6.06.2013 ДИС-упражнение II - подготовка за контролно

1 неявни функции

$$z = xf\left(\frac{z}{y}\right)$$

$$xz'_x + yz'_y = z$$

$$F(x,y,z) = z - xf\left(\frac{z}{y}\right) = 0$$

$$\begin{cases} F'_z = 0 \\ F = 0 \end{cases}$$

$$\begin{cases} 1 - x \cdot f' \cdot \frac{1}{y} \neq 0 \\ z - xf\left(\frac{z}{y}\right) = 0 \end{cases} \qquad z'_x : f'$$

f' - непр. производна

$$\begin{cases} z'_x = f\left(\frac{y}{z}\right) + xf'\frac{z'_x}{y} & \cdot \text{ * условието да } \exists \text{ неявна} \\ z'_x = \left(1 - \frac{x}{y}f'\right) = f\left(\frac{z}{y}\right) & \cdot x \quad \text{и да е диф трябва да } \equiv \end{cases}$$

$$\begin{cases} z'_y = xf'\frac{z'_y y - z}{y^2} \\ \left(1 - \frac{x}{y}f'\right)z'_y = -\frac{xz}{y^2}f' & \cdot y \end{cases}$$

$$a\left(xz'_x + yz'_y\right) = xf\left(\frac{z}{y}\right) - \frac{x \cdot z}{y}f' = z\left(1 - \frac{x}{y}f'\right) = a$$

по усл. $a \neq 0$

$\cancel{q(xz'_x + yz'_y) = xf\left(\frac{z}{y}\right)}$ $\quad f'\left(\frac{z}{y}\right)$ - сложна f

$$z''_{xy} = z''_{xy}\,a + z'_x\left(-x\,\frac{f''\,\frac{z'_y y - z}{y^2}\,y - f'}{y^2}\right) = f'\,\frac{z'_y \cdot y - z}{y^2}$$

$$\begin{cases} x = u + v \\ y = u^2 + v^2 \\ z = u^3 + v^3 \end{cases} \qquad z'_x \qquad \begin{matrix} u(x, y) \\ v(x, y) \end{matrix}$$

$$\begin{cases} z'_x = 3u^2 u'_x + 3v^2 v'_x \\ 1 = u'_x + v'_x \\ 0 = 2u u'_x + 2v v'_x \end{cases} \qquad u'_x = 1 - v'_x$$

$$u'_x = \frac{\begin{vmatrix} 1 & 1 \\ 0 & 2v \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2u & 2v \end{vmatrix}} = \frac{2v}{2v - 2u} = \frac{v}{v - u}$$

$$F(x, y, u, v) = u + v - x = v$$
$$F(x, y, u, v) = u^2 + v^2 - y = u$$

$$\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix} \neq 0$$

$$\begin{vmatrix} 1 & 1 \\ 2u & 2v \end{vmatrix} = 2v - 2u \neq 0$$

$$v'_x = \frac{\begin{vmatrix} 1 & 1 \\ 2u & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2u & 2v \end{vmatrix}} = \frac{-2u}{2v-2u} = \frac{u}{u-v}$$

$\Rightarrow z'_x =$ заместване

II) Смяна на променливите $\qquad y(x)$

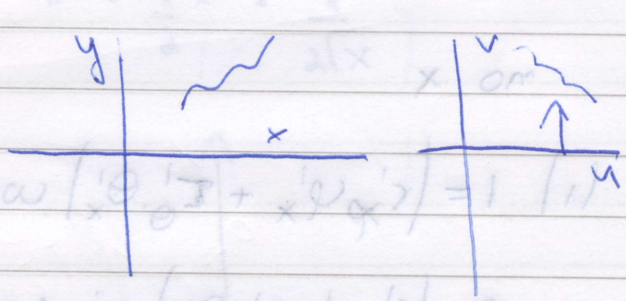
$$(1+x^2)^2 y'' + 2x(1+x^2)y' + y = 0 \qquad x = f(u,v), \quad y = g(u,v)$$

(1) $x = \mathrm{tg}\, t$ старата $y(x)$

$y = u$ $\quad u(t) \leftarrow$ новата

$y(x) = u(\mathrm{arctg}\, x)$ в (1) $t = t(x)$

$y(x) = u(t(x))$



сменяме едната равнина с другата

$y'(x) = \dot u = u'_t$

$y'(x) = u'_t t'$

(1)' $1 = \frac{1}{\cos^2 t} t' \qquad t' = \cos^2 t$

$y'(x) = \overset{\dot u}{u'_t} \cos^2 t$

$y''(x) = (u''_t \cos^2 t + u'_t 2\cos t \sin t)t'$ — получено

$y''(x) = u''_t \cos t - u'_t \cos t \sin t$

$$\frac{1}{\cos^2 t}(\ddot u \cos^4 t - 2\dot u \cos^2 t \sin t) + \frac{2\sin t}{\cos^2 t}(\dot u \cos^2 t) + u = 0$$

$\ddot u + u = 0 \qquad u(t) = A\cos t + B\sin t$

$y(x) = A\cos(\mathrm{arctg}\, x) + B\sin(\mathrm{arctg}\, x)$

задача

$y z'_x - x z'_y = 0$ в сферични координати

$z(x,y)$

(1) $x = r\cos\varphi\sin\theta$

(2) $y = r\sin\varphi\sin\theta$

$z = r\cos\theta$

$(x, y, z) \to (z, \varphi, \theta)$, нека $r(\varphi,\theta)$

$x = (r(\varphi,\theta) \ldots$ | $\varphi(x,y)$

$y = \{r\,\varphi\,\theta \ldots$ | $\theta(x,y)$

по x

(1) $1 = (r'_\varphi\varphi'_x + r'_\theta\theta'_x)\cos\varphi\sin\theta - r\sin\varphi\,\varphi'_x\sin\theta + r\cos\varphi\cos\theta\,\theta'_x$

$0 = (r'_\varphi\varphi'_x + r'_\theta\theta'_x)\sin\varphi\sin\theta + r\cos\varphi\,\varphi'_x\sin\theta + r\sin\varphi\cos\theta\,\theta'_x$

$z'_x = (r'_\varphi\varphi'_x + r'_\theta\theta'_x)\cos\theta - r\sin\theta\cdot\theta'_x$

задача $z'_x z''_{yy} - z'_y z''_{xy} = 0$

независими y, z

$x = w$

$y = u$ $\quad w(u,v) \to u(x,y) \to v(x,y)$

$z = v$

III Локални екстремуми

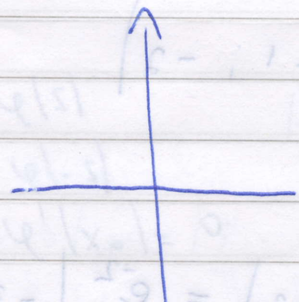
$z'_x = 0$ точките, които удовлетв. тази система
$z'_y = 0$ са стационарни

$$\begin{vmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{vmatrix}$$ минори

$\Delta_2 = z''_{xx} z''_{yy} - z''^{\,2}_{xy}$ II = 0 допълнителни

$\Delta_2 = 0$

$\sqrt{x}$ $\quad x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$



<u>задача</u> $z(x,y) = \frac{e^y}{\sqrt{x}}(1-x^2-y^2)$

$$z'_x = -\frac{e^y\,1}{x\,2\sqrt{x}}(1-x^2-y^2) + \frac{e^y}{\sqrt{x}}(-2x) =$$

$$= -\frac{e^y}{2x\sqrt{x}}(1-x^2-y^2) + \frac{e^y}{\sqrt{x}}2x =$$

$$= -\frac{e^y}{\sqrt{x}}\left(\frac{1-x^2-y^2}{2x} + 2x\right) = -\frac{e^y}{2x\sqrt{x}}(1-x^2-y^2+4x^2) =$$

$$= -\frac{e^y}{2x\sqrt{x}}(3x^2-y^2+1)$$

$$z'_y = e^y\sqrt{x}(1-x^2-y^2) - \frac{e^y}{\sqrt{x}}(-2y) = e^y\left(-\sqrt{x} - x^2\sqrt{x} - y^2\sqrt{x} + \frac{2y^2}{\sqrt{x}}\right) =$$

$$= e^y(-x - x^4 - y^2x + 2yx) = e^y x(-x^3 - 1 - y^2 + 2y)$$

$$\begin{cases} -3x^2 + y^2 = 1 \\ x^2 + y^2 + 4y = 1 \end{cases}$$

$$-4x^2 - 2y = 0$$

$$2y = -2x^2$$

$$-3x^2 + 4x^4 - 1 = 0$$

$$4x^4 - 3x^2 - 1 = 0$$

$$D = 9 + 16 = 5^2$$

$$x_1^2 = \frac{3-5}{8} = -\frac{1}{4}$$

$$x_2^2 = \frac{3+5}{8} = 1$$

$$x_{1,2} = \pm 1$$

$$y = -2.$$

$$(1, -2)$$

$$(-1, -2)$$

$$z''_{xx}(1,-2) = \frac{e^{-2}}{2}(-3x^2 + y^2 - 1)'' = \frac{e^{2}}{2}(-6)$$

$$z''_{xy}(1,-2) = \frac{e^{-2}}{2}(-4)$$

$$z''_{yy}(1,-2) = \frac{e^{-2}}{1}(-2+4) = 2e^{-2}$$

$$\Delta < 0$$

$$\begin{vmatrix} \frac{e^{-2}}{2}(-6) & \frac{e^{-2}}{2}(-4) \end{vmatrix}$$

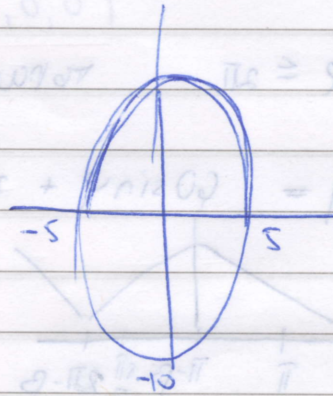
т. $(-1, -2) \notin DC$

$\Rightarrow (1, -2)$ - седловидна т.

задача

$$f(x,y) = x^2 + \frac{y^2}{4} - 12x + 8y \qquad (6,-16)$$

$$4x^2 + y^2 \le 100$$



$$f'_x = 2x - 12 = 0$$

$$f'_y = \frac{y}{2} + 8 = 0 \qquad y = \sqrt{100-4x^2}$$

$$y = -16 \qquad y = \sqrt{100-4x^2}$$

$$f(x, \sqrt{100-4x^2})$$

$$-5 \le x \le 5$$

$$\varphi'(x) \qquad \varphi(5)$$

$$\varphi(-5)$$

$$\varphi'(x_0) = 0$$

$$\begin{array}{l|l} x = a r\cos\varphi & x = 5r\cos\varphi \\ y = b r\sin\varphi & y = 10\sin\varphi \\ & r = 1 \end{array}$$

$$Q(x) = f(x, -\sqrt{100-4x^2})$$

$$Q'(x_1) = 0$$

заместване в условието

$$4x^2 + y^2 = 100$$

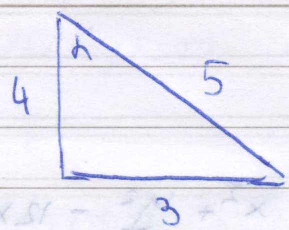
$$x = 5\cos\varphi \qquad 0 \le \varphi \le 2\pi$$

$$y = 10\sin\varphi$$

$$h(\varphi) = 25\cos^2\varphi + \frac{100\sin^2\varphi}{4} - 60\cos\varphi + 80\sin\varphi =$$

$$= -60\cos\varphi + 80\sin\varphi + 25 \qquad 0 \le \varphi \le 2\pi$$

$5|\cos\alpha\sin\varphi - \sin\alpha\cos\varphi| = 5\sin|\varphi - \alpha|$

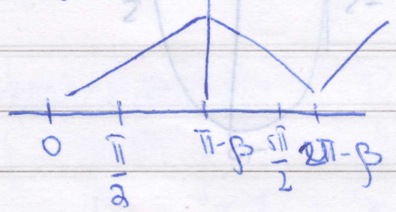


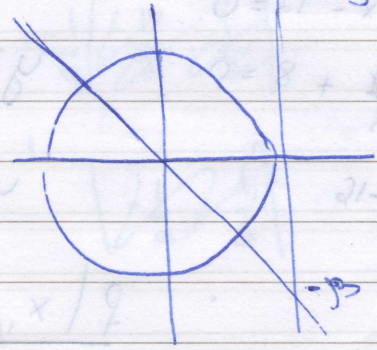
абсурдния метод →

⟶

$0 \le \varphi \le 2\pi$ търсим НГС $h(\varphi)$

$h'(\varphi) = 60\sin\varphi + 80\cos\varphi = 60\cos\varphi(\operatorname{tg}\varphi + \frac{4}{3})$





$h(0)$

$h(2\pi)$

$h(\pi-\beta)$

$h(2\pi-\beta)$

2 начин $f(x,y) = x^2 + \frac{y^2}{4} - 12x + 8y$

$4x^2 + y^2 = 100$

$g(x,y) = f(x,y) + \lambda(4x^2 + y^2 - 100)$

$$\begin{cases} g'_x = 2x - 12 + 8x\lambda = 0 \\ g'_y = 2y + 8 + 2y\lambda = 0 \\ 4x^2 + y^2 = 100 \end{cases}$$

$x(1+4\lambda) = 6$

$y(1+4\lambda) = -16$

$x = \frac{6}{1+4\lambda}$

$y = \frac{-16}{1+4\lambda}$

$\left(\frac{1}{1+4\lambda}\right)^2 (144+256) = 100$

$1+4\lambda = \pm 2$ $\lambda = 2$ $x=3$ $y=-8$

$\lambda =$

Задача

$u = 3x^2 + 5y^2 - 2z^2$

щом елипсата е затворена
по В-В е компактно множество

$\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} \leq 1$ $\Rightarrow$ има НМ и НГС

$u'_x = 6x$
$u'_y = 10y$ анулират се само в $(0,0,0)$
$u'_z = -4z$ $\Rightarrow u(0,0,0) = 0$

$\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} = 1$

$x = 3\cos\varphi \sin\theta$
$y = 5\sin\varphi \sin\theta$
$z = 2\cos\theta$
$0 \leq \varphi \leq 2\pi$
$0 \leq \theta \leq \pi$

замествам u_x, u_y, u_z в u и се получава $h(\varphi, \theta)$

$F(x,y,z) = 3x^2 + 5y^2 - 2z^2 + \lambda\left(\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} - 1\right)$ Тh Лагранж

$$\begin{cases} F'_x = 6x + \lambda \frac{2x}{9} = 0 & x(\lambda + 27) = 0 & x = 0 & \lambda = -27 \\ F'_y = 10y - \lambda\frac{2y}{25} = 0 & y(\lambda+125) = 0 & y = 0 & \lambda = -125 \\ F'_z = -\cancel{4z} + \lambda\frac{z}{2} = 0 & z(-8+\lambda) = 0 & z = 0 & \lambda = 8 \\ \frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} = 1 \end{cases}$$

1сл. $\lambda = -27$	2сл. $\lambda = -125$	3сл. $\lambda = 8$
$y = 0$	$x = 0$	$x = 0$
$z = 0$	$z = 0$	$y = 0$
$x = \pm 3$	$y = \pm 5$	$z = \pm 2$
$u = 27$	$u = 125$	$u = 8$

6.06.2013г. ДИС - упражнения

1. диф на неявни f
2. смяна на променливите
3. лок екстремуми
4. абсолютни екстремуми
5. множители на Лагранж
6. сходимост на $\Sum$
7. събиране на ст. редове и разливане на f в $\Sum$

Редове на Фурие

ако имаме f(x)

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx$$

n = 0, ...

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx$$

Th ↓

ако имаме интервал

$$\omega - \int_a^b f(x) dx = \int_{a+\omega}^{b+\omega} f(x) dx$$

$$\int_0^{\omega} f(x) dx = \int_a^{a+\omega} f(x) dx$$

f(x) - нечетна $\int_{-a}^{a} f(x) dx = 0$ f(x) - четна $\int_{-a}^{a} f(x) dx = 2 \int_0^a f(x) dx$

Th Ако $f(x)$ е частично гладка (в краен брой к диференц.) то редът на Фурие е сх (в рег не к диференцуема) и неговата сума е $f(x)$

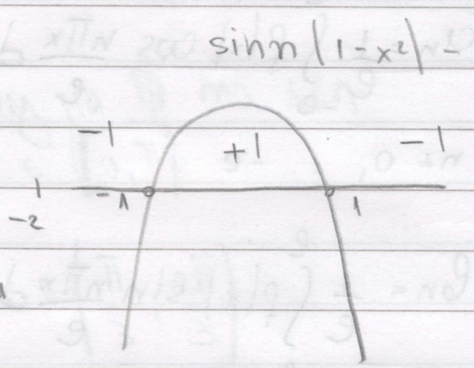
В точките, в които f е прекъсната (ср. аритметично на лявата и дясната граница)

$$\frac{\lim\limits_{\substack{x\to x_0\\ x<x_0}} f(x) + \lim\limits_{\substack{x\to x_0\\ x>x_0}} f(x)}{2}$$

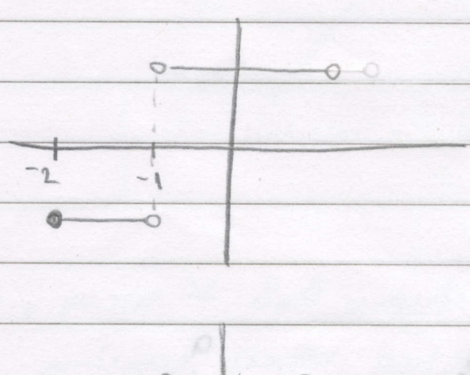
Задачи

$$f(x) = \underset{(\text{знак})}{\operatorname{sign}}(1-x^2) \underset{[-2,1]}{=}$$

$\operatorname{sign}(1-x^2)$ - графика



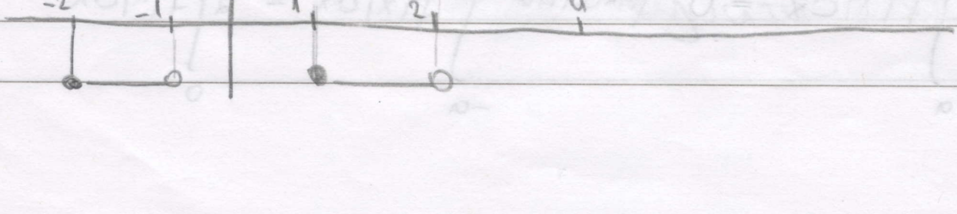
$$f(x) = \operatorname{sign}(1-x^2)\underset{[-2,1]}{=} \begin{cases} -1 & -2 \le x \le -1 \\ 1 & -1 < x < 1 \end{cases}$$



$2l = 3$ интервала в които е дефинирана

пренасяме f със същите интервали

- симетрична относно Ox



теории

$$b_n = \frac{1}{l}\int_{-l}^{l} \overbrace{f(x)}^{\text{четна - заради симетрията}} \underbrace{\sin\frac{n\pi x}{l}}_{\text{нечетна}}\,dx$$

четна · нечетна = н

$$\int_{-a}^{a} f(x)\,dx = 0 \Rightarrow b_n = 0$$

смятаме a_n

$$a_n = (2)\frac{2}{3}\int_0^{3/2} f(x)\cos\frac{2n\pi x}{3}\,dx =$$ ще се разложи на $\int$ от

от 0 до 1 и от 1 до $\frac{3}{2}$

защото сме взели $\frac{1}{2}$ от интервала

$$a_0 = \frac{4}{3}\int_0^{3/2} f(x)\,dx = \frac{4}{3}\left(\int_0^1 1\,dx + \int_1^{4/3} -1\,dx\right) = \frac{4}{3}\left(1 - \frac{1}{2}\right) = \frac{2}{3} = a_0$$

$$a_n = \frac{4}{3}\int_0^{3/2} f(x)\cos\frac{2n\pi x}{3}\,dx = \frac{4}{3}\left(\int_0^1 f(x)\cos\frac{2n\pi x}{3} + \int_1^{2/3} f(x)\cos\frac{2n\pi x}{3}\,dx\right) =$$

$$= \frac{4}{3}\int_0^1 \frac{1}{3}\cos\frac{2n\pi x}{3}\,dx - \frac{4}{3}\int_1^{3/2}\cos\frac{2n\pi x}{3}\,dx =$$

$$= \frac{4}{3}\left(\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\Big|_0^1 - \frac{3}{2\pi}\sin\frac{2n\pi x}{3}\Big|_1^{3/2}\right) = \frac{4}{3}\left(2\sin\frac{2n\pi}{3}\cdot\frac{3}{2n\pi}\right) = \frac{4\sin\frac{2n\pi}{3}}{n\pi}$$

при кратни на $n = 0$

при кратни $n+1 = 1$

при кратни $n+2 = -1$

Задача

$f(x) = x$ какво озн да и разв по sin

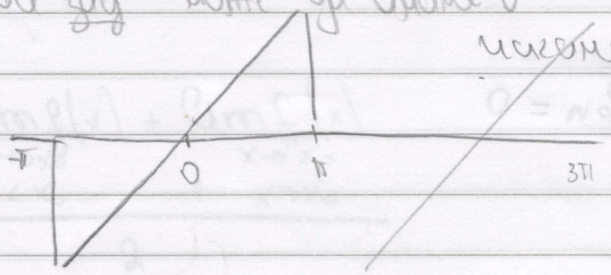
$f(x) = \{$

$\rightsquigarrow$

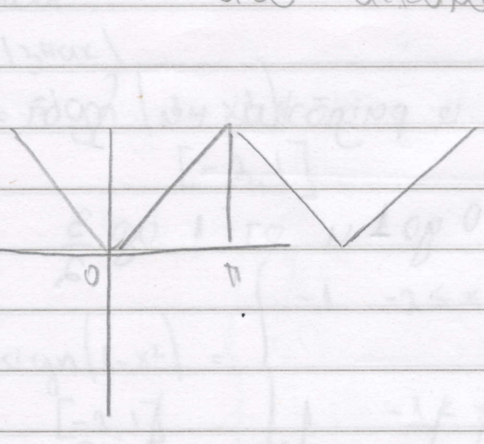
какви зад може да имаме ↓

по sin

искаме коефициент от a_n = да са = 0

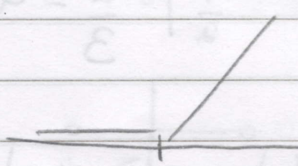


ако искаме да разв по cos



коеф от $b_n = 0$

по sin и cos



в интервала $[0, 2\pi]$

взимаме в $0, 2\pi = x$

и след това по периодичност

задача

$$f(x) = x \quad [-\pi, \pi]$$

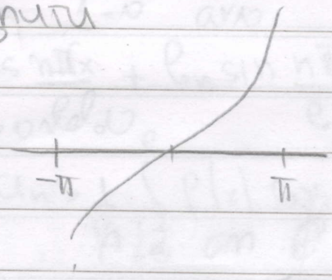
$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = \frac{2}{n\pi}\int_{0}^{\pi} x \sin nx\,dx\,n = \frac{2}{\pi n}\int_{0}^{\pi} x\,d\cos nx =$$

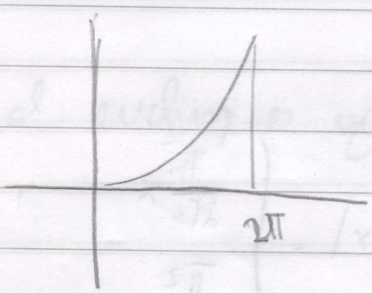
$$= \frac{-2}{n\pi}\left(x\cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx\,dx \right) = -\frac{2}{n\pi} \cos n\pi + \underbrace{\int_0^{\pi} \cos nx\,dnx}_{0} = \frac{2(-1)^n}{n}$$

задачи от изпити

$$f(x) = x^3$$



$$f(x) = x^3 \quad [0, 2\pi]$$

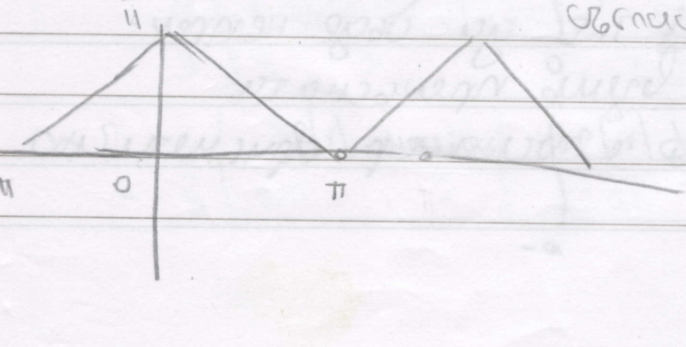


задача

а) да се развие $f(x) = \begin{cases} \pi + x, & [-\pi, 0] \\ \pi - x, & [0, \pi] \end{cases}$

б) за кой x получената ред е сх

в) сумата на реда



Непрекъсната навсякъде

съгласно Th реда на Фурие във $\forall$ точка е сх

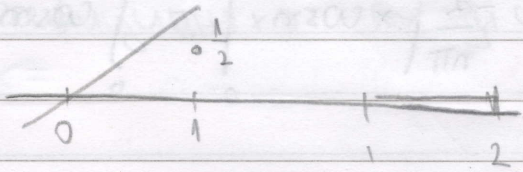
и във всяка точка сумата

му е $f(x)$

За домашно!

Да се развие $f(x) = \begin{cases} x & 0 \le x < 1 \\ a & x = 1 \\ 0 & 1 < x \le 2 \end{cases}$ в ред на Ф

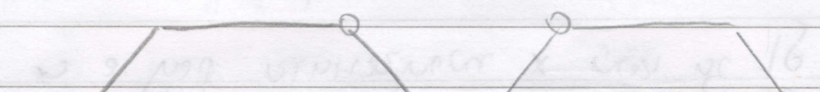
Да се определи a, при която сумата на реда на Фурие да е $= f(x)$ във всяка точка



ако $a = \frac{1}{2}$ реда за $\forall x$ съвпада с f

задача да се развие в ред на Ф по sin

$$f(x) = \begin{cases} \frac{\pi}{2\sqrt{3}} x, & x \in [0, \pi] \\ \frac{\pi^2}{6\sqrt{3}}, & x \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right) \\ \frac{\pi}{2\sqrt{3}}(\pi - x), & x \in \left(\frac{2\pi}{3}, \pi\right] \end{cases}$$



ако искаме Ф трябва да вземем f т.е да бъде непрек.

ако я вземем прекъснати сумата е точно ср. аритметично