}, t_n)\, dt_n \right) d\tilde{t} =$$

Обратно Фубини

$$\varphi(t) = \varphi(\tilde{t}, t_n) = \begin{pmatrix} \tilde{t} \\ \varphi_n(\tilde{t}, t_n) \end{pmatrix}$$

$$= \int_{\Delta t} f(\varphi(t)) |\det \varphi'(t)| \chi_{D_t}(t)\, dt =$$

$$= \int_{D_t} f(\varphi(t)) |\det \varphi'(t)|\, dt \quad :))$$

Лема 4 $D\tilde{\tau}$, D_t, D_x отв. в $\mathbb{R}^n$
огр.

$$D\tilde{\tau} \xrightarrow{\varphi} D_t \xrightarrow{\psi} D_x$$

Формулата за смяна е вярна за φ и ψ
=> е вярна и за $\psi \circ \varphi$

$$\int\limits_{D_x} f(x)\,dx \overset{\text{смяна } \psi}{=} \int\limits_{D_t} \underbrace{f(\psi(t)) \cdot |\det \psi'(t)|}_{(f\circ\psi)\,|\det \psi'|}\,dt =$$

смяна φ

$$= \int\limits_{D\tilde{\tau}} (f\circ\psi) \cdot |\det \psi'| \cdot (\varphi(\tilde{\tau})) \cdot |\det \varphi'(\tilde{\tau})|\,d\tilde{\tau} =$$

$$= \int\limits_{D\tilde{\tau}} f(\psi(\varphi(t))) \cdot |\det \psi'(\varphi(\tilde{\tau})) \cdot \det \varphi'(\tilde{\tau})|\,d\tilde{\tau} =$$

$$= \int\limits_{D\tilde{\tau}} f(\psi(\varphi(\tilde{\tau}))) \, |\det (\psi'(\varphi(\tilde{\tau})) \cdot \varphi'(t)|\,d\tilde{\tau}$$

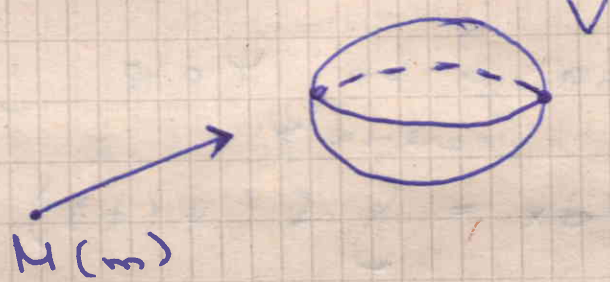
$$= \int\limits_{D\tilde{\tau}} f((\psi\circ\varphi)(\tilde{\tau})) \cdot |\det(\psi\circ\varphi)'(\tilde{\tau})|\,d\tilde{\tau}$$

:))

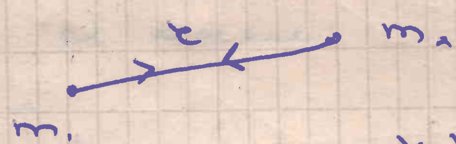
13. 11. 2013г.

Упражнение

Имаме материална точка $M(x, y, z)$ с маса m



Имаме тяло V с плътност $\rho(x, y, z)$.



$$|F| = \gamma \frac{m_1 m_2}{r^2}$$

$$\vec{F} = m\gamma \iiint\limits_V \frac{x-\xi}{r^3} \rho(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$$

където

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

$$\vec{F} = m\gamma \iiint\limits_V \frac{\begin{pmatrix} \xi \\ \zeta \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{r^3} \rho(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$$

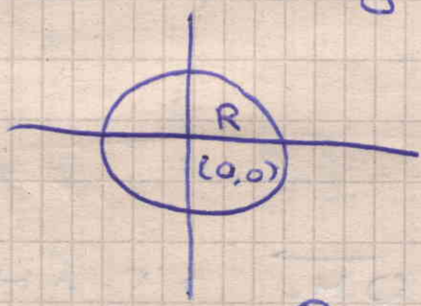
Криви в $\mathbb{R}^2$, $\mathbb{R}^3$

Крива в $\mathbb{R}^2$, $\mathbb{R}^3$ наричаме едномерно множество (с ПЖ мярка от $\mathbb{R}^1$)

2 основни начина за дефиниране на крива в $\mathbb{R}^2$
1. чрез уравнение
2. параметрично !!!

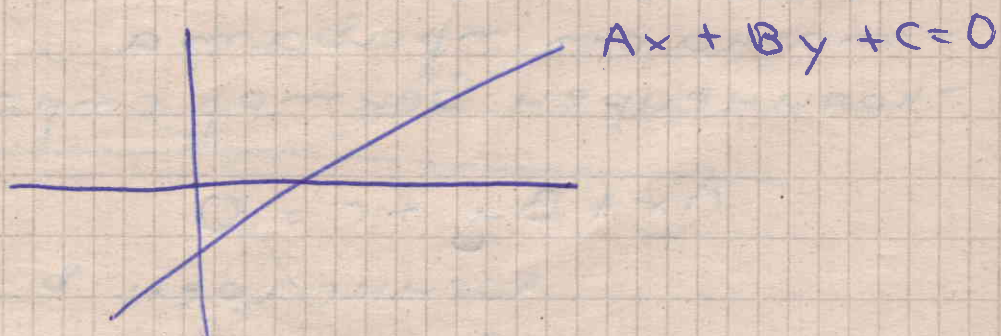
– в $\mathbb{R}^3$
1. параметрично
2. като пресечница на 2 повърхнини

Примери: $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=R^2\}$



$Ax+By+C=0$ права

$y = ax + b$



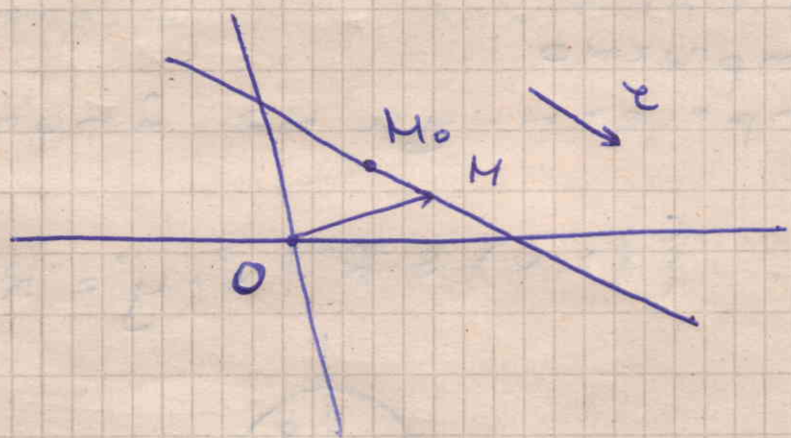
$$x^2+y^2=R^2 \xrightarrow{} \text{параметър}$$

$$\begin{cases} x = \rho\cos\varphi = R\cos\varphi \\ x = \rho\sin\varphi = R\sin\varphi \end{cases}$$

$$\rho^2 = R^2 \quad \Rightarrow \rho = R$$

$$\varphi \in \{0; 2\pi\}$$

Казваме, че сме параметризирали окръжността с радиус R чрез ъгъла φ.



$$\overrightarrow{OM} = \overrightarrow{OM_0} + r.\ell M$$

Параметрично задаване на права в $\mathbb{R}^2$

- точка от правата
- колинеарен вектор с правата

$$Ax + By + c = 0$$

колинеарен в-р $(-B, A)$

нормален в-р (A, B)

$$\begin{cases} x = x_0 + \lambda(-B) \\ y = y_0 + \lambda A \end{cases}$$

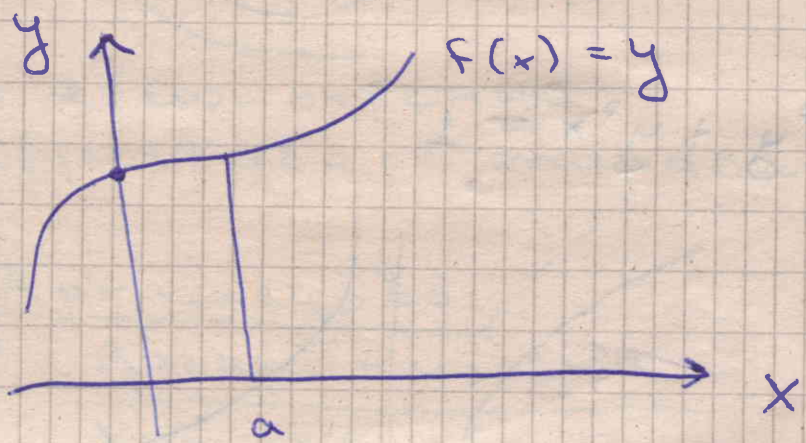
Пример: $2x + 3y = 5 = a \qquad (1, 1) \in a$

Колинеар в-р $(-3, 2)$

У-е: $\begin{cases} x = \cancel{-3 + x} \quad 1 - 3\lambda \\ y = \cancel{2 + x} \quad 1 + 2\lambda \end{cases}$

! Параметризиране на графика на ф-я

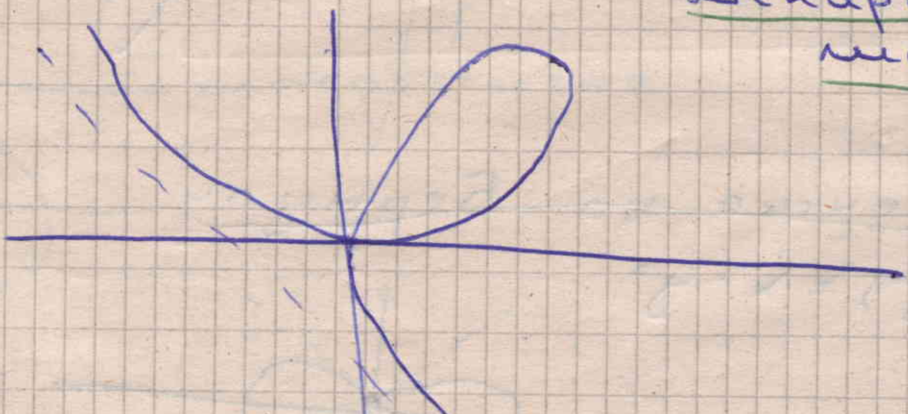
$x \in [a, b]$



$$\begin{cases} x = t \\ y = f(t) \end{cases}, \quad t \in [a, b]$$

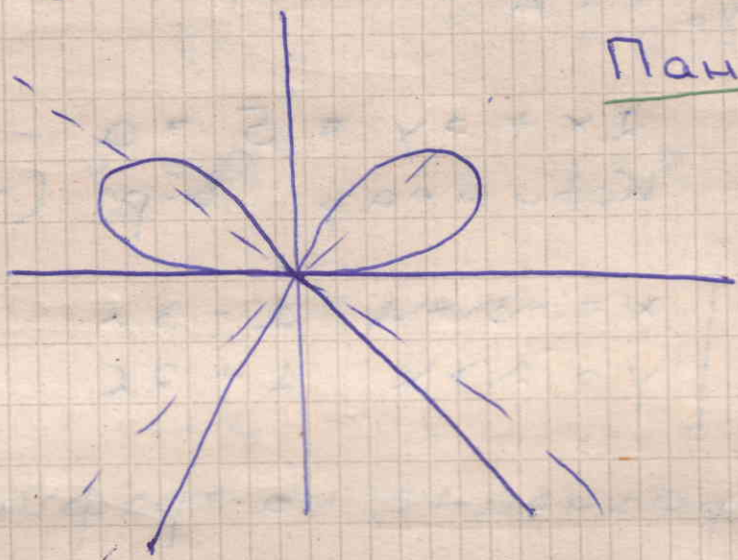
$x^3 + y^3 = xy$

Декартов лист

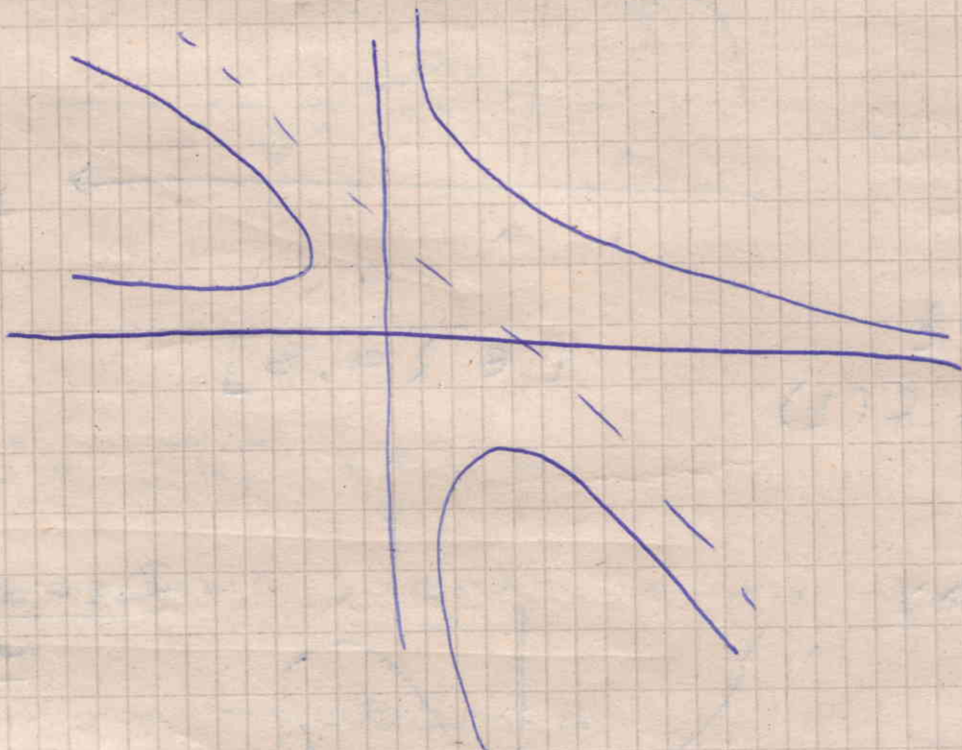


$$y(x^2-y^2)=x^4$$

Панделка



$$x^2y+y^2x=1$$

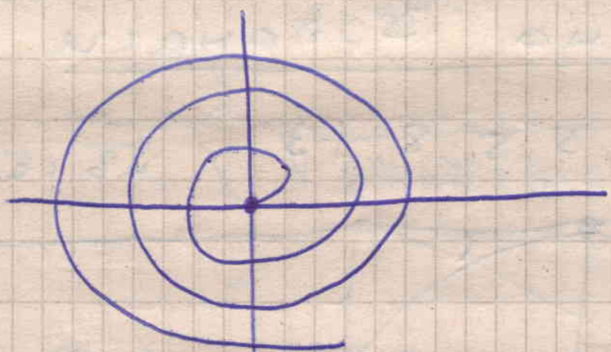


Лемненска на Бернули

$$(x^2+y^2)^2=x^2-y^2$$

$\boxed{\text{Спирали}}$

$$x = \rho(\varphi)\cos\varphi \qquad , \rho(\varphi) \geq 0$$
$$y = \rho(\varphi)\sin\varphi$$



Обикновено ρ-монотонна

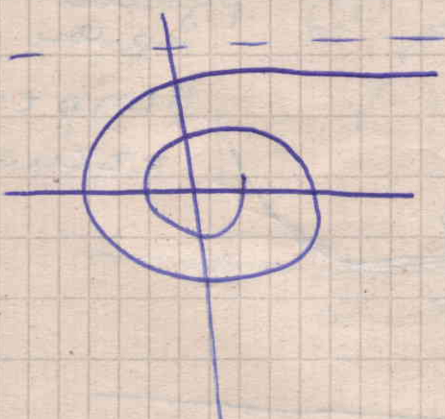
ρ-растяща (развива се спиралата)

$\rho(\varphi) = \varphi$ Архимедова

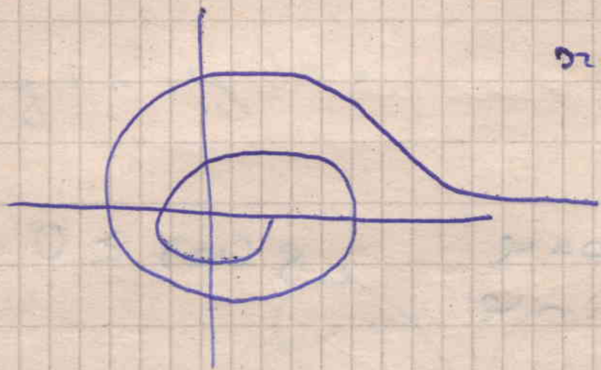
$\rho(\varphi) = e^{\varphi}$ Логаритмична

$\rho(\varphi) = \dfrac{1}{\varphi}$ Хиперболична

$\rho(\varphi) = \dfrac{1}{\sqrt{\varphi}}$ Жезъл



хиперболична

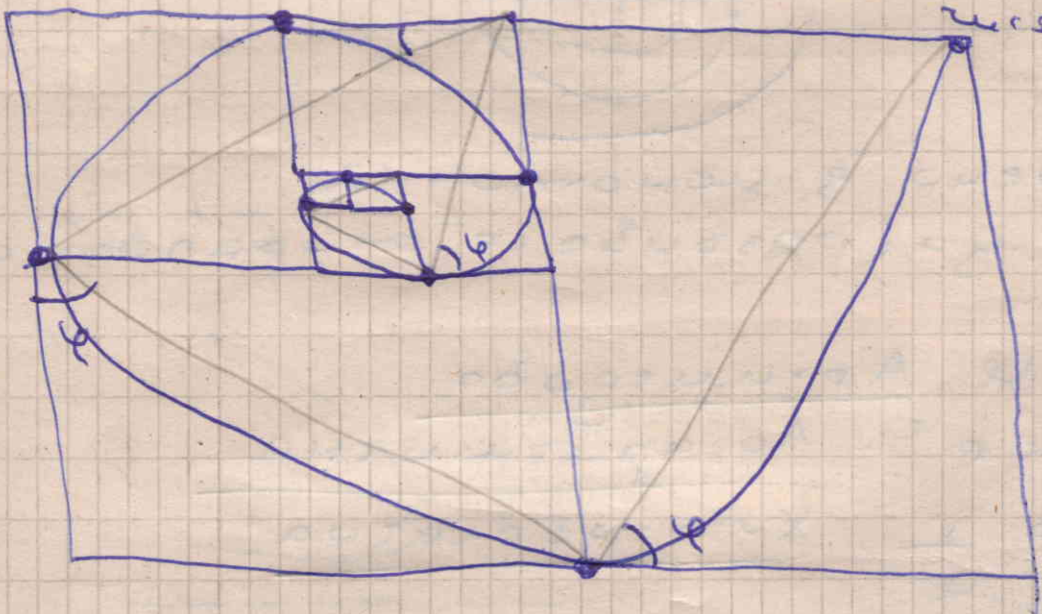


жезъл

Спирала на Фибоначи

1, 1, 2, 3, 5, 8, 13

кв. със страна ч число



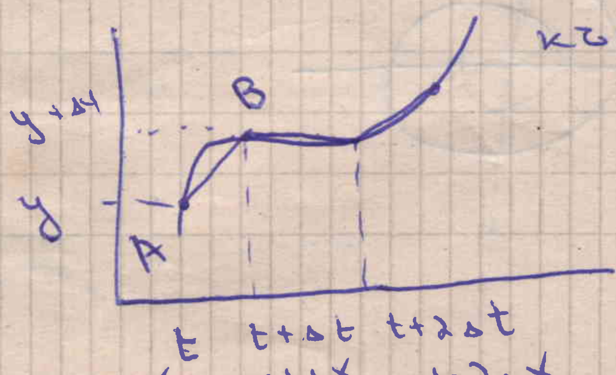
$\frac{a_{n+1}}{a_n} \to \varphi$ ратани :))

Дължина на крива

ректиференцируеми криви

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

непре-къснати



$$AB = \sqrt{(\Delta x)^2 + (\Delta y)^2}\,\frac{\Delta t}{\Delta t} =$$

$$= \sqrt{\frac{x + \Delta x - x}{(\Delta t)^2} + \frac{y + \Delta y - y}{(\Delta t)^2}}\;\Delta t$$

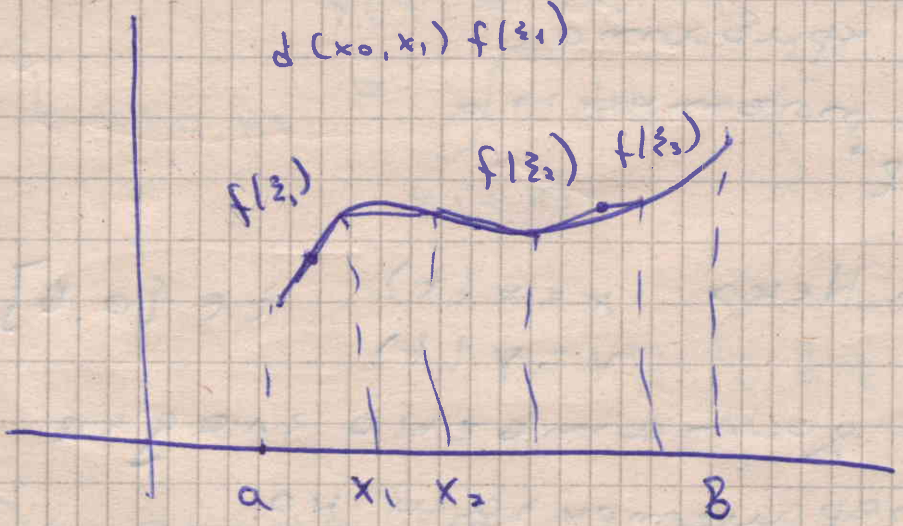
$$\sum \text{отсечки} \longrightarrow \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

графика на ф-я $y = y(x)$

$\ell\,[a, в]$

$$в\ \ell\,[a, в] = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

Криволинеен интеграл от I род



имаме материална нишка (кабел, връв)

Във $\forall$ точка имаме зададена функц. стойност (напр. плътност)

жица
сняг - неравномерно
колко сняг?

Искаме да сметнем сумарното количество сняг?

Начупваме кривата

Във $\forall$ участък вземаме стойност на ф-ята

$$I = \sum_{i=0}^{n-1} f(\xi_i)\, \underbrace{d(x_i, x_{i+1})}_{\text{дължина}}$$

$$\longrightarrow \int_{\gamma} f\, dl$$

γ е кривата

f е плътността

„dl"

<u>**Твърдение**</u>: Нека $x = x(t)$, $y = y(t)$, $t \in [a, b]$

е достатъчно гладка параметризация на кривата.

Нека $\overset{(x,y)}{f}$ е непрекъсната в/у γ.
Тогава $\exists$

$$\exists \int_\gamma f\,d\ell = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

В частност, ако $f \equiv 1$

$$\int_\gamma d\ell = L_\gamma [a, b]$$

Задачи

Зад. 1 Пресметнете дължината (обиколката) на окръжност с радиус r=R

1. параметризираме окръжността
$x(t) = R\cos t$
$y(t) = R\sin t$ $\quad t \in [0, 2\pi]$

2. $$L_{k(0,R)} = \int_{k(0,R)} d\ell = \int_0^{2\pi} 1 \cdot \sqrt{(-R\sin t)^2 + (R\cos t)^2}\,dt =$$

$$= R \int_0^{2\pi} dt = 2\pi R$$

2) Дължина на елипса с радиуси a и b, $a > b$

$$x = a\cos t \qquad t \in [0, 2\pi]$$
$$y = b\,\overset{\sin}{\cancel{\cos}}\,t$$

$$L = \int_0^{2\pi} 1 \cdot \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt =$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2 - b^2\sin^2 t}\, dt =$$

$$= \int_0^{2\pi} \sqrt{b^2 + (a^2 - b^2)\sin^2 t}\, dt$$

$$= \boxed{b\int_0^{2\pi} \sqrt{1 + k^2\sin^2 t}\, dt}$$

$$k = \sqrt{\frac{a^2 - b^2}{b^2}}$$

При $a = b \to$ окр. $k = 0$

При $a \neq b$ $k \neq 0$

Интегралът не може да се сметне експлицитно.

Този интеграл наричаме елиптичен интеграл от I род

В ред може иначе не

Криви в $\mathbb{R}^3$

① параметрично $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in \{a, b\}$

② като пресечница на 2 повърхнини

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

$\begin{cases} m\ddot{x} = F_x \\ m\ddot{y} = F_y \\ m\ddot{z} = F_z \end{cases}$ ч-я на д.е на мат. точки $(x(t), y(t), z(t))$ под действие на сила F

Параметризиране на права в $\mathbb{R}^3$
Искаме точка и колинеарен вектор (λ, μ, ν)

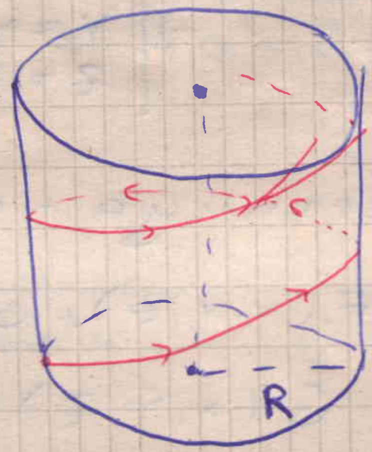
$$\begin{cases} x(t) = x_0 + t\lambda \\ y(t) = y_0 + t\mu \\ z(t) = z_0 + t\nu \end{cases}$$

Спирали в $\mathbb{R}^3$

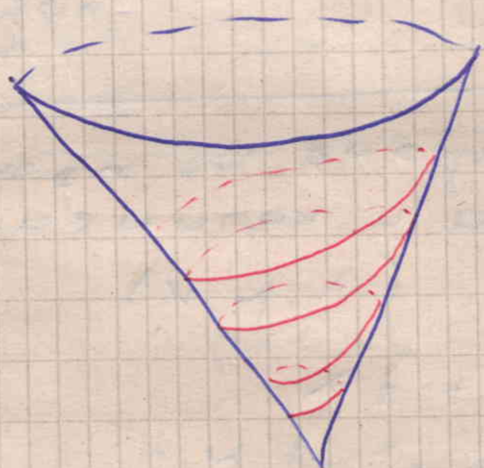
$$\begin{cases} x = \rho(\varphi)\cos\varphi \\ y = \rho(\varphi)\sin\varphi \\ z = k(\varphi) \end{cases}$$

Ако $\left| \begin{array}{l} z = \varphi \\ \rho = R \end{array} \right.$ — по цилиндър

<u>витлова линия</u>



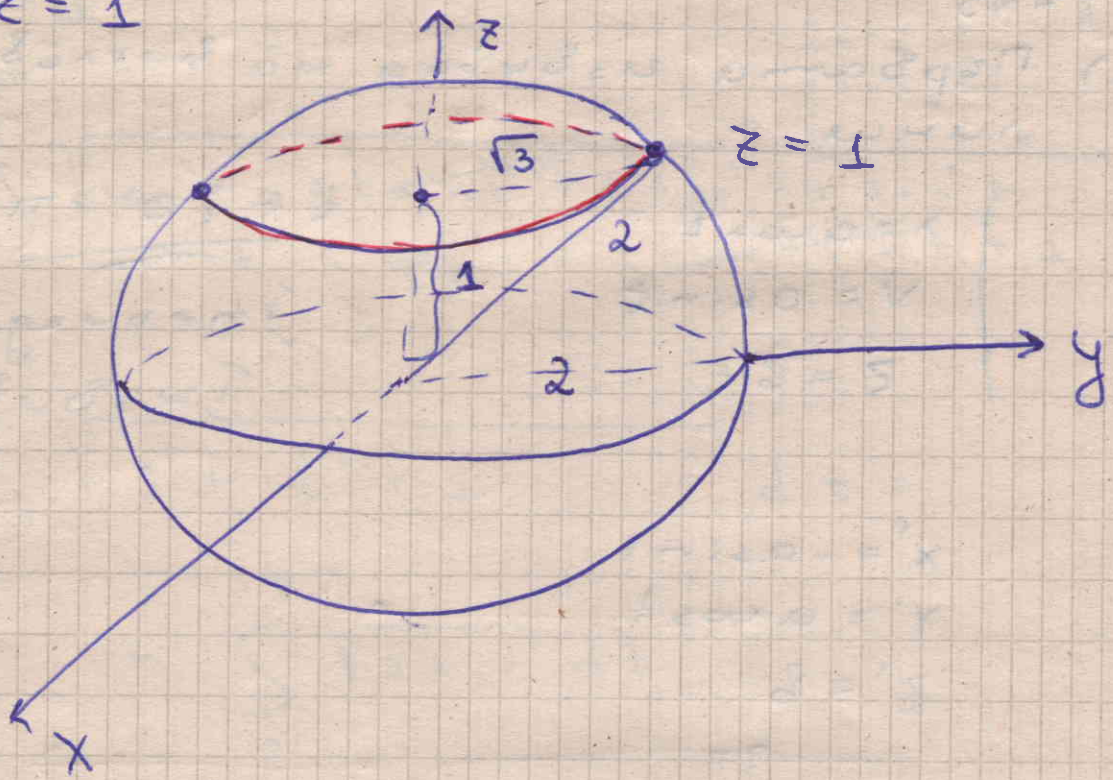
Ако $\left| \begin{array}{l} z = \varphi \\ \rho = \varphi \end{array} \right\}$ конична ~~вин~~ витлова линия



$\left| \begin{array}{l} x = \varphi\cos\varphi \\ y = \varphi\sin\varphi \\ z = \varphi \end{array} \right.$ $\subset$ конуса $x^2+y^2=z^2$

$\underline{x^2+y^2=z^2}$ Конус

Кривата е зададена като пресечни
на сферата с радиус 2 и равнина
та $z = 1$



$$\begin{cases} x^2 + y^2 + z^2 = 4 \\ z = 1 \end{cases}$$

$$\begin{cases} x^2 + y^2 = 3 \\ z = 1 \end{cases}$$

$$x = \sqrt{3}\cos\varphi \qquad \varphi \in [0, 2\pi]$$
$$y = \sqrt{3}\sin\varphi$$
$$z = 1$$

Задачи

1.4 | 445

а) Първата извивка на витловата линия

$$\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$$

$t \in [0; 2\pi]$

1 период-
1 извивка

$x' = -a\sin t$

$y' = a\cos t$

$z' = b$

$$l = \int_0^{2\pi} 1 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, d\,t =$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2}\, dt = \sqrt{a^2+b^2} \cdot 2\pi$$

б) 1 извивка на коничната витлова линия

$x = t\cos t$

$y = t\sin t$

$z = t$

$0 \le t \le 2\pi$

$x' = \cos t - t\sin t$

$y' = \sin t + t\cos t$

$z' = 1$

$$l = \int_0^{2\pi} \sqrt{\cos^2 t - \cancel{2t\sin t\cos t} + t^2\sin^2 t + \sin^2 t + t^2\cos^2 t + \cancel{2t\cos t\sin t} + 1}\, dt =$$

$$= \int_0^{2\pi} \sqrt{1 + t^2 + 1}\, dt =$$

$$= \int_0^{2\pi} \sqrt{2 + t^2}\, dt$$

?
$$\begin{cases} x^2 + y^2 + z^2 = 2^2 \\ x + y + z = 1 \end{cases}$$

параметриза-
ция

$$\boxed{\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ (x^2+y^2)^2 &= x^2 - y^2 \end{aligned}}$$

!

~~x = a cos t~~

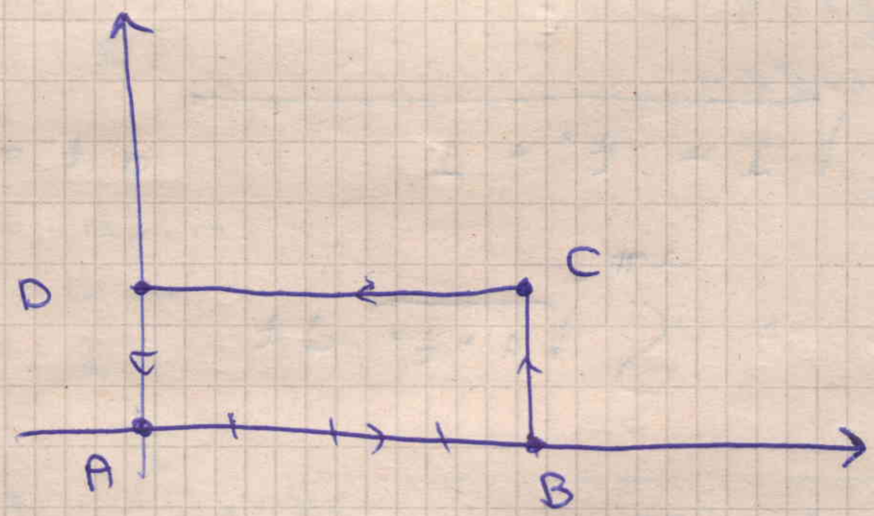
~~x = a~~

$x = 4$

$y = -3$

$z = 0$

16. $I = \int\limits_L xy\,dl$, където L е контурът на правоъгълник с върхове A(0,0), B(4,0), C(4,2), D(0,2)



Да забележим, че в условието не е указано в коя посока обикаляме правоъгълника
Повърхнинния интегралот I род не зависи от начина на обхождане, т.е. при ∀ 2 еднозначни параметризации, стойността не се променя. Кривата, в/у която интегрираме, е непр в т. A, B, C, D - не е гладка

$$I = \underbrace{\int\limits_{AB} xy\,dl}_{I_1} + \underbrace{\int\limits_{BC} xy\,dl}_{I_2} + \underbrace{\int\limits_{CD} xy\,dl}_{I_3}$$

I_4

$$+\int_{OA} xy\,dl \qquad \vec{AD}(4,0)$$

AB: $\begin{cases} x = 0 + \lambda 4 \\ y = 0 + 0\cdot\lambda \end{cases}$

$\lambda \in \{0, 1]$

BC: $\begin{cases} x = 4 + \lambda\cdot 0 \\ y = 0 + \lambda\cdot 2 \end{cases}$

CD: $\begin{cases} x = 4 + \lambda(-4) \\ y = 2 + \lambda\cdot 0 \end{cases}$

DA: $\begin{cases} x = 0 + \lambda\cdot 0 \\ y = 2 + \lambda\cdot(-2) \end{cases}$

$$I_1 = \int_0^1 4\lambda\cdot 0\,\sqrt{4^2+0^2}\,d\lambda = 0$$

$$I_2 = \int_0^1 4\cdot 2\lambda\,\sqrt{4}\,d\lambda = \cancel{16}\,8\lambda^2\Big|_0^1 = 8$$

$$I_3 = \int_0^1 (4-4x)\cdot 2\sqrt{16+0^2}\,dx =$$

$$= 16\int_0^1 (1-x)\,dx =$$

$$= -\frac{32}{2}(1-x)^2\Big|_0^1 =$$

$$= 16$$

$$I_4 = \int_0^1 0 \cdot \ldots = 0$$

$$\Rightarrow I = \sum_{i=1}^{4} I_i = 8 + 16 = 24 =$$

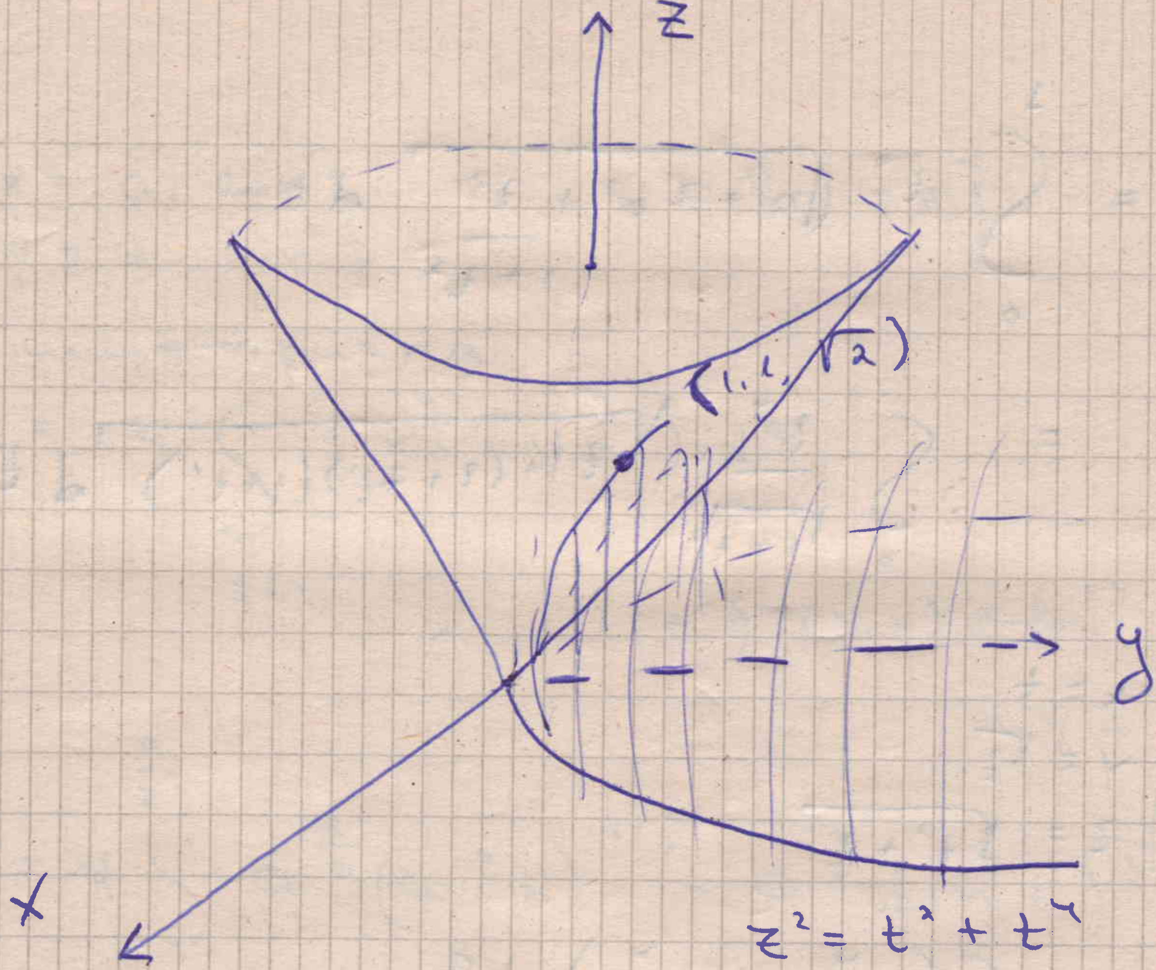
масата на нишката с плътността ф-я xy

Зад.

$$I = \int_L x^2\,dl$$

L е кривата, ~~получена при пресичането на повърхнините~~ лежаща м/у $(0,0,0)$ $(1,1,\sqrt{2})$

$$\left|\begin{array}{l} x^2+y^2=z^2 \\ y^2=x \end{array}\right.$$



$$z^2 = t^2 + t^4$$

$$\begin{cases} y = t \\ x = t^2 \\ z = \sqrt{t^2 + t^4} = t\sqrt{1+t^2} \end{cases}$$

$t \in [0, 1]$

$$I = \int_0^1 t^4 \sqrt{1^2 + (2t)^2 + \left(\sqrt{1+t^2} + \frac{t \cdot 2t}{2\sqrt{1+t^2}}\right)^2}\, dt =$$

$$= \int_0^1 t^4 \sqrt{1 + 4t^2 + 1 + t^2 + \frac{t^2}{1+t^2} + 2t^2}\, dt =$$

$$= \int_0^1 t^4 \sqrt{2+7t^2+\frac{t^2}{1+t^2}}\,dt$$

$$= \int_0^1 \frac{t^4}{\sqrt{1+t^2}} \sqrt{t^2+(2+7t^2)(1+t^2)}\,dt$$

$$\left|\begin{array}{l} x = t \\ y = \sqrt{t} \\ z = \sqrt{t^2+t} \end{array}\right.$$

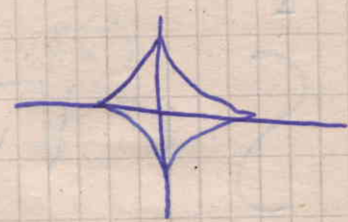
$\boxed{1.7}$ $$\int_L \left(x^{\frac{4}{3}} + y^{\frac{4}{3}}\right) dl$$

$$L: x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$ $\underline{a > 0}$

$$|x|^k + |y|^k = a^k \longrightarrow k = 1$$

$k > 1$

$k < 1$



$x = a \cdot \cos^3 t \qquad t \in \{0, 2\pi]$

$y = a \cdot \sin^3 t$

симетрична

$$I = 4 \int_{L_1} \left(x^{\frac{4}{3}} + y^{\frac{4}{3}} \right) dl$$

$L_1 = \{ (x, y) \in L \mid x, y \geq 0 \}$

$$I = 4 \int_0^{\frac{\pi}{2}} a^{\frac{4}{3}} (\cos^3 t)^{4/3} + a^{\frac{4}{3}} (\sin^3 t)^{\frac{4}{3}} \, dt$$

$$\sqrt{(3a\cos^2 t \sin t)^2 + (a\, 3\sin^2 t \cos t)^2} =$$

$$= 4 \cdot 3 \cdot a^{\frac{4}{3}} \cdot a \int_0^{\frac{\pi}{2}} (\cos^4 t + \sin^4 t) \sin t \cos t$$

$$= 12 a^{\frac{7}{3}} \left[\int_0^{\frac{\pi}{2}} \cos^5 t \sin t \, dt + \int_0^{\frac{\pi}{2}} \sin^5 t \cos t \, dt \right]$$

$$= 12 a^{\frac{7}{3}} \left[- \int_0^{\frac{\pi}{2}} \cos^5 t \, d\cos t + \int_0^{\frac{\pi}{2}} \sin^5 t \, d\sin t \right]$$

$$= 4 a^{\frac{7}{3}}$$

...

$$x^k + y^k = a^k$$

$$x = a\cos^{\frac{2}{k}}\varphi$$

$$y = a\sin^{\frac{2}{k}}\varphi$$

13.11 Лекция

Th Продължение:

D_t, $D_x \subset \mathbb{R}^n$
отв. и огр.

$\varphi : D_t \longrightarrow D_x$
дифеом.

$f : D_x \longrightarrow \mathbb{R}$
интегрусма по Риман

$\operatorname{supp} f \subset D_x$

Тогава $(f \circ \varphi) \cdot |\det \varphi'|$ е интегр. в D_t и

$$\int\limits_{D} f(x)\, dx = \int\limits_{D_t} f(\varphi(t))\, |\det \varphi'(t)|\, dt$$

Лема 5 : $\varphi : D_t \to D_x$
D_t, D_x отв. в $\mathbb{R}^n$, φ
дифеоморфизъм

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \to \begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$$

лин. изобр. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ортогон.

$\det = \pm 1$

$t_0 \in D_t$

$\exists$ околност U на t_0,

$U \subset D_t$,

такава че $\varphi|_U$ се представа като композиция на прости дифеоморфизми (с точност до разместване на координатите)

Д-во: (индукция по бр. на коорд. които разместват φ) - k

$\varphi(t) =$

$$\begin{aligned}
x_1 &= \varphi_1(t_1, \dots, t_n) \\
x_2 &= \varphi_2(t_1, \dots, t_n) \\
&\dots \\
x_{k-1} &= \varphi_{k-1}(t_1, \dots, t_n) \\
x_k &= \varphi_k(t_1, \dots, t_n) \\
x_{k+1} &= t_{k+1} \\
&\vdots \\
x_n &= t_n
\end{aligned}$$

(t_0)

$$\varphi'(t) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial t_1} & \frac{\partial \varphi_1}{\partial t_2} & \dots & \cancel{\frac{\partial \varphi_1}{\partial t_n}} \; \frac{\partial \varphi_1}{\partial t_k} & \frac{\partial \varphi_1}{\partial t_{k+1}} & \dots & \frac{\partial \varphi_1}{\partial t_n} \\ \frac{\partial \varphi_k}{\partial t_1} & \frac{\partial \varphi_k}{\partial t_2} & \dots & \frac{\partial \varphi_k}{\partial t_k} & \frac{\partial \varphi_k}{\partial t_{k+1}} & \dots & \frac{\partial \varphi_k}{\partial t_n} \\ & 0 & & & 1 & \ddots & 0 \\ & & & & 0 & & 1 \end{pmatrix}$$

dt

$$\det \varphi'(t) = \det \begin{vmatrix} \frac{\partial \varphi_1}{\partial t_1} & \dots & \frac{\partial \varphi_1}{\partial t_k} \\ \vdots & & \\ \frac{\partial \varphi_k}{\partial t_1} & \dots & \frac{\partial \varphi_k}{\partial t_k} \end{vmatrix} (t_0) \neq 0$$

$\psi(t)$ Б.О.О.

$$\det \begin{vmatrix} \frac{\partial \varphi_1}{\partial t_1} & \dots & \frac{\partial \varphi_1}{\partial t_{k-1}} \\ \frac{\partial \varphi_{k-1}}{\partial t_1} & \dots & \frac{\partial \varphi_{k-1}}{\partial t_{k-1}} \end{vmatrix} (t_0) \neq 0$$

$\psi(t)$

$$\left| \begin{array}{l} y_1 = \varphi_1(t_1, \dots, t_n) \\ \vdots \\ y_{k-1} = \varphi_{k-1}(t_1, \dots, t_n) \\ y_k = t_k \\ y_{k+1} = t_{k+1} \\ \vdots \\ y_n = t_n \end{array} \right.$$

$\det \neq 0$

$\Rightarrow \exists$ U околност на t_0
V ок. на $\psi(t_0) = y_0$

$\varphi|_U$ е дифеом. м/у U и V

$\neq 0$

$\psi^{-1} : V \to U$

$$\boxed{\eta = \varphi \circ \psi^{-1}}$$

прост дифеоморфизъм

$$\begin{tikzcd}
t \arrow[r, "\psi"] \arrow[d, "\varphi"] & y \arrow[l, bend left, "\psi^{-1}"] \\
x \in D_x &
\end{tikzcd}$$

0

$$\begin{cases}
x_1 = \varphi_1(\psi^{-1}(y)) = t_1 \\
\vdots \\
x_{k-1} = \varphi_{k-1}(\psi^{-1}(y)) = t_{k-1} \\
x_k = \varphi_k(\psi^{-1}(y)) \\
x_{k+1} = t_{k+1} \\
\vdots \\
x_n = t_n
\end{cases}$$

$\Rightarrow \eta$ е прост дифеоморф.

$\psi(\psi^{-1}(t)) = t$

прост

разместване повече от $k-1$ координати

$$\boxed{\varphi = \eta \circ \psi}$$

0

$\varphi(t) = \eta(\psi(t)) = \varphi(\psi^{-1}(\psi(t))) =$
$= \varphi(t)$

Д-во на Th :

$$K_t = \operatorname{supp}(f \circ \varphi) \, |\det \varphi'| \subset D_t$$

компакт

(проверено :)))

$\forall t \in K_t \;\; \exists \, \delta(t) > 0$:

φ се представя (с

точност до разместване)

като композиция на

прости дифеом

$B_{\delta(t)}(t)$

$$\bigcup_{t \in K_t} B_{\frac{\delta(t)}{2}}(t) \supset K_t$$

отворени

кълба

$t_1, \ldots t_k \in K_t$

$$\bigcup_{i=1}^{k} B_{\frac{\delta(t_i)}{2}}(t_i) \supset K_t$$

$$\delta = \frac{1}{2} \min \{ \delta(t_1), \ldots \delta(t_k) \} > 0 .$$

Δ_t компакт

$$\left.\begin{array}{l} \Delta \cap K_t \neq \emptyset \\ \operatorname{diam} \Delta < \delta \end{array}\right\} \Rightarrow \exists\, i \in \{1, 2, \dots k\}$$

т.че

$$\Delta \subset B_{\delta(t_i)}(t_i)$$

Взимаме

$$t' \in \Delta \cap K_t \Rightarrow \exists\, i \in \{1, \dots, k\} \quad t' \in B_{\frac{\delta(t_i)}{2}}(t_i)$$

$$t \in \Delta \Rightarrow \|t - t'\| \leq \operatorname{diam} \Delta < \delta \leq \frac{1}{2}\delta(t_i)$$

$$\|t - t_i\| \leq \|t - t'\| + \|t' - t_i\| < \frac{1}{2}\delta(t_i) + \frac{1}{2}\delta(t_i) = \delta(t_i)$$

$$\Rightarrow \quad \Delta \subset B_{\delta(t_i)}(t_i)$$

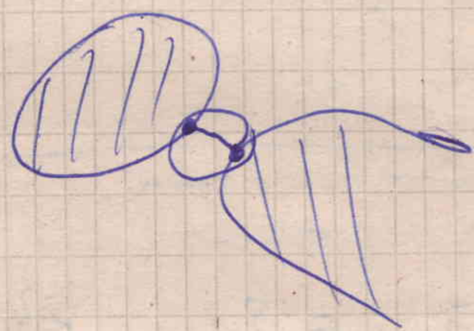
:))

$$d = \operatorname{dist}(K_t, \mathbb{R}^n \setminus D_t) > 0$$

Разстояние м/у м-ва

$$A, B \subset \mathbb{R}^n$$

$$\operatorname{dist}(A, B) = \inf \left\{ \|x-y\| : \begin{array}{l} x \in A \\ y \in B \end{array} \right\}$$



Ако A е компакт
B затворено
$A \cap B = \varnothing$

$$\Rightarrow \operatorname{dist}(A, B) > 0$$

$$\operatorname{dist}(A,B) + \frac{1}{n} > \operatorname{dist}(A,B)$$

$$\Rightarrow \exists \; x_n \in A, \; y_n \in B$$

$$\|x_n - y_n\| < \operatorname{dist}(A,B) + \frac{1}{n}$$

$$\{x_n\}_{n=1}^{\infty} \subset A$$

$$\{y_n\}_{n=1}^{\infty} \subset B$$

минимизираща редица

$$dist(A,B) \leq \|x_n - y_n\| < dist(A,B) + \frac{1}{n}$$

$$x_{n_k} \xrightarrow[k \to \infty]{} x_0 \in A$$

подред. $\{y_{n_k}\}_{k=1}^{\infty} \subset B \cap$ кълбо с ц. т. от A $\overline{B}(x_0, dist(A,B)+2)$

$$y_{n_{k_m}} \xrightarrow[m \to \infty]{} y_0 \in B \qquad x_{n_{k_m}} \xrightarrow[m \to \infty]{} x_0 \in A$$

$$dist(A,B) \leq |x_{n_{k_m}} - y_{n_{k_m}}| \leq dist.. + \frac{1}{n_{k_m}}$$

$$n \to \infty$$

$$\|x_0 - y_0\| = dist(A,B)$$

$$\Delta_t \supset D_t$$

$\Pi = \{\Delta_i\}_{i=1}^{i_0}$ разб. на Δ_t

$$d(\Pi) < \min\{\delta, d\}$$

Б.О.О. $\{\Delta_i\}_{i=1}^{i_1}$ са паралел. от Π, за които

$$\Delta_i \cap K_t \neq 0$$

$\Rightarrow \forall i \in \{1, 2, \dots, i_1\}$:

$$\Delta_i \subset D_t$$

$$\left.\begin{array}{l} t \in \Delta_i \\ t' \in \{\Delta_i \cap K_t\} \end{array}\right\}$$

$$\|t - t'\| < d$$

$$\Rightarrow t \in \mathbb{R}^n \setminus D_t$$

за $\forall i = \{1, \dots, i_1\}$: $\Delta_i \subset B_{\delta(t_j)}(t_j)$

$$\int_{D_t} (f \circ \varphi)(t) . |\det \varphi'(t)| dt =$$

$$= \int_{\Delta_t} (f \circ \varphi) |\det \varphi'(t)| \chi_{D_t}(t) dt$$

$$= \sum_{i=1}^{i_1} \int_{\Delta} f \circ \varphi(t) |\det \varphi'(t)| \chi_{D_t}(t) dt$$

$$= \sum_{i=1}^{i_1} \int_{\Delta_i} (f \circ \varphi)(t) |\det \varphi'(t)| dt$$

лема 1

$$\varphi(\Delta_i) = E_i \subset \bar{E}_i \subset D_x$$

изм. по Л-Ж

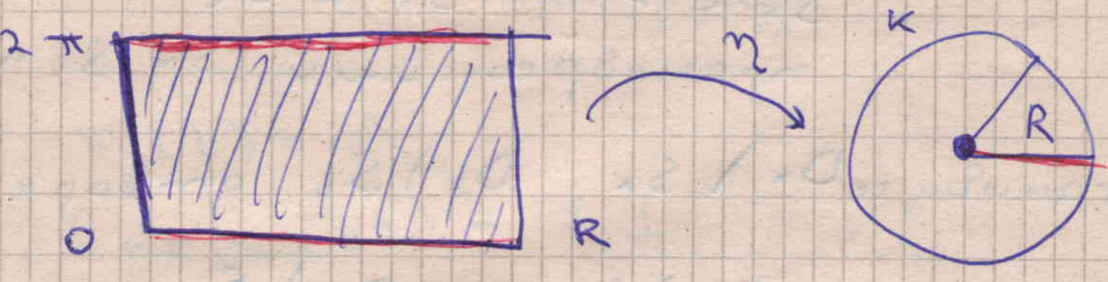
$$= \sum_{i=1} \int_{E_i} f(x)\,dx = \int_{\cancel{D_x}\,E} f(x)\,dx$$

$$\mu_n(E_i \cap E_j) = 0 \quad \forall i \neq j$$

$$= \int_E f(x)\,dx \qquad E = \bigcup_{i=1} E_i, \quad E \subset D_x$$

$$\operatorname{supp} f \subset E$$

$$= \int_{D_x} f(x)\,\chi_{D_x}(x)\,dx = \int_{D_x} f(x)\,dx$$



$$D \begin{cases} 0 \le \rho \le R \\ 0 \le \varphi \le 2\pi \end{cases}$$

не биекция

$$\eta(\rho, \varphi) = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \end{pmatrix}$$

$$K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$$

$$\mathring{\Delta} \left| \begin{array}{l} 0 < \rho < R \\ 0 < \varphi < 2\pi \end{array} \right.$$

$$E = \partial K \cup \{(x,0) : x \in [0,R]\}$$

$K \setminus E$

$\eta : \mathring{\Delta} \longrightarrow K \setminus E$ биекция м/у отворени $\odot$

дифеом.

$$\det \eta'(\rho,\varphi) = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix} = \rho$$

Th D_x, D_t измерими по П-Ж в $\mathbb{R}^n$

$S_x \subset D_x$, $S_t \subset D_t$

пренебрежими S_x, S_t по Лебег

и такива, че $D_x \setminus S_x$, $D_t \setminus S_t$ отворени

$\varphi : D_t \setminus S_t \longrightarrow D_x \setminus S_x$

е дифеоморфизъм

$f : D_x \to \mathbb{R}$ е интегруема по Риман

$|\det \varphi'(t)|$ е ограничена $D_t \setminus S_t$

Тогава $(f\circ\varphi)\cdot|\det\varphi'|$ е интегр. в $D_t\setminus S_t$ и

$$\int_{D_t\setminus S_t}(f\circ\varphi)(t)|\det\varphi'(t)|\,dt = \int_{D_x} f(x)\,dx$$

Ако $|\det\varphi'|$ е дефинирана и огр. в S_t, то

$$\int_{D_x} f(x)\,dx = \int_{D_t}(f\circ\varphi)(t)\,|\det\varphi'(t)|\,dt$$

$\varphi(t) = t_0 + At$

A-ортогонална матрица

$\varphi'(t) = A$

$\det(\varphi'(t)) = \pm 1$

$|\det\varphi'(t)| = 1$

$$\int_{D_x} f(x)\,dx = \int_{D_t} f(\varphi(t))\,dt$$

Мярката на П-Ж е инвариантна относно еднаквостите.

K

$\rho(x_1, x_2, x_3)$ - плътност

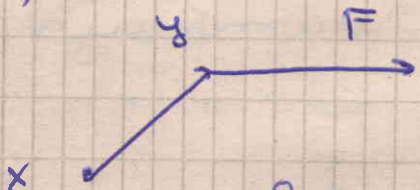
$\rho: K \to \mathbb{R}$
непр.

$\{E_i\}_{i=1}^{i_0}$

$\sum_{i=1}^{i_0} \rho(\xi_i)\mu_3(E_i)$

$\cancel{m} = m(K) = \iiint_K \rho(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3$

$\vec{F} = (0,0,1)$



въртящ момент
$\vec{F} \times (y - x)$

(x_0, y_0, z_0)

$(x, y, z) \in K$

$\iiint_K \left[(0,0,1) \times (x - x_0, y - y_0, z - z_0)\right] \rho(x,y,z)\, dx\, dy\, dz = (0,0,0)$

в една точка

$$\begin{matrix} 0 & 0 & 1 \\ x-x_0 & y-y_0 & z-z_0 \end{matrix}$$

$(\cancel{y_0 - y},$ $\quad (-(y-y_0), \; x-x_0, \; 0)$

$$\begin{cases} = -\iiint_K (y-y_0)\,\rho(x,y,z)\,dx\,dy\,dz = 0 \\ \iiint_K (x-x_0)\,\rho(x,y,z)\,dx\,dy\,dz = 0 \\ 0 = 0 \end{cases}$$

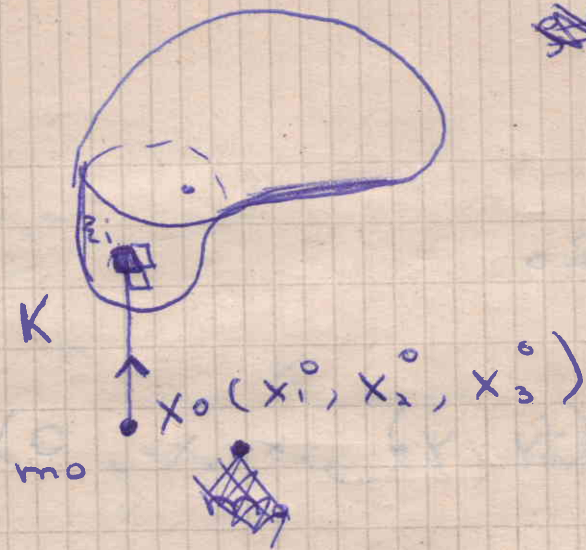
$$+\iiint_K y\,\rho(x,y,z)\,dx\,dy\,dz = +y_0 \underbrace{\iiint_K \rho(x,y,z)\,dx\,dy\,dz}_{m}$$

$$\Rightarrow \quad y_0 = \frac{1}{m(K)} \iiint_K y\,\rho(x,y,z)\,dx\,dy\,dz$$

$$x_0 = \frac{1}{m(K)} \iiint_K x\,\rho(x,y,z)\,dx\,dy\,dz$$

$F = (1, 0, 0)$ за z :))

$\rho(x_1, x_2, x_3)$



Δ_i парченца

$\xi_i \in \Delta_i$

$\cancel{\xi_i - x^0}$

$\cancel{\|\xi_i - x^0\|}$

$$\frac{\xi_i - x^0}{\|\xi_i - x^0\|} \cdot m_0 \rho(\xi_i) \mu_3(\Delta_i) \underbrace{\varkappa}_{\text{грав. const}} \cdot \frac{1}{\|\xi_i - x^0\|^2}$$

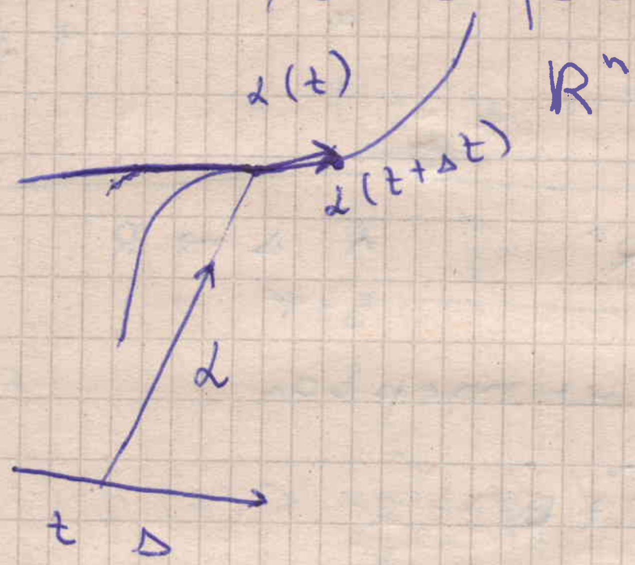
$$m_0 \varkappa \sum_{i=1}^{i_0} \rho(\xi_i) \frac{\xi_i - x^0}{\|\xi_i - x_0\|^3} \mu_3(\Delta_i) \longrightarrow$$

$\cancel{m_0 \varkappa \iiint_K \rho(x_1, x_2, x_3)}$

$$\longrightarrow m_0 \varkappa \iiint_K \rho(x) \overbrace{\frac{x - x^0}{\|x - x^0\|^3}}^{\text{в-р}} dx$$

$$F = \begin{pmatrix} m_0 \varkappa \iiint_K \rho(x_1, x_2, x_3) \dfrac{x_1 - x_1^0}{\|x - x^0\|^3} dx_1 dx_2 dx_3 \\ - - - - \\ - - - - \end{pmatrix}$$

Криволинейни интеграли



Δ - отв. интервал

$\alpha : \Delta \to \mathbb{R}^n$ векторна ф-я на скаларен аргумент

$\Gamma = \{ \alpha(t) : t \in \Delta \}$ траектория
крива α - нетрек.

$$\lim_{\Delta t \to 0} \frac{\alpha(t+\Delta t) - \alpha(t)}{\Delta t} =: \alpha'(t)$$

$$\dot{\alpha}(t)$$

има допир. към кривата

α e гладка в $[a, b]$

~~[illegible]~~

$$\underset{\Gamma = \alpha[a,b]}{\ell(\Gamma)} = \int_a^b \|\dot\alpha(t)\| dt =$$

$$= \int_a^b \sqrt{\dot\alpha_1^2(t) + \dot\alpha_2(t)^2 + \dot\alpha_3^2(t)}\, dt$$

$\alpha, \beta : \Delta \longrightarrow \mathbb{R}^n$ диф. в Δ $\qquad \lambda : \Delta \to \mathbb{R}$ диф.

Δ отв. интервал

1) $(\alpha + \beta)^\cdot = \dot\alpha + \dot\beta$

2) $(<\alpha, \beta>)^\cdot = <\dot\alpha, \beta> + <\alpha, \dot\beta>$

3) $n = 3$

$(\alpha \times \beta)^\cdot = \dot\alpha \times \beta + \alpha \times \dot\beta$

4) $(\lambda \cdot \alpha)^\cdot = \dot\lambda \alpha + \lambda \dot\alpha$

по координатно

Д-во 3)

$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

$\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$

$$\alpha(t) \times \beta(t) = \Big(\alpha_2^{(t)} \beta_3^{(t)} - \alpha_3^{(t)} \beta_2^{(t)}\Big),$$

$$\alpha_3(t)\beta_1(t) - \alpha_1(t)\beta_3(t),$$

$$\alpha_1(t)\beta_2(t) - \alpha_2(t)\beta_1(t)\Big)$$

$$\Big(\alpha(t) \times \beta(t)\Big)^{\cdot}_1 = \dot{\alpha}_2(t)\beta_3(t) - \alpha_2 \beta_3(t) -$$

$$- \dot{\alpha}_3(t)\beta_2(t) - \alpha_3^{(t)}\dot{\beta}_2(t) =$$

$$= \underline{\left(\dot{\alpha}_2 \beta_3(t) - \dot{\alpha}_3 \beta_2\right)} +$$

$$\underline{\left(\alpha_2 \dot{\beta}_3 - \alpha_3 \dot{\beta}_2\right)}$$

D-vo 2)

$$\langle \alpha(t) \beta(t) \rangle = \sum_{i=1}^{n} (\alpha_i \beta_i)^{\cdot} = \sum_{i=1}^{n} \big(\dot{\alpha}_i(t) \cdot \beta_i + \alpha_i \dot{\beta}_i\big) =$$

$$= \sum_{i=1}^{n} \dot{\alpha}_i \beta_i + \sum_{i=1}^{n} \alpha_i \dot{\beta}_i =$$

$$= \langle \dot{\alpha}, \beta \rangle + \langle \alpha, \dot{\beta} \rangle$$

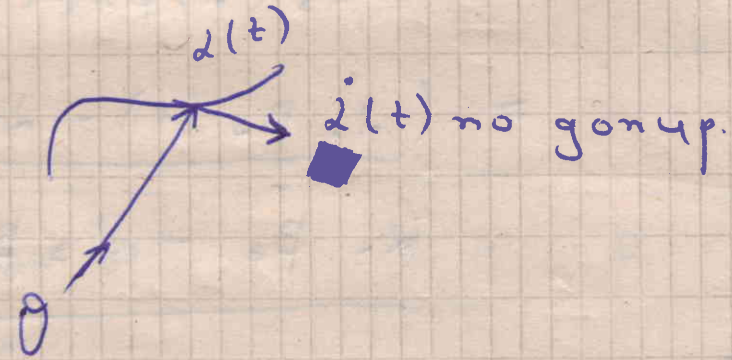
$$\|\alpha(x)\|^{\cdot} = \left(\langle \alpha, \alpha \rangle\right)^{\frac{1}{2}} =$$

$$= \frac{1}{2} \langle \alpha, \alpha \rangle^{-\frac{1}{2}} \cdot \big(\langle \dot{\alpha}, \alpha \rangle +$$

$$+ \langle \alpha, \dot{\alpha} \rangle\big) =$$

$$= \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{\langle \alpha, \alpha \rangle}} \cdot \cancel{2} \langle \dot{\alpha}, \alpha \rangle =$$

$$= \frac{\langle \dot{\alpha}, \alpha \rangle}{\sqrt{\langle \alpha, \alpha \rangle}} = \frac{\langle \dot{\alpha}, \alpha \rangle}{\| \alpha \|}$$



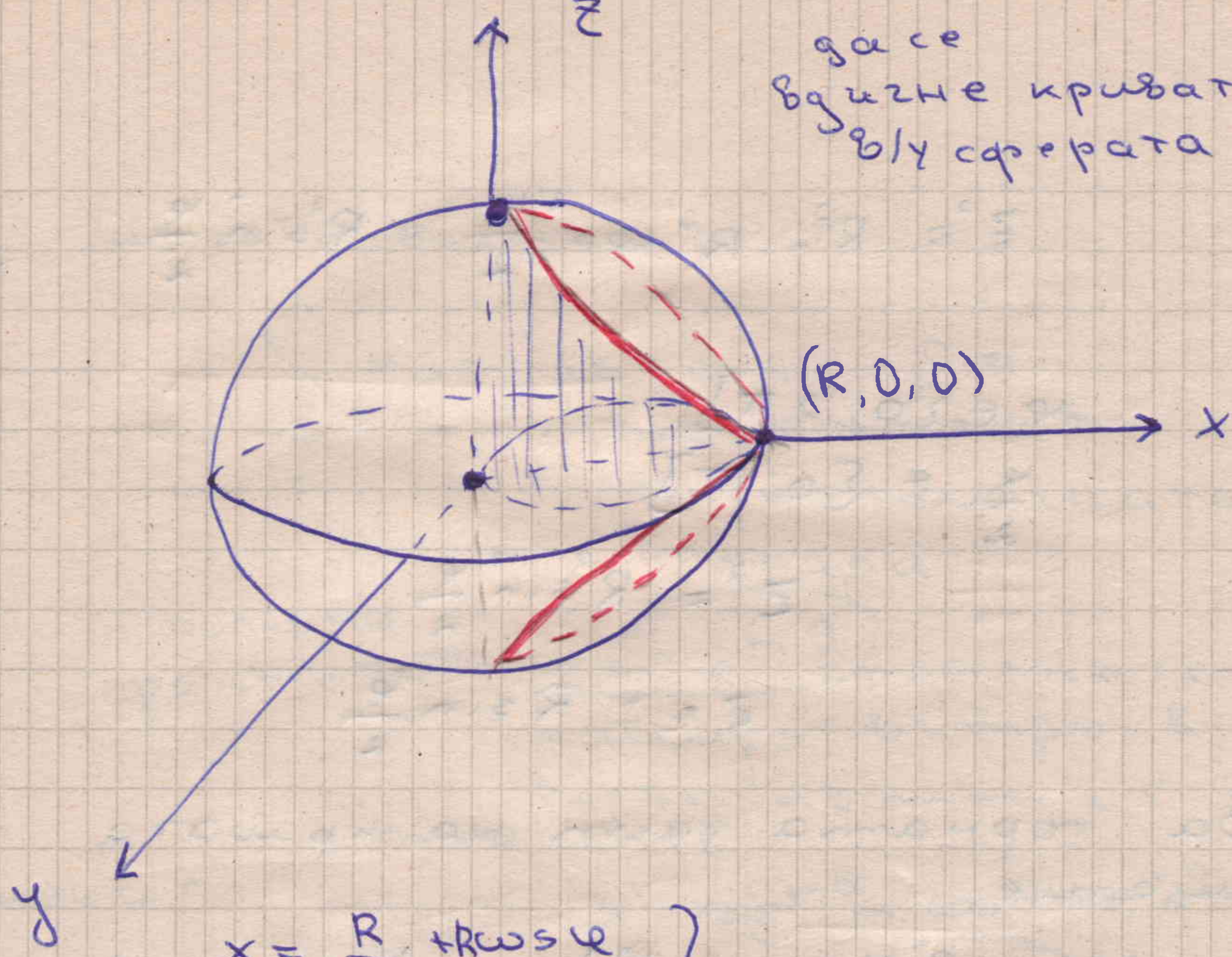
20.11.2013г.

Упражнение

Крива на Вивиани

Да се пресметне дължината на кривата

$$\begin{cases} x^2 + y^2 + z^2 = R^2 & (1) \\ \left(x - \frac{R}{2}\right)^2 + y^2 = \left(\frac{R}{2}\right)^2 & (2) \end{cases}$$



да се
вдигне кривата
в/у сферата

$$\left.\begin{aligned} x &= \frac{R}{2} + \frac{R}{2}\cos\varphi \\ y &= \frac{R}{2}\sin\varphi \end{aligned}\right\}$$ удовлетворили сме (2)

Заместване в (1), за да получим

$$z = z(\varphi)$$

$$\left(\frac{R}{2}(1+\cos\varphi)\right)^2 + \left(\frac{R}{2}\sin\varphi\right)^2 + z^2 = R^2$$

$$1 + 2\cos\varphi + \cancel{\cos^2\varphi + \sin^2\varphi} + 4\frac{z^2}{R^2} = 4$$

$$\cancel{4z^2 = 3 - 2\cos\varphi}$$

$$\cancel{z = \sqrt{3 - 2\cos\varphi}}$$

$$z = R^2 - \frac{R^2}{2}\left(2\cos^2\frac{\varphi}{2}\right) =$$

$$z^2 = R^2 - R^2\cos^2\frac{\varphi}{2} = R^2\sin^2\frac{\varphi}{2}$$

$$\varphi \in \{0; 2\pi\}$$

$$\frac{\varphi}{2} \in \{0; \pi\}$$

$$z^2 = R^2\sin^2\frac{\varphi}{2}$$

$$z = \pm R\sin\frac{\varphi}{2}$$

За горната част на кривата имаме

$$\left| \begin{array}{ll} x = \frac{R}{2}(1+\cos\varphi) & , x' = -\frac{R}{2}\sin\varphi \\ y = \frac{R}{2}\sin\varphi & y' = \frac{R}{2}\cos\varphi \\ z = \frac{R}{2}\sin\frac{\varphi}{2} & z' = \frac{R}{2}\cdot\cos\frac{\varphi}{2}\cdot\frac{1}{2} \end{array} \right.$$

Дължината на кр. на Вивиани е

$$2\int_0^{2\pi} \sqrt{\frac{R^2}{4}\sin^2\varphi + \frac{R^2}{4}\cos^2\varphi + \frac{R^2}{16}\cos^2\frac{\varphi}{2}}\, d\varphi$$

$$= 2\int_0^{2\pi} \frac{R}{2}\sqrt{1 + \frac{1}{4}\cos^2\frac{\varphi}{2}}\, d\varphi$$

Универсална субституция

<u>зад.</u> Да се пресметне

$\int_c x\,ds$, където с е онази част от логаритмичната спирала

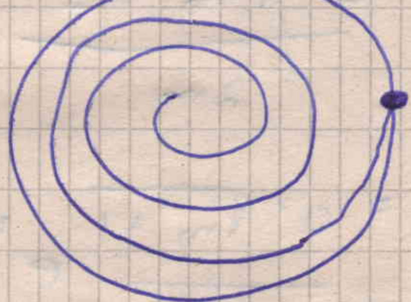
полярно разстояние — $\boxed{r = ae^{m\theta}}$, която се намира вътре в окръжността $r = 2a$

$m > 0$ const

Решение

$x' = a(e^{m\theta}.m.\cos\theta + e^{m\theta}\sin\theta)$

$$\left|\begin{array}{l} x = ae^{m\theta}\cos\theta \\ y = ae^{m\theta}\sin\theta \\ \theta \in \{0; \frac{\ln 2}{m}] \end{array}\right.$$



$2\cancel{a} = \cancel{a}e^{m\theta}$ } където се пресичат

Решаване спрямо параметъра

$m\theta = \ln 2$

$\theta = \frac{\ln 2}{m} > 0$

$$\int_c x\,ds = \oint_c \cancel{x} = \int ae^{m\theta}\cos\theta\sqrt{(ame^{m\theta}\cos\theta - ae^{m\theta}\sin\theta)^2}$$

$$+\left(a m e^{m\theta}\sin\theta + a e^{m\theta}\cos\theta\right)^2\, d\theta =$$

$$= \int_0^{\frac{\ln 2}{m}} a e^{m\theta}\cos\theta \sqrt{(a m e^{m\theta})^2 + (a\cancel{m}e^{m\theta})^2}\, d\theta$$

$$= \int_0^{\frac{\ln 2}{m}} a e^{m\theta} \cdot a e^{m\theta}\sqrt{m^2+1}\, d\theta =$$

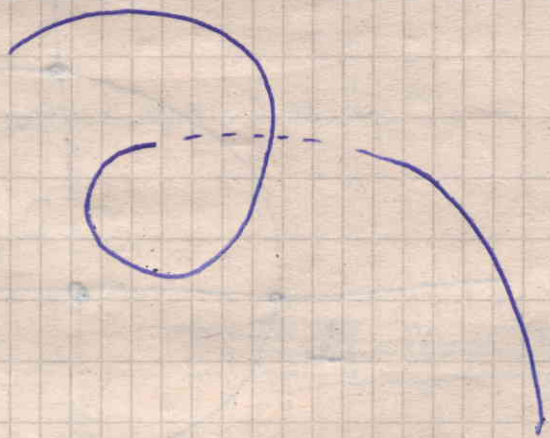
$$= \frac{a^2\sqrt{m^2+1}}{2m}\int_0^{\frac{\ln 2}{m}} e^{2m\theta}\, d\,2m\theta =$$

$$= \frac{a^2\sqrt{m^2+1}}{2m}\, e^{2m\theta}\Big|_{\theta=0}^{\frac{\ln 2}{m}} = \frac{a^2\sqrt{m^2+1}}{2m}\left(e^{2\ln 2}-1\right)$$

$$= \frac{3}{2}\,\frac{a^2\sqrt{m^2+1}}{m}$$

Физична интерпретация на криволинейния интеграл от I род

Нека имаме материална нишка с плътност $\rho(x, y, z)$



линейна маса -

$$M_L = \int_L \rho \, ds$$

$$\boxed{\text{Център на тежестта}}$$

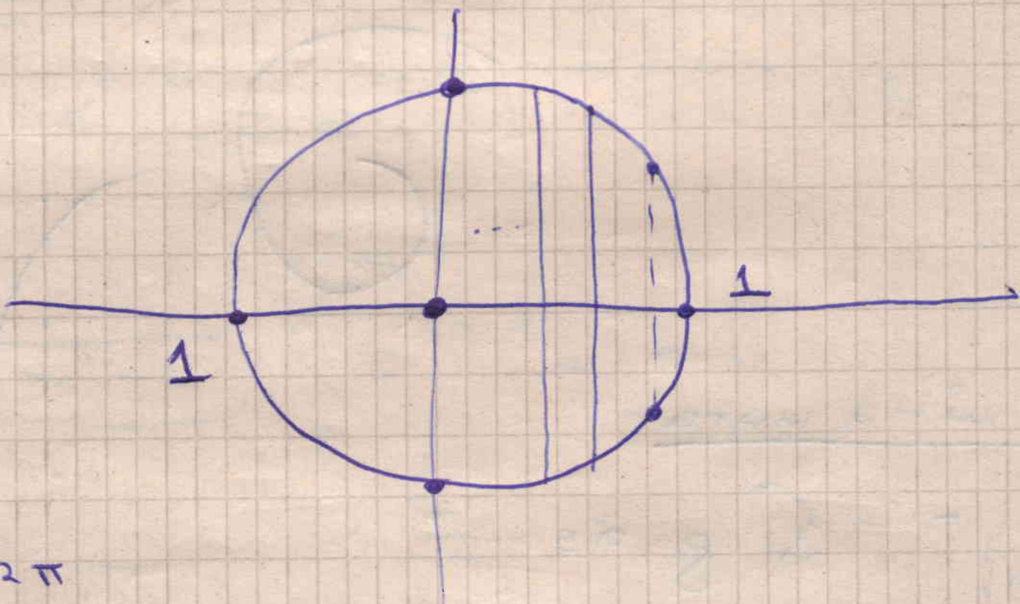
$$x_G = \frac{1}{M} \int_L x \rho \, ds$$

$$y_G = \frac{1}{M} \int_L y \rho \, ds$$

$$z_G = \frac{1}{M} \int_L z \rho \, ds$$

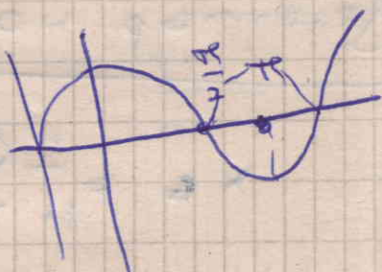
Зад. Пресметнете центъра на тежестта

$$x = a\cos t$$
$$y = a\sin t$$
$$\rho(x,y) = |x|$$



$$M = \int_0^{2\pi} |a\cos t| \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt =$$

$$= a^2 \int_0^{2\pi} |\cos t|\, dt =$$



$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} |\cos t\, dt|\, dt =$$

$$= 2a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt = -2a^2 \sin t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4a$$

$$x_G = \frac{1}{4|a|^2}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} a\cos t\,|a\cos t|\cdot a =$$

$$= \frac{a^3}{4a^2}\cdot 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt =$$

$$= \frac{a}{2}\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2t + 1)\,dt =$$

$$= \frac{a}{4}\,\frac{1}{2}\,\overset{0}{\sin 2t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}\;\; \frac{a}{4}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= \frac{a^3}{4a^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt - \frac{a^3}{4a^2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2 t\,dt = 0$$

Потенциал

1. Градиент

Нека $u = u(x, y, z)$

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \text{ - градиент}$$

Във ∀ точка от графиката на ф-ята по направлението на градиента тя расте най-бързо

$$u = x^2 + y^2$$
$$\nabla u = (2x, 2y)$$

$$u = x + e^y + zx$$
$$\nabla u = (1, e^y, x)$$

$$u = \|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$$

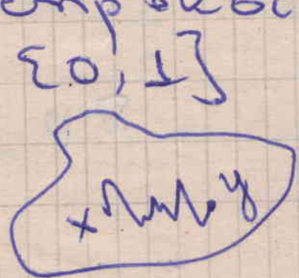
$$\nabla u = \left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}, \frac{\cancel{2}y}{\cancel{2}\sqrt{}}, \frac{\cancel{2}z}{\cancel{2}\sqrt{}} \right) =$$

$$= \frac{1}{u}(x,y,z)$$

Деф. Област в $\mathbb{R}^2$ $D \subseteq \mathbb{R}^2$, е област, ако е
- отворено
- свързано

Деф D е свързано, ако за ∀ 2 точки $\forall x,y \in D$ $\exists$ непрекъсната крива $\gamma(t)$, $t \in \{0,1]$

$$\gamma(0) = x$$
$$\gamma(1) = y$$

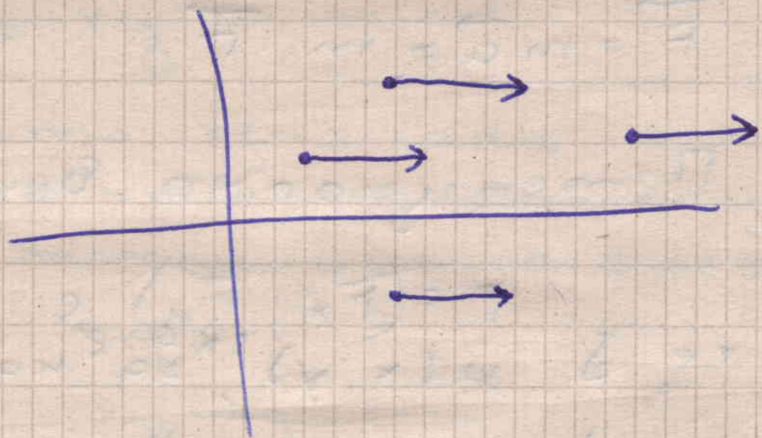


Деф. Едносвързана област
$D \subseteq \mathbb{R}^2$ е едносвързана област, ако за $\forall$ затворена крива в D, вътрешността на кривата също е в D. (няма дупки) (=) допълнението ѝ да е свързано

Пример:

не е едносвързана

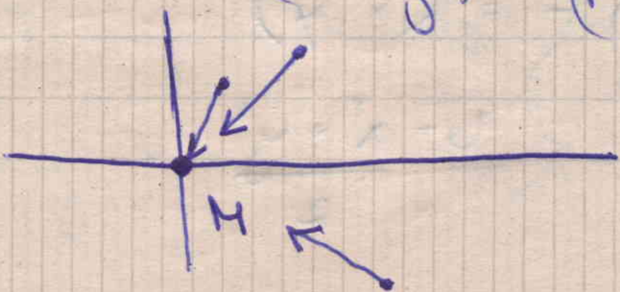
Деф. Векторно поле в $\mathbb{R}^2$
$\forall$ непрекъсната ф-я $F: \mathbb{R}^2 \to \mathbb{R}^2$ се нарича векторно поле



$F(x,y) = (1,0)$

център на тежестта

$$F(x,y) = -\gamma M \left(\frac{x}{(\sqrt{x^2+y^2})^3}, \frac{y}{(\sqrt{x^2+y^2})^3} \right)$$



Движението на материална точка се определя от векторното поле, в което тя се намира според закона на Нютон

$$\begin{cases} m\ddot{x} = F_x(t, x, \dot{x}) \\ m\ddot{y} = F_y(t, x, \dot{x}) \end{cases}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix} = \begin{pmatrix} V_{x_0} \\ V_{y_0} \end{pmatrix}$$

$$\vec{F} = m\vec{a} = m \cdot \ddot{\vec{r}}$$

<u>Деф.</u> Потенциално векторно поле $F = (F_x, F_y)$ е такова поле, че $\exists$ $u(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$, за което

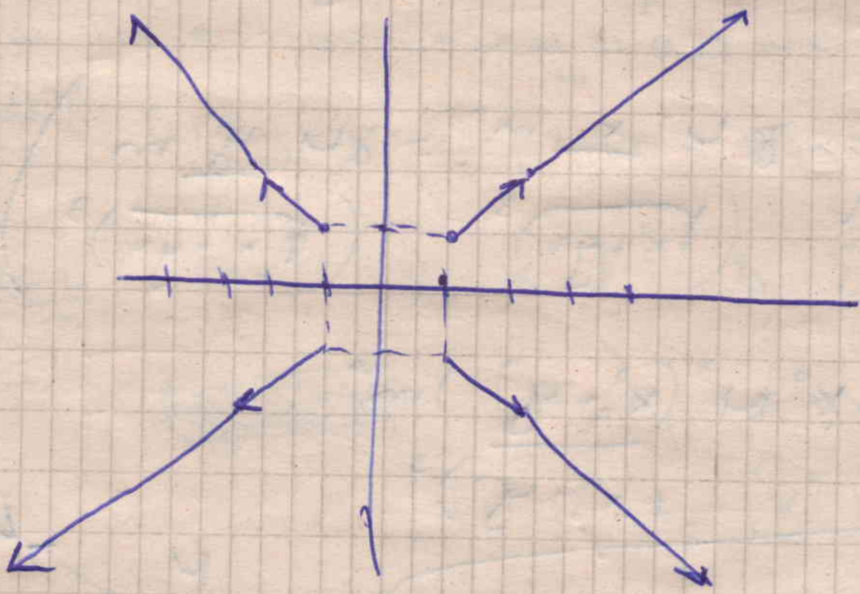
$$\frac{\partial u}{\partial x} = F_x, \quad \frac{\partial u}{\partial y} = F_y, \text{ т.е.}$$

$$\nabla u = F$$

<u>Пример:</u> 1) $\vec{F} = (x, y)$

$$u = \frac{x^2 + y^2}{2}$$

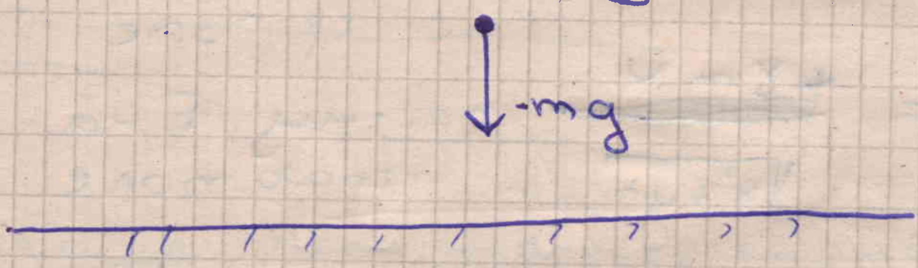
$$\frac{\partial u}{\partial x} = x, \quad \frac{\partial u}{\partial y} = y$$



2) $\vec{F} = (y, x)$

$$u(x,y) = xy$$

В случая, когато една сила е потенциална, намираме потенциал и наричаме потенциална енергия.

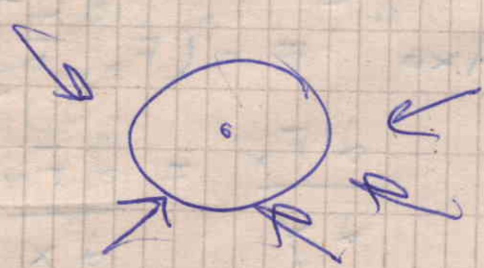


$F = (0, -mg)$

$u = -mgy$

$u_x = 0$

$u_y = -mg$



Ако в т. (0,0) има тяло с маса M, то създава гравитационно поле със сила

$$F = \left(-\gamma M \frac{x}{\left(\sqrt{x^2+y^2}\right)^3} m,\ -\gamma M \frac{y}{\left(\sqrt{x^2+y^2}\right)^3} m \right)$$

$$F^2 = \gamma^2 M^2 \frac{(x^2+y^2)}{(x^2+y^2)^3} m^2$$

$$\boxed{\Rightarrow |F| = \frac{\gamma M}{\sqrt{x^2+y^2}}}$$

M - маса Земя

m - маса на точка, поставена в това гр-поле

$$u(x,y) = \frac{\pm\gamma m M}{\sqrt{x^2+y^2}}$$ потенц. E на грав. поле

?. Кога едно поле е потенциално?

<u>НУ</u> Ако $F = (F_x, F_y) \in C^1$ е потенциално, то

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

Това е критерий за проверка. Ако равенството не е изпълнено, то полето не е потенциално.

НУ е **ДУ** в едносвързана област!!!

Примери: $F = (xy, x^2)$ в $[0,1]^2$

$(F_1)'_y = (xy)'_y = x$

$(F_2)'_x = (x^2)' = 2x$

$x \neq 2x \Rightarrow$ не е пот.

Полето

$(2xy, x^2)$ е потенциално в

$[0,1]^2$ с потенциал

$u(x,y) = x^2 y$

Твърдение: Нека материална точка се движи в потенциално поле с потенциал $u(x,y)$.

Тогава е в сила ЗЗЕ

$$\boxed{m\frac{x^2+y^2}{2} - u(x,y) = C}$$

D-80. Имаме

$$\begin{cases} m\ddot{x} = F_x = \dfrac{\partial u}{\partial x} & (\cdot \dot{x}) \\ m\ddot{y} = F_y = \dfrac{\partial u}{\partial y} & \cdot 1(\dot{y}) \end{cases} +$$

$$m(\ddot{x}\dot{x} + \dot{y}\ddot{y}) = \dot{x}\cdot\frac{\partial u}{\partial x} + \dot{y}\cdot\frac{\partial u}{\partial y}$$

$$\frac{d}{dt}\left(m\frac{\dot{x}^2+\dot{y}^2}{2}\right) = \frac{d}{dt}\left(u(x(t), y(t))\right)$$

$$\frac{d}{dt}\left(\frac{m(\dot{x}^2+\dot{y}^2)}{2} - u(x,y)\right) = 0$$

$$E_к + E_п = Const$$

$$E_к = \frac{m(\dot{x}^2+\dot{y}^2)}{2}, \quad E_п = -u$$

Система, в която е в сила ЗЗЕ, се нарича <u>консервативна</u>

$\boxed{\text{Алгоритъм за намиране на потенциал}}$

Имаме $F = (F_1, F_2)$ в едносвързана област

$$1.\ ?\ F'_{1y} = F'_{2x} \xrightarrow[\text{не}]{\text{не}} \text{няма п-л}$$

да $\downarrow$

2. Интегриране по x F_1. Получаване

$$u(x,y)=\int_{x_0}^{x} F_1(\xi,y)\,d\xi + A(y)$$

Диференциране по y и приравняване резултата към F_2.
Получаваме ДУ за $A(y)$
След решение п-лае определен с точност до const.

<u>Пример:</u> $F=(y^2+2xy,\ x^2+2xy)$

$$u(x,y)=\int(y^2+2xy)\,dx + A(y) =$$

$$= xy^2+yx^2+A(y)$$

$$\frac{\partial u}{\partial y}=F_2$$

$$\cancel{2xy}+\cancel{x^2}+A'(y)=\cancel{x^2}+\cancel{2xy}$$

$$\Rightarrow A(y)=\text{const}$$

и п-ла е

$$u(x,y)=xy^2+yx^2$$

Потенциал в $\mathbb{R}^3$

$F = (F_1, F_2, F_3)$ е пот., ако

$$F = \nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

В случая на $\mathbb{R}^3$ НУ

Ако F е потенциално, то е вярно

$$\begin{cases} (F_1)'_y = (F_2)'_x \\ (F_2)'_z = (F_3)'_y \\ (F_3)'_x = (F_1)'_z \end{cases}$$

$$F_1 = \frac{\partial u}{\partial x} \qquad F_2 = \frac{\partial u}{\partial y}$$

$$(F_1)'_y = \frac{\partial^2 u}{\partial x \partial y} = (F_2)'_x = \frac{\partial^2 u}{\partial y \partial x}$$

В случая на $\mathbb{R}^3$ интегрираме по едната променлива, след което получаваме потенциал с точнос до ф-я на другите 2. Тази ф-я намиране, като наложим останалите 2 условия последователно, т.е. (което се свежда до решаване на 2 диференциални у-я)

Пример: $F = (3x^2y^2z + 3x^2, 2x^3yz, x^3y^2 + 3z^2)$ в $\{0.1\}$

1) $(F_1')_y \overset{?}{=} (F_2)'_x \iff$

$$6x^2yz = 6x^2yz \checkmark$$

$(F_2)'_z = (F_3)'_y \iff$

$$2x^3y = 2x^3y \checkmark$$

$(F_3)'_x = (F_1)'_z \iff$

$$3x^2y^2 = 3x^2y^2 \checkmark \text{ да}$$

$$u(x,y,z) = \int F_1\,dx + G(y,z) =$$

$$= \int (3x^2y^2z + 3x^2)\,dx + G(y,z) =$$

$$= x^3y^2z + x^3 + G(y,z)$$

$u'_y = 2x^3yz$

$$2x^3yz + 0 + G'_y(y,z) = 2x^3yz$$

$$G'_y = 0 \Rightarrow G = G(z)$$

$$u'_z = x^3 y + 3z^2 \Rightarrow$$

$$\cancel{x^3 y^2} + 0 + G'(z) = \cancel{x^3 y^2} + \cancel{3z^2}$$

$$G(z) = z^3 + C$$

$$u(x,y,z) = x^3 y^2 z + x^3 + z^3 + C$$

<u>Зад.</u> $F = (3x^2 + 2xy,\ x^2 + 2y + z,\ y + 3z^2)$

$$x^3 + x^2 y + y^2 + z^3$$

$$u(x,y,z) = z^3 + x^3 + x^2 y + y^2 + yz$$

Дом.: Намерете потенциала

$$F(x,y,z) = \left(e^{\frac{y}{z}},\ \frac{e^{\frac{y}{z}}(z+1)}{z} + z e^{yz},\right.$$

$$\left. - \frac{e^{\frac{y}{z}}(x+y)y}{z^2} + y e^{yz} + e^{-z}\right)$$

в $[1,2]^3$

20.11

Лекция

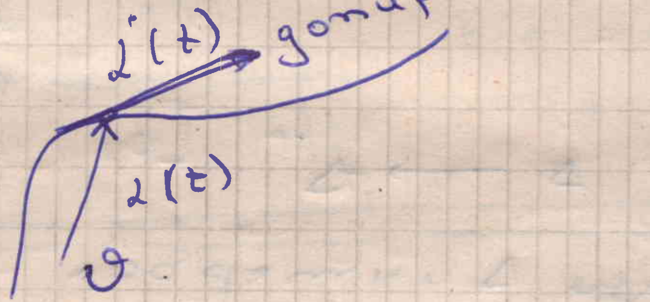
$\alpha : \Delta \longrightarrow \mathbb{R}^n$

$n = 2, 3$

Δ интервал в $\mathbb{R}$

α - диф.

$$\dot{\alpha}(t) = \lim_{\Delta t \to 0} \frac{\alpha(t + \Delta t) - \alpha(t)}{\Delta t}$$

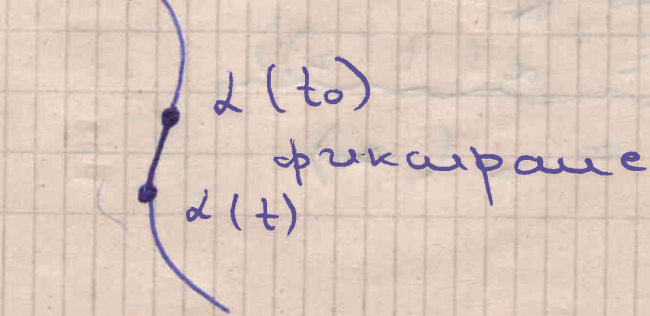


$\alpha : \Delta \to \mathbb{R}^n$

$\alpha \in C^1(\Delta)$ регулярна крива

$\dot{\alpha} \neq 0$

$\Rightarrow \Gamma = \alpha(\Delta)$ е регулярна гладка крива



$$S(t) = \begin{cases} +\rho(\Gamma_{\alpha(t_0)}, \alpha(t)) & \text{ако } t > t_0 \\ -\rho(\Gamma_{\alpha(t_0)}, \alpha(t)) & \text{ако } t < t_0 \end{cases}$$

$$= \int_{t_0}^{t} \|\dot\alpha(\tilde\tau)\| \, d\tilde\tau$$

$$S(t) = \int_{t_0}^{t} \|\dot\alpha(\tilde\tau)\| \, d\tilde\tau$$ параметър

$$\dot s(t) = \|\dot\alpha(t)\| > 0$$ непрекъсн. диф. строго растяща

$$S : \Delta \longrightarrow \Delta'$$

S биекция Δ' интервал

$$\Delta \underset{t}{\overset{s}{\rightleftarrows}} \Delta'$$

t - обратна биекция

(1) $s(t(s)) = s \quad \forall s \in \Delta'$
$t(s(t)) = t \quad \forall t \in \Delta$

$$\tilde\alpha : \Delta' \longrightarrow \mathbb{R}^n$$

$$\tilde\alpha(s) = \alpha(t(s))$$

1-диф. спрямо ест. парам

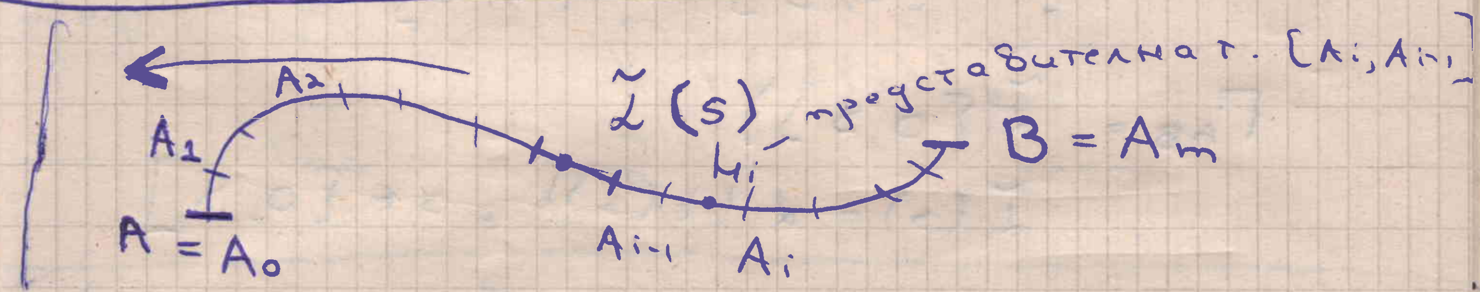
$$\tilde\alpha(\Delta') = \alpha(\Delta)$$

$$\tilde{\alpha}'(s) = \dot{\alpha}(t(s)) \cdot t'(s)$$

от ① $\dot{s}(t(s)) \cdot t'(s) = 1$

$$t'(s) = \frac{1}{\dot{s}(t(s))} = \frac{1}{\cancel{\|\dot{\alpha}(t)\|}\ \|\dot{\alpha}(t(s))\|}$$

$$\Rightarrow \tilde{\alpha}'(s) = \frac{\dot{\alpha}(t(s))}{\|\dot{\alpha}(t(s))\|}$$ в-р по допир. с дължина 1



$\alpha : \{a, b\} \longrightarrow \mathbb{R}^n \quad n = 2, 3$

$\alpha \in C^1, \ \dot{\alpha} \neq 0$

$\Gamma_{AB} = \alpha(\{a, b\})$

крива
с нач. A и кр. B
⊙ от точки

$$f : \Gamma_{AB} \longrightarrow \mathbb{R}$$

$$\int_{\Gamma_{AB}} f$$

$$s(t) = \int_a^t \| \dot{\lambda}(\tau) \| \, d\tau$$

$$S = \int_a^b \| \dot{\lambda}(\tau) \| \, d\tau$$

$$s : [a, b] \to [0, S]$$

$$t : [0, S] \to [a, b]$$

$$\Gamma_{AB} = \tilde{\lambda}([0, S])$$

$$\tilde{\lambda}(s) = \lambda(t(s)) \ , \ s \in [0, S]$$

$$\pi : \quad s = 0 < s_1 < s_2 < \dots < s_m = S$$

$$A_i := \tilde{\lambda}(s_i) \quad i = 0, 1, \dots, m$$

$$\overset{\frown}{A_{i-1}A_i} \ni M_i = \tilde{\lambda}(\xi_i) \quad i = 1, \dots, n$$

$$\xi_i \in [s_{i-1}, s_i]$$

$$m \sim \sum_{i=1}^{m} f(M_i) \cdot \ell(\overset{\frown}{A_{i-1}A_i}) =$$

Риманова Σ

$$= \sum_{i=1}^{m} f(\tilde{\alpha}(\xi_i)) \cdot (s_i - s_{i-1}) =$$

$$= \sigma_{f \circ \tilde{\alpha}}(\Pi, \xi) \xrightarrow[d(\Pi) \to 0]{} \int_{\Gamma_{AB}} f$$

криволин.

$\int$ от I род

$$\int_0^S f(\tilde{\alpha}(s))\,ds$$

Деф.: $$\int_{\Gamma_{AB}} f := \int_0^S f(\tilde{\alpha}(s))\,ds$$

$$\int_{\Gamma_{AB}} f := \int_0^S f(\tilde{\alpha}(s))\,ds = s = s(t), \; t \in [a, b]$$

$$= \int_a^b f(\alpha(t(s(t)))\,\dot{s}(t)\,dt =$$

$$= \int_a^b f(t)\,\|\dot{\alpha}(t)\|\,dt$$

Свойства: 1) не зависят от посоката (смяна на променливите)

$$\Gamma_{AB} = \alpha([a,b])$$

$$\Gamma_{BA} = \gamma([a,b])$$

$$\gamma(\tilde{t}) = \alpha(a+b-\tilde{t})$$

$$\tilde{t} = a$$

$$\gamma(a) = \alpha(b) = B$$

$$\gamma(b) = \alpha(a) = A$$

$$\tilde{t} \in [a,b] \longleftrightarrow t = a+b-\tilde{t} \in [a,b]$$

$$\int_{\Gamma_{BA}} f = \int_a^b f(\gamma(\tilde{t})) \cdot \|\dot{\gamma}(\tilde{t})\| \, d\tilde{t}$$

$$t = a+b-\tilde{t}$$

$$\tilde{t} = a+b-t$$

$$= \int_b^a f(\gamma(a+b-t)) \cdot \|\dot{\gamma}(a+b-t)\| (-1)\, dt =$$

$$= \int_a^b f(\alpha(t)) \, \|\dot{\alpha}(t)\| \, dt =$$

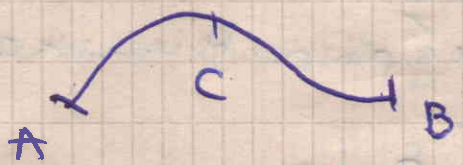
$$\gamma(a+b-t) = \alpha(a+b-(a+b-t)) = \alpha(t)$$

$\dot\gamma(\tilde t)(a+b-\tilde t)(-1)$

$\dot\gamma(a+b-t) = \dot\alpha(t).(-1)$

$= \int_{\Gamma_{AB}} f$

2) адитивност



$$\int_{\Gamma_{AB}} f = \int_{\Gamma_{AC}} f + \int_{\Gamma_{CB}} f$$

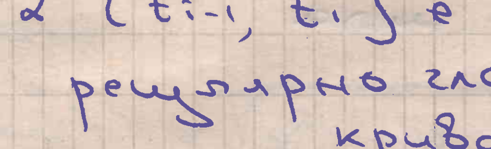
Деф. $\alpha : [a,b] \to \mathbb{R}^n$ непрекъснато

$a = t_0 < t_1 < \dots < t_k = b$

$\alpha\big|_{[t_{i-1}, t_i]} \in C^1$ и $\dot\alpha\big|_{[t_{i-1}, t_i]} \neq 0$

т.е. $\Gamma_i = \alpha[t_{i-1}, t_i]$ е регулярно гладка крива

$\Gamma = \alpha([a,b])$ частично гладка крива



$$\int_\Gamma f = \sum_{i=1}^{k} \int_{\Gamma_i} f$$

Δ

Физични приложения

- маса на материална нишка

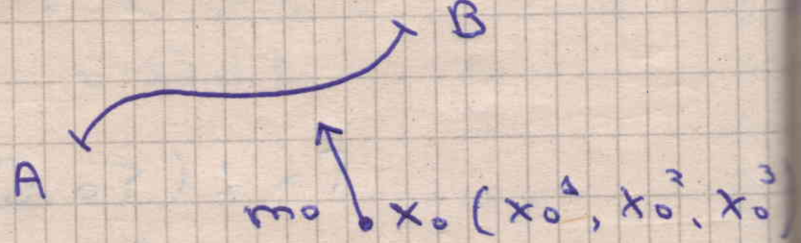
$$\int_\Gamma \rho \, ds$$

- център на масите

$$x_0 = \frac{1}{m}\int_\Gamma x \cdot \rho(x,y)\, ds$$

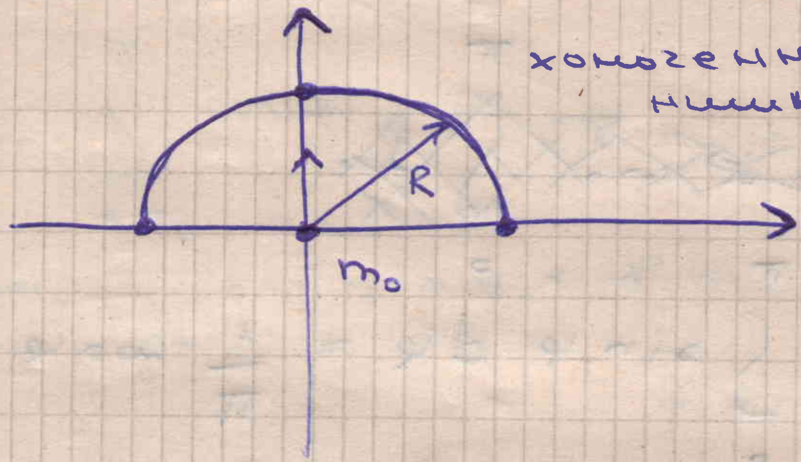
$$y_0 = \frac{1}{m}\int_\Gamma y \cdot \rho(x,y)\, ds$$

- грав. сила



$$F = \varkappa m_0 \int_\Gamma \rho(x) \frac{x - x_0}{\|x - x_0\|^3}\, ds$$

Зад.



хомогенна нишка

$\rho \equiv 1$

$$\alpha(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\varphi \in [0, \pi]$$

$$m = \int_{\Gamma} 1\, ds = \int_0^{\pi} 1 \cdot \|\dot{\alpha}(\varphi)\|\, d\varphi =$$

$$\dot{\alpha}(\varphi) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

$$= \int_0^{\pi} \sqrt{\sin^2\varphi + \cos^2\varphi}\, d\varphi = \pi$$

$$x_0 = \frac{1}{\pi} \int_0^{\pi} (\cos\varphi) \cdot 1 \sqrt{\sin^2\varphi + \cos^2\varphi}\, d\varphi =$$

$$= 0$$

$$\cancel{y_0 = \frac{1}{\pi} \int_0^{\pi} \cos\varphi \cdot 1 \sqrt{\sin^2\varphi + \cos^2\varphi}\, d\varphi = \frac{1}{\pi} - \sin\varphi \Big|_0^{\pi}}$$

$$y_0 = \frac{1}{\pi} \int_0^{\pi} \sin\varphi \, d\varphi = \frac{1}{\pi} -\cos\varphi \Big|_0^{\pi} =$$

$$= \frac{2}{\pi}$$

$$F = \kappa m_0 \int_\Gamma \rho(x,y) \cdot \frac{(x,y)}{\|x,y\|^3} \, ds =$$

$$x_0 = (0,0)$$

$$= \kappa m_0 \int_\Gamma 1 \cdot \frac{(x,y)}{\left(\sqrt{x^2+y^2}\right)^3} \cdot ds =$$

$$x = \cos\varphi$$
$$y = \sin\varphi$$

$$= \kappa m_0 \int_0^{\pi} 1 \cdot \frac{(\cos\varphi, \sin\varphi)}{1} \cdot \sqrt{\cos^2\varphi + \sin^2\varphi} \, d\varphi$$

$$F_1 = \varkappa m_0 \int_0^{\pi} \cos\varphi \, d\varphi = -\varkappa m_0 \sin\varphi \Big|_0^{\pi} = 0$$

$$F_2 = \varkappa m_0 \int_0^{\pi} \sin\varphi \, d\varphi = \varkappa m_0 \cos\varphi \Big|_0^{\pi} =$$

$$= \varkappa m_0 (-1 - 1) =$$

$$= -2\varkappa m$$

$$\varkappa m_0 \left(\int_0^{\pi} \cos\varphi \, d\varphi \int_0^{\pi} \sin\varphi \, d\varphi \right) =$$

$$= \varkappa m_0 \left(\sin \Big|_0^{\pi}, -\cos\varphi \Big|_0^{\pi} \right) = (0, 2\varkappa m_0)$$

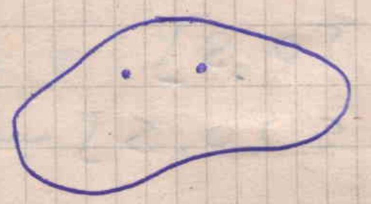
с радиус $R \; \left(0, \frac{2\varkappa m_0}{R}\right)$

Криволинейни интеграли от II род

Деф. $\Omega \subset \mathbb{R}^n$

Казваме, че Ω е област, ако Ω е отворено и свързано.

Деф. Векторно поле.

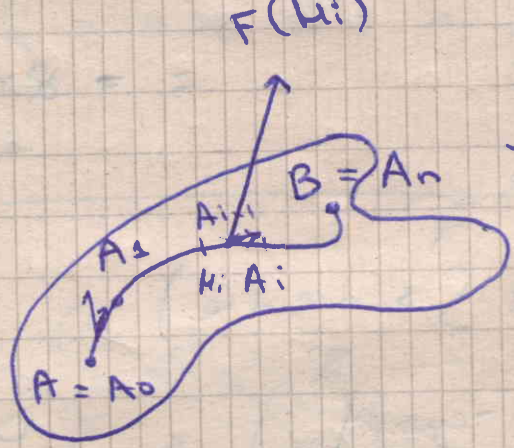


$F : \Omega \longrightarrow \mathbb{R}^n$

$\tilde{F}(x) = (x; F(x))$

свързан вектор

$$\text{работа} = \left\langle \vec{F}, \frac{\vec{AB}}{\|\vec{AB}\|} \right\rangle \cdot \|\vec{AB}\|$$



$\Omega \subset \mathbb{R}^n \ (n = 2, 3)$

Ω област

$F : \Omega \longrightarrow \mathbb{R}^n$

векторно поле

$$\alpha : [a, b] \longrightarrow \Omega$$

$$\Gamma_{AB} = \alpha([a, b]) \subset \Omega$$

$$\alpha \in C^1, \ \dot{\alpha} \neq 0$$

$$s(t) = \int_0^t \|\dot{\alpha}(\tau)\| \, d\tau$$

$$s \in [0, S]$$

$$t(s) \in [0, S] \longrightarrow [a, b]$$

$$\tilde{\alpha}(s) = \alpha(t(s))$$

$$s_0 = 0 < s_1 < \ldots < s_m = S$$

$$\tilde{\alpha}(s_i) = A_i \qquad i = 0, 1, \ldots, m$$

$M_i \in \overset{\frown}{A_{i-1}, A_i}$

$M_i = \tilde{\mathcal{L}}(\xi_i) \quad , \quad \xi_i \in [s_{i-1}, s_i]$

$i = 1, \dots, m$

работата $\quad < F(M_i)$

Проектираме

$\tau(M)$ единичен вектор по допирателната към Γ в т. M по посока на нар. на пор.

$$\sum_{i=1}^{m} < F(M_i), \tau(M) > . (s_i - s_{i-1}) =$$

$$= \sum_{i=1}^{m} < F(\tilde{\mathcal{L}}(\xi_i)), \tilde{\mathcal{L}}'(\xi_i) > (s_i - s_{i-1}) \xrightarrow[d(\Pi)\to 0]{}$$

$$\tilde{\mathcal{L}}'(s) = \tau(\tilde{\mathcal{L}}(s))$$

$$\xrightarrow[d(\Pi)\to 0]{} \int_0^S < F(\tilde{\mathcal{L}}(s)). \tilde{\mathcal{L}}'(s) > ds$$

$$\cancel{\int_{\Gamma_{AB}} (F, dr)} \qquad \int_{\Gamma_{AB}} < F, dr > := \int_0^S < F(\tilde{\mathcal{L}}(s)), \tilde{\mathcal{L}}'(s) > ds$$

криев. $\int$ от
II род от
полето F по Γ_{AB}

$$\int_{\Gamma_{AB}} \langle F, dr \rangle = \int_0^S \langle F(\tilde{\alpha}(s)), \tilde{\alpha}'(s) \rangle \, ds$$

$$s = s(t) = \int_0^t \|\dot{\alpha}(\tilde{\tau})\| \, d\tilde{\tau} =$$

$$t \in [a, b]$$

$$= \cancel{\int_a^b} \int_a^b \langle F(\alpha(t))$$

$$\tilde{\alpha}(s) = \alpha(t(s))$$

$$\tilde{\alpha}'(s) = \dot{\alpha}(t(s)) \cdot t'(s) =$$

$$= \frac{\dot{\alpha}(t(s)) \cdot 1}{\|\dot{\alpha}(t(s))\|}$$

$$= \int_a^b \langle F(\alpha(t)), \frac{\dot{\alpha}(t)}{\cancel{\|\dot{\alpha}(t)\|}} \rangle \cdot \cancel{\|\dot{\alpha}(t)\|} \, dt =$$

$$= \int_a^b \langle F(\alpha(t)), \dot{\alpha}(t) \rangle \, dt \quad (*)$$

сходно означение $\int\limits_{\Gamma_{AB}} F_1 dx_1 + F_2 dx_2 + F_3 dx_3$

$n = 3$

$$\alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \end{pmatrix}$$

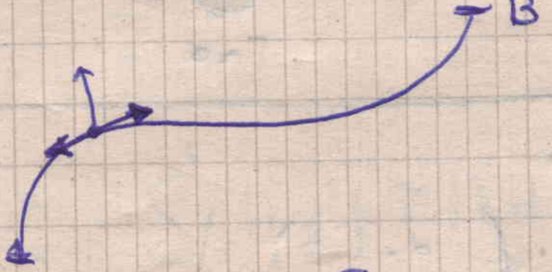
$$\dot{\alpha}(t) = \lim_{\Delta t \to 0} \frac{\alpha(t+\Delta t) - \alpha(t)}{\Delta t} =$$

$$= \begin{pmatrix} \lim\limits_{\Delta t \to 0} \frac{\alpha_1(t+\Delta t) - \alpha_1(t)}{\Delta t} \\ \lim\limits_{\Delta t \to 0} \frac{\alpha_2(t+\Delta t) - \alpha_2(t)}{\Delta t} \\ \ldots \end{pmatrix}$$

$$= \begin{array}{c} \begin{pmatrix} \dot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \\ \dot{\alpha}_3(t) \end{pmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_2 \end{array}$$

$$* = \int_a^b F_1(\alpha_1(t), \alpha_2(t), \alpha_3 t)\, \dot{\alpha}_1(t) + F_2(\alpha_1(t), \alpha_2(t), \alpha_3(t))\, \dot{\alpha}_2(t) + F_3(\alpha_1(t), \alpha_2(t), \alpha_3(t))\, \dot{\alpha}_3(t) \, dt$$

Свойства: 1) При смяна посоката на обикаляне



$$\int_{\Gamma_{AB}} \langle F, dr \rangle = - \int_{\Gamma_{BA}} \langle F, dr \rangle$$

$$\Gamma_{AB} = \alpha([a,b]), \quad t \in [a,b]$$
$$\alpha(t)$$

$$\Gamma_{BA} = \gamma([a,b])$$
$$\gamma(\tilde{t}) = \alpha(a+b-\tilde{t})$$

$$\int_{\Gamma_{BA}} F\,dr = \int_a^b \langle F(\gamma(\tilde{t})), \dot{\gamma}(\tilde{t}) \rangle \, d\tilde{t} =$$

$$= \int_a^b \langle F(\alpha(a+b-\tilde{t})), \dot{\alpha}(a+b-\tilde{t})(-1) \rangle \, d\tilde{t} =$$

смяна $t = a+b-\tilde{t}$
$\tilde{t} = a+b-t$

$$= - \int_b^a \langle F(\alpha(t)), \dot{\alpha}(t) \rangle (-1) \, dt =$$

$$= - \int_a^b \langle F(\alpha(t)), \dot{\alpha}(t) \rangle \, dt =$$

$$= - \int_{\Gamma_{AB}} \langle F, dr \rangle$$

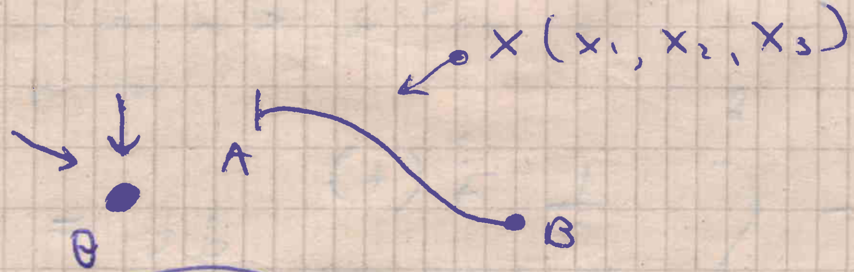
2) адитивност

$$c \in \Gamma_{AB} \Rightarrow \int_{\Gamma_{AB}} \langle F, dr \rangle =$$

$$= \int_{\Gamma_{AC}} \langle F, dr \rangle + \int_{\Gamma_{CB}} \langle F, dr \rangle$$

крив. $\int$ от II род в/у частично гладки криви

Примери: Гравитационно поле



$$-\frac{x}{\|x\|} \, \varkappa \cdot m \cdot m \cdot \frac{1}{\|x\|^2}$$

$$F(x) = -\frac{x}{\|x\|^3}$$

F-поле $F(x) = -\frac{x}{\|x\|^3}$

$\mathbb{R}^3 \setminus \{\theta\}$ - област

$\theta \notin \Gamma_{AB} = \alpha([a,b])$

$A = \alpha(a)$, $B = \alpha(b)$

$\alpha \in C^1$

$$\int_{\Gamma_{AB}} \langle F, dr \rangle = \int_a^b \langle -\frac{\alpha(t)}{\|\alpha(t)\|^3}, \dot{\alpha}(t) \rangle dt =$$

$$= -\int_a^b \frac{\langle \alpha(t), \dot{\alpha}(t) \rangle}{\|\alpha(t)\|^3} dt$$

$f(t) = \langle \alpha(t), \alpha(t) \rangle = \|\alpha(t)\|^2$

$\dot{f}(t) = \langle \dot{\alpha}(t), \alpha(t) \rangle +$

$+ \langle \alpha(t), \dot{\alpha}(t) \rangle =$

$= 2 \langle \alpha(t), \dot{\alpha}(t) \rangle$

$$-\int_a^b \frac{\frac{1}{2}\dot{f}(t)}{f(t)^{\frac{3}{2}}} dt =$$

$$= -\frac{1}{2}\int_a^b \cancel{\frac{df(t)}{f(t)^{-3}}}\, f(t)^{-3/2}\, df(t) =$$

$$= -\frac{1}{2}\left.\frac{f(t)^{-\frac{1}{2}}}{-\frac{1}{2}}\right|_a^b = \left.\frac{1}{\sqrt{\|\alpha(t)\|}}\right|_a^b = \frac{1}{\|B\|} - \frac{1}{\|A\|}$$

само от начало и край, не от пътя

произволна крива

консервативни векторни полета

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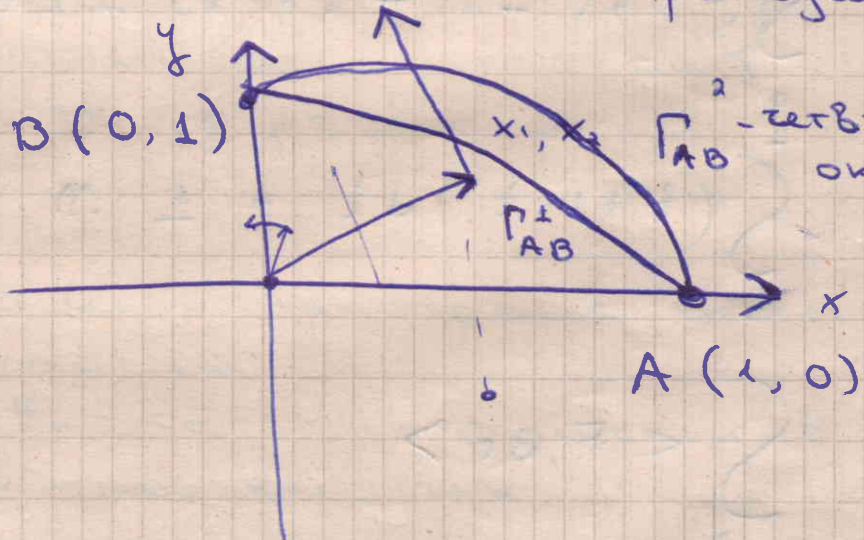
2) $\Omega = \mathbb{R}^2$

$$F(x_1, x_2) = (-x_2, x_1)$$

ротация $\frac{\pi}{2}$



① $\int_{\Gamma^1_{AB}} \langle F, dr \rangle$

$$(1,0) + t\,((0,1) - (1,0)) =$$

$$= (1,0) + t(-1,1) =$$

$$= (1-t,\ t)$$

$\alpha(t) = \begin{pmatrix} 1-t \\ t \end{pmatrix}$

$t \in [0,1]$

$$\Rightarrow \int_{\Gamma_{AB}^1} \langle F, dr \rangle = \int_0^1 \langle F(\alpha(t)), \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle dt =$$

$$\dot{\alpha}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$F(1-t, t) = (-t, 1-t)$

$$= \int_0^1 ((-t)(-1) + (1-t) \cdot 1)\, dt =$$

$$= \int_0^1 (t + 1 - t)\, dt = 1 \quad :))$$

2) $$\int_{\Gamma_{AB}^2} \langle F, dr \rangle$$

$$\alpha(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \qquad \varphi \in [0, \tfrac{\pi}{2}]$$

$$\dot{\alpha}(\varphi) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}, \quad \varphi \in [0, \tfrac{\pi}{2}]$$

$$F(\alpha(\varphi)) = F(\cos\varphi, \sin\varphi) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

$F(\ell(\varphi)) . \dot{\ell}(\varphi) = -\sin\varphi(-\sin\varphi) +$

$+ \cos\varphi\cos\varphi = 1$

$$\Rightarrow \int_{\Gamma_{AB}^{2}} \langle F, dr \rangle = \int_0^{\frac{\pi}{2}} \cancel{1} . \langle F(\ell(\varphi)), \dot{\ell}(\varphi)\rangle d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} 1 \, d\varphi = \frac{\pi}{2}$$

27.11.2013 г.

Упражнение

Криволинеен интеграл от II род

физика
- → работа на сили при преместване на мат. точка
- → разлика $E_k - E_0$ - кинетични E
- → разлика в пот. енергии

компл. анализ
- → интегрира се по крива

формула на Грин → физика и ЧДУ

Деф. Нека Γ е гладка крива с параметризация $(x(t), y(t))$, $t \in [a, b]$ и нека функциите $f(x,y)$ и $g(x,y)$ са непр. в/у Γ. Криволинеен и. в. р. на полето $F = (f(x,y), g(x,y))$ в/у Γ бележим с

$$\int_{\Gamma_в} f(x,y)\,dx + g(x,y)\,dy = \int_a^b \Big[f(x(t), y(t)) \cdot x'(t) + g(x(t), y(t)) \cdot y'(t) \Big] dt$$

$$\int_{\Gamma_в} f(x,y)\,dx + g(x,y)\,dy = \int_\Gamma \langle F, dr \rangle$$

$F = (f, g)$

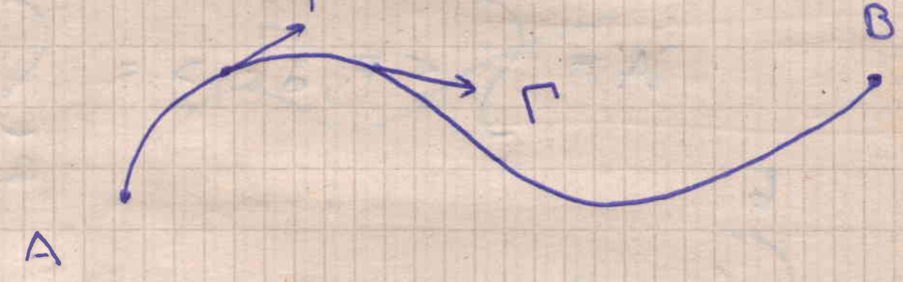
$\int_\Gamma \langle F, dr \rangle \to$ скаларното произведение на $F = \begin{pmatrix} f \\ g \end{pmatrix}$ и $dr = \begin{pmatrix} dx \\ dy \end{pmatrix}$

! В сайта на Теми ДИ ~ 9-тема

!!! Да се види !!!

Физична интерпретация:

- Мат. точка се движи под действие на сила $\overrightarrow{F} = (f, g)$ от т. A до т. B по крива Г



Работата, която извършва тази сила

$$= \int_{\Gamma} \langle F, dr \rangle$$

dr ↓ допир. в-р във ∀ точка

---

2 закон на Нютон казва

$$\overrightarrow{F} = m\overrightarrow{a}$$

$$\overrightarrow{a} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix}$$

$x(t), y(t)$ - траектория

$\dot{x}(t), \dot{y}(t)$ - моментна скорост

$(\ddot{x}(t), \ddot{y}(t))$ - ускорение

Движението се описва от

$$\begin{cases} m\ddot{x}(t) = f(x,y) \\ m\ddot{y}(t) = g(x,y) \end{cases},$$

$t \in [a,b]$

$$A = \int_{\gamma} \langle F, d\gamma \rangle = \int_a^b f(x(t),y(t))\cdot\dot{x}(t) + g(x(t),y(t))\cdot\dot{y}(t)\, dt$$

$$= \int_a^b (m\ddot{x}\dot{x} + m\ddot{y}\dot{y})\, dt =$$

$$= m\int_a^b \frac{d}{dt}\left(\frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2}\right) dt =$$

$$= \left(\frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2}\right)\Bigg|_{t=a}^{t=b} =$$

$$= E_k(B) - E_k(A)$$

$$E_k = \frac{mv^2}{2}$$

$$E_k = \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} = \frac{mv^2}{2} \quad :))$$

Свойства:

1) Адитивност. Ако $\Gamma = \Gamma_1 \cup \Gamma_2$, и Γ_1 и Γ_2 имат обща точка

$$\int_{\Gamma} \langle F, dr \rangle = \int_{\Gamma_1} \langle F, dr \rangle + \int_{\Gamma_2} \langle F, dr \rangle$$

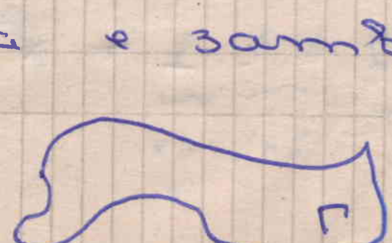
Когато кривата е негладка в краен бр. точки, я разбиваме на $k+1$ гладки криви

2) Криволинейният интеграл от II род зависи от посоката, т.е. има значение дали параметризираме кривата от т. A към т. B или от т. B към т. A. Обикновено посоката на обикаляне бележим в долен индекс $\Gamma_{\overset{\frown}{AB}}$

$$\int_{\Gamma_{AB}} \langle F, dr \rangle = - \int_{\Gamma_{BA}} \langle F, dr \rangle$$

Γ

Когато кривата е затворена



К. И. В. Р. бележки:

$$\oint_{\Gamma} \langle F, dr \rangle$$

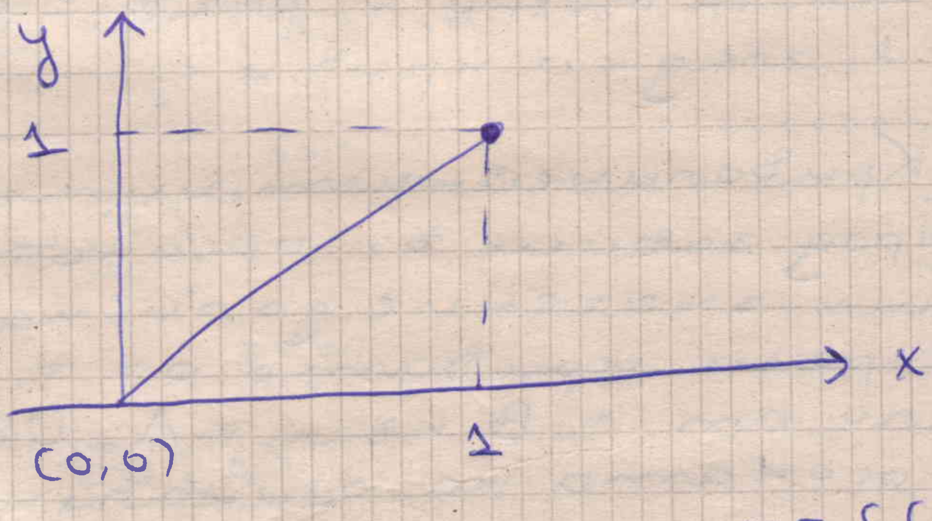
$\boxed{\text{Зад. 2.2/451}}$ Пресметнете

$$\int_L 2x\cancel{d}y\,dx + x^2 dy$$

а) $L: y = x$, $0 \le x \le 1$

б) $L: y = x^2$, $0 \le x \le 1$

в) $L: y = x^3$, $0 \le x \le 1$



$y = f(x)$

$\left| \begin{array}{l} x = t \\ y = f(\end{array} \right.$

а) $\left| \begin{array}{l} x = t \\ y = t \end{array} \right.$ $t \in [0; 1]$

$$I = \int_0^1 \Big[2 \cdot \underbrace{t \cdot t}_{2xy} \cdot \underbrace{1}_{\substack{dx \\ \shortparallel \\ x'}} + \cancel{}\, t^2 \cdot 1 \Big]\, dt =$$

$$= t^2 \Big|_0^1 + \frac{t^3}{3} \Big|_0^1 = 1 + \frac{1}{3} = \frac{4}{3}$$

$$= t^3 \Big|_0^1 = 1$$

б) $$I = \int_0^1 \left[2t \cdot t^2 \cdot 1 + t^2 \cdot 2t\right] dt =$$

$$\begin{cases} x = t \\ y = t^2 \end{cases} \qquad t \in [0, 1]$$

$$= \int_0^1 4t^3 \, dt = t^4 \Big|_0^1 = 1$$

в) $$\begin{cases} x = t \\ y = t^3 \end{cases}$$

$$I = \int_0^1 (2t^4 \cdot 1 + t^2 \cdot 3t^2) \, dt =$$

$$= \cancel{2t^5} \int_0^1 5t^4 \, dt = t^5 \Big|_0^1 = 1$$

$F(2xy, x^2)$ $\qquad$ $u_x = 2xy$

$u = x^2 y$ $\qquad$ $u_y = x^2$

$u(1,1) = 1^2 . 1 = 1$
$u(0,0) = 0 . 0 = 0$

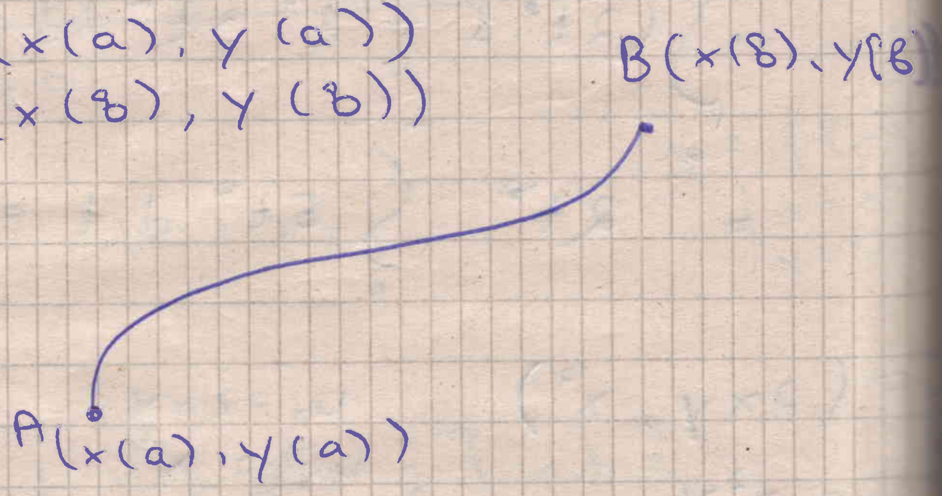
$$u(1,1) - u(0,0) = 1 - 0 = 1$$

Т8. Нека F е потенциално и Γ_{AB} е гладка крива. Тогава криволинейното интегриране не зависи от Γ, а зависи само от стойността на потенциала в последната точка и в първата точка, т.е.

$$\int_{\Gamma_{AB}} \langle F, dr \rangle = u(B) - u(A)$$

Д-во: $\Gamma: \begin{matrix} x(t) \\ y(t) \end{matrix} \quad t \in \{a, b\}$

$A(x(a), y(a))$
$B(x(b), y(b))$



$$u(x,y) \quad F=(F_1,F_2)$$
$$u'_x = F_1$$
$$u'_y = F_2$$

$$I = \int_a^b F_1 x' + F_2 \cdot y' =$$

$$= \int_a^b \Big(u'_x(x(t),y(t)) \cdot x'(t) + u'_y(x(t),y(t)) \cdot y'(t)\Big) dt =$$

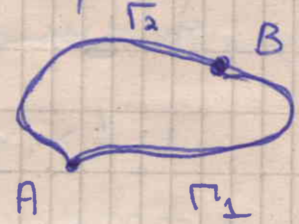
→ Верижно правило chain rule

$$= \int_a^b \frac{d}{dt}\big(u(x(t),y(t))\big) =$$

$$= u(x(b),y(b)) - u(x(a),y(a)) =$$
$$= u(B) - u(A)$$

<u>Следствие</u>: Ако F е потенциално, то крив. инт. в/у затв. крива $= 0$

$$\oint_\Gamma \langle F, dr \rangle = 0$$



$$\oint_\Gamma = \underbrace{\int_{\Gamma^1_{AB}} + \int_{\Gamma^2_{BA}}}_{\substack{u(B)-u(A)+\\ +u(A)-u(B) \\ = 0}}$$

2.22 а) $\int_L 3x^2y^2\,dx + 2x^3y\,dy$

$L: x = e^{\sqrt{t}}$

$y = \sin^3 \frac{\pi t}{2}$

$t \in [0, 1]$

$\begin{cases} u_x = 3x^2y^2 \\ u_y = 2x^3y \end{cases}$ НУ:

$$(3x^2y^2)'_y = (2x^3y)'_x$$

$$\parallel$$

$$6x^2y \overset{?}{=} 6x^2y$$

ф-иите са безкрайно гладки

$u(x,y) = x^3y^2$

$I = u(x(1), y(1)) - u(x(0), y(0)) =$

$= u(e^{\sqrt{1}}, \sin^3 \frac{\pi}{2}) - u(e^{\sqrt{0}}, \sin^3 0) =$

$= u(e, 1) - u(1, 0) =$

$= e^3 - 0 = e^3$:))

Когато имаме К.И.В.Р първо проверяваме дали полето е потенциално, а чак след това пресметаме по стандартния начин, ако е необходимо.
Когато има потенциал

$$\int_{\Gamma} \langle F, dr \rangle = u(B) - u(A) =$$
$$= E_k(B) - E_k(A)$$

кин. пот.

$$\Rightarrow E_k(B) + (-u(B)) =$$
$$= E_k(A) + (-u(A))$$

кин. пот.

2.19 а) Пресметнете

$$\int_L x\,dx + y\,dy + (x+y-1)\,dz,$$

където L е отсечката, свързваща $(1,1,1)$ и $(2,3,4)$

не е потенциално - НУ

$x' = 1$ $\quad x(t) = 1 + t.(2-1) = 1+t \quad \overrightarrow{AB}(1,2,3)$

$y' = 2$ $\quad y(t) = 1 + t.(3-1) = 1+2t$

$z' = 3$ $\quad z(t) = 1 + t.(4-1) = 1+3t$

$t \in [0,1]$

$$I = \int_0^1 \left((1+t)\cdot 1 + (1+2t)\cdot 2 + (1+3t)\cdot 3 \right) dt =$$

$$= \int_0^1 (1+t+2+4t+3+9t)\,dt =$$

$$= \int_0^1 6+14t\,dt =$$

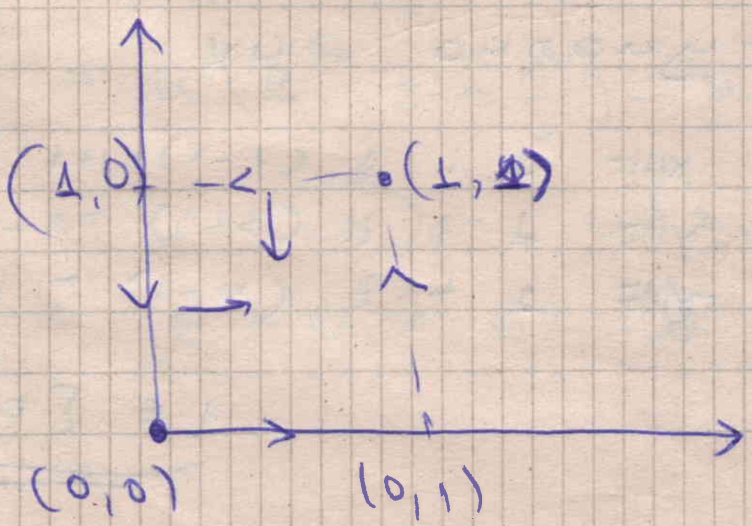
$$= (6t+7t^2)\Big|_0^1 =$$

$$= 13$$

$\boxed{3.22\ 3)}$ Пресметнете

$$\oint_{C_2} \cos x \sin y\,dx + \sin x \cos y\,dy$$

C_2 е едини. кв. със срещуположни върхове $(0,0)$ и $(1,1)$



Щом не е дад. посоката на обикаляне, стандартно се приема обратна на часовниковата стрелка, която се нарича правилна.
(наляво ни е вътрешността)

?. потенциално
$$u_{(x,y)} = \sin x \sin y \text{ е п.-л.} \Rightarrow I = 0$$

2.19б)
$$\int yz\,dx + zx\,dy + xy\,dz$$

$$L: x = a\cos t$$
$$y = a\sin t$$
$$z = \frac{bt}{2\pi} \qquad t \in [0; 2\pi]$$

$u(x,y,z) = xyz$ е потенциал

$$u(a,0,b) - u(a,0,0) = 0$$

Формула на Грин

Th (Грин) Нека L е частично гладка проста затв. крива. Нека P, Q са непр. диф. в $D \cup L$, където D е вътр. на L.
Тогава

$$\oint_L P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Пример

$$\oint_{\partial C_1} x^2\,dx + y^2\,dy \qquad C_1 - \text{ед. окр.}$$

$$\shortparallel$$

$$\iint_{\langle C_1 \rangle} \left(\frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial y} \right) dx\,dy = \iint 0\,dx\,dy$$

$$\shortparallel$$

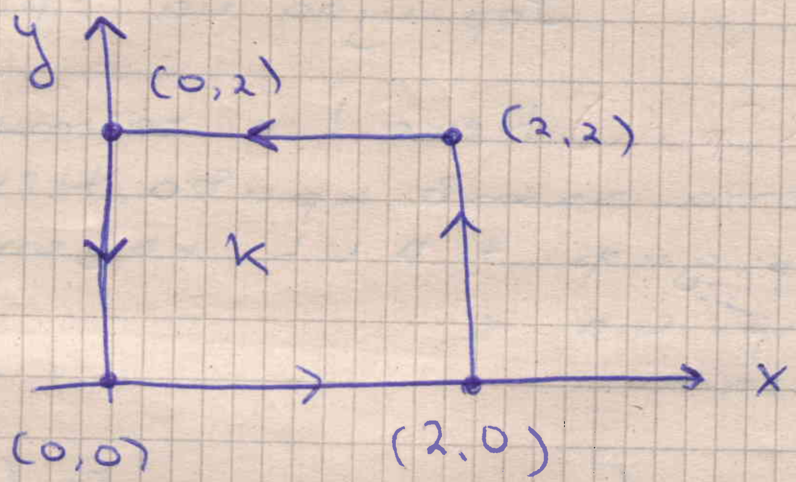
$$0$$

Зад. Пресметнете

$$\oint_L \overset{P}{y^2}\,dx + \overset{Q}{x}\,dy, \quad L \text{ е квадрат}$$

с върхове

(0, 0)
(2, 0)
(2, 2)
(0, 2)



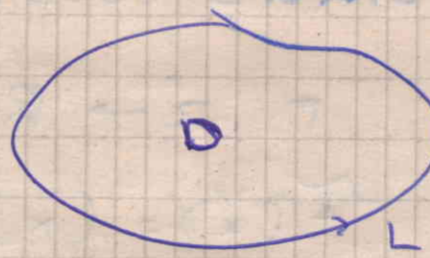
$$\oint_L y^2 dx + x\,dy = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =$$

$$= \iint_K (1 - 2y)\,dx\,dy$$

$$= \int_0^2 \left(\int_0^2 (1 - 2y)\,dy \right) dx =$$

$$= \int_0^2 (y-y^2)\Big|_0^2 dx = \int_0^2 (2-4)\,dx = -4$$

Зад. Нека L е частично гладка проста затв. крива, която загражда множество D.



Док., че

$$\boxed{S_D = \frac{1}{2}\oint x\,dy - y\,dx}$$

Д-во: $\frac{1}{2}\left(\oint x\,dy - y\,dx\right) =$

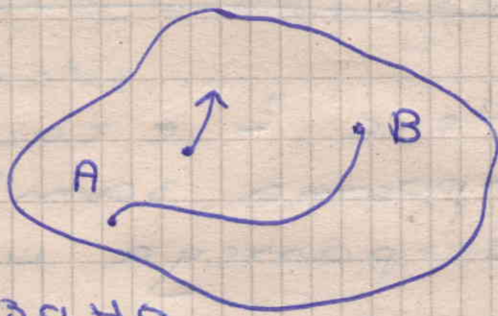
$$= \frac{1}{2}\oint \left(\underbrace{-y}_{P}\,dx + \underbrace{x}_{Q}\,dy\right) =$$

$$= \frac{1}{2}\iint_D \left(\cancel{dx}\ \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) =$$

$$= \frac{1}{2}\iint_D 2\,dx\,dy = \iint_D dx\,dy = S(D)$$

Лекция

$\Omega \subset \mathbb{R}^n$
$(n = 2, 3)$



<u>област</u> - отв. и свързано

$$F : \Omega \to \mathbb{R}^n$$

$$\mathcal{F}(x) = (x ; F(x))$$

$\alpha : [a, b] \to \Omega$

$\alpha \in C^1$

$\Gamma_{AB} = \alpha([a, b])$

$A = \alpha(a)$ $\qquad\qquad \alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \end{pmatrix}$

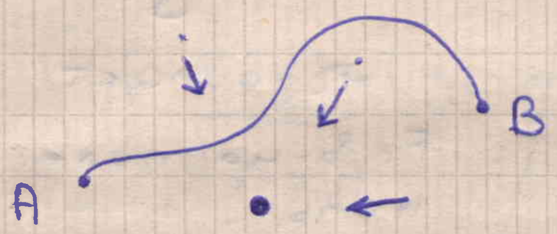
$B = \alpha(b)$

$$\int_{\Gamma_{AB}} \langle F, dr \rangle = \int_a^b \langle F(\alpha(t)), \dot{\alpha}(t) \rangle \, dt$$

$$\int_{\Gamma_{AB}} F_1 dx_1 + F_2 dx_2 + F_3 dx_3 + \dots =$$

$$= \int_a^b F_1(\alpha(t)) \dot{\alpha}_1(t)$$
$$+ F_2(\alpha(t)) \cdot \dot{\alpha}_2(t) +$$

$$+ F_3(\alpha(t)) \cdot \dot{\alpha}_3(t))\, dt$$



$$F(x) = -\frac{x}{\|x\|^3} \quad \text{в} \quad \mathbb{R}^3$$

гравитационно поле

$$\Omega = \mathbb{R}^3 \setminus \{0\}$$

$$\int_{\Gamma_{AB}} \langle F, dr \rangle = \frac{1}{\|B\|} - \frac{1}{\|A\|}$$

Д

$$u(x) = \frac{1}{\|x\|} = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\frac{\partial u}{\partial x_1} = -\frac{1}{2} (x_1^2 + x_2^2 + x_3^2)^{-\frac{3}{2}} \cdot 2x_1 = \frac{-x_1}{\|x\|^3}$$

$$\frac{\partial u}{\partial x_2} = -\frac{x_2}{\|x\|^3}$$

$$\frac{\partial u}{\partial x_3} = -\frac{x_3}{\|x\|^3}$$

$F \equiv \operatorname{grad} u$

Дом.: в ~~едно~~ област

ако има потенц, той е единствен с точност до const

Ω - област

F в-но поле в Ω

непр.

u_1, u_2 - пот F в Ω

$\Rightarrow u_1 - u_2 \equiv const$

Деф. $\Omega \subset \mathbb{R}^n$ област

$F : \Omega \to \mathbb{R}^n$

векторно поле

Казваме, че $u : \Omega \to \mathbb{R}$ е потенциал за F, ако $F \equiv \operatorname{grad} u$, т.е.

$$F_1 = \frac{\partial u}{\partial x_1} \quad F_2 = \frac{\partial u}{\partial x_2}, \ldots \quad F_n = \frac{\partial u}{\partial x_n}$$

Полето F се нарича потенциално, когато $\exists$ $u : \Omega \to \mathbb{R}$ и е ~~потенциал за F в Ω.~~

изт.

компакт $K \subset \mathbb{R}^3$

плътност

$\rho : K \to [0, +\infty)$

непр.

$\mathbb{R}$

$\Omega = \mathbb{R}^3 \setminus K$

$$u(x) = \iiint\limits_K \frac{\rho(y)}{\|y - x\|} \, dy$$

$$\frac{d}{d\alpha} \int_a^b f(x, \alpha) \, dx = \int_a^b f'_\alpha (x, \alpha) \, dx$$

x

$() \ k ()$

$$\|y-x\| = \left((y_1-x_1)^2 + (y_2-x_2)^2 + (y_3-x_3)^2\right)^{\frac{1}{2}}$$

$$\frac{\partial u}{\partial x_1}(x) = \iiint_K P(y) \cdot \frac{1}{2} \cdot \|y-x\|^{-3} \cdot 2(y_1-x_1)(-1)\,dy$$

$$= - \iiint_K \frac{P(y)}{\|y-x\|^3}(y_1-x_1)\,dy$$

$$\operatorname{grad} u(x) = - \iiint_K \frac{y-x}{\|y-x\|^3} P(y)\,dy$$

Th $\Omega \subset \mathbb{R}^n$, Ω област

$F: \Omega \to \mathbb{R}^n$

непрекъснато в-но поле в Ω

Тогава F е потенциално в-но поле $\iff \int_{\Gamma_{AB}} \langle F, dr \rangle$ не зависи от кривата

$\Gamma_{AB} \subset \Omega$, а само от A и B.

$\Rightarrow$) $F \equiv \operatorname{grad} u$

$u: \Omega \to \mathbb{R}$

$\Gamma_{AB} \subset \Omega$

$\Gamma_{AB} = \alpha([a,b])$, $\alpha: [a,b] \to \Omega$

$\alpha \in C^1$

$\alpha(a) = A$

$\alpha(b) = B$

$$\int_{\Gamma_{AB}} \langle F, dr \rangle =$$

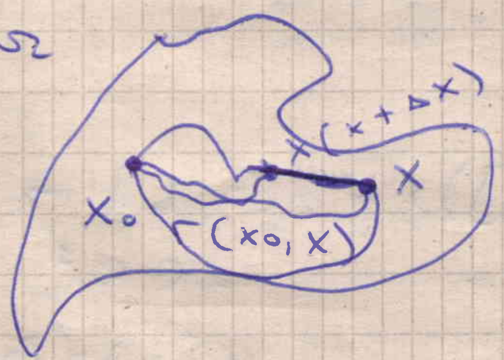
$$= \int_a^b \Big(F_1(\alpha(t)).\dot\alpha_1(t) + F_2(\alpha(t)).\dot\alpha_2(t) + F_3((\alpha(t)).\dot\alpha_3(t)\Big)\,dt =$$

$$= \int_a^b \Big(\frac{\partial u}{\partial x_1}(\alpha_1(t), \alpha_2(t), \alpha_3(t)).\dot\alpha_1(t) + \frac{\partial u}{\partial x_2}(\alpha_1(t), \alpha_2(t), \alpha_3(t)).\dot\alpha_2(t) + \frac{\partial u}{\partial x_3}(\alpha_1(t), \alpha_2(t), \alpha_3(t)).\dot\alpha_3(t)\Big)\,dt =$$

$$= \int_a^b \frac{d}{dt}\Big(u(\alpha_1(t), \alpha_2(t), \alpha_3(t))\Big)\,dt =$$

$$= u(\alpha(t))\Big|_a^b = u(\alpha(b)) - u(\alpha(a)) = u(B) - u(A)$$

$\Leftarrow$



частично гладка не излиза от Ω

$\Gamma_{x_0 x} \subset \Omega$

част. гл.

$$u(x) = \int_{\Gamma_{x_0 x}} \langle F, dr \rangle$$

деф-ята е коректна от това, че интегралът не зависи от пътя.

$$\Delta x(\Delta x_1, 0, 0)$$

$$\frac{\partial u}{\partial x_1}(x) = \lim_{\Delta x_1 \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x_1} =$$

$$= \lim \frac{\int_{\Gamma_{x_0 x}} \langle F, dr \rangle + \int_{\substack{L_{x, x+\Delta x} \\ \text{отсечка}}} \langle F, dr \rangle - \int_{\Gamma_{x_0 x}} \langle F, dr \rangle}{\Delta x_1} =$$

парам. отсечката

$L_{x, x+\Delta x}$:

$$\alpha(t) = x + t\Delta x =$$

$$= (x_1 + t\Delta x_1, x_2, x_3)$$

$$t \in [0, 1]$$

$$\dot{\alpha}(t) = (\Delta x_1, 0, 0) = \Delta x$$

$$F(\alpha(t)) \cdot \dot{\alpha}(t) = F_1(x_1 + t\Delta x_1, x_2, x_3)\Delta x_1 + F_2(\ )0 + F_3(\ )0$$

Th
ср. ст-ти
F_1-непр.

$$= \lim_{\Delta x_1 \to 0} \frac{\int_0^1 F_1(x_1 + t\Delta x_1, x_2, x_3) \cancel{\Delta x_1}\, dt}{\cancel{\Delta x_1}} =$$

$$= \lim_{\Delta x_1 \to 0} F_1(x_1 + \theta \Delta x_1, x_2, x_3)$$

$$0 \leq \theta_{\Delta x_1} \leq 1$$

! F_1 - непрекъсната

$$= F_1(x_1, x_2, x_3) = F_1(x)$$

F_1 непр.

Аналогично $\frac{\partial u}{\partial x_2}(x) = F_2(x)$

$$\frac{\partial u}{\partial x_3}(x) = F_3(x)$$

Th $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$

$F : \Omega \to \mathbb{R}^n$

непр. векторно поле

TFAE:

(а) F потенциално поле

(б) $\int_{\Gamma_{AB}} \langle F, dr \rangle$, $\Gamma_{AB} \subset \Omega$ част. гладка не зависи от Γ_{AB}, а само от A и B

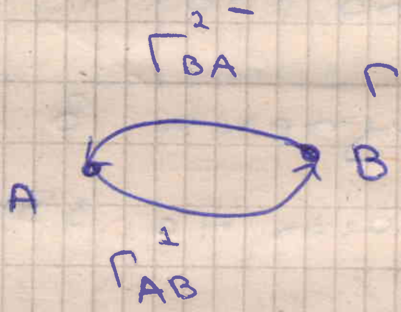
(в) $\int_{\Gamma} \langle F, dr \rangle = 0 \quad \forall\, \Gamma$ затв. в Ω

(г) $\int_{L_{AB}} \langle F, dr \rangle$, $L_{AB} \subset \Omega$ не
начупена
зависи от L_{AB},
а само от A и B

(д) $\int_{L} \langle F, dr \rangle = 0 \quad \forall\, L$ затв. нач. в Ω

е) $\int_{L} \langle F, dr \rangle = 0 \quad \forall\, L$ проста затв.
(несе
самопресич.) нач. в Ω

(а) $\Leftrightarrow$ (б)
(б) $\Leftrightarrow$ (в)



$$\int_{\Gamma^1_{AB}} \langle F, dr \rangle = \int_{-\Gamma^2_{BA}} \langle F, dr \rangle = -\int_{\Gamma^2_{BA}} \langle F, dr \rangle$$

$$\int_{\Gamma^1_{AB}} + \int_{\Gamma^2_{BA}} = 0$$

$$\Gamma = \Gamma^1_{AB} + \Gamma^2_{BA}$$

Th
ср. ст-ти
F_1-непр.

$$= \lim_{\Delta x_1 \to 0} \frac{\int_0^1 F_1(x_1 + t\Delta x_1, x_2, x_3) \cancel{\Delta x_1}\, dt}{\cancel{\Delta x_1}} =$$

$$= \lim_{\Delta x_1 \to 0} F_1(x_1 + \theta \Delta x_1, x_2, x_3)$$

$$0 \leq \theta_{\Delta x_1} \leq 1$$

! F_1- непрекъсната

$$= F_1(x_1, x_2, x_3) = F_1(x)$$

F_1 непр.

Аналогично $\dfrac{\partial u}{\partial x_2}(x) = F_2(x)$

$$\frac{\partial u}{\partial x_3}(x) = F_3(x)$$

Th $\Omega \subset \mathbb{R}^n \ (n = 2, 3)$

$F : \Omega \to \mathbb{R}^n$

непр. векторно поле

TFAE:

(а) F потенциално поле

(б) $\int_{\Gamma_{AB}} \langle F, dr \rangle$, $\Gamma_{AB} \subset \Omega$ част. гладка не зависи

от Γ_{AB}, а само от A и B

(в) $\int_{\Gamma} \langle F, dr \rangle = 0 \quad \forall\, \Gamma$ затв. в Ω

(г) $\int_{L_{AB}} \langle F, dr \rangle$, $L_{AB} \subset \Omega$ не начупена

зависи от L_{AB},

а само от A и B

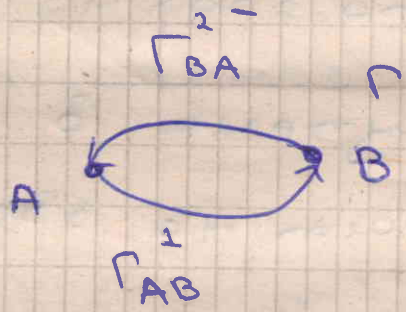
(д) $\int_{L} \langle F, dr \rangle = 0 \quad \forall\, L$ затв. нач. в Ω

(е) $\int_{L} \langle F, dr \rangle = 0 \quad \forall\, L$ проста затв.

(несе самопресич.) нач. в Ω

(а) $\Leftrightarrow$ (б)

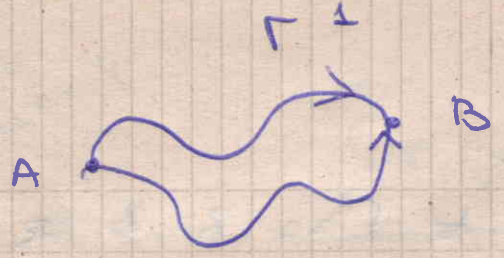
(б) $\Leftrightarrow$ (в)



$$\int_{\Gamma^1_{AB}} \langle F, dr \rangle = \int_{-\Gamma^2_{BA}} \langle F, dr \rangle = -\int_{\Gamma^2_{BA}} \langle F, dr \rangle$$

$$\int_{\Gamma^1_{AB}} + \int_{\Gamma^2_{BA}} = 0$$

$\Gamma = \Gamma_{AB} + \Gamma^2_{BA}$



(б) => (г)

(г) => (а)

$$u(x) = \int_{L_{x_0 x}} \langle F, dr \rangle$$

нач + отс.

(г) (=> (д)   адит + знаци

(д) => (е) о.к. :))

? (е) => (д)

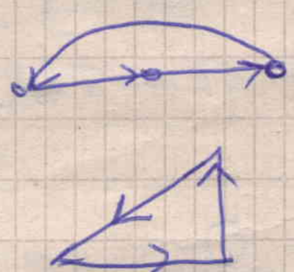
Индукция по броя на звената

L - затв. начупена в Ω

$n = |L|$ брой на звената на L

$n = 2$

$n = 3$



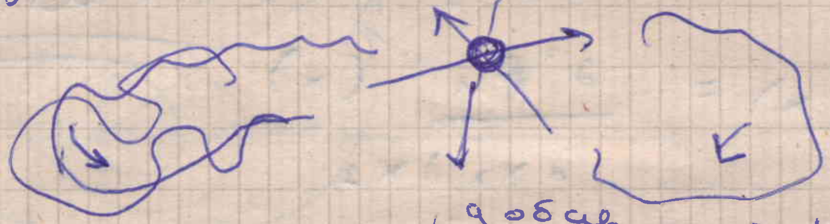
Доп., че $\forall$ затв. начупена с не повече от n звена $\int = 0$

n+1 звена -> ако е проста о.к.

Ако не е $\rightarrow$ има точка на самопресичане

n звена

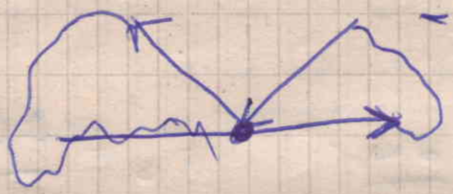
$\vee|$



$\leq n$ звена

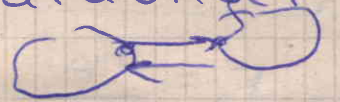
добавяме т-на пресичане

2 затв. начупени



$\rightarrow \int = 0$:))

адитивност



НУ за потенциалност

$\Omega \subset \mathbb{R}^n$

област

$F \in C^1(\Omega, \mathbb{R}^n)$

гладко

Ако F е потенциално, значи същ.

$u : \Omega \rightarrow \mathbb{R}$

$$F_i = \frac{\partial u}{\partial x_i} \quad \text{в } \Omega$$

$n = 2$

$$F_1(x) = \frac{\partial u}{\partial x_1}(x) \qquad F_2(x) = \frac{\partial u}{\partial x_2}(x)$$

$x \in \Omega$

$$\frac{\partial F_1}{\partial x_2}(x) = \frac{\partial^2 u}{\partial x_2 \partial x_1}(x)$$

$$\|$$ **Th Шварц**

$$\frac{\partial F_2}{\partial x_1}(x) = \frac{\partial^2 u}{\partial x_2 \partial x_1}(x)$$

F - пот. и гладко в $\Omega \subset \mathbb{R}^2 \Rightarrow \boxed{\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}}$

в Ω

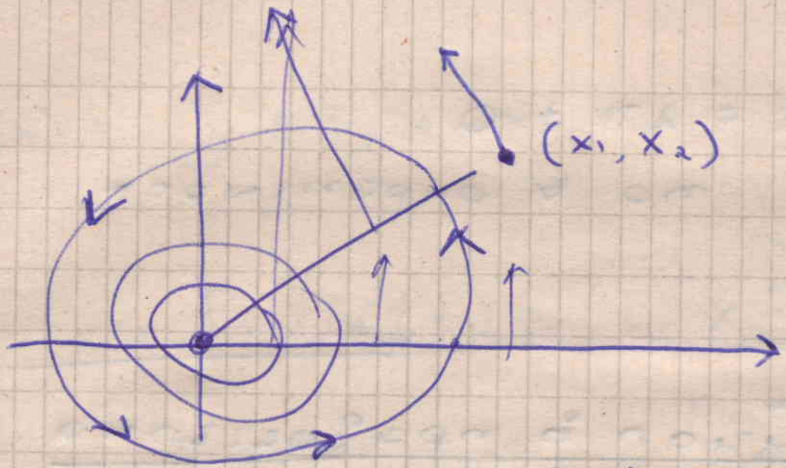
$n = 3$

$$F_1 = \frac{\partial u}{\partial x_1} \qquad F_2 = \frac{\partial u}{\partial x_2} \qquad F_3 = \frac{\partial u}{\partial x_3}$$

$$\boxed{\begin{aligned} &\frac{\partial F_1}{\partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1} = \frac{\partial F_2}{\partial x_1} \\ &\frac{\partial F_1}{\partial x_3} = \frac{\partial F_3}{\partial x_1} = \frac{\partial^2 u}{\partial x_1 \partial x_3} \\ &\frac{\partial F_2}{\partial x_3} = \frac{\partial F_3}{\partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_3} \end{aligned}}$$

Пример: $F(x_1, x_2) = \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right)$

$$\mathbb{R}^2 \setminus \{0\} = \Omega$$



$$\|F(x)\| = \frac{1}{\|x\|}$$

$$F'_{1 x_2} = \frac{-1(x_1^2+x_2^2) - (-x_2)\cdot 2x_2}{(x_1^2+x_2^2)^2} =$$

$$= \frac{-x_1^2 - x_2^2 + 2x_2^2}{(x_1^2+x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2}$$

$$F'_{2 x_1} = \frac{(x_1^2+x_2^2) - x_1 \cdot 2x_1}{(x_1^2+x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2}$$

НУ е изп. $\quad \dot{\alpha} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix}$

Окръжност

$G \qquad \alpha(\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} \quad \varphi \in \{0, 2\pi\}$

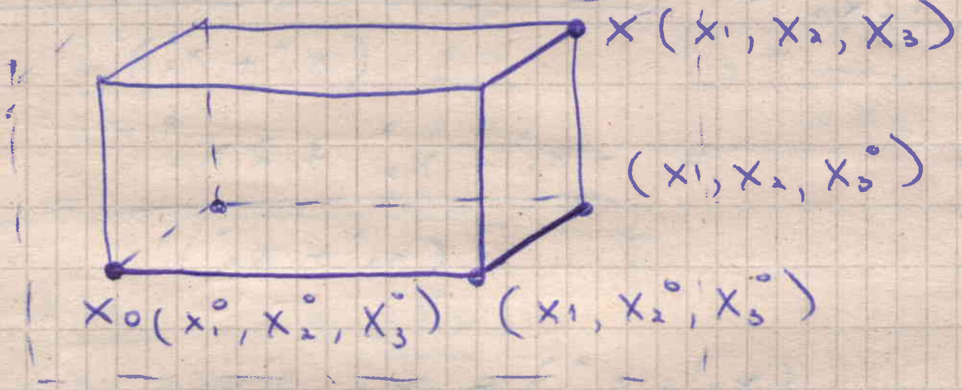
$$\int_G \langle F, dr \rangle = \int_0^{2\pi} -\frac{\cancel{r}\sin\varphi}{r^{\cancel{2}}}\cdot(-r\sin\varphi) + \frac{\cancel{r}\cos\varphi}{\cancel{r^2}}\cdot r\cos\varphi) \, d\varphi$$

$$= \int_0^{2\pi} 1 \, d\varphi = 2\pi \neq 0$$

по $\forall$ окръжност

Потенциал в правоъгълна област

$$\Omega \equiv (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \subset \mathbb{R}^3$$



$x_0 = (x_1^0, x_2^0, x_3^0) \in \Omega$

$x = (x_1, x_2, x_3) \in \Omega$

F гладко в-но поле в Ω и

$$\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$$

$$\frac{\partial F_1}{\partial x_3} = \frac{\partial F_3}{\partial x_1}$$

$$\frac{\partial F_2}{\partial x_3} = \frac{\partial F_3}{\partial x_2} \quad \text{в } \Omega$$

$$u(x) = \int_{L_{x_0 x}} \langle F, dr \rangle$$

параметр. на начупената

$$\Gamma_1 : \alpha(t) = (t, x_2^0, x_3^0)$$

t от x_1^0 до x_1

$$\Gamma_2 : \beta(t) = (x_1, t, x_3^0)$$

t от x_2^0 до x_2

$$\Gamma_3 : \gamma(t) = (x_1, x_2, t)$$

t от x_3^0 до x_3

$$L_{x_0 x} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

$\dot{\alpha}(t) = (1, 0, 0)$

$$u(x) = \int_{L_{x_0 x}} \langle F, dr \rangle = \int_{x_1^0}^{x_1} F_1(t, x_2^0, x_3^0)\, dt$$

$$+ \int_{x_2^0}^{x_2} F_2(x_1^0, t, x_3^0)\, dt +$$

$$+ \int_{x_3^0}^{x_3} F_3(x_1^0, x_2^0, t)\, dt$$

$$\frac{\partial u}{\partial x_1}(x) = F_1(x_1, x_2^0, x_3^0)$$

$$+\int_{x_2^0}^{x_2} \cancel{F_2}\, \frac{\partial F_2}{\partial x_1}(x_1, t, x_3^0)\, dt \;+$$

$$+\int_{x_3^0}^{x_3} \frac{\partial F_3}{\partial x_1}(x_1, x_2, t)\, dt =$$ примитивна $F_1(x_1, .., x_3^0)$

$$= F_1(x_1, x_2^0, x_3^0) + \int_{x_2^0}^{x_2} \frac{\partial F_1}{\partial x_2}(x_1, t, x_3^0)\, dt +$$

$$+ \int_{x_3^0}^{x_3} \frac{\partial F_1}{\partial x_3}(x_1, x_2, t)\, dt =$$

$$= F_1(x_1, x_2^0, x_3^0) + F_1(x_1, \overset{t}{\cancel{x_2}}, x_3^0)\Big|_{x_2^0}^{x_2}$$

$$+ F_1(x_1, x_2, \overset{t}{\cancel{x_3}})\Big|_{x_3^0}^{x_3} =$$

$$\frac{\partial u}{\partial x_1}(x) = F_1(x_1, x_2^0, x_3^0) + \underline{F_1(x_1, x_2, x_3^0)}$$

$$- F_1(x_1, x_2^0, x_3^0) +$$

$$+ F_1(x_1, x_2, x_3) - \overline{F_1(x_1, x_2, x_3^0)} =$$

$$= F_1(x_1, x_2, x_3) = F_1(x)$$

$$\frac{\partial u}{\partial x_2}(x) = F_2(x_1, x_2, x_3^0) + \int_{x_3^0}^{x_3} \frac{\partial F_3}{\partial x_2}(x_1, x_2, t)\, dt$$

$$\frac{\partial F_3}{\partial x_2} \;\text{"}\; \frac{\partial F_2}{\partial x_3} \text{ ЯЧ}$$

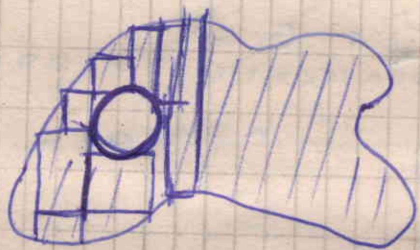
$$= F_2(x, x_2, x_3^0) + F_2(x_1, x_2, t)\Big|_{x_3^0}^{x_3} =$$

$$= F_2(x, x_2, x_3^0) + F_2(x_1, x_2, x_3) -$$

$$- F_2(x_1, x_2, x_3^0) =$$

$$= F_2(x_1, x_2, x_3) = F_2(x).$$

$$\frac{\partial u}{\partial x_3} = F_3(x_1, x_2, x_3) = F_3(x) \quad :))$$



Th на Жордан

Γ непрекъсната проста затв. крива в $\mathbb{R}^2$

$\alpha : [a, b] \to \mathbb{R}^2$

непр.

$\alpha(a) = \alpha(b)$

няма самопресичане

$\Rightarrow \mathbb{R}^2 = \Omega_\Gamma \cup \Omega_\Gamma' \cup \Gamma$

$\Omega_\Gamma, \Omega_\Gamma'^{\circ}$ - области

Ω_Γ - огр. , Ω'_Γ неогр.

$$\Gamma = \partial\Omega_\Gamma = \partial\Omega'_\Gamma$$

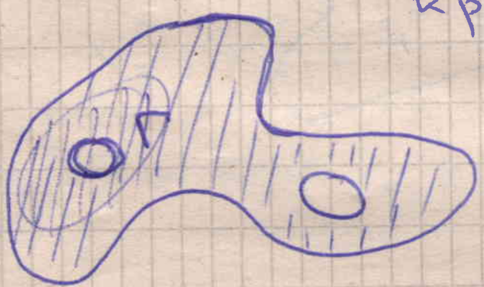
Деф. $\Omega \subset \mathbb{R}^2$, Ω област

Ω се нарича едносвързана (няма дупки), ако за всяка нетр. проста затв. крива $\Gamma \subset \Omega$ е вярно, че

$\Omega_\Gamma \subset \Omega$

областта, заградена от кривата

Ω



Th Нека Ω е едносвързана област в равнината в $\mathbb{R}^2$ и F е гладко в-но поле в Ω.

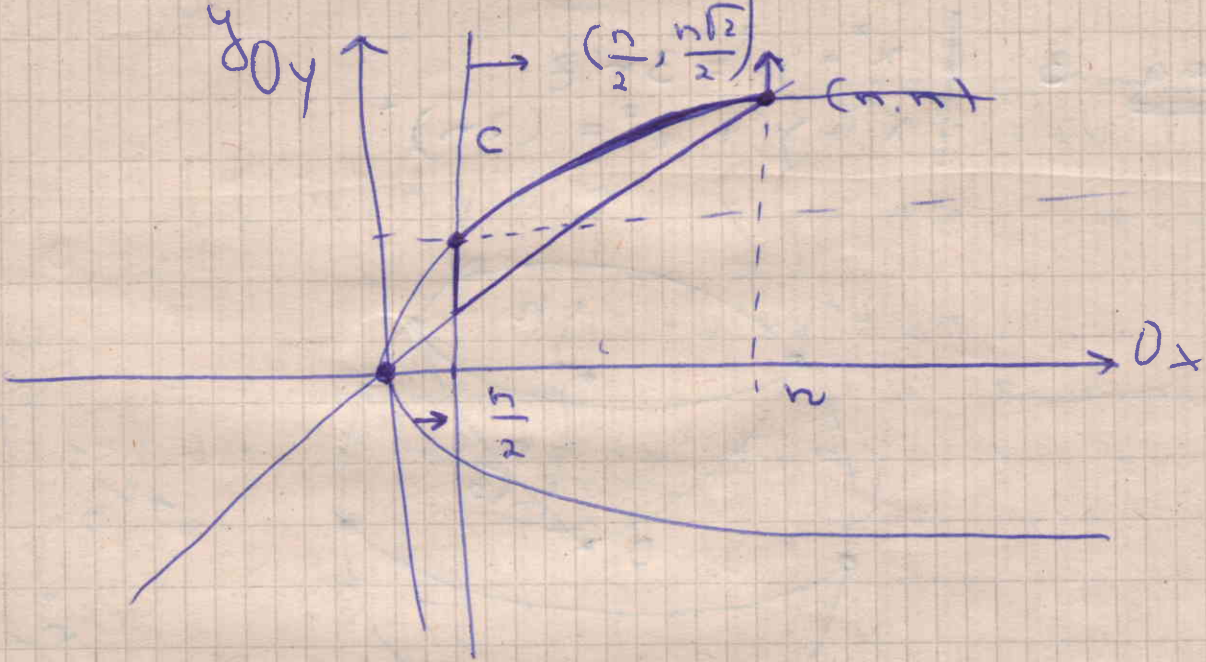
Ако $\dfrac{\partial F_1}{\partial x_2} = \dfrac{\partial F_2}{\partial x_1}$ в Ω, то F е потенциално.

04.12.2013г.

Упражнение

От Домашното:

) $D = \{ (x,y) \in \mathbb{R}^2 \mid y^2 \leq xn \text{ } \& \text{ } \frac{n}{2} \leq x \leq y \}$



$x = n$

$y^2 = n^2 \quad y = \pm n$

$C: \left| \begin{array}{l} x = \frac{n}{2} \\ y^2 = xn \end{array} \right.$ $\quad y^2 = \frac{n \cdot n}{2}$

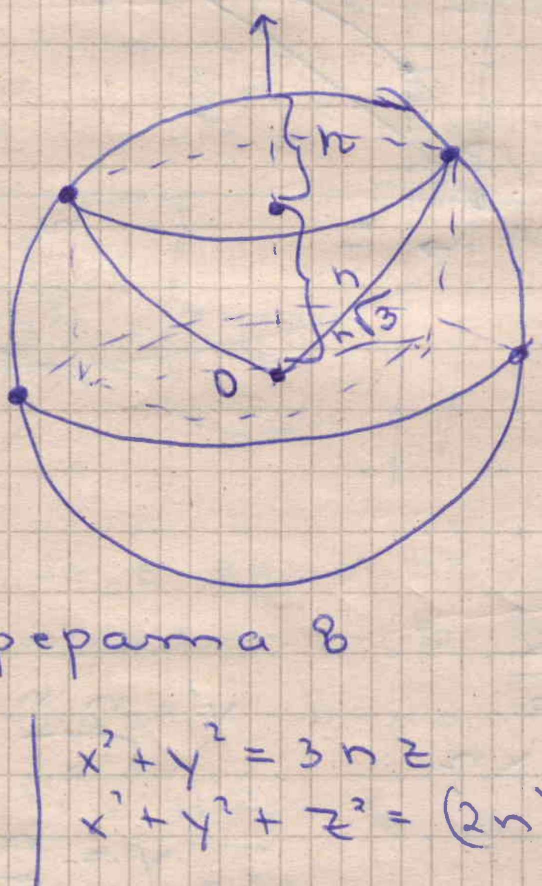
$y = \frac{n\sqrt{2}}{2}$

$D = \left\{ (x,y) \;\middle|\; \begin{array}{l} \frac{n}{2} \leq x \leq n \\ x \leq y \leq \sqrt{nx} \end{array} \right.$

$$D_1 = \left\{ (x, y) \;\middle|\; \begin{array}{l} \frac{r}{2} \le y \le \frac{r\sqrt{2}}{2} \\ \frac{r}{2} \le x \le y \end{array} \right\}$$

$$D_2 = \left\{ (x, y) \;\middle|\; \begin{array}{l} \frac{r\sqrt{2}}{2} \le y \le r \\ \sqrt{r^2 - y^2} \le x \le y \end{array} \right\} \qquad D = D_1 \cup D_2$$

Зад. 3 $\left| \begin{array}{l} x^2 + y^2 = 3rz \\ x^2 + y^2 + z^2 = (2r)^2 \end{array} \right.$



Пар. сече сферата в

$$\left| \begin{array}{l} x^2 + y^2 = 3rz \\ x^2 + y^2 + z^2 = (2r)^2 \end{array} \right.$$

$$z^2 + 3rz - 4r^2 = 0$$

$$z^2 + 3rz - 3r^2 - r^2 = 0$$

$z = r$

$z = -4r$ не

$z \geq 0$ $\quad (z - r)(z + r) + 3r(z - r) = 0$

$$(z - r)(z + 4r) = 0$$

$$\Rightarrow \left| \begin{array}{l} z = r \\ x^2 + y^2 = 3r^2 \end{array} \right.$$

По-малкото тяло е цилиндрично

$$T = \left\{ (x, y, z) \;\middle|\; \begin{array}{l} x^2+y^2 \le 3r^2 \\ \dfrac{x^2+y^2}{3r} \le z \le \sqrt{4r^2-(x^2+y^2)} \end{array} \right.$$

$$V_T = \iiint\limits_T dx\,dy\,dz = \iint\limits_{x^2+y^2 \le 3r^2} \int^{\sqrt{4r^2-(x^2+y^2)}} dz \; dx\,dy =$$

Пол. смяна

$$= \iint\limits_{x^2+y^2 \le 3r^2} \sqrt{4r^2-(x^2+y^2)} - \frac{x^2+y^2}{3r} \, dx\,dy =$$

$$= \int_0^{2\pi} \int_0^{r\sqrt{3}} \rho \left(\sqrt{4r^2-\rho^2} - \frac{\rho^2}{3r} \right) d\rho \, d\varphi =$$

$$= 2\pi \left[-\frac{1}{2} \int_0^{r\sqrt{3}} \sqrt{4r^2-\rho^2} \, d(4r^2-\rho^2) - \int_0^{r\sqrt{3}} \frac{\rho^3}{3r} d\rho \right]$$

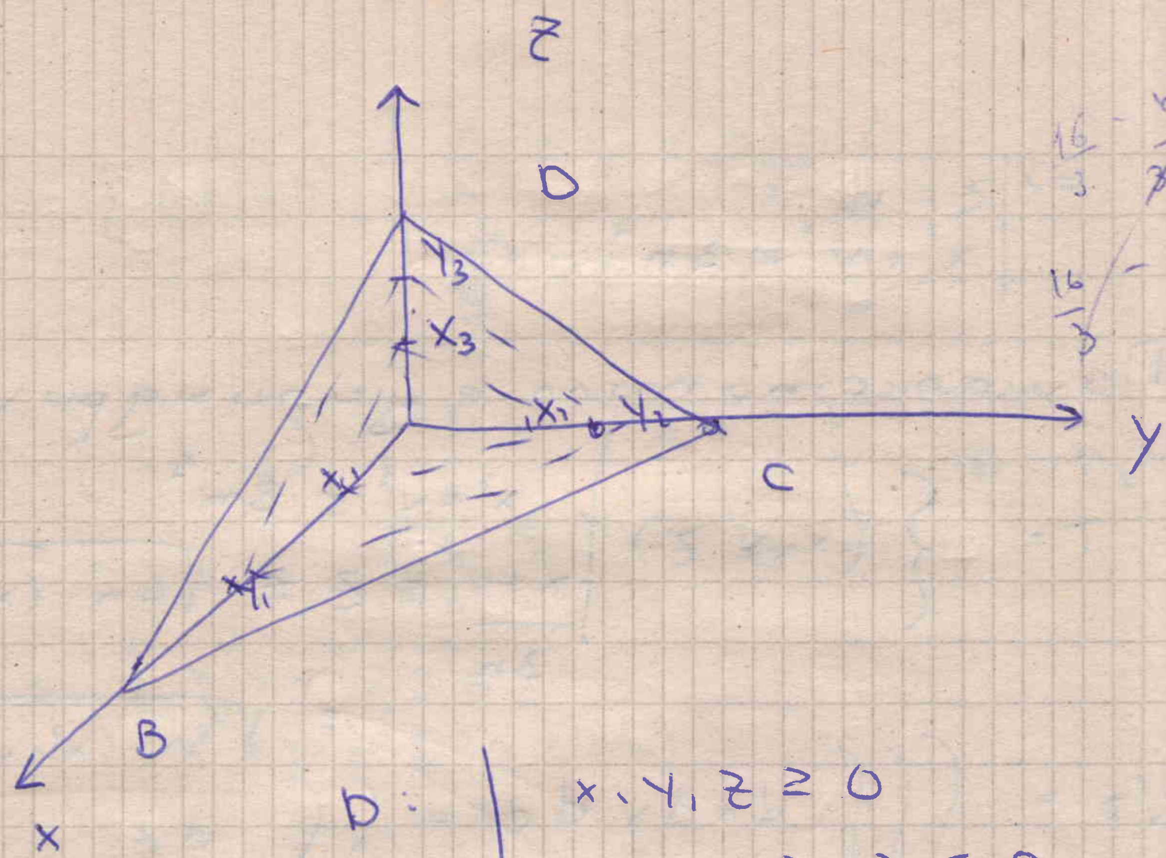
$$2\pi \left\{ -\frac{1}{\cancel{2}} \frac{\sqrt{4r^2-\rho^2}^{\,3}}{3/\cancel{2}} \Bigg|_0^{r\sqrt{3}} - \frac{\rho^4}{4 \cdot 3r} \Bigg|_0^{r\sqrt{3}} \right]$$

$$\left\{ \frac{r^2}{3} - \frac{r^{\cancel{8}} \cdot \cancel{9}}{4 \cdot \cancel{3}r} \right)$$

$$2\pi \left\{ \cancel{\frac{4}{2} - \frac{8}{2}} \right.$$

$$\frac{16}{3} - \frac{3}{4} = \frac{64-9}{12} \qquad -\frac{16}{3} \, \frac{3}{4} \Rightarrow \frac{64+9}{12}$$

зад. 4



$$D: \left| \begin{array}{l} x, y, z \geq 0 \\ x + y + z - 3 \leq 0 \end{array} \right.$$

$0 \leq x+y+z < 1 \qquad \{x+y+z\} = 0 \qquad D_1$

$1 < x+y+z \leq 2 \qquad \{x+y+z\} = 1$

$2 \leq x+y+z < 3 \qquad \{x+y+z\} = 2$

D_1 е пир. $AX_1X_2X_3$

D_2 е пир.

D_3 е пир.

$$I = \iiint_{D_1} e^{-0}\,dx\,dy\,dz + \iiint_{D_2} e^{-1}\,dx\,dy\,dz + \iiint_{D_3} e^{-2}\,dx\,dy\,dz$$

$\frac{16}{3} - \frac{h^4 \cdot 3^3}{3h \cdot 4} \quad \frac{16}{3} - \frac{h^3 3}{4} = \frac{16 A}{3}$

$\frac{16}{3} - \frac{h^4 \cdot 3^3}{3h}$

$\frac{16}{3}$

M

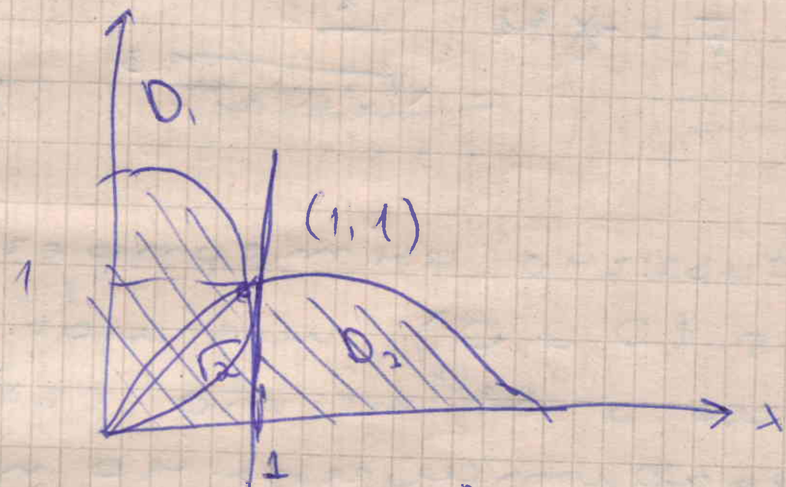
т. (x_0, y_0, z_0) до равн.

$$Ax + By + Cz + D = 0$$

$$d(M, \alpha) = \frac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}}$$

Зад. 2

$$I = \iint_D |y|\,dx\,dy = 4\iint_{D_1} y\,dx\,dy$$



$$A: \quad (x-1)^2 + y^2 = 1$$

$$D_1 = \left\{ (x, y) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 + \sqrt{1 - x^2} \end{array} \right.$$

$$D_2 = \left\{ (x, y) \;\middle|\; \begin{array}{l} 1 \le x \le 2 \\ 0 \le y \le \sqrt{1 - (x-1)^2} \end{array} \right.$$

$$\frac{1}{4} I = I_1 + I_2$$

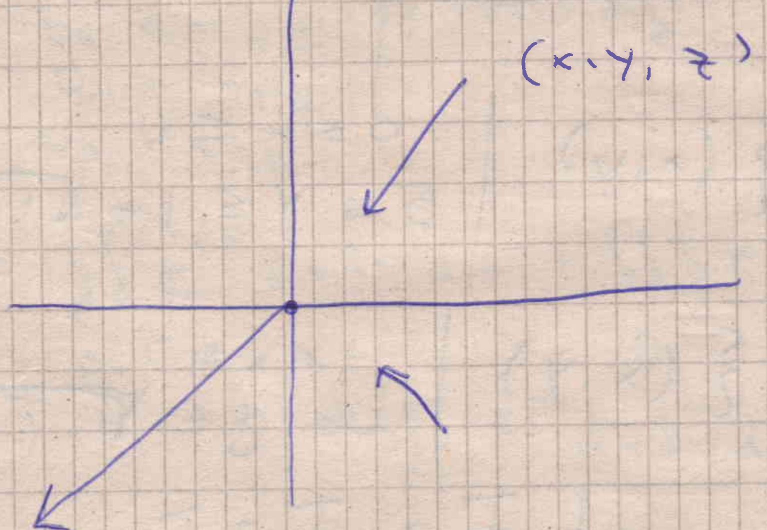
$$I_1 = \int_0 \int_0^{1 + \sqrt{1 - x^2}} y\,dy\,dx$$

зад. 2.22 g) $-\gamma M \int\limits_L \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}^{\,3}}$

$$x = e^t$$
$$y = \cancel{2}t \qquad 0 \le t \le 1$$
$$z = e^t$$

$$\vec{F} = -\gamma M \left(\frac{x}{\sqrt{x^2+y^2+z^2}^{\,3}}, \frac{y}{\sqrt{}}, \frac{z}{\sqrt{}} \right)$$

Физична интерпретация: в т. (0,0,0) има мат. точка с маса M, която създава гравитационно поле $\vec{F}$.



$\int$ = работа, която ще извърши гравит. сила за преместване на точка с маса 1 от т. $(e^0, 0, e^0)$ до $(e^1, 1, e^1)$ по съответната крива. Понеже полето е потенциално, всъщност пътя няма значение.

! $\nearrow u$ потенциална ф-я на полето $= \dfrac{\gamma M}{\sqrt{x^2+y^2+z^2}}$

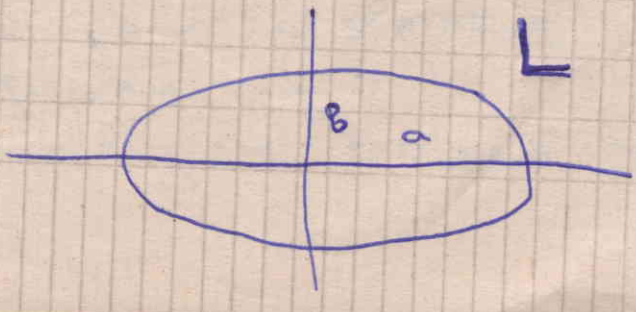
Да проверим, че $u_x = F_x$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\gamma M}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-\gamma M x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$\int = u(1) - u(0) =$$
$$= u(e^1, 1, e^1) - u(e^0, 0, e^0) =$$
$$= \gamma M\left[\frac{1}{\sqrt{2e^2+1}} - \frac{1}{\sqrt{2}}\right]$$

γ - гравит. const

68) $\displaystyle\oint_L y^2\,dx + x\,dy \qquad L: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Тн. (Грийн)

$$\oint I = \iint_{\substack{D=\langle L\rangle \\ \text{вътр. на } L}} \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y^2)$$

$$= \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} (1-2y)\,dx\,dy \qquad \begin{array}{l} x = a\rho\cos\varphi \\ y = b\rho\sin\varphi \end{array}$$

$$I = \int_0^{2\pi}\int_0^1 ab\rho\,(1-2b\rho\sin\varphi)\,d\rho\,d\varphi =$$

$$= \int_0^{2\pi}\left[ab\frac{\rho^2}{2}\Big|_0^1 - 2ab^2\sin\varphi\,\frac{\rho^3}{3}\right]d\varphi =$$

$$= \int_0^{2\pi}\left(\frac{ab}{2} - \frac{2}{3}ab^2\sin\varphi\right)d\varphi =$$

$$= \pi ab + \frac{2}{3}ab^2\cos\varphi\Big|_0^{2\pi} =$$

$$= \pi ab$$

II н. Директно. Параметризиране елипсата

$$x = a\cos\varphi$$

$$y = b\sin\varphi$$

$$I = \int_0^{2\pi} \left[b^2\sin^2\varphi\,(-a\sin\varphi) + a\cos\varphi\,(b\cos\varphi) \right] d\varphi$$

$$= ab\int_0^{2\pi} (\cos^2\varphi - \sin^3\varphi\, b)\, d\varphi$$

$$\boxed{\text{Алгоритъм за крив. инт. от. II род}}$$

$$\oint_L P\,dx + Q\,dy$$

) Проверяваме дали полето е потенциално. Ако да - $I = 0$

) Ако не, проверяваме условията на Th Грийн (дали P и Q са непр. диф. по ∀ напр. в $D \cup L$, и диф. в D

$D = \langle L \rangle$)

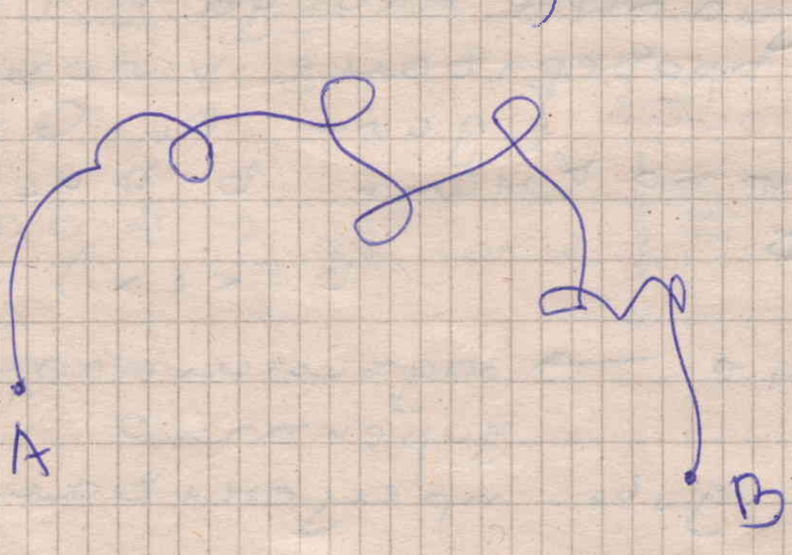
Ако не $\longrightarrow$ задължително директно

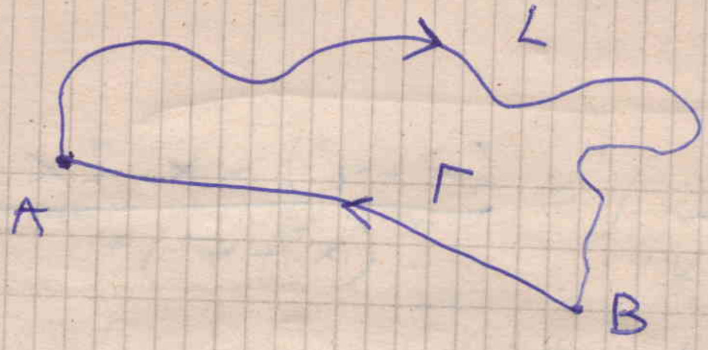
Ако са удовл. преценяваме как

ще е по-лесно да сметаме

$$\int_L P\,dx + Q\,dy$$ → можем да помислим дали да не затворим кривата, като по този начин ще сведем пресмятането на К.И. към пресмятане на двоен и пресмятане на др. криволинеен, съответно в/у кривата, с която сме допълнили. Това може да се окаже по-лесно, особено в случаите, когато L е сложна и/или $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ - просто.

не може параметризаци





$$I_L = \int\limits_L P\,dx + Q\,dy =$$

$$= \underbrace{\oint\limits_{L\cup\Gamma} P\,dx + Q\,dy}_{\text{Грийн}} - \int\limits_{\Gamma} P\,dx + Q\,dy$$

! $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$

Зад. 3.7 !

$$\oint\limits_L \underbrace{\frac{-y}{x^2+y^2}}_{=P}\,dx + \underbrace{\frac{x}{x^2+y^2}}_{=Q}\,dy$$

а) $L: (x-2)^2 + y^2 = 1$

б) $x^2 + y^2 = r^2$

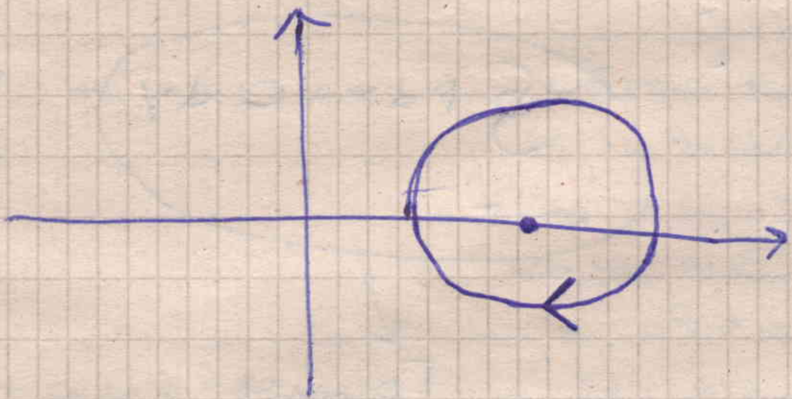
$F'_{1y} = F'_{2x}$

$$\left(\frac{-y}{x^2+y^2}\right)'_y = \frac{-1(x^2+y^2) + y \cdot 2y}{(x^2+y^2)^2} =$$

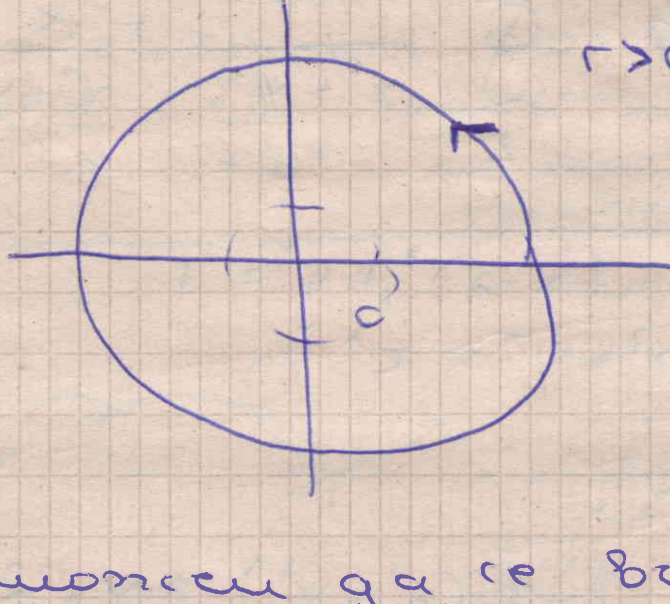
$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\left(F'_{2x}\right) = \left(\frac{x}{x^2+y^2}\right)' = \frac{(x^2+y^2) - x.2x}{(x^2+y^2)^2} =$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$



в а) имаме едносвързана област и $I = 0$



$r > 0$

$I = \int_{\Gamma \cup c} - \int_c$

$= 0$ Грийн

в б) не можем да се възползваме от потенциала, нито да приложим Грийн

Сметане директно $\quad x = r\cos\varphi$
$\quad y = r\sin\varphi$

$$I = \int_0^{2\pi} \left(\frac{-r\sin\varphi\,(-r\sin\varphi)}{r^2} + \frac{r\cos\varphi\, r\cos\varphi}{r^2} \right) d\varphi =$$

$$= \int_0^{2\pi} \frac{r^2}{r^2}\, d\varphi = \boxed{2\pi}$$

→ резидуи на ф-я

! Връзка м/у К.И.В.Р и комплексен интеграл

Неформално имаме

$$f(z) = f(x+iz) -$$

комплексно значна функция.

$$f(x+iy) = a(x,y) + b(x,y)i$$

Нека имаме крива L в $\mathbb{R}^2$.

$$\int_L f(z)\,dz = \int_L \big((a(x,y) + ib(x,y))\big)(dx + d\,iy) =$$

$$\overbrace{= \int_L a(x,y)\,dx - b(x,y)\,dy}^{\text{реалнозначен К.И.В.Р.}} +$$

$$+ i \int_L \underbrace{(b(x,y))\,dy + a(x,y)\,dy}$$

В. 3.7 а) дали има функция $f(z)$, за която ако

$$I = \int_L f(z)\,dz, \text{ то}$$

реалната или Im част на I да са интерпр. в 3.7 а)

Формули на Грийн → ползват се в ЧДУ (УМФ)

$$1) \iint_D u \Delta v = \oint u \frac{\partial v}{\partial x}\,dy - \cancel{}u\frac{\partial v}{\partial y}\,dx - \iint_D \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,dx\,dy$$

$$\parallel \iint_D \langle \operatorname{grad} u, \operatorname{grad} v \rangle$$

u, v - достатъчно гладки в $D \cup \partial D$

$\Delta v = v_{xx} + v_{yy}$ ⟶ оператор на Лаплас

Д-80: За инт. $\oint \underbrace{-u\frac{\partial v}{\partial y}}_{P} dx + \underbrace{u\frac{\partial v}{\partial x}}_{Q} dy$

да приложим Грийн

$$I = \iint_D \left(\left(u\frac{\partial v}{\partial x}\right)'_x - \left(-u\frac{\partial v}{\partial y}\right)'_y \right) dxdy =$$

$$= \iint_D \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} \right) dxdy =$$

$$= \iint_D u\Delta v\, dxdy$$

$$= \iint_D u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) dxdy + \iint_D \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dxdy$$

) Симетрична ф-ла на Грийн

$$\iint_D (u\Delta v - v\Delta u)\, dxdy =$$

$$= \oint \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) dy - \left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right) dx$$

Прилагаме 1) за u, v и v, u

(*) $$\iint_D u \Delta v = \oint u \frac{\partial v}{\partial x} dy - u \frac{\partial v}{\partial y} dx - \iint_D \langle \nabla u, \nabla v \rangle dx dy$$

(**) $$\iint v \Delta u \, dx dy = \oint v \frac{\partial u}{\partial x} dy - v \frac{\partial u}{\partial x} dx - \iint_D \langle \nabla v, \nabla u \rangle dx dy$$

Вадим (**) от (*) :))

<u>Зад. 3.17</u> Нека u има непрекъснати производни до втори ред в/у затв. крива L и областта D, която тя заграждa. Нека $\Delta u = 0$ в D и $u|_L = 0$

Тогава $u = 0$.

Д-во: Да приложим 1) за

$u = v$

$$\underset{D}{\iint} u \Delta u \, dx\,dy = \underset{L}{\oint} u \frac{\partial u}{\partial x} dy - u \frac{\partial u}{\partial y} dx =$$

$$= - \underset{D}{\iint} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dx\,dy$$

$$\underset{D}{\iint} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dx\,dy = 0 \Rightarrow$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0$$

$$\frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 0$$

$u = c$ и 0 b/y границата

При същите условия ако f и g са непрекъснати, задачата

Th за единственост

$$\begin{cases} \Delta u = f, & (x,y) \in D \\ u|_L = g & (x,y) \in L \end{cases}$$

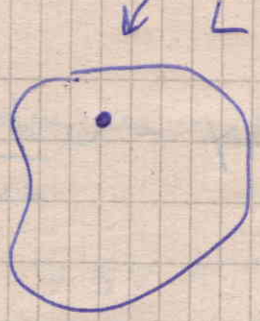
има единствено решение.

Д-во: Нека u_1 и u_2 са 2 реш.

$\Delta(u_1 - u_2) = f - f = 0 \Rightarrow u_1 - u_2 = 0$

$(u_1 - u_2)|_L = g - g = 0$:))

само чрез ст-те на кривата на 2 променливи

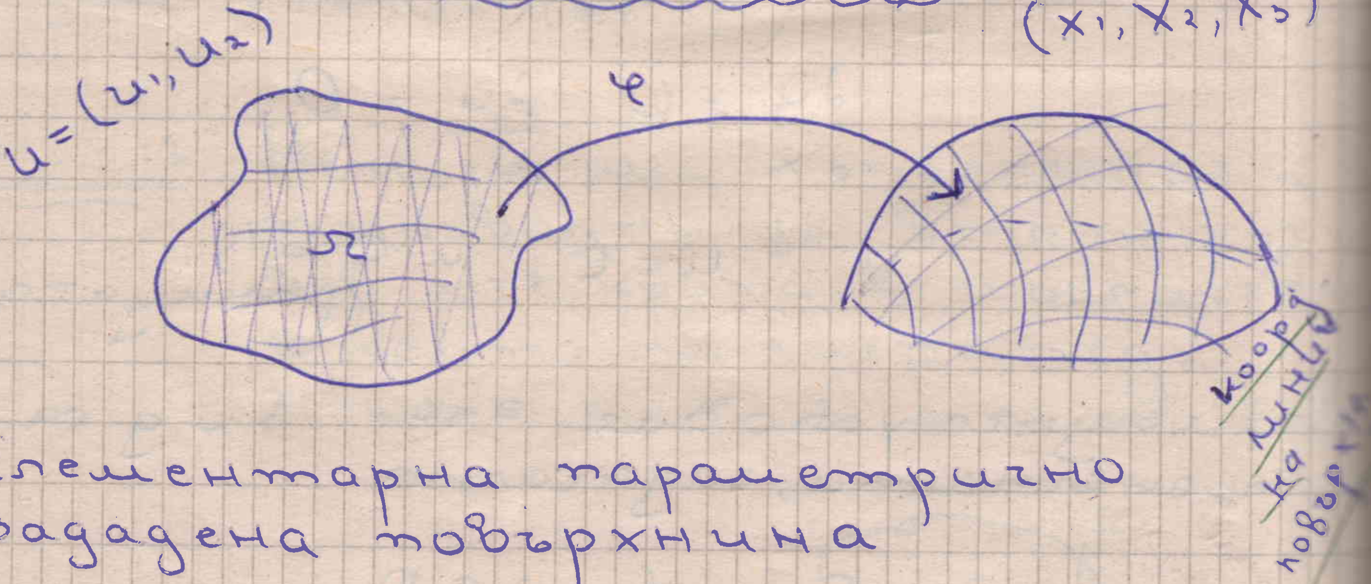


!!! $\Delta u = 0$ се нарича <u>хармонична</u>

Лекция

04.12

Повърхнинни интеграли



Елементарна параметрично зададена повърхнина

$$\varphi : \Omega \longrightarrow \mathbb{R}^3$$

Ω област в $\mathbb{R}^2$

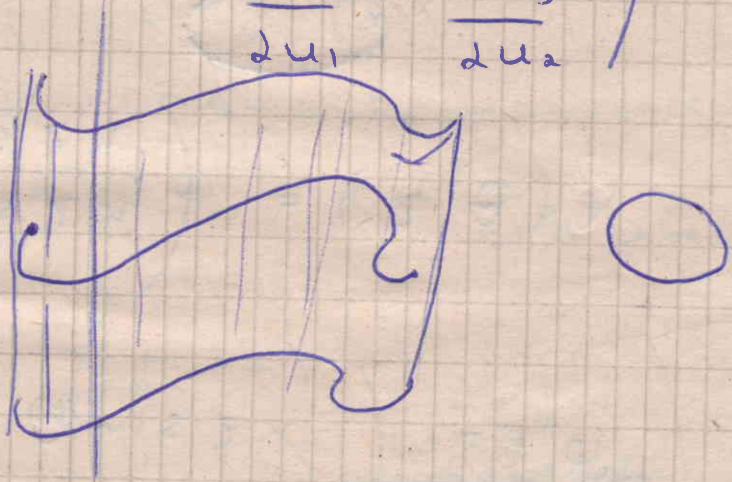
$$\varphi \in C^1(\Omega, \mathbb{R}^3)$$

$$rg\, \varphi' = 2$$

ранг на матрицата от производни

$$\varphi(u) = \begin{pmatrix} \varphi_1(u_1, u_2) \\ \varphi_2(u_1, u_2) \\ \varphi_3(u_1, u_2) \end{pmatrix}$$

$$\varphi'(u) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u_1} & \frac{\partial \varphi_1}{\partial u_2} \\ \frac{\partial \varphi_2}{\partial u_1} & \frac{\partial \varphi_2}{\partial u_2} \\ \frac{\partial \varphi_3}{\partial u_1} & \frac{\partial \varphi_3}{\partial u_2} \end{pmatrix} (u_1, u_2)$$



Примери:

$\alpha(t)$, $t \in \Delta$

Δ отворен интервал

α регулярна гладка крива плоска

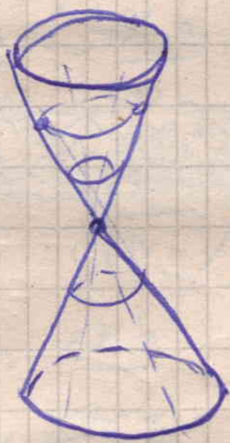
$$\varphi(t, z) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ z \end{pmatrix}$$

$(t, z) \in \Delta \times \mathbb{R}$

$$\varphi' = \begin{pmatrix} \dot{\alpha}_1(t) & 0 \\ \dot{\alpha}_2(t) & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{matrix} \dot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \end{matrix}$$

Понеже $i(t) \neq 0$

$\Rightarrow$ кривата е регулярна



$$\varphi(\theta, z) = \begin{pmatrix} kz\cos\theta \\ kz\sin\theta \\ z \end{pmatrix}$$

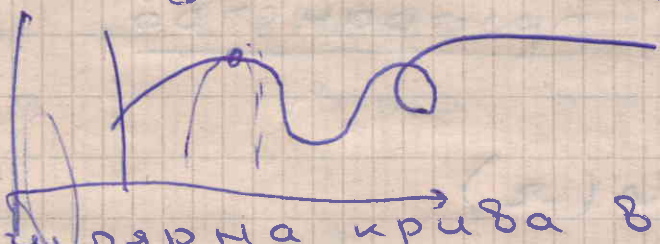
$$\varphi' = \begin{pmatrix} -kz\sin\theta & k\cos\theta \\ kz\cos\theta & k\sin\theta \\ 0 & 1 \end{pmatrix}$$

$$-k^2 z \;,\; kz\cos\theta \;,\; -kz\sin\theta$$

$$z=0 \quad (0,0,0)$$

Ротация

Ротационно тяло



регулярна крива в горната полуравнина

$\alpha(t) = (\alpha_1(t), \alpha_2(t))$

плоска гладка регулярна крива

$\alpha_2(t) > 0 \qquad t \in \Delta$ отв. инт.

$$\varphi(t,\theta) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t)\cos\theta \\ \alpha_2(t)\sin\theta \end{pmatrix}$$

$$\varphi'^{(t,\theta)} = \begin{pmatrix} \alpha_1(t) & 0 \\ \alpha_2(t)\cos\theta & -\sin\theta\,\alpha_2(t) \\ \alpha_2(t)\sin\theta & \alpha_2(t)\cos\theta \end{pmatrix}$$

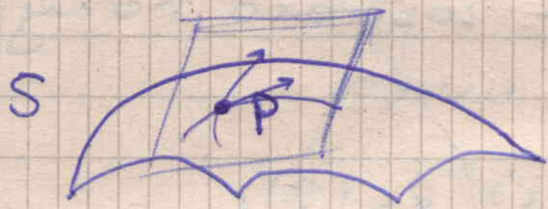
минори

$-\alpha_1(t)\sin\theta\,\alpha_2(t)$ $\qquad \cancel{\dot\alpha_2\alpha_2\cos^2\theta + \dot\alpha_2\alpha_2\sin^2\theta}$

$\alpha_1(t)\,\alpha_2(t)\cos\theta$ $\qquad \dot\alpha_2\alpha_2$

Допирателно пространство

$S = \varphi(\mathfrak{r})$



гладка крива в повърхнината
допирателен в-р

Θ от допир. в-ри - допир. п-во

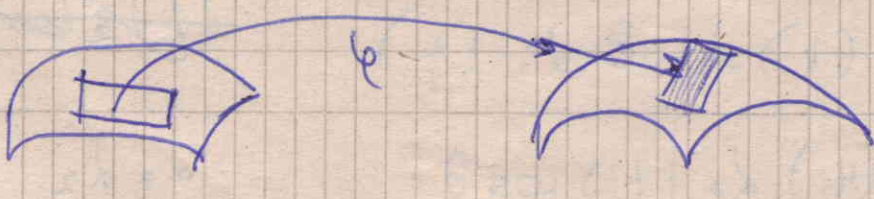
S_p - допирателно п-во към S в т. p

$S_p = \{ (p; v) \in S \times \mathbb{R}^3$: съществува

$\beta \in C^1(\Delta, S)$

Δ отв. интервал, съдържащ
$t_0 \in \Delta$

$$\left.\begin{array}{l} \beta(t_0) = p \\ \dot{\beta}(t_0) = v \end{array}\right\}$$



$\varphi : \Omega \to \mathbb{R}^3$

$d\varphi(u;W) = \left(\varphi(u);\ \omega_1 \frac{\partial \varphi}{\partial u_1}(u) + \omega_2 \frac{\partial \varphi}{\partial u_2}(u)\right)$

изображение свързан в-р

$\cancel{d\varphi : \Omega \times \mathbb{R}^2}$

$d\varphi : \Omega \times \mathbb{R}^2 \longrightarrow S \times \mathbb{R}^3$

$\omega(\omega_1, \omega_2)$

$$\frac{\partial \varphi}{\partial u_1} = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u_1}(u) \\ \frac{\partial \varphi_2}{\partial u_1}(u) \\ \frac{\partial \varphi_3}{\partial u_1}(u) \end{pmatrix} \qquad \cancel{\frac{\partial \varphi_1}{\partial u_1} \ \frac{\partial \varphi_1}{\partial u_2}}$$

$$\frac{\partial \varphi}{\partial u_2}(u) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u_2}(u) \\ \frac{\partial \varphi_3}{\partial u_2}(u) \end{pmatrix}$$

$\Omega \ni \alpha(t) \quad t \in \Delta$
гладка
крива

$\beta(t) = \varphi(\alpha(t)) \in S \quad \forall\, t \in \Delta$

$(\beta(t);\ \dot\beta(t))$

$$\dot\beta(t) = \frac{\partial \varphi}{\partial u_1}(\alpha(t)) \cdot \dot\alpha_1(t) + \frac{\partial \varphi}{\partial u_2}(\alpha(t)) \cdot \dot\alpha_2(t)$$

~~Ако~~ $d\varphi(\alpha(t), \dot{\alpha}(t)) = (\beta(t); \dot{\beta}(t))$

Th $p = \varphi(u)$

$S_p = d\varphi(u; \mathbb{R}^2)$ за $\forall\, \omega$

$\quad \parallel$

$\left\{ (p; v) : v \in \underbrace{\operatorname{span} \ell\left(\frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_2}(u) \right)}_{\text{линейна обвивка}} \right\}$

$\left(\varphi(u), \omega_1 \frac{\partial \varphi}{\partial u_1}(u) + \omega_2 \frac{\partial \varphi}{\partial u_2}(u) \right) \in d\varphi(u; \mathbb{R}^2)$

произволен ел.

$\omega = (\omega_1, \omega_2)$

Взимаме $u + t\omega$ права в $\mathbb{R}^2$

$t \in (-\delta, \delta)$

ω-отворено

$p = \varphi(u)$

$\dot{\alpha}(t) = (\omega_1, \omega_2)$ $\in S_p$

$d\varphi(\alpha(0), \dot{\alpha}(0)) = (\beta(0)\ \dot{\beta}(0))$,

където $\beta(t) = \varphi(u + t\omega)$

гладка крива $\in S$

Обратно,

$$S = \varphi(\Omega)$$

<u>Лема</u>: $\varphi : \Omega \longrightarrow \mathbb{R}^3$

$\varphi \in C^1$, $\operatorname{rg} \varphi' = 2$

$u_0 \in \Omega$

$\Rightarrow \exists$ околност U на u_0 (в $\mathbb{R}^2$) и околност V на $p = \varphi(u_0) \in S$ (в $\mathbb{R}^3$), такива че

$\varphi|_U : U \to V \cap S$ е биекция. (Обратната биекция е гладка)

$(\varphi_U)^{-1}$ е непр.

(даже $(\varphi_U)^{-1}$ ~~на~~ е рестрикция в/у S на гладко изобр. деф. във V)

$\operatorname{rg} \varphi'(u_0) = 2$

$\cancel{\varphi'(u_0)}$ $\qquad \dfrac{\partial \varphi_1}{\partial u_1}(u_0) \quad \dfrac{\partial \varphi_1}{\partial u_2}(u_0)$

$$\varphi'(u_0) = \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial u_1}(u_0) & \dfrac{\partial \varphi_1}{\partial u_2}(u_0) \\ \dfrac{\partial \varphi_2}{\partial u_1}(u_0) & \dfrac{\partial \varphi_2}{\partial u_2}(u_0) \\ \dfrac{\partial u_3}{\partial u_1}(u_0) & \dfrac{\partial \varphi_3}{\partial u_2}(u_0) \end{pmatrix}$$

Б.о.о. $\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial u_1} & \dfrac{\partial \varphi_1}{\partial u_2} \\ \dfrac{\partial \varphi_2}{\partial u_1} & \dfrac{\partial \varphi_2}{\partial u_2} \end{vmatrix} \neq 0$

Съществува комп. U над u_0 и

ок. V на $(p_1, p_2) =$

$= (\varphi_1(u_0), \varphi_2(u_0))$,

такива че

$$x_1 = \varphi_1(u_1, u_2)$$
$$x_2 = \varphi_2(u_1, u_2)$$

$$\Leftrightarrow$$

$$u_1 = \psi_1(x_1, x_2)$$
$$u_2 = \psi_2(x_1, x_2)$$

в $U \times V'$

ψ_1, ψ_2 са гладки

$$(x_1, x_2, x_3) \in S$$

$$x_3 = \varphi_3(\psi_1(x_1, x_2), \psi_2(x_1, x_2))$$

$$V = \underset{\substack{\cap \\ \mathbb{R}^2}}{V'} \times \mathbb{R}$$

$$\left(\varphi\big|_U\right)^{-1}(x_1, x_2, x_3) = \begin{pmatrix} \psi_1(x_1, x_2) \\ \psi_2(x_1, x_2) \end{pmatrix}$$

<u>Продължение на Д-вото</u>:

$(p; v) \in S_p$

$\beta : \Delta \to S$

$t_0 \in \Delta$ отв. инт.

$\beta(t_0) = p$

$\dot\beta(t_0) = v$

U на u

V на p

Δ по-малко така, че $\beta(t) \in V \cap S$

$\forall t \in \Delta$.

$\alpha(t) = (\varphi_{|U}^{-1})(\beta(t)) \in \Omega$

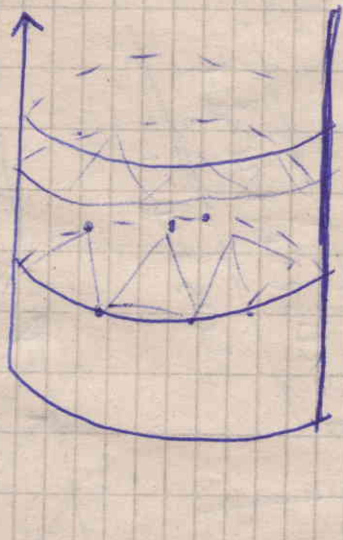
$\beta(t) = \varphi(\alpha(t))$

$(\beta(t_0); \dot\beta(t_0)) = d\varphi(\alpha(t_0), \boxed{\dot\alpha(t_0)})$

$\alpha(t_0) = u_0$

W

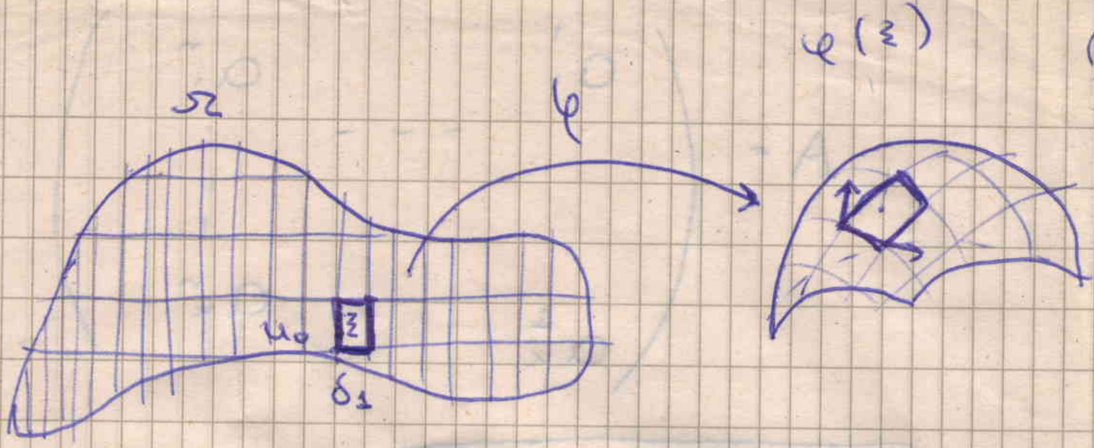
//



m+1 хор. окр.

правилен

n-ъгълници



$$\frac{\partial\varphi}{\partial u_1}(u_0)\delta_1, \frac{\partial\varphi}{\partial u_2}(u_0)\delta_2$$

допир. п-во

успоредник

a, b - вектори

Детерм. на Грам

$$\begin{vmatrix} \langle a,a\rangle & \langle a,b\rangle \\ \langle b,a\rangle & \langle b,b\rangle \end{vmatrix} = \|a\times b\|^2$$

$$a_1, a_2, \ldots, a_n \in \mathbb{R}^k$$

$$n \le k$$

$\cos 75^\circ = \cos(45^\circ+30^\circ) = \frac{\sqrt2}{2}\cdot\frac{\sqrt3}{2} - \frac{\sqrt2}{2}\cdot$

$$(\mu_n(P(a_1, \ldots a_n)))^2 = \begin{vmatrix} \langle a_1,a_2\rangle & \ldots & \langle a_1,a_n\rangle \\ \langle a_2,a_1\rangle & \ldots & \langle a_2,a_n\rangle \\ \vdots & & \\ \langle a_n,a_1\rangle & \ldots & \langle a_n,a_n\rangle \end{vmatrix}$$

$a_1(a_{11}^1, a_{12}^1, \ldots a_{1k}^1)$

$a_2(a_{21}^2, a_{22}^2, \ldots a_{2k}^2)$

$\vdots$

$a_n(a_{n1}^n \ldots a_{nk}^n)$

A - матрица

$$A = \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \vdots & & \vdots \\ a_k^1 & & a_k^n \end{pmatrix}$$

$$\boxed{\sqrt{\det(A^T A)}}$$

$$\mu_2(S) \sim \sum_{i=1}^{10} \sqrt{\det(\varphi'(\xi_i)^T \varphi'(\xi_i))}\ \mu_2(\Delta_i)$$

мцето на повърхнината

$\varphi : \Omega \to \mathbb{R}^3$ $\quad K \subset \Omega$

$\varphi \in C^1$ $\quad$ измерим компакт

? $S = \varphi(K)$

$$\mu_2(s) = \iint_K \sqrt{\det(\varphi'(u)^T \varphi'(u))}\, du$$

$\varphi : \Omega \to \mathbb{R}^k$

$\Omega \subset \mathbb{R}^n$ $\quad$ φ гладко

$n \leq k$

$K \subset \Omega$

измерим компакт

$$\boxed{\Rightarrow \mu_n(\varphi(K)) = \int_K \sqrt{\det(\varphi'^T(u)\varphi'(u))}\, du}$$

1) $n=1$

$$\varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \vdots \\ \varphi_k(t) \end{pmatrix}$$

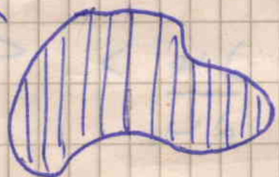
$$\varphi'(t) = \begin{pmatrix} \varphi_1'(t) \\ \vdots \\ \varphi_k'(t) \end{pmatrix}$$

$$\varphi'(t)^T \cdot \varphi'(t) = \langle \varphi'(t), \varphi'(t) \rangle$$

$$\sqrt{\det(\langle \varphi'(t), \varphi'(t) \rangle)} = \|\varphi'(t)\|$$

2) $n = k$ $\quad\varphi$ - идентитет

K



Ако не е идентитет, φ - гладко

$$\varphi(K)$$

$$\mu_n(\varphi(K)) = \int_{\varphi(K)} 1 . dx$$

$$\int_K |\det \varphi'(u)| \, du$$

Цилиндър

$\alpha(t)$ - регул. плоска гладка $\quad t \in [a, b]$

$z \in [c, d]$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t))$$

$$\varphi(t, z) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ z \end{pmatrix}$$

$$K = [a, b] \times [c, d]$$

$$n = 2, k = 3$$

$$\varphi'(u) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u_1} & \dots \\ \dots & \dots \end{pmatrix}$$

$$\varphi'(u)^T \varphi'(u) = \begin{pmatrix} \langle \frac{\partial \varphi}{\partial u_1}, \frac{\partial \varphi}{\partial u_1} \rangle, & \langle \frac{\partial \varphi}{\partial u_2}, \frac{\partial \varphi}{\partial u_1} \rangle \\ \langle \frac{\partial \varphi}{\partial u_1}, \frac{\partial \varphi}{\partial u_2} \rangle, & \langle \frac{\partial \varphi}{\partial u_2}, \frac{\partial \varphi}{\partial u_2} \rangle \end{pmatrix}$$

коеф. на I-вата квадратична ф-ма

$$E = \langle \frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_1}(u) \rangle$$

$$F = \langle \frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_2}(u) \rangle$$

$$G = \langle \frac{\partial \varphi}{\partial u_2}(u) \frac{\partial \varphi}{\partial u_2}(u) \rangle$$

$$\mu_2(S) = \iint_K \sqrt{EG - F^2}\, du$$

$$\varphi'(t,z) = \begin{pmatrix} \dot{\alpha}_1(t) & 0 \\ \dot{\alpha}_2(t) & 0 \\ 0 & 1 \end{pmatrix}$$

$E = \dot{\alpha}_1(t)^2 + \dot{\alpha}_2(t)^2$

$F = 0$ коорд. линии се пресичат под прав ъгъл

$G = 1$

$$\mu_2(S) = \iint\limits_{[a,b]\times[c,d]} \sqrt{\dot{\alpha}_1(t)^2 + \dot{\alpha}_2(t)^2}\, dt\, dz =$$

$$= (d-c) \int_a^b \|\dot{\alpha}(t)\|\, dt =$$

$$= (d-c)\, l(\Gamma)$$

Ротационна пов.

$\alpha(t) = (\alpha_1(t), \alpha_2(t))$

$\alpha_2(t) > 0$ $\quad t \in [a,b]$

$$\varphi(t,\theta) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t)\cos\theta \\ \alpha_2(t)\sin\theta \end{pmatrix}$$

$(t,\theta) \in [a,b] \times [0, 2\pi]$

$$\varphi' = \begin{pmatrix} \dot{\alpha}_1 & 0 \\ \dot{\alpha}_2\cos\theta & -\alpha_2\sin\theta \\ \dot{\alpha}_2\sin\theta & \alpha_2\cos\theta \end{pmatrix}$$

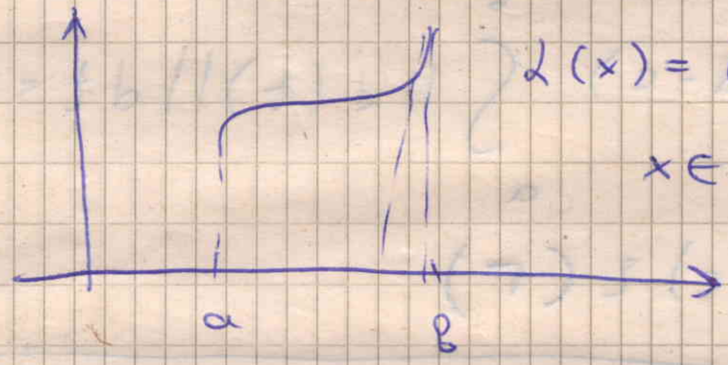
$$E = \dot{\alpha}_1(t)^2 + \dot{\alpha}_2(t)^2$$

$$F = 0 \qquad \text{ортог. коорд. линии}$$

$$G = \alpha_2(t)^2$$

$$\mu_n(S) = \iint\limits_{[a,b]\times[0,2\pi]} \sqrt{\alpha_2^2(t)\left(\dot{\alpha}_1(t)^2 + \dot{\alpha}_2(t)^2\right)}\, dt =$$

$$= \int_0^{2\pi}\int_a^b \alpha_2(t)\, \|\dot{\alpha}\|\, dt$$



$$\alpha(x) = (x, f(x))$$

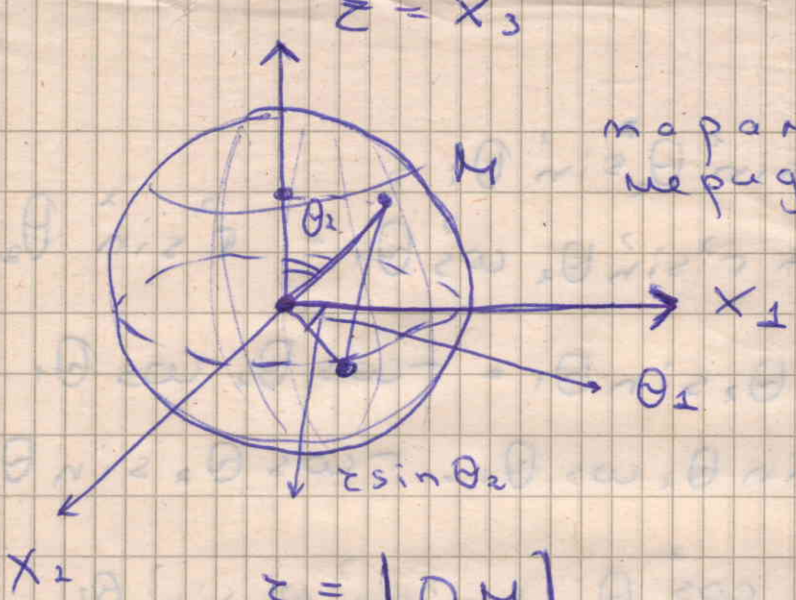
$$x \in [a, b]$$

$$\dot{\alpha}(x) = (1, f'(x))$$

$$\|\dot{\alpha}(x)\| = \sqrt{1 + (f'(x))^2}$$

$$2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx$$

<u>Сфера:</u>



паралели
меридиани

$$r = |OM|$$
$$\theta_2 \in [0, \pi]$$
$$x_3 = r\cos\theta_2$$

$$x_1 = r\sin\theta_2\cos\theta_1$$
$$x_2 = r\sin\theta_2\sin\theta_1$$
$$x_3 = r\cos\theta_1$$

$$\theta_1 \in [0; 2\pi]$$
$$\theta_2 \in [0; \pi]$$

$$\varphi(\theta_1, \theta_2) = \begin{pmatrix} r\sin\theta_2\cos\theta_1 \\ r\sin\theta_2\sin\theta_1 \\ r\cos\theta_2 \end{pmatrix}$$

$$\varphi' = \begin{pmatrix} -r\sin\theta_2 \cancel{\cos}\sin\theta_1 & r\cos\theta_2\cos\theta_1 \\ r\sin\theta_2\cos\theta_1 & r\cos\theta_2\sin\theta_1 \\ \cancel{r}\cdot 0 & -r\sin\theta_2 \end{pmatrix}$$

8

$$E = r^2 \sin^2\theta_2 \sin^2\theta_1 + \\ + r^2\sin^2\theta_2 \cos^2\theta_1 = r^2 \sin^2\theta_2$$

$$F = -r\sin\theta_2\sin\theta_1 \cdot r\cos\theta_2\cos\theta_1 \\ + r\sin\theta_2\cos\theta_1 \cdot r\cos\theta_2\sin\theta_1 = 0$$

$$G = r^2\cos^2\theta_2\cos^2\theta_1 + r^2\cos^2\theta_2\sin^2\theta_1 \\ + r^2\sin^2\theta_2 = r^2$$

$$\sqrt{EG - F^2} = \sqrt{r^2\sin^2\theta_2 \cdot r^2} = \\ = \underline{\underline{r^2\sin\theta_2}}$$

$$\mu_2(S) = \iint\limits_{[0,2\pi]\times[0,\pi]} r^2\sin\theta_2 \, d\theta_1 \, d\theta_2 =$$

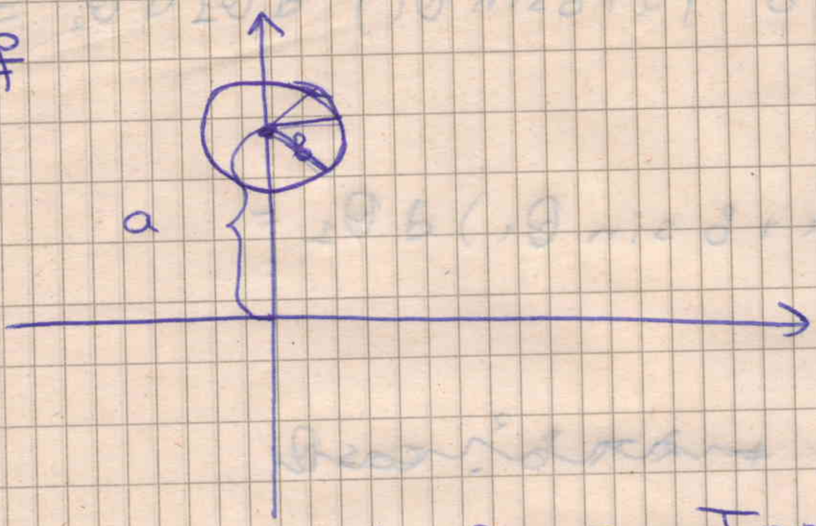
$$= r^2\int_0^{2\pi} d\theta_1 \cdot \int_0^{\pi} \sin\theta_2 \, d\theta_2 =$$

$$= 2\pi r^2 (-\cos\theta_2)\Big|_0^{2\pi} = 4\pi r^2$$

? Конус

Тор

Top



$0 < b < a$ Top

$$\alpha(\theta_1) = (b\cos\theta_1 ,\ a + b\sin\theta_1)$$

$$\varphi(\theta_1, \theta_2) = \begin{pmatrix} b\cos\theta_1 \\ (a + b\sin\theta_1)\cos\theta_2 \\ (a + b\sin\theta_1)\sin\theta_2 \end{pmatrix}$$

$$\theta_1, \theta_2 \in [0; 2\pi]$$

$$\varphi'(\theta_1, \theta_2) = \begin{pmatrix} -b\sin\theta_1 & 0 \\ b\cos\theta_1\cos\theta_2 & -(a+b\sin\theta_1)\sin\theta_2 \\ b\sin\theta_2\cos\theta_1 & (a+b\sin\theta_1)\cos\theta_2 \end{pmatrix}$$

$$E = b^2\sin^2\theta_1 + b^2\cos^2\theta_1\cos^2\theta_2 + \dots = b^2$$

$$F = 0$$

$$G = (a + b\sin\theta_1)^2$$

(10)

$$\int_0^{2\pi}\int_0^{2\pi} \sqrt{b^2 (a+b\sin\theta_1)}\, d\theta_1\, d\theta_2 =$$

$$= 2\pi b \int_0^{2\pi} (a+b\sin\theta_1)\, d\theta_1 =$$

$$= a\,2\pi b \cdot 2\pi \cancel{+ 2\pi b^2 \cdot \cos\theta_1}$$

$$= 4\pi^2 a b$$