27.II.2013г. Диференциално и интегрално смятане II част
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(1) $\lim\limits_{\substack{x \to 0 \\ x \geq 0}} \dfrac{\sqrt{2x+1} - \sqrt[3]{1+3x}}{x^2} = A(x)$ Пиано

$$(1+x)^{\alpha} = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + o(x^2)$$

$$A(x) = \frac{\cancel{1} + \cancel{\frac{1}{2} \cdot 2x} - \frac{1}{8} \cdot 4x^2 \quad \cancel{-1} - \cancel{\frac{1}{3} \cdot (+3x)} + \frac{1}{9} \cdot 9x^2 + o(x^2)}{x^2} =$$

$\binom{\frac{1}{2}}{0} = 1$ $\quad\binom{\frac{1}{3}}{0} = 1$

$\binom{\frac{1}{2}}{1} = \frac{1}{2}$ $\quad\binom{\frac{1}{3}}{1} = \frac{1}{3}$

$\binom{\frac{1}{2}}{2} = -\frac{1}{8}$ $\quad\binom{\frac{1}{3}}{2} = -\frac{1}{9} = \dfrac{\frac{1}{3} \cdot \left(-\frac{2}{3}\right)}{2}$

при $1-3x$

$$= \frac{2x + \frac{1}{2}x^2 + o(x^2)}{x^2} \Longrightarrow \text{няма граница}$$

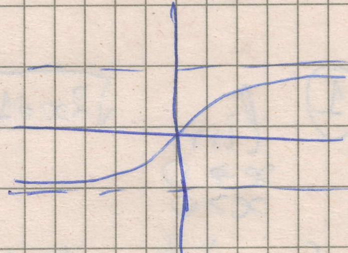
при $1+3x$

$$= \frac{\frac{1}{2}x^2 + o(x^2)}{x^2} \longrightarrow \frac{1}{2} + \frac{o(x^2)}{x^2} \longrightarrow \frac{1}{2}$$

VI.2.1

~~Пример~~ ② $(2\operatorname{arctg} x - \pi)\, x \quad x \to \infty$

$2\operatorname{arctg} x \to 0$, $x \to \infty$



$= \dfrac{2\operatorname{arctg} x - \pi}{\frac{1}{x}} =$

$= \dfrac{2 \cdot \frac{1}{1+x^2}}{-\frac{1}{x^2}} = \dfrac{-2x^2 \cdot 1}{1+x^2} = -2 \to -2$

$f(x)^{g(x)} = y =$ $\qquad \ln y = g(x) \cdot \ln f(x) \to A$

$= e^{g(x)\ln f(x)}$ $\qquad e^{\ln y} = e^{A}$

③ $f(x) = \sqrt[3]{4x - x^2}$ $\qquad$ лок. екстрем.

1) Най-голяма стойност или абсолютни екстремуми (неравенства)

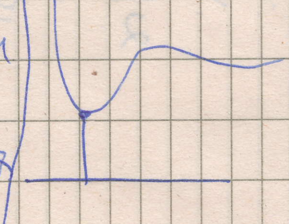
$\underset{\cancel{x \in M}}{f(x)} \geq A$ (намерете най-малката стойн. в това множество M)

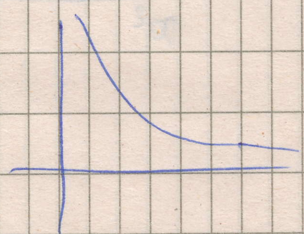
$f(x) \geq f(x_0) \qquad f(x) \geq \inf_{x \in M} f(x)$

~~2)~~

2) локални екстремуми

3) графика на функция





VI. 2.2

И в трите случая първо се намират критичните точки: (~~крайните точки~~)

за I, II, III:

1) краища на D_f (деф. множ.) a ——— b
2) $f'(x)$ не съществува
3) $f'(x) = 0$
4) $f''(x)$

всичките 4 точки за III) графика на ф-я

I $\left.\begin{array}{l} f(P) \\ f(-\infty) \\ f(+\infty) \end{array}\right\}$ ако множ. е $(-\infty; +\infty)$

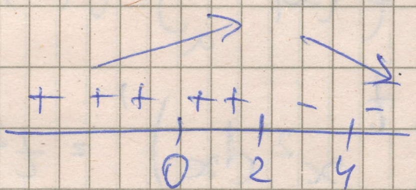
II знаци на f'

$$f'(x) = \frac{1}{3}\,\frac{(4-2x)}{\sqrt[3]{(4x-x^2)^2}}$$

$-\infty; +\infty$

$0, 4$

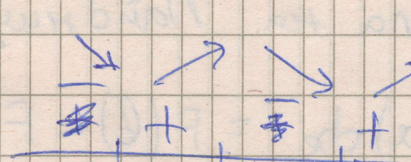
2



$\Rightarrow$ 1 лок. екстремум (max)

(4) $f(x) = \sqrt[3]{(4x-x^2)^2}$

$$f'(x) = \frac{2}{3}\,\frac{(4-2x)}{\sqrt[3]{(4x-x^2)}}$$



5 Определени интеграли

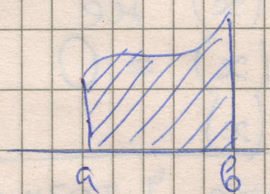
неопределен интеграл

$$F(x) = \int f(x)\,dx + C \text{ =(число)}$$

определен интеграл

$$\int_a^b f(x)\,dx = \text{(функция)}$$

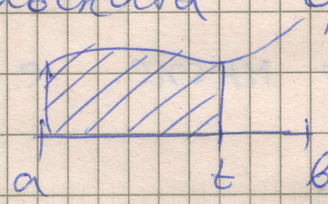
$$F'(x) = f(x)$$



Теорема на Лайбниц - Нютон

Нека $f(x)$ - непрекъсната ф-я

$$\int_a^t f(x)\,dx = \cancel{F(t)}$$



$f(x)$ $[a,b]$ $\quad \varphi(t) = \int_a^t f(x)\,dx \quad \varphi'(t) = f(t)$

$$\left(\int_0^t x^2\,dx\right)' = t^2$$

Формула на Лайбниц - Нютон $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F(x) = \int f(x)\,dx \qquad \frac{x^3}{3} \qquad A = \int_0^t x^2\,dx = \frac{t^3 - 0}{3} = \frac{t^3}{3}$

$$A' = t^2$$

VI. 2. 4

(1) $\int_2^3 \frac{dx}{x^4-1}$ $\quad \frac{1}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$

$x=1 \quad A=\frac{1}{4}$

$x=-1 \quad B=-\frac{1}{4}$ $\quad x=i \quad 1=(Ci+D)(-2)$

$C=0 \quad D=-\frac{1}{2}$

$0 = A + B + C \Rightarrow C=0$

$x=0 \quad 1=A-B-D \Rightarrow D=-\frac{1}{2}$

$\frac{1}{x^4-1} = \frac{1}{4}\cdot\frac{1}{x-1} - \frac{1}{4}\cdot\frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{1+x^2}$

$F(x) = \int \frac{1}{x^4-1}dx = \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) - \frac{1}{2}\operatorname{arctg} x$

$= \ln\sqrt[4]{\frac{x-1}{x+1}} - \frac{1}{2}\operatorname{arctg} x$

$\int_2^3 \frac{dx}{x^4-1} = \ln\sqrt[4]{\frac{1}{2}} - \frac{1}{2}\operatorname{arctg} 3 - \ln\sqrt[4]{\frac{1}{3}} + \frac{1}{2}\operatorname{arctg} 2$

(2) $\int_0^1 \frac{x^4\,dx}{(1+x^2)^2}$

$\int \frac{x^4\,dx}{(1+x^2)^2} = \frac{1}{2}\int x^3 (1+x^2)^{-2}\,d(x^2+1) =$

$= -\frac{1}{2}\int x^3\,d(x^2+1)^{-1} = -\frac{1}{2}\cdot\frac{x^3}{1+x^2} + \frac{1}{2}\int \frac{3x^2\,dx}{x^2+1} =$

$= -\frac{1}{2}\cdot\frac{x^3}{1+x^2} + \frac{3}{2}\int dx - \frac{3}{2}\int\frac{dx}{x^2+1}$

$\int \frac{x^4\,dx}{(1+x^2)^2} = -\frac{1}{2}\cdot\frac{x^3}{1+x^2} + \frac{3}{2}x - \frac{3}{2}\,\mathrm{arctg}\,x$

$\int_0^1 \frac{x^4}{(1+x^2)^2}\,dx = -\frac{1}{4} + \frac{3}{2} - \frac{3}{2}\cdot\frac{\pi}{4}$

(3) $\int_0^1 x \arcsin x\,dx$

$\int x\arcsin x\,dx = \frac{1}{2}\int \arcsin x\,dx^2 =$

$-1 \le x \le +1$

$= \frac{1}{2}x\arcsin x - \frac{1}{2}\int\frac{x^2\,dx}{\sqrt{1-x^2}}$

$-1 < x < 1$

VI.2.6

1) $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

2) $x = \sin t$

3) $x^2(1-x^2)^{-\frac{1}{2}} = \cancel{x^2}\, x(x^{-2}-1)^{-\frac{1}{2}}$

$\text{нел}: x^{-2}-1 = t^2$ ✓

$\frac{2+1}{2} - \frac{1}{2} = 1 \qquad \frac{1+1}{-2} = -1$

4) $\sqrt{1-x^2} = xt+1$ ✓

$\sqrt{ax^2+bx+c} = \pm\sqrt{a}x \pm t$
$\pm xt \pm \sqrt{c}$
$(x-x_1)t$

$(x+1)t$

$\frac{\sqrt{1-x^2}+1}{x} = t$

$\cancel{x}\; xt-1 \geq 0$

$\cancel{1}-x^2 = x^2t^2+2xt+\cancel{1}$

$x^2(t-1)(t+1)+2xt = 0$

$x(x(t-1)(t+1)+2t) = 0$

$x_1 = 0 \qquad x_2 = \frac{-2t}{(t^2-1)}$

$0 \qquad 1$

$\sqrt{1-x^2} = \varphi(t)$

dx

VI 2.7

28.II.2013r. (1) 1) $\int_0^1 x \arcsin x \, dx = \frac{\pi}{8}$

$$\int x \arcsin x \, dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \cancel{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x}$$

$-1 \le x \le 1$ $\qquad -1 < x < 1$

$$I = \int \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx = \arcsin x - \int \sqrt{1-x^2}\,dx =$$

$$= \arcsin x - x\sqrt{1-x^2} + \underbrace{\frac{1}{2}\int \frac{-x^2\,dx}{\sqrt{1-x^2}}}_{-I}$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2}$$

$$\varphi(\varepsilon) = \int_0^\varepsilon x \arcsin x = \frac{\varepsilon^2}{2}\arcsin\varepsilon - \frac{1}{4}\arcsin\varepsilon + \frac{1}{4}\sqrt{1-\varepsilon^2}$$

$0 < \varepsilon < 1$

0 ε 1

$< \varepsilon \le 1$

$\varepsilon \to 1$

$$\int_0^1 x \arcsin x \, dx = \frac{\pi}{4} - \frac{\pi}{8} + 0 = \frac{\pi}{8}$$

VI 2-8

2) $\int_{-\frac{1}{2}} x \arcsin x \, dx = -\frac{\arcsin x \cdot \cos 2\arcsin x}{4} + \frac{1}{8}\sin 2\arcsin x$

$-1 \le x \le 1 \qquad \arcsin x = t \qquad \sin t = x$

$-1 \le x \le 1 \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$\underline{I} = \frac{2}{2 \cdot 2}\int \sin t \cdot t \cdot \cos t \, dt \cdot 2 = -\frac{1}{4}\int t \, d\cos 2t + \frac{1}{4 \cdot 2}\underbrace{\int \cos 2t \, dt \cdot 2}_{\sin 2t}$

$\int_0^1 x \arcsin x \, dx = \frac{\pi}{8} + 0 = \frac{\pi}{8}$

1) $\int \frac{x^2}{\sqrt{1-x}} dx \qquad \sqrt{1-x^2} = \pm(\pm x \cdot t \pm 1)$

$\cancel{x \cdot t + 1} \qquad xt + 1$

$\cancel{-xt - 1}$

$xt = \sqrt{1-x^2} - 1 \qquad t = \frac{\sqrt{1-x^2} - 1}{x} = \frac{-x}{\sqrt{1-x^2} + 1}$

$1 - x^2 = x^2 t^2 + 2xt + 1 \qquad x = \frac{-2t}{1+t^2}$

$0 \le x \le 1$

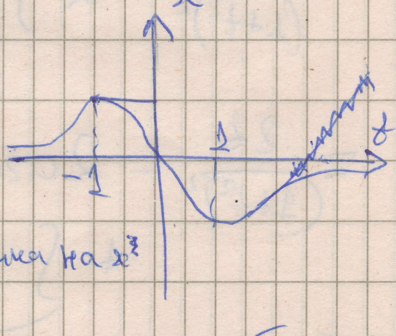
$x = -\frac{2t}{1+t^2}$

$x' = -\frac{4t^2 - 1 - 2t^2 - t^4}{(1+t^2)^2} = -\frac{-t^4 + 2t^2 - 1}{(1+t^2)^2} = 2\frac{(t^2-1)}{(1+t^2)^2}$

$0 \le x \le 1$

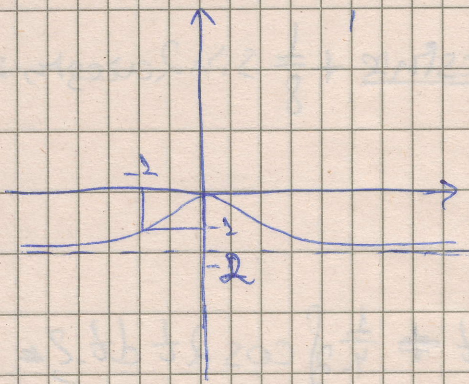
$t \le -1 \qquad -1 \le t \le 0$

$xt = -\frac{2t^2}{1+t^2}$



графика на x'

VI.2.9



графика на xt

$xt \geq -1$

$-1 \leq t \leq 0$

$xt + 1 \geq 0$

$0 \leq x \leq 1$ $\quad t = \frac{-x}{\sqrt{1-x^2}+1}$

$-1 \leq t \leq 0$ $\quad x = \frac{-2t}{1+t^2}$

$\sqrt{1-x^2} = xt + 1$

$\int \frac{x^2}{\sqrt{1-x^2}} dx =$

$\sqrt{1-x^2} = -\frac{2t^2}{1+t^2} + 1 = \frac{1-t^2}{1+t^2}$

$= 2 \int \frac{4t^2}{(1+t^2)^2} \cdot \cancel{\frac{1+t^2}{1-t^2}} \cdot \frac{\cancel{t^2-1}^{\,-1}}{(1+t^2)^2} = -8 \int \frac{t^2}{(1+t^2)^3} dt =$

$= -4 \int \frac{t\, dt^2}{(1+t^2)^3} = 2 \int t\, d\frac{1}{(1+t^2)^2} = \frac{2t}{(1+t^2)^2} - 2 \int \frac{(1+t^2-t^2)dt}{(1+t^2)^2} =$

$= \frac{2t}{(1+t^2)^2} - 2 \int \frac{1}{1+t^2} dt + 2 \int \frac{t^2 dt}{(1+t^2)^2} =$

$= \frac{2t}{(1+t^2)^2} - 2 \operatorname{arctg} t - \int t\, d\frac{1}{1+t^2} = \frac{2t}{(1+t^2)^2} - 2 \operatorname{arctg} t - \frac{t}{1+t^2} +$

$+ \underbrace{\int \frac{1\, dt}{1+t^2}}_{\operatorname{arctg} t}$

VI. 2.10

$0 \leq x \leq 1$ $\quad x = 1 \Rightarrow t = -1$

$t = \dfrac{-x}{\sqrt{1+x^2}+1}$ $\quad \int x \arcsin x \, dx = \dfrac{\pi}{4} + \left(\dfrac{\pi}{2} - \dfrac{\pi}{4}\right) \cdot \dfrac{1}{2} = \dfrac{\pi}{8}$

(3) $\displaystyle\int_0^{\pi} \frac{dx}{2+\cos x}$ $\quad$ (4) $\displaystyle\int_0^{\pi} \frac{dx}{2+\cos^2 x}$

$\operatorname{tg} \frac{x}{2} = t$

(3) $\displaystyle\int \frac{dx}{2+\cos x}$ $\quad F(\sin x, \cos x)$ $\quad F(-\sin x, -\cos x)$

$\operatorname{tg} x = t$ $\quad \sin x = \pm \dfrac{t}{\sqrt{1+t^2}}$ $\quad \cos x = \pm \dfrac{1}{\sqrt{1+t^2}}$

$\operatorname{tg} \frac{x}{2} = t$ $\quad \sin x = \dfrac{2t}{1+t^2}$ $\quad \cos x = \dfrac{1-t^2}{1+t^2}$

$-\pi < x < \pi$

$x = 2 \operatorname{arctg} t$ $\quad x - \frac{\pi}{2} = t$ $\quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$0 < x < \pi$ $\quad x = t + \frac{\pi}{2}$

$$\int_0^{\pi} \frac{dx}{2+\cos x} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{2-\sin t} \qquad dx = \frac{2\,dt}{1+t^2}$$

$$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} = \int \frac{2}{3+t^2}\,dt = \frac{2}{3}\sqrt{3} \int \frac{\frac{t}{\sqrt{3}}}{1+\left(\frac{t}{\sqrt{3}}\right)^2} =$$

$$= \frac{2\sqrt{3}}{3} \operatorname{arctg} \frac{t}{\sqrt{3}}$$

VI.2.11

$$\int_0^\varepsilon \frac{dx}{2+\cos x} = \frac{2\sqrt{3}}{3}\,\operatorname{arctg}\frac{\operatorname{tg}\frac{\varepsilon}{2}}{\sqrt{3}} = \frac{\pi}{2}\cdot\frac{2\sqrt{3}}{3}$$

$\varepsilon \to \pi$

$$\int_0^\pi \frac{dx}{2+\cos x} = \frac{2\sqrt{3}}{3}\cdot\frac{\pi}{2}$$

$0 \quad \frac{\pi}{2} \quad \pi$

$\operatorname{tg} x = t$

06.III.2013r.

① $\int_0^\pi \frac{dx}{2+\cos x}$ ② $\int_0^\pi \frac{dx}{2+\cos^2 x}$ $\int_a^b f(x)\,dg(x) = f(x)\cdot g(x)\Big|_a^b -$

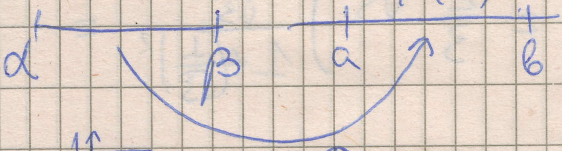
$\operatorname{tg}\frac{x}{2} = t$ $\operatorname{tg} x = t$ $\int_a^b g(x)\,df(x)$

$\int_a^b f(x)\,dx$ $x = \varphi(t)$

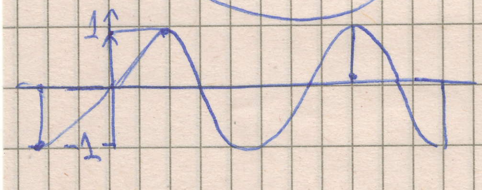
$\int f(x)\,dx$ $a = \varphi(\alpha)$, $b = \varphi(\beta)$ $\alpha \quad \beta$ $a \quad b$

$x = \varphi(t)$

$t = \psi(x)$



$\varphi(t) \in [a, b]$



$x = \sin t$ $0 = \varphi(\alpha)$ $\alpha = 0$

$x \in [0, 1]$ $1 = \varphi(\beta)$ $\beta = \frac{\pi}{2}$

$x \in [-1, 1]$ $-1 = \varphi(\alpha)$ $\alpha = -\frac{\pi}{2}$

$1 = \varphi(\beta)$ $\beta = \frac{5\pi}{2}$

$t \in \left[-\frac{\pi}{2}, \frac{5\pi}{2}\right]$

VII.2.12

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi)\,d\varphi$$

(1) $x = 2\,\mathrm{arctg}\, t$

$0 \to 0$

$\pi \to$

$\mathrm{arctg}\, t < \frac{\pi}{2}$

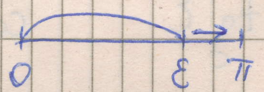
$2\,\mathrm{arctg}\, t < \pi$

$$\int_0^\varepsilon \frac{dx}{2+\cos x} = \int_0^\beta \frac{2}{\left(2+\frac{1-t^2}{1+t^2}\right)} \frac{dt}{(1+t^2)} =$$

$\alpha = 0$

$\beta = tg\,\frac{\varepsilon}{2}$

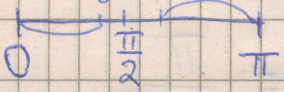
$\varepsilon < \pi$



$$= 2\int_0^\beta \frac{dt}{3+t^2} = \frac{2}{\sqrt3}\int_0^\beta \frac{d\frac{t}{\sqrt3}}{1+\left(\frac{t}{\sqrt3}\right)^2} = \frac{2}{\sqrt3}\,\mathrm{arctg}\,\frac{t}{\sqrt3}\Big|_0^\beta =$$

$$= \frac{2}{\sqrt3}\,\mathrm{arctg}\,\frac{tg\frac{\varepsilon}{2}}{\sqrt3} \longrightarrow \frac{\pi}{\sqrt3}$$

(2) $tg\,x = t$



$$\int_0^\pi \frac{dx}{2+\cos^2 x} = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}$$

$x - \frac{\pi}{2} = t$

$$\int_0^\pi \frac{dx}{2+\cos^2 x} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{2+\sin^2 t} =$$

$x = t + \frac{\pi}{2}$

$\cos\left(t+\frac{\pi}{2}\right) = -\sin t$

$$= 2\int_0^{\frac{\pi}{2}} \frac{dt}{2+\sin^2 t} =$$

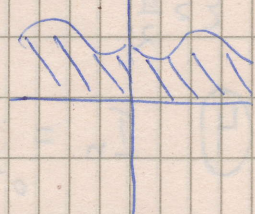
$\sin t = \frac{\pm 2t}{\sqrt{1+t^2}}$

I $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$f(x)$ е четна

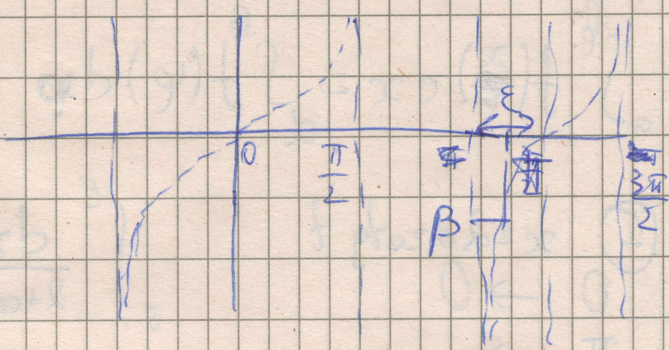
$\int_{-a}^{0} f(x)\,dx$

$x = -t$



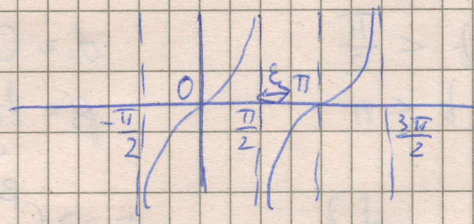
$f(x)$ е нечетна 0

(3) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{dx}{2+\cos^2 x} =$



$\operatorname{tg} x = t \qquad \varepsilon < t \le \pi$

$\beta = \operatorname{tg}\varepsilon \qquad \cos x = \pm \dfrac{1}{\sqrt{1+\operatorname{tg}^2 t}}$



$$= \int_{\operatorname{tg}\varepsilon}^{0} \frac{1}{\left(2+\frac{1}{1+t^2}\right)} \cdot \frac{dt}{(1+t^2)} = \qquad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$= \int_{\operatorname{tg}\varepsilon}^{0} \frac{dt}{3+2t^2} = \frac{\sqrt{3}}{3\sqrt{2}} \int_{\operatorname{tg}\varepsilon}^{0} \frac{d\frac{t\sqrt{2}}{\sqrt{3}}}{\left(\frac{t\sqrt{2}}{\sqrt{3}}\right)^2} = \frac{\sqrt{3}}{3} \Big|_{\operatorname{tg}\varepsilon}^{0} =$$

$$= -\frac{\sqrt{3}}{3\sqrt{2}} \operatorname{arctg} \frac{\sqrt{2}}{\sqrt{3}} \operatorname{tg}\varepsilon = \frac{\pi}{2\sqrt{6}}$$

$\operatorname{tg}\varepsilon \to -\infty$

$\varepsilon \to \frac{\pi}{2}$

$\varepsilon > \frac{\pi}{2}$

(4) $\displaystyle I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \cancel{\sin^n x \cdot x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x\, d\sin^n x}$

$$= -\int_0^{\frac{\pi}{2}} \sin^{n-1} x\, d(\cos x) = -\sin^{n-1} x \cdot \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\, d\sin^{n-1} x =$$

$$= \underbrace{-\sin^{n-1} x \cos x \Big|_0^{\frac{\pi}{2}}}_{0} + \int_0^{\frac{\pi}{2}} \cos x\,(n-1) \cdot \sin^{n-2} x \cdot \cos x\, dx =$$

VI.2.14

$$= (n-1)\int_0^{\frac{\pi}{2}} \underbrace{(1-\sin^2 x)}_{\cos^2 x} \cdot \sin^{n-2} x \, dx = (n-1)\int_0^{\frac{\pi}{2}} (1-\sin^2 x)\sin^{n-2} x \, dx$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$

$$I_{2n-2} = \frac{2n-3}{2n-2} I_{2n-4}$$

$$\vdots$$

$$I_2 = \frac{1}{2} I_0$$

$$I_{2n} = \frac{(2n-1)!!}{2^n \cdot n!} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$$

$$\int_a^b dx = b-a$$

$$I_{2n-1} = \frac{2n-2}{2n-1} I_{2n-3}$$

$$I_{2n-3} = \frac{2n-4}{2n-3} I_{2n-5}$$

$$\vdots$$

$$I_3 = \frac{2}{3} I_1$$

$$I_{2n-1} = \cancel{\frac{(2n-2)!!}{(2n-1)!!}} = \frac{2^n (n-1)!}{(2n-1)!!} \cdot \underset{1}{\underset{\parallel}{I_1}} =$$

$$\int_0^{\frac{\pi}{2}} \sin^{2n-2} x \, dx \geq \int_0^{\frac{\pi}{2}} \sin^{2n-1} x \, dx \geq \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$$

$$0 < \sin x < 1$$

$$0 < f \leq g \qquad \int_a^b f \leq \int_a^b g$$

VI. 2.15

$$\frac{(2n-3)!!}{2^{n-1}(n-1)!}\cdot\frac{\pi}{2} \ge \frac{2^n(n-1)!}{(2n-1)!!} \ge \frac{(2n-1)!!}{2^n \cdot n!}\cdot\frac{\pi}{2}$$

$$\frac{\pi}{2} \ge \frac{2\left(2^{n-1}(n-1)!\right)^2}{(2n-1)\left((2n-3)!!\right)^2} \ge \frac{(2n-1)}{2n}\cdot\frac{\pi}{2}$$

$$\frac{\pi}{2}$$

$$\lim_{n\to\infty} \frac{2\left(2^{n-1}(n-1)!\right)^2}{(2n-1)\left((2n-3)!!\right)^2} = \frac{\pi}{2}$$

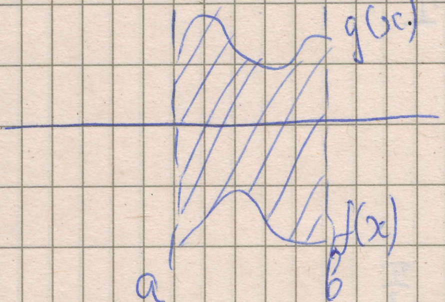
~~Лица.~~

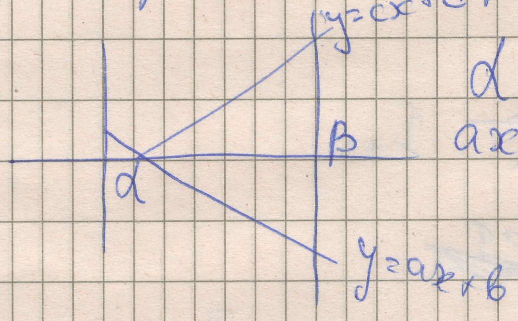
Лица.

I Вариант - Криволинеен трапец

$a \le x \le b$

$f(x) \le y \le g(x)$ > множеството от тях е криволинеен трапец

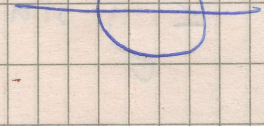


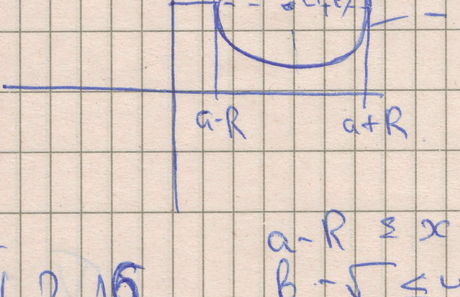


$\alpha \le x \le \beta$

$ax+b \le y \le cx+d$

$y = b \pm \sqrt{R^2-(x-a)^2}$





$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$(x-a)^2 + (y-b)^2 = R^2$$

$a-R \le x \le a+R$

$b-\sqrt{} \le y \le b+\sqrt{}$

VI.2.16

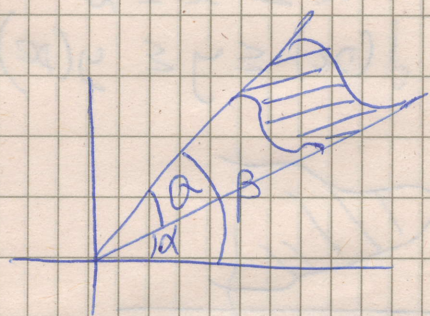
$$S = \int_a^b \left(g(x) - f(x)\right) dx$$

(II) Вариант – криволинеен сектор

$$\alpha \le \theta \le \beta$$

$$0 \le \rho_1(\theta) \le \rho \le \rho_2(\theta)$$

$$0 \le \rho$$

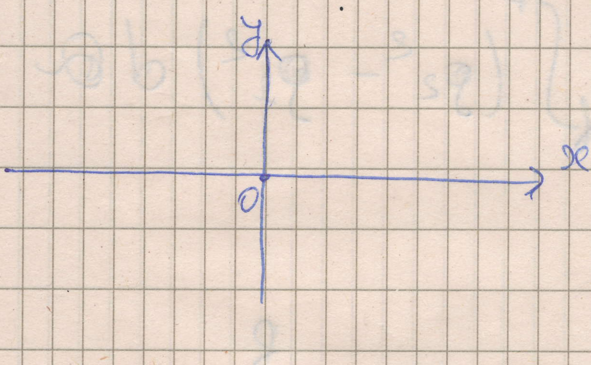


$$S = \frac{1}{2} \int_\alpha^\beta \left(\rho_2^2 - \rho_1^2\right) d\theta$$

(1) Да се пресметне лицето на фигурата в I квадрант окръжност: $x^2+y^2=3a^2$
и ограничено от параболите:

$x^2 = 2ay$

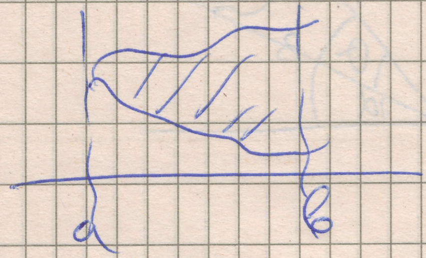
$y^2 = 2ax$



07.III.2013г.

Лице на криволинеен трапец

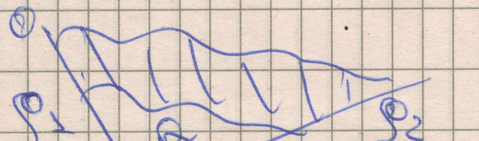
$$G \begin{cases} a \leq x \leq b \\ f(x) \leq y \leq g(x) \end{cases}$$



$$S = \int_a^b (g(x) - f(x))\, dx$$

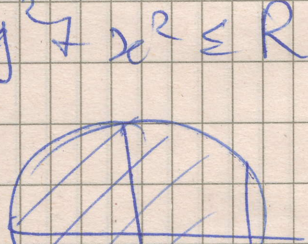
Лице на криволинеен сектор

$$\alpha \leq \theta \leq \beta$$
$$\rho_1(\theta) \leq \rho \leq \rho_2(\theta)$$



$$S = \frac{1}{2} \int_\alpha^\beta (\rho_2^2 - \rho_1^2)\, d\theta$$

$$y^2 + x^2 \leq R^2$$



$$y_1 = \sqrt{R^2 - x^2} \qquad S = 2 \int_{-R}^{R} \sqrt{R^2 - x^2}\, dx =$$

$$y_2 = \sqrt{R^2 - x^2} \qquad = 4 \int_0^R \sqrt{R^2 - x^2}\, dx$$

VI.2.18

$${}_0\int^{\varepsilon} \sqrt{R^2-x^2}\,dx$$

$${}_0\int^{\varepsilon} \sqrt{R^2-x^2}\,dx = x\sqrt{R^2-x^2}\Big|_0^{\varepsilon} - {}_0\int^{\varepsilon} \frac{x^2+R^2-R^2}{\sqrt{R^2-x^2}} = \varepsilon\sqrt{R^2-\varepsilon^2} -$$

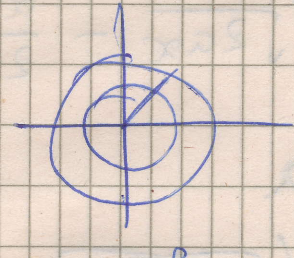
$$I + \frac{R^2}{R}\,{}_0\int^{\varepsilon} \frac{dx}{\sqrt{1-\left(\frac{x}{R}\right)^2}}$$

$$I = \frac{1}{2}\left(\varepsilon\sqrt{R^2-\varepsilon^2} + R^2 \arcsin\frac{x}{R}\Big|_0^{\varepsilon}\right) \xrightarrow{\varepsilon \to R}$$

$$\xrightarrow{\varepsilon \to R} \frac{1}{2}R^2 \arcsin 1 = \frac{R^2\pi}{4}$$

$0 \le \Theta \le 2\pi$

$0 \le \rho \le R$

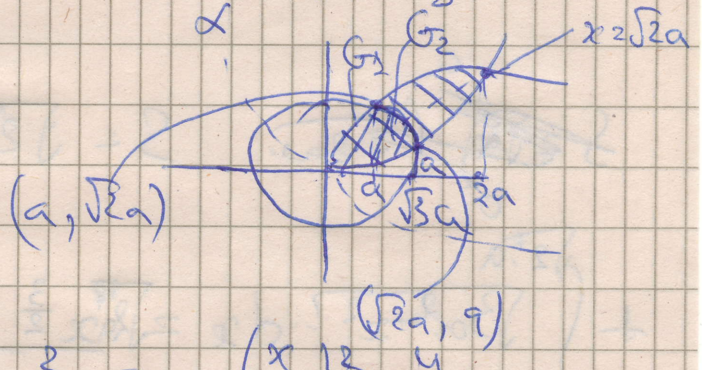


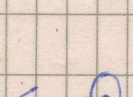
$$S = \frac{1}{2}\int_0^{2\pi} R^2\, d\Theta = \frac{2\pi}{2}R^2$$

$$S = \frac{1}{2}\int_{\alpha}^{\beta} (\rho_2^{\,2}-\rho_1^{\,2})\,d\Theta$$

1

$$\begin{cases} x^2+y^2=3a^2 \\ x^2=2ay \\ y^2=2ax \end{cases} \qquad a>0$$





~~$2ay\sqrt{2a}$~~ $\quad$ ~~$2ay+y^2=3a^2$~~

~~$y^2+2ay-3a^2=0$~~

~~$D=a^2+$~~

$$\left(\frac{x}{y}\right)^2 = \frac{y}{x}$$

$$\left(\frac{y}{x}\right)^3 = 1$$

$y=x \qquad x^2=2ax \qquad x(1-2a)=0$

VII.2.19

$y^2 = 2ax$
$y^2 = 2ay$

$y^2 - 2ay = 0$
$y(y - 2a) = 0$
$y = 0 \quad y = 2a$

$x^2 - 2ax = 0$
$x(x - 2a) = 0$
$x = 0 \quad x = 2a$

$\cancel{2x^2 = 3a^2}$
$\cancel{2x^2 - 3a^2 = 0}$
$\cancel{x^2 = \frac{3}{2} a^2}$
$\cancel{x = \pm \sqrt{\frac{3}{2}}\, a}$

$x^2 + 2ax - 3a^2 = 0$
$D = a^2 + 3a^2 = 4a^2$
$x_{1,2} = \frac{-a \pm 2a}{\cancel{\quad}} \to a, \; -3a$

$G_1 \begin{cases} 0 \le x \le a \\ \frac{x^2}{2a} \le y \le \sqrt{2ax} \end{cases}$

$G_2 \begin{cases} a \le x \le \sqrt{2}a \\ \frac{x^2}{2a} \le y \le \sqrt{3a^2 - x^2} \end{cases}$

$$S_1 = \int_0^a \left(\sqrt{2ax} - \frac{x^2}{2a} \right) dx$$

$$S_2 = \int_a^{\sqrt{2}a} \left(\sqrt{3a^2 - x^2} - \frac{x^2}{2a} \right) dx$$

$\cancel{S = \int_0^a} \left(\sqrt{2ax} \right.$

$$S = \sqrt{2a} \int_0^a \sqrt{x}\, dx - \int_0^{\sqrt{2}a} \frac{x^2}{2a} dx +$$

$$+ \int_a^{\sqrt{2}a} \sqrt{3a^2 - x^2}\, dx = \sqrt{2a}\, x^{\frac{3}{2}} \cdot \frac{2}{3} \Big|_0^a - \frac{1}{2a} \frac{x^3}{3} \Big|_0^{\sqrt{2}a} + I =$$

VI.2.20

$\cancel{I = 3}\ \cancel{I} = \frac{2\sqrt{2}\, a^{\frac{5}{2}}}{3} - \frac{\sqrt{2}\, a^2}{3} + I =$

$=$

$I = \int_a^{\sqrt{2}a} \sqrt{3a^2 - x^2}\, dx = \left(\sqrt{3a^2-x^2}\right)\cdot x \Big|_a^{\sqrt{2}a} - \int_a^{\sqrt{2}a} x \, d\sqrt{3a^2-x^2} =$

$= \left(\sqrt{3a^2-x^2}\right)\cdot x \Big|_a^{\sqrt{2}a} - \int_a^{\sqrt{2}a} \frac{-x^2+3a^2-3a^2}{\sqrt{3a^2-x^2}}\, dx =$

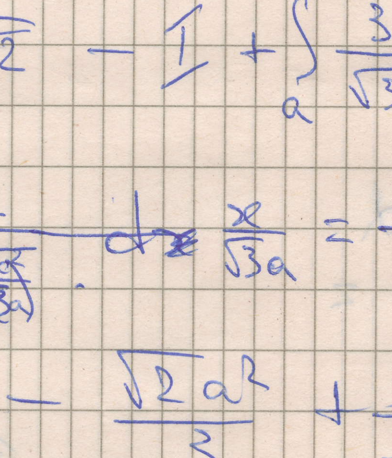
$\cancel{= \left(\sqrt{3a^2-x^2}\right) x \Big|_a^{\sqrt{2}a} - \int}$

$= \cancel{\sqrt{2}\, a^2} - \cancel{a^2\sqrt{2}} - I + \int_a^{\sqrt{2}a} \frac{3a^2}{\sqrt{3a^2-x^2}}\, dx =$

$= -I + \frac{3a^2}{\sqrt{3}a}\cdot\sqrt{3}a \int_a^{\sqrt{2}a} \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}a}\right)^2}}\, d\frac{x}{\sqrt{3}a} = -I + 3a^2 \cdot \arcsin\left(\frac{x}{\sqrt{3}a}\right)\Big|_a^{\sqrt{2}a} =$

$S = \frac{2\sqrt{2}\,\sqrt{a^5}}{3} - \frac{\sqrt{2}\,a^2}{3} + \frac{3}{2}a^2\left(\underset{\frac{\pi}{2}}{\arcsin}\underbrace{\sqrt{\frac{2}{3}}}_{\beta} - \underset{2\alpha}{\arcsin}\frac{1}{\sqrt{3}}\right) =$

$\frac{\pi R^2}{2\pi}\alpha$



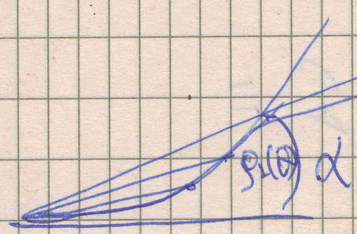
$\frac{R^2\alpha}{2}$

$\sin\beta = \frac{\sqrt{2}}{\sqrt{3}}$

$\sin\alpha = \frac{1}{\sqrt{3}}$

$\underbrace{\sin^2\beta}_{\cos^2\alpha} + \sin^2\alpha = 1$

VI. 2. 21



$$\rho = \frac{2a\sin\theta}{\cos^2\theta}$$

$$0 \le \theta \le \alpha$$

$$0 \le \rho \le \rho_1(\theta)$$

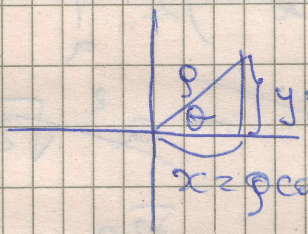
$\sin\alpha = \frac{1}{\sqrt3}$ $\qquad \alpha = \arcsin\frac{1}{\sqrt3}$

$$\arccos\frac{1}{\sqrt3} = \frac{\pi}{2} - \arcsin\frac{1}{\sqrt3}$$

$$S = \frac{\frac{\pi}{4} - \arcsin\frac{1}{\sqrt3}}{2}\ 3a^2$$

$x^2 = 2ay$

$$S_2 = \frac{1}{2}\int_0^\alpha \frac{4a^2\sin^2\theta}{\cos^4\theta}\,dt =$$



$y = \rho\sin\theta$

$x = \rho\cos\theta$

$$= \frac{1}{2}\,tg^3\frac{\theta}{3}\Big|_0^\alpha$$

$\rho^2\cos^2\theta = 2a\rho\sin\theta$

$$I = 2a^2\,\frac{tg^3\theta}{3}\Big|_0^\alpha =$$

$$\rho = 2a\,\frac{\sin\theta}{\cos^2\theta}$$

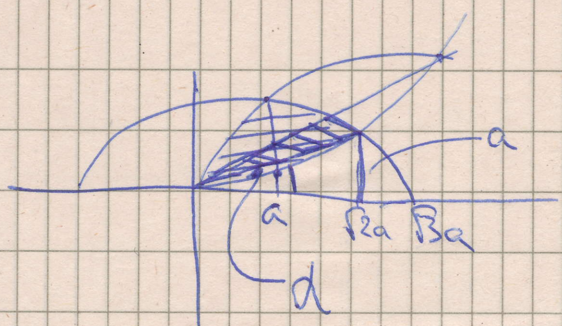
$$= \frac{2a^2}{3}\,\frac{\sin^3\alpha}{\left(\sqrt{1-\sin^2\alpha}\right)^3} =$$

$\sin\alpha = \frac{1}{\sqrt3}$

$$= \frac{2a^2}{3}\cdot\frac{1}{3\sqrt3\cdot\left(\sqrt{\frac{2}{3}}\right)^3} = \cancel{\frac{2a^2}{3}\frac{1}{\sqrt2}} = \cancel{\frac{\sqrt2 a^2}{6}} \qquad \frac{1}{2\sqrt2}\cdot\frac{2a^2}{3}$$

$$S = \left(\frac{\pi}{4} - \arcsin\frac{1}{\sqrt3}\right)\cdot 3a^2 + \frac{2}{2\sqrt2}\cdot\frac{2a^2}{3}$$

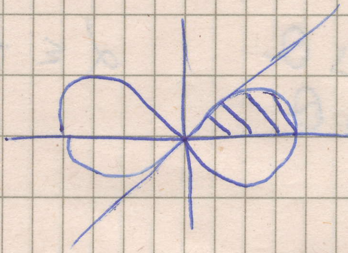
VI. 2.22



$\alpha = \arcsin \frac{1}{\sqrt{3}}$

Д.Р. 5/12 от Глава 5.

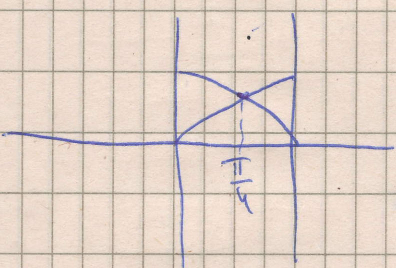
$(x^2+y^2)^2 = a^2(x^2-y^2)$ Лемнискатата на Бернули (лемнискатата на Бернули)



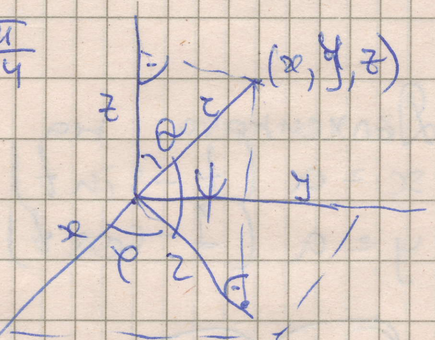
$x = \rho\cos\theta$ $\quad 0 \le \theta \le \frac{\pi}{2}$

$y = \rho\sin\theta$

$\rho^{\cancel{4}2} = a^2\cancel{\rho^2}(\cos^2\theta - \sin^2\theta) = a^2\cos 2\theta$



$0 \le \theta \le \frac{\pi}{4}$



$\rho^2 = a^2\cos 2\theta$

$0 \le \theta \le \frac{\pi}{4}$

$0 \le \rho \le \rho_1$ $\quad S_{\text{лемнискатата}} = ?$

13.III.2013 г.

Д.Л.(2) $\int_0^1 \sqrt{x^2+1}\,dx$

$\begin{cases} y = f(x) \\ a \le x \le b \end{cases}$ ∼

$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \quad \alpha \le t \le \beta$

$\rho = \rho(\theta) \qquad \begin{cases} x = \rho(\theta)\cos\theta \\ y = \rho(\theta)\cos\theta \end{cases} \qquad \alpha \le \theta \le \beta$

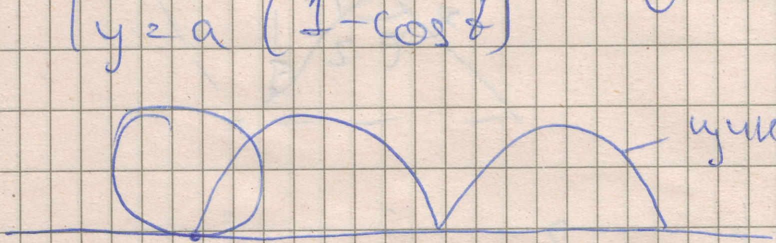
$$l = \int_a^b \sqrt{1+f'^2}\,dx = \int_\alpha^\beta \sqrt{\varphi'^2+\psi'^2}\,dt =$$

$$= \int_\alpha^\beta \sqrt{\rho^2+\rho'^2}\,d\theta$$

(1)

(3) Дължината на циклоидата

$\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases} \qquad 0 \le t \le 2\pi$



$$l = a\int_0^{2\pi} \sqrt{(1-\cos t)^2+\sin^2 t}\,dt = a\int_0^{2\pi} \sqrt{1-2\cos t+1}\,dt =$$

VI.2.2

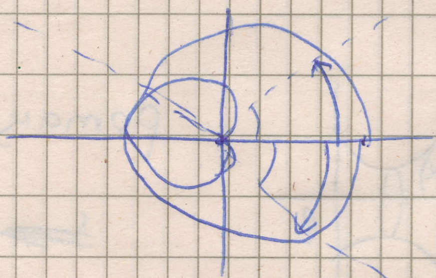
$$= \sqrt{2}a \int_0^{2\pi} \sqrt{1-\cos t}\, dt = \sqrt{2}a \int_0^{2\pi} \sqrt{2\sin^2 \frac{t}{2}}\, dt =$$

$$= 2a \int_0^{2\pi} \sin \frac{t}{2}\, dt = -4a \cos \frac{t}{2} \Big|_0^{2\pi} = 8a$$

② $\rho = a\sin^4 \frac{\theta}{4} \qquad 0 \leq \theta \leq 4\pi$

~~$\theta \quad \pi \quad \theta$~~

$\theta \quad -\theta$



$$\int_\alpha^\beta \sqrt{\rho^2 + \rho'^2}\, d\theta$$

$$l = \int_0^{4\pi} \sqrt{a^2 \sin^8 \frac{\theta}{4} + \left(4a\sin^3 \frac{\theta}{4} \cdot \cos \frac{\theta}{4} \cdot \frac{1}{4}\right)^2}\, d\theta =$$

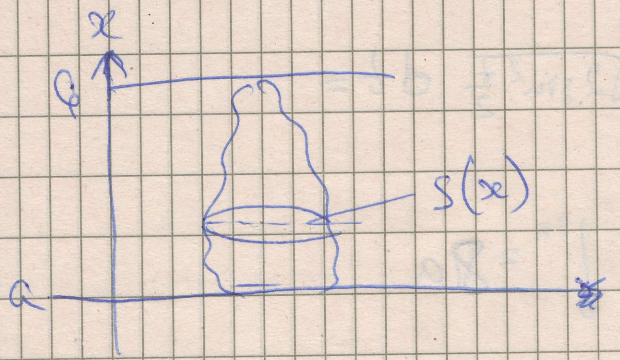
$$= a\int_0^{4\pi} \sqrt{\sin^8 \frac{\theta}{4} + \sin^6 \frac{\theta}{4} \cos^2 \frac{\theta}{4}}\, d\theta =$$

$$= a\int_0^{4\pi} \sqrt{\sin^6 \frac{\theta}{4}}\, d\theta = 4a \int_0^{4\pi} \sin^3 \frac{\theta}{4}\, d\frac{\theta}{4} =$$

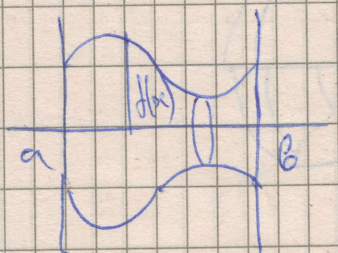
$$= -4a \int_0^{4\pi} \left(1-\cos^2 \frac{\theta}{4}\right) d\cos \frac{\theta}{4} =$$

$$= -4a \cos \frac{\theta}{4} \Big|_0^{4\pi} + \frac{4a\cos^3 \frac{\theta}{4}}{3} \Big|_0^{4\pi} = 8a - \frac{8a}{3} = \frac{16}{3}a$$

VI.2.25



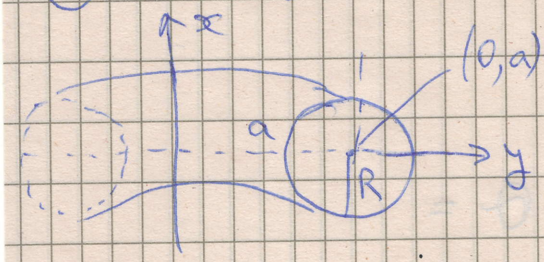
$V = \int_a^b S(x)\,dx$ – формула на Каролиер



ротационно тяло

$$V = \pi \int_a^b f^2(x)\,dx$$

(5) ~~V~~ Обемът на геврек (тор)



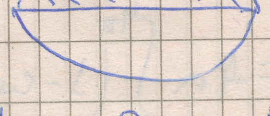
$a > R$

$$x^2 + (y-a)^2 = r^2$$

$$(y-a)^2 = r^2 - x^2$$

$$y = a \pm \sqrt{r^2 - x^2}$$

$$V = \pi \int_{-R}^{R} \left(a + \sqrt{r^2 - x^2}\right)^2 - \left(a - \sqrt{r^2 - x^2}\right)^2 dx =$$

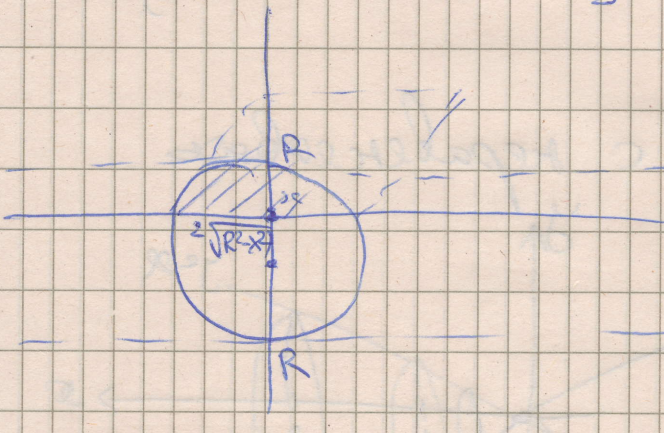


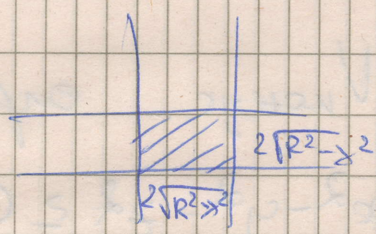
$$= \pi \int_{-R}^{R} 4a\sqrt{r^2 - x^2}\,dx = 4a\pi \int_{-R}^{R} \sqrt{r^2 - x^2}\,dx = 2a\pi r^2$$

VI 2.26

5.1~~6~~2 – 5.16 една за лице и една за дължина

(4) V. на оназИ част от пространството, която е част от 2 кюнеца, под прав ъгъл.



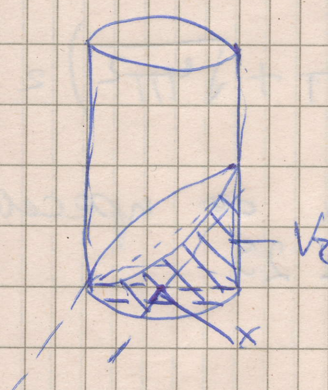


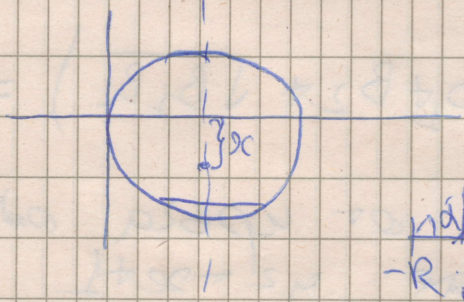
$$\int_{-R}^{R} f(x)\,dx$$

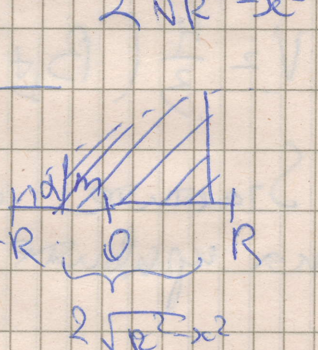
$$4\left(R^2 - x^2\right) =$$

$$\cancel{8}\int_0^R \cancel{4}\left(R^2 - x^2\right) dx = 8R^2 x - \frac{\cancel{8}}{3}x^3 \Big|_0^R = \frac{16}{3}R^3$$

(5) Прав кръгов цилиндър в/у равнина







$$\cancel{tg\alpha} = \frac{m}{\cancel{R}\cdot\sqrt{R^2-x^2}}$$

$$m_1 = \cancel{tg\alpha}\left(R - \sqrt{R^2-x^2}\right) tg\alpha \;\Big|\; m_2 = R + \sqrt{R^2 - x^2}\; tg\alpha$$

VI. 2.27

$S_x = R\,tg\alpha\, 2\sqrt{R^2-x^2}$

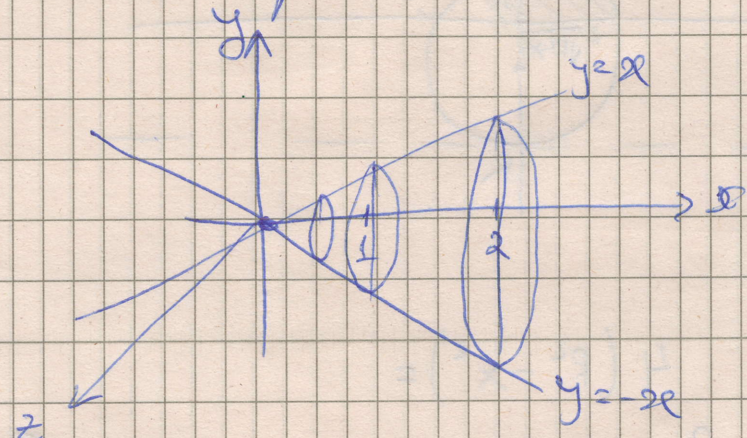
$V = \int_{-R}^{R} R\,tg\alpha\, 2\sqrt{R^2-x^2}\,dx = \pi R^{3} tg\alpha$

(6) $V_{конус}$ определен с неравенствата

$x^2 - y^2 - z^2 \geq 0$

$1 \leq x \leq 2$

$x^2 = y^2 + z^2$



$z=0 \quad x^2-y^2=0$

$V = \pi \int_1^2 x^2\,dx = \frac{x^3}{3}\pi \Big|_1^2 = \frac{8}{3}\pi - \frac{1}{3}\pi = \frac{7}{3}\pi$

$V = \frac{h}{3}\left(B_1 + B_2 + \sqrt{B_1 B_2}\right) = \frac{1}{3}\left(\pi + 4\pi + \sqrt{4\pi^2}\right) =$

(7) S частите от кръга образувани от пресичане на кривите $y^2 = x+1$ $x^2+y^2 = 25$

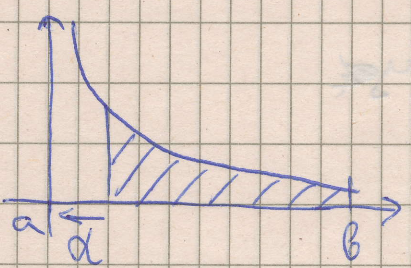
VI. 2.28

14.III.2013 г.

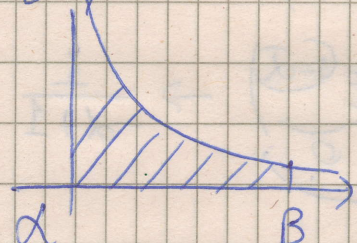
$f(x)$

непрекъсната в $[a, b]$

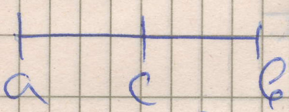
$\int_a^b f(x)\,dx$



$[a, b]$

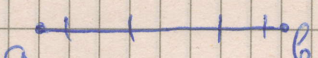


$$\lim_{\alpha\to a}\int_\alpha^b f(x)\,dx = \int_a^b f(x)\,dx$$



$\int_a^c$ $\int_c^b$

$$\lim_{p\to\infty}\int_a^p f(x)\,dx = \int_a^\infty f(x)\,dx$$



$\int_{a+\varepsilon_1}^{b-\varepsilon_2}$

1) пресмятане на несобствен интеграл

2) определяне на сходящи и разходящи интеграли

1) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ $\int_0^\varepsilon \frac{dx}{\sqrt{1-x^2}} = \arcsin\varepsilon \xrightarrow[\varepsilon\to 1]{} \arcsin 1 = \frac{\pi}{2}$

сходящ и е равен на $\frac{\pi}{2}$

2) $\int_0^1 \frac{dx}{x^\alpha}$ $\cancel{\int_\varepsilon^1 \frac{dx}{x^\alpha} =}$

$${}_{\varepsilon}\int^{1} \frac{dx}{x} = \begin{cases} \ln 1 - \ln\varepsilon & \alpha = 1 \\ \frac{x^{-\alpha+1}}{-\alpha+1}\Big|_{\varepsilon}^{1} = \cancel{} & \end{cases} \qquad \frac{\frac{1}{-\alpha+1}\cdot\frac{1}{x^{\alpha-1}}}{\frac{1}{\alpha+1}\cdot x^{-\alpha+1}}\Big|_{0}^{1}$$

1) $-\infty$

2) $\frac{1}{-\alpha+1}\left(1 - \frac{1}{\varepsilon^{\alpha-1}}\right)$ разходящ $\Big\{ -\infty$ разходящ

3) $\frac{1}{-\alpha+1}\underbrace{\left(1 - \underbrace{\varepsilon^{1-\alpha}}_{0}\right)}_{1} \to \frac{1}{-\alpha+1}$ сходящ

$${}_{a}\int^{b} \frac{dx}{(x-a)^{\alpha}}$$

$\alpha \geq 1$ разходящ

$\alpha < 1$ сходящ

$${}_{1}\int^{\infty} \frac{dx}{x^{\alpha}}$$

$1 \quad \varepsilon \to \infty$

$\alpha < 0$

$\alpha > 0$

3 $${}_{1}\int^{\varepsilon} \frac{dx}{x^{\alpha}} = \Big\{ \cancel{\ln x}\Big|_{1}^{\varepsilon}$$

$$= \begin{cases} \ln\varepsilon - \ln 1 & \alpha = 1 \\ \frac{x^{-\alpha+1}}{-\alpha+1}\Big|_{1}^{p} & \alpha > 1 \\ & \alpha < 1 \end{cases}$$

$\frac{1}{-\alpha+1}\cdot\frac{1}{x^{\alpha-1}}\Big|_{1}^{p}$

$\frac{1}{-\alpha+1}\, x^{-\alpha+1}$

$+\infty$ разходящ

0 сходящ

$+\infty$ разходящ

$${}_{0}\int^{b} \frac{dx}{x^{\alpha}} \qquad \cancel{{}_{a}\int^{b}\frac{dx}{(x-a)^{\alpha}}}$$

$\alpha \geq 1$ разх.

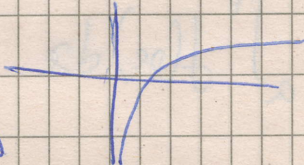
$\alpha < 1$ сход.

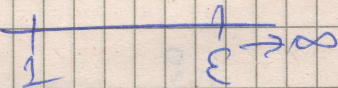
$${}_{1}\int^{\infty} \frac{dx}{x^{\alpha}}$$

$\alpha \leq 1$ разходящ

$\alpha > 1$ сходящ

VI. 2.30





④ $\int_2^{\infty} \frac{dx}{x \ln x}$

$$\int_2^{p} \frac{d \ln x}{\ln x} = \ln \ln x \Big|_2^{p} = \ln \ln p - \ln \ln 2 \to +\infty$$

разх.

$$\int_0^{\infty} e^x \sin x \, dx$$

$$I = \int_0^{\varepsilon} e^x \sin x \, dx = \int_0^{\varepsilon} \sin x \, de^x = e^x \sin x \Big|_0^{\varepsilon} - \int_0^{\varepsilon} e^x d\sin x =$$

$$= e^{\varepsilon} \sin^{\varepsilon} + \int_0^{\varepsilon} e^x \cos x \, dx = e^{\varepsilon} \sin^{\varepsilon} + \int_0^{\varepsilon} \cos x \, de^x =$$

$$= e^{\varepsilon} \sin^{\varepsilon} \mp e^x \cos x \Big|_0^{\varepsilon} - \int_0^{\varepsilon} \cos x \, de^x =$$

$$= e^{\varepsilon} \sin^{\varepsilon} \mp e^x \cos x \Big|_0^{\varepsilon} + \int_0^{\varepsilon} e^x d\cos x =$$

$$= e^{\varepsilon} \sin^{\varepsilon} - \cos^{\varepsilon} - e^{\varepsilon} + 1 - \int_0^{\varepsilon} e^x \sin x$$

$$I_{\varepsilon} = \frac{1}{2}\left(e^{\varepsilon}\underbrace{(\sin\varepsilon - \cos\varepsilon)}_{\varepsilon \to \infty} + 1\right)$$ — няма граница разходящ

⑤ $e^{\varepsilon}(\sin\varepsilon - \cos\varepsilon)$ Д.П. x_n' x_n'' $f(x_n') \to A$

$\varepsilon \to \infty$ $f(x_n'') \to B$

$x_n \to \infty$

$f(x_n)$

$x_n \to \infty$

$f(x_n) \to \infty$

⑥ $\int_0^{\frac{\pi}{2}} \ln\sin x\,dx =$

$\int_\varepsilon^{\frac{\pi}{2}} \ln\sin x\,dx = \ln\sin x \cdot x\Big|_\varepsilon^{\frac{\pi}{2}} - \int_\varepsilon^{\frac{\pi}{2}} x\,d\ln\sin x =$

$= \ln\sin x \cdot x\Big|_\varepsilon^{\frac{\pi}{2}} - \int_\varepsilon^{\frac{\pi}{2}} x\frac{1}{\sin x}\cdot\cos x\,dx =$

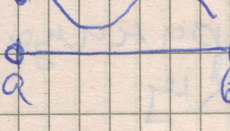
$= \cancel{\ln\sin\varepsilon\cdot\varepsilon} = -\ln\sin\varepsilon\cdot\varepsilon - \int_\varepsilon^{\frac{\pi}{2}} \underbrace{x\frac{1}{\sin x}}_{\to 1}\cos x\,dx$

$\varepsilon\to 0$ $\quad \varepsilon\ln\sin\varepsilon =$ $\quad x\ln x$

$= \frac{\ln\sin\varepsilon}{\frac{1}{\varepsilon}} = -\frac{1}{\sin\varepsilon}\cdot\cos\varepsilon\cdot\varepsilon^2 = -\frac{1}{\sin\varepsilon}\cdot\cos\varepsilon\cdot\varepsilon^2 =$

$= -\varepsilon \to 0$

$x\ln x$ $\quad -\frac{\ln\frac{1}{x}}{\frac{1}{x}} \to 0$



$\int_0^{\frac{\pi}{2}} \frac{x\cos x}{\sin x}\,dx$ – сходится

⑦ $\int_0^{\infty} \frac{\ln x\,dx}{1+x^2}$ $\quad \int_0^1 \frac{\ln x\,dx}{1+x^2} = \int_\varepsilon^1 \frac{\ln x}{1+x^2}\,dx =$

$= \cancel{\int_\varepsilon^1 \frac{d\ln x}{1+x^2}} = \int \ln x\; d\,\mathrm{arctg}\,x\Big|_\varepsilon^1 -$

VI.232

$$-\int_{\varepsilon}^{1} \operatorname{arctg} x \, d\ln x = \underbrace{-\ln\varepsilon \operatorname{arctg}\varepsilon}_{0} - \underbrace{\int_{\varepsilon}^{1} \frac{\operatorname{arctg} x}{x} dx}_{\text{сходящ}} =$$

$$= -\underbrace{\varepsilon\ln\varepsilon}_{0} \cdot \underbrace{\frac{\operatorname{arctg}\varepsilon}{\varepsilon}}_{1} \; \cancel{\text{сходящ}} \; - \underbrace{\int_{\varepsilon}^{1} \frac{\operatorname{arctg} x}{x} dx}_{\text{сходящ}} \rightarrow$$

$\rightarrow$ сходящ

$$\int_{1}^{p} \frac{\ln x \, dx}{1+x^2} = -\int_{1}^{\frac{1}{p}} \frac{\ln \frac{1}{t}}{1+\frac{1}{t^2}} \frac{1}{t^2} dt =$$

$t = \frac{1}{x}$ $\qquad x = \frac{1}{t}$

$dx = -\frac{1}{t^2}$

$$= \int_{\frac{1}{p}}^{1} \frac{\ln\frac{1}{t} \cdot \frac{1}{t^2}}{1+\frac{1}{t^2}} dt = -\int_{\frac{1}{p}}^{1} \frac{\ln t}{t^2+1} dt \rightarrow \int_{0}^{1} \frac{\ln t}{1+t^2} dt$$

$p \to \infty$

$\frac{1}{p} \to 0$

$$\int_{1}^{\infty} \frac{\ln x}{1+x^2} dx = -\int_{0}^{1} \frac{\ln x}{1+x^2} dx$$

$$\int_{0}^{\infty} = \underbrace{\int_{0}^{1} + \int_{1}^{\infty}}_{\text{сходящи}} = 0$$

$$\int_a^b f(x)\,dx = \int_a^c + \int_c^b$$

сходящ = сход. + сход.
разходящ = сход. + разх.
разходящ = разх. + сход.
разходящ = разх. + разх.

20.III.2013г.

Интеграл от неправилни функции

$$\int_a^b \frac{dx}{(x-a)^\alpha}$$

$\alpha < 1$ сход.
$\alpha \geq 1$ разх.

~~Интеграл от~~ безкраен интегр.
~~функции~~

$$\int_\alpha^\infty \frac{dx}{x^\alpha}$$

$\alpha > 1$ сход.
$\alpha \leq 1$ разход.

Методи:
смяна на променливата
~~чите~~
интегриране по части

- $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ особена точка $\to 1$

$x = \sin t$ $\int_0^{\frac{\pi}{2}} dt$ ...

- $\int_1^\infty \frac{\sin x}{x}\,dx$

$$\int_1^p \frac{\sin x}{x}\,dx = -\frac{\cos x}{x}\Big|_1^p - \int_1^p \frac{\cos x}{x^2}\,dx =$$

$$= -\frac{\cos p}{p} + \cos 1 \quad (\text{има граница за } p \to \infty)$$

VI.2.34

• $\int_0^{\frac{\pi}{2}} \ln \sin x \, dx = \ldots$ (интегриране по части) $=$

$= x \ln \sin x \Big|_{\varepsilon}^{\frac{\pi}{2}} - \int_{\varepsilon}^{\frac{\pi}{2}} \frac{x}{\sin x} \cdot \cos x \, dx$

$$\boxed{x \ln x \xrightarrow[x \to 0]{} 0}$$

Теорема 1: $\int_a^b f(x)\,dx$; т. a - особена;

$\int_a^b |f(x)|\,dx$ - има особ. в същата точка.

Ако втория интеграл е сход., то I също е сходяща. Обратното не е вярно.
(I) Нарича се абсолютно сходящ.
Ако $\int |\ldots| dx$ е разходящ, а I е сходящ.

Теорема 2: Ако $f(x)$ и $g(x)$ са неограничени функции и $0 \le f(x) \le g(x)$ - вярно за $\forall x$ (без точка a) ~~вярно за $\forall x$(~~ и $\int g(x)$ е сход., то и $\int f(x)$ е сходящ. Ако $\int f(x)$ е разходящ $\Rightarrow \int g(x)$ е разходящ

Пример: $\int_1^{b} \frac{\cos x}{x^2} dx \qquad 0 \le \left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2} \Rightarrow \int$ е сход.

Теорема 3: $f(x)$ и $g(x)$ са неотрицателни и е известна сходимостта на $\int g(x)$

Образ $\frac{f(x)}{g(x)} \xrightarrow[x \to a \text{ (особена точка)}]{} ?$

Ако има $\lim A > 0 \Rightarrow f \sim g$

Ако $\lim = 0$ и $\int g(x)$ е сходящ
$\Rightarrow \int f(x)$ е сходящ

Ако $\lim = \infty$ и $\int g(x)$ е разходящ
$\Rightarrow \int f(x)$ е разходящ

① $\int_0^1 \frac{\arcsin x}{x^\alpha} dx$. За кои стойности на α
$\int$ е сходящ или разходящ?

$\frac{f(x)}{g(x)} = \varphi(x)$ $\quad f(x) = g(x) \cdot \varphi(x)$

$\varphi(x) \to A$ (да има граница)

$\frac{\arcsin x}{x^\alpha} = \frac{\arcsin x}{x} \cdot \frac{1}{x^{\alpha-1}}$

$\frac{\arcsin x}{x} \xrightarrow[x \to 0]{} 1$

$\alpha - 1 < 1$
$\alpha - 1 > 1$

$\int_0^\infty \frac{\operatorname{arctg} x}{x^\alpha} dx$ $\quad \frac{\operatorname{arctg} x}{x} \quad \frac{1}{x^{\alpha-1}}$

$\frac{\operatorname{arctg} x}{x} \to 1$

$\frac{f(x)}{g(x)} = \varphi(x)$

$f(x) = \varphi(x) g(x)$

$\alpha - 1 < 1$ $\quad \alpha < 2$ е сходящ в 0

$\frac{\operatorname{arctg} x}{x^\alpha} = \frac{\operatorname{arctg} x}{\cancel{x}} \cdot \frac{1}{x^\alpha}$

$\operatorname{arctg} x \xrightarrow[x \to \infty]{} \frac{\pi}{2}$

VI. 2.36

при $\alpha > 1$ е сходящ
при $\alpha \leq 1$ е разходящ
$\alpha \in (1, 2)$

Д.Р. Ръководство по математически анализ II част
страница 64

(2) $\int_0^5 \frac{dx}{x^2 + \sqrt[3]{x}} =$

$$\frac{1}{x^2 \cdot \sqrt[3]{x}} = \frac{1}{x^{\frac{1}{3}}} \cdot \frac{1}{x^{\frac{1}{3}+1}} = \frac{1}{3}$$

$\frac{1}{x^{\frac{1}{3}+1}} \to 1$

тъй като степента е $\frac{1}{3}$

(3) $\int_0^\infty e^{-x} x^{\beta} dx$

$e^{-x} \quad x^{\beta}$

при $\beta \neq 0$

$$\frac{e^{-x}}{x^{-\beta}} \sim \frac{1}{x^{-\beta}}$$

$\beta > -1 \to \int$ е сходящ
$\beta \leq -1 \to \int$ е разходящ

$$e^{-x} x^{\beta} = \left(\frac{x^{\beta}}{e^{x}}\right) \to 0$$

e^{x} - най-бързо растяща функция

$$e^{-x} x^{\beta} = \frac{x^{\beta} x^{2}}{e^{x} x^{2}} = \frac{x^{\beta+2}}{e^{x}} \cdot \frac{1}{x^2}$$

$\frac{x^{\beta+2}}{e^{x}} \to 0$

$$\frac{x^{\beta+\frac{1}{2}}}{e^{x}} \cdot \frac{1}{x^{\frac{1}{2}}}$$

$\frac{x^{\beta+\frac{1}{2}}}{e^{x}} \to 0$

при $\beta > -1$ $\int$ е сходящ

(4) $\int_0^1 x^\alpha \ln\sin x \, dx$

$x^\alpha \ln\sin x = \frac{x}{x} \cdot \frac{x^\alpha}{\sin x} \cdot \sin x \cdot \ln\sin x \sim$

$\sim \underbrace{\frac{1}{x^{-\alpha}}}_{\text{сходящ}} \cdot \underbrace{\sin x \ln\sin x}_{\downarrow\, 0}$

$1-\alpha < 1$ сходящ

$\alpha > 0$ — $\int$ е сходящ

$\alpha = 0$ сходящ

$\alpha < 0$ разходящ

$\frac{\ln\sin x}{x^\beta} = \frac{1}{x^\beta} \underbrace{|\ln\sin x|}_{\downarrow\, \infty}$

ако $-\alpha = \beta \geq 1$, то $\int$ е разходящ

при $\alpha \geq 0$ е сходящ

$\alpha \leq -1$ е разходящ

$\int_0^1 x^\alpha |\ln\sin x| dx = \int_\varepsilon^1 |\ln\sin x| \, \alpha \, dx^{\alpha-1}$

$\frac{1}{\alpha+1} \ln\sin x \Big|_\varepsilon^1 \cdot x^{\alpha+1} - \int_\varepsilon^1 \frac{1}{\alpha+1} \cdot x^{\alpha+1} \, d(\ln\sin x) =$

$= \frac{\ln\sin 1}{2} - \frac{\ln\sin\varepsilon \cdot \varepsilon^{\alpha-1}}{2}$

VI .2.38

21.III.2013г. $f(x)$ - неограничена функция

$\int_a \frac{dx}{(x-a)^\alpha}$ $\int^\infty \frac{dx}{x^\alpha}$

безкраен интеграл

$\alpha \leq 1$ сход. $\alpha > 1$ сходящ

$\alpha \geq 1$ разход. $\alpha \leq 1$ разходящ

$0 < f(x) = g(x)\,\varphi(x)$ $\varphi(x) \to 0,\ A \neq 0,\ \infty$

$g \to f$ сход. сход.

$f \sim g$

$g \sim f$

разх. сход.

(1) $\int_0^\theta \frac{d\varphi}{\sqrt{\cos\varphi - \cos\theta}}$ $0 < \varphi < \theta < \frac{\pi}{2}$

$\cos\varphi > \cos\theta$ $\frac{1}{\sqrt{\cos\varphi - \cos\theta}} = \frac{1}{\sqrt{2\sin\frac{\varphi+\theta}{2} \cdot \frac{\sin\frac{\theta-\varphi}{2}}{\frac{\theta-\varphi}{2}} \cdot \frac{\theta-\varphi}{2}}}$ $\to 1$

$= f(\varphi) \cdot \frac{1}{\sqrt{\theta-\varphi}}$ $\frac{1}{\sqrt{\cos\varphi - \cos\theta}} \sim \frac{1}{(\theta-\varphi)^{\frac{1}{2}}}$

$\downarrow$ сходящ $\leftarrow$ сходящ

$A \neq 0$

$\cos\varphi - \cos\theta = (\varphi - \theta)(-\sin\xi)$

$\varphi \quad \xi \quad \theta$

$\frac{1}{\sqrt{\cos\varphi - \cos\theta}} = \frac{1}{\sin^{\frac{1}{2}}\xi} \cdot \frac{1}{(\theta-\varphi)^{\frac{1}{2}}}$

$\downarrow$

$\frac{1}{\sqrt{\sin\theta}} \neq 0$

VI.2.39

② $\int_0^{\infty} \frac{\ln x}{|x-1|^{\alpha}} dx \quad \alpha > 0$

$0 \quad 1 \quad \infty$

$\frac{\ln x}{|x-1|^{\alpha}} \sim \ln x$

сход. сход.

$\int_{\varepsilon}^{1} \ln x\, dx = x \ln x \Big|_{\varepsilon}^{1} - \int_{\varepsilon}^{1} x\, d \ln x =$

$= \underbrace{-\varepsilon \ln \varepsilon}_{\varepsilon \to 0 \;\to\; 0} - (1-\varepsilon) \qquad \int_0^1 \ln x\, dx = -1$ сходящ

$x^{\varepsilon} \ln x \xrightarrow[x \to 0]{} 0 \qquad \ln x$

$\varepsilon > 0$

$\ln x = x^{\varepsilon} \ln x \cdot \frac{1}{x^{\varepsilon}} \qquad x^{\frac{1}{2}} \ln x \cdot \frac{1}{x^{\frac{1}{2}}} = \ln x$

$\to 0 \qquad \to 0$

$\frac{\ln x}{|x-1|^{\alpha}} = \underbrace{\frac{\ln x}{|x-1|}}_{\to\, 1 \neq 0} \cdot \frac{1}{|x-1|^{\alpha-1}} \sim \frac{1}{|x-1|^{\alpha-1}}$

$\alpha - 1 < 1$

$\frac{\ln(x+1)}{x} \xrightarrow[x \to 0]{} 1$

$0 < \alpha < 2$ сходящ

$0 < \alpha \leq 1$ расходящ

$\boxed{1 \leq \alpha < 2 \text{ сходящ}}$

$\frac{\ln y}{y-1} \xrightarrow[y \to 1]{} 1$

$\frac{\ln x}{|x-1|^{\alpha}} \sim \frac{\ln x}{x^{\alpha} \underbrace{\frac{|x-1|^{\alpha}}{x^{\alpha}}}_{\to 1}} = \frac{\ln x}{x^{\alpha}}$

VI 2.40

$$\int_2^{p} \frac{\ln x}{x^\alpha}\,dx \qquad \int_2^{p} \frac{\ln x}{x}\,dx = \frac{\ln^2 p}{2} - \frac{\ln^2 2}{2} \to \infty$$

$0 \le f \le g$

разх. $\Leftarrow$ разх.

сход. $\Leftarrow$ сход.

$$\frac{\ln x}{x} \le \frac{\ln x}{x^\alpha}$$

разход. $\alpha < 1$

$$\frac{\ln x}{x^\alpha} = \underbrace{\frac{\ln x}{x^{\alpha-\beta}}}_{\to 0} \cdot \frac{1}{x^\beta}$$

сход. $\beta > 1$

$\alpha > 1$

$1 \quad \beta \quad \alpha$

(3) $\int_0^{\infty} \frac{\sin x}{x^\alpha}\,dx$ – абсолютно сходящ

$$\frac{\sin x}{x^\alpha} = \frac{\sin x}{x} \cdot \frac{1}{x^{\alpha-1}} \sim \frac{1}{x^{\alpha-1}}$$

$\frac{\sin x}{x} \to 1$

$\alpha - 1 < 1$

$\alpha < 2$

абсолютни стойности

$$\underbrace{\left|\frac{\sin x}{x}\right|}_{\text{абсолютно сходящ}} \le \underbrace{\frac{1}{x^\alpha}}_{\alpha > 1}$$

$1 < \alpha < 2$ абсолютно сходящ

$0 < \alpha < 1$ условно сходящ

$$\int_1^{p} \frac{\sin x}{x^\alpha}\,dx = -\frac{\cos x}{x^\alpha}\Big|_1^p - \alpha \underbrace{\int_1^p \cos x \frac{1}{x^{\alpha+1}}\,dx}_{\text{сходящ}}$$

$$\int_1^p \frac{\sin x}{x^\alpha} d.$$

$$\left|\frac{\cos x}{x^{\alpha+1}}\right| \le \underbrace{\frac{1}{x^{\alpha+1}}}_{\text{сходящ}} \qquad -\frac{\cos p}{p^\alpha} + \cos 1$$

$$0 < \left|\frac{\cos p}{p^\alpha}\right| < \frac{1}{p^\alpha}$$

$$0$$

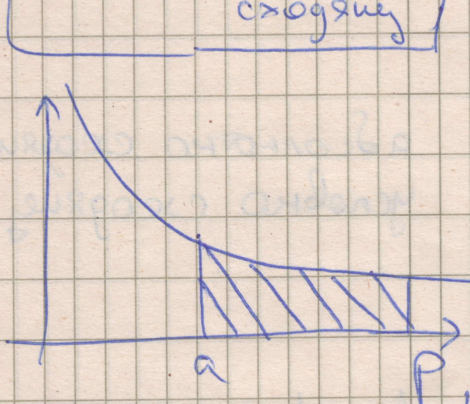
$$\boxed{0 < \alpha < 1}$$

$$\underbrace{\left|\frac{\sin x}{x^\alpha}\right|}_{\text{разходящ}} \longrightarrow \frac{\sin^2 x}{x^\alpha} = \underbrace{\frac{1 - \cos 2x}{2x^\alpha}}_{\text{разходящ}}$$

$$\underbrace{\frac{1}{2x^\alpha}}_{\text{разходящ}}$$

Д. Р. $\underbrace{\dfrac{\cos 2x}{x^\alpha}}_{\text{сходящ}}$

$$\frac{1}{2x^\alpha} - \frac{\cos 2x}{x^\alpha}$$



$$f(x) > 0$$

$$\int_a^\infty$$

Интеграл на Френел

$$\int_0^\infty \sin x^2 \, dx \qquad \int_0^\infty \cos x^2 \, dx$$

VI. 2.42

④ $\int_0^p \sin x^2 \, dx = \int_0^{p^2} \sin t \, d\sqrt{t} = \frac{1}{2}\int_0^{p^2} \sin t \frac{1}{\sqrt{t}} \, dt$ условно сходящ

полагане: $x^2 = t$

$x^2 \sqrt{t}$

⑤ $\int_0^\infty e^{-t} t^{x-1} \, dt = \Gamma(x)$ (гама от x), когато интеграла е сходящ

⑥ $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} \, dt$

$$\int_0^p e^{-t} t^{n-1} \, dt = -\int_0^p t^{n-1} \, de^{-t} =$$

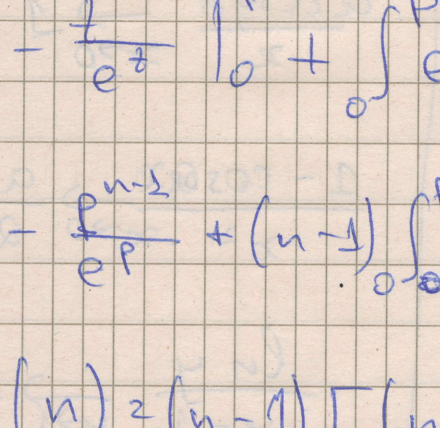
$$= -\frac{t^{n-1}}{e^t}\Big|_0^p + \int_0^p e^{-t} \, dt^{n-1} = -\frac{t^{n-1}}{e^t}\Big|_0^p + (n-1)\int_0^p e^{-t} t^{(n-2)} \, dt =$$

$$= -\frac{p^{n-1}}{e^p} + (n-1)\int_0^p e^{-t} t^{(n-2)} \, dt$$

$\Gamma(n) = (n-1)\,\Gamma(n-1)$ $\qquad$ $\Gamma(n) = (n+1)! \, \underbrace{\Gamma(1)}_{1}$

$\Gamma(2) = 1 \cdot \Gamma(1)$

$\Gamma(1) = \int_0^\infty e^{-t} \, dt = -e^{-t}\Big|_0^\infty = 1$ $\qquad$ $\boxed{\Gamma(n) = (n-1)!}$



Контролно: интеграли, несобствени, собствени, сходящи, разходящи, лица, обеми.

VI.2.43

27.III.2013г.

(1) $\int_0^\theta \frac{d\varphi}{(\cos\varphi - \cos\theta)^\alpha}$

(2) $\int_0^\theta \frac{dx}{x^\alpha(e^x - e^{-x})}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$

$\frac{1}{1-x} = 1 + x + x^2 + \dots$

$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6)$

$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1} x + \dots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$\cancel{(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1} x}$

VI 2.44

$\frac{e^x - 1}{x} \xrightarrow[x \to 0]{} 1$

$\frac{\ln(1+x)}{x} \xrightarrow[x \to 0]{} 1$

$\frac{\sin x}{x} \xrightarrow[x \to 0]{} 1$

$\frac{\arcsin x}{x} \xrightarrow[x \to 0]{} 1$

$\frac{\operatorname{arctg} x}{x} \xrightarrow[x \to 0]{} 1$

$\frac{1 - \cos ax}{x^2} \xrightarrow[x \to 0]{} \frac{a^2}{2}$

$\frac{\ln y}{y - 1} \xrightarrow[y \to 1]{} 1$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$

$\frac{1}{1-x} = 1 + x + x^2 + \dots$

$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6)$

$$e^{x} - e^{-x} = \cancel{1} + \frac{x}{1!} + \ldots - \cancel{1} + \frac{x}{1!} + o(x) =$$

$$= x\left(1 + \frac{o(x)}{x}\right)$$

$$\frac{1}{x^{a}\left(e^{x} - e^{-x}\right)} = \frac{1}{x^{a+1}\underbrace{\left(1 + \frac{o(x^2)}{x}\right)}_{\to 1}} \sim \frac{1}{x^{a+1}}$$

(3) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right)^{p} d\theta$$

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + o\left((x-a)^2\right)$$

$$\frac{1\cos\theta - 1\sin\theta}{\sqrt{2}} \cdot \sqrt{2} = \sqrt{2}\sin\underbrace{\left(\frac{\pi}{4} - \theta\right)}_{t}$$

$$\cos\theta - \sin\theta = f\left(\frac{\pi}{4}\right) = \left(\theta - \frac{\pi}{4}\right)\underbrace{\left(-\sin\xi - \cos\xi\right)}_{-\sqrt{2}}$$

$-\frac{\pi}{4} \quad \xi \quad \theta$

(4) $$\int_{0}^{\infty} \frac{\ln x}{|x-1|^{\alpha}}\, dx$$

$x \to \infty$

$\ln x \quad x^{\alpha} \quad e^{x}$

$\alpha > 0$

$$x^{\varepsilon}\ln x \xrightarrow[x \to 0]{} 0$$

VI 2.45

(5) $\displaystyle\int_0^{\infty} \frac{\ln x}{1+x^2}\,dx = 0$ $\displaystyle\int_1^{\infty}$ $\displaystyle\int_0^{1}$

$x = \frac{1}{t}$

(6) $\displaystyle\int_0^{\frac{\pi}{2}} \ln \sin x\, dx$

$\ln \sin x \cdot \sin^{\varepsilon} x \xrightarrow{} \frac{x^{\varepsilon}}{\sin^{\varepsilon} x} \cdot \frac{1}{x^{\varepsilon}}$ $\to 0$

сход.

$f(x) = \varphi(x) \cdot g(x)$

$\varphi(x) \to 0$, $g(x)$ сход.

$\displaystyle I_1 = \int_{\varepsilon}^{\frac{\pi}{2}} \ln \sin x\, dx = \cancel{\ln \sin x \cdot x \Big|_{\varepsilon}^{\frac{\pi}{2}} \int_{\varepsilon}}$

$x = \frac{\pi}{2} - t \qquad t = \frac{\pi}{2} - x$

$\displaystyle = -\int_{\frac{\pi}{2}-\varepsilon}^{0} \ln \sin\left(\frac{\pi}{2} - t\right) d t = \cancel{-\ln \sin\left(\frac{\pi}{2} - t\right) \cdot t \Big|_{\frac{\pi}{2}-\varepsilon}^{0}} +$

$d\left(\frac{\pi}{2} - t\right) = -1\, dt$

$\displaystyle \cancel{+ \int_{\frac{\pi}{2}-\varepsilon}^{0} t\, d \ln \sin\left(\frac{\pi}{2} - t\right) = -\ln \sin\left(\frac{\pi}{2} - t\right) \cdot t \Big|_{\frac{\pi}{2}}}$

$\displaystyle I_1 = \int_0^{\frac{\pi}{2}-\varepsilon} \ln \cos t\, dt$

VI. 2.46

$$\int_0^{\frac{\pi}{2}} \ln\cos \cancel{x}\, d\cancel{x}$$

$$I_1 + I_2 = \int_0^{\frac{\pi}{2}} \ln \frac{\sin \cancel{2x}}{2}\, dx$$

$$\int_0^{\frac{\pi}{2}} \ln \sin \cancel{2x}\, dx = \frac{\pi}{2} \ln 2$$

$\cancel{2x = t}$ $2x = t$ $\qquad \frac{1}{2} \int \ln \sin t\, dt - \frac{\pi}{2} \ln 2$

$$\int_0^{\frac{\pi}{2}} \ln \sin x\, dx + \int_{\frac{\pi}{2}}^{\pi} \ln \sin x\, dx$$

$x = \pi - t$ $\qquad -\int_{\frac{\pi}{2}}^{0} \ln \sin(\pi - t)\, dt$

$t = \pi - x$

$$I_1 + I_2 = I_1 - \frac{\pi}{2} \ln 2$$

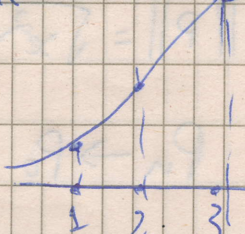
Контролно: Петък 05.IV.2013г. 10:00г. 210 ФХФ

(7) $\int_0^{\infty} \sin x^2\, dx$ условно сходящ $\qquad$ (8) $\int_0^{\infty} \frac{\sin x}{x^\alpha}\, dx$

$x^2 = t$

(9) $\int_0^{\infty} e^{-t} t^{x-1}\, dt = \Gamma(x)$ – гама функция

$x > 0$



$\Gamma(n) = (n-1)\,\Gamma(n-1) \qquad \Gamma(n) = (n-1)!$

$\vdots$

$\Gamma(2) =$

VI. 2.47

(10) $\int_0^1 x^4 \ln x \, dx$

(11) $\int_0^\infty \cancel{\operatorname{arctg}} \frac{\operatorname{arcctg} x}{x^2} \, dx$

(12) $\int_0^\infty \frac{x^p}{2+x^q} \operatorname{arctg} x \, dx$

(13) $\int_0^\infty \frac{\left(x - \ln \frac{(4x+1)^{\frac{1}{2}}}{(4x+1)^{\frac{1}{4}}}\right)^2}{(x^3+x^p)\left(\sqrt{1+4x^2} - \cos 2x\right)^q} \, dx$

$p > 3$

$p \leq 3$

$\ln(\quad)^{\frac{1}{4}} = \frac{1}{4} \ln(\quad)$

$x\left(\sqrt{\frac{1}{x^2} + 4} - \frac{\cos 2x}{x}\right)$

Функции на две променливи

(x, y)

$\rho(P, Q) =$ $\quad P(x_1, y_1)$, $\quad Q(x_2, y_2)$

$= \sqrt{(x_1 - x_2)^2 - (y_1, y_2)^2}$

$\|P\| = \sqrt{x^2 + y^2}$

$P_n \to P_0 \qquad \rho(P_n, P_0) \to 0$

$x_n \to x_0$

$y_n \to y_0$

f - непрекъсната, ако за $\forall P_n$

$f(P) \to f(P_0)$

VI. 2.48

$f(P)$

$\varepsilon > 0 \to \delta$ за $\forall P \in k(P_1, \delta)$

$\rho(P, P_0) < \delta \Rightarrow |f(P) - f(P_0)| < \varepsilon$

$$f = \begin{cases} \frac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

$P_n' \quad P_n''$

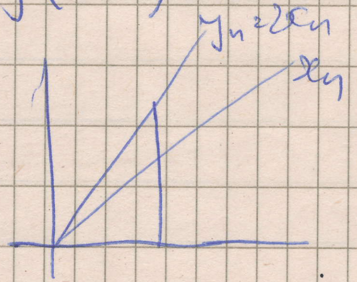
$\searrow \; \swarrow$

P_0

$f(P_n') \to A$

$f(P_n'') \to B \neq A$

$f(P_n) =$



$f(P_n) = \frac{1}{2}$

$x_n = y_n \to 0$

$f(P_n') = \frac{2}{5}$

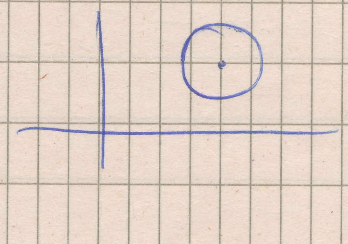
$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

Да се док., че не е диференц. в $(0,0)$

$xy = \frac{x^2+y^2}{2}$

$f'_x \quad x^2+y^2 \neq 0$

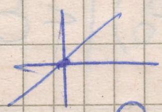
$f(x,0) \equiv 0$



$$f'_x = \begin{cases} \frac{2xy^3}{(x^2+y^2)^2} & x^2+y^2 \neq 0 \\ 0 & (0,0) \end{cases}$$

VI.2.19

$$f'_y = \begin{cases} \dfrac{yx^4 - x^2y^2}{(x^2+y^2)^2} & x^2+y^2 \neq 0 \\ 0 & (0,0) \end{cases}$$



$x_n = y_n \to 0$

$f'_x(x_n, y_n) = \frac{1}{2} \not\to 0$ прекъсната в $(0,0)$

$f(x_n, y_n) = 0$

Диференцируемост на функция на две променливи

$$f(x,y) = f(x_0,y_0) + (x - x_0) f'_x(x_0,y_0) + (y - y_0) f'_y(x_0,y_0) + o(\rho)$$

28.III.2013г.

1) $$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

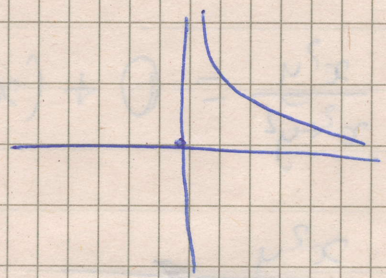
$$f'_x = \begin{cases} \dfrac{y(x^2+y^2) - xy\,2x}{(x^2+y^2)^2} = \dfrac{-x^2y + y^3}{(x^2+y^2)^2} \\ 0 & x^2+y^2 = 0 \end{cases} \qquad \cancel{\left(\dfrac{xy}{x^2+y^2}\right)'_y = \dfrac{\quad}{(x^2+y^2)^2}}$$

$f(x, 0) \equiv 0$

$$f'_y = \begin{cases} \dfrac{-y^2x + x^3}{(x^2+y^2)^2} \\ 0 & x^2+y^2 = 0 \end{cases}$$

VI.2.50

$f''_{xy}(0,0) = \cancel{\ }$



$$f'_x(0,y) = \begin{cases} \frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

$$\underset{p}{f(x,y)} = f\underset{p_0}{(x_0,y_0)} + (x-x_0) f'_x(x_0,y_0) + (y-y_0) f'_y(x_0,y_0) +$$

$$+ o(\rho(p, p_0))$$

$$\frac{f(x) - f(x_0)}{x - x_0} \xrightarrow[x \to x_0]{} f'(x_0)$$

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = \varphi(x)$$

$$\frac{\varphi(x)}{x - x_0} \longrightarrow 0$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$$

② $$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

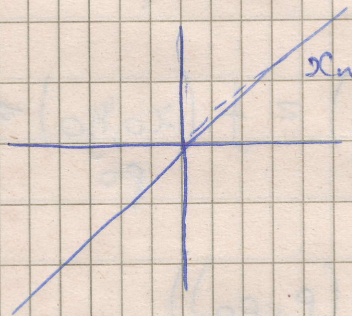
$$f'_x = \begin{cases} \frac{2xy^3}{(x^2+y^2)^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

$$f'_y = \begin{cases} \frac{x^4 - x^2 y^2}{(x^2+y^2)^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

$$\cancel{f} \quad \frac{x^2y}{x^2+y^2} = 0 + (x-x_0)\underbrace{f'_x(0,0)}_{=0} + (y-y_0)\cdot 0 + \frac{x^2y}{x^2+y^2}$$

$$\varphi = \frac{x^2y}{(x^2+y^2)\sqrt{x^2+y^2}} \qquad \frac{o(\rho)}{\rho} \xrightarrow[\rho\to 0]{} 0$$

$$\varphi(x_n, y_n) = \frac{x_n^3}{2x_n^2\cdot\sqrt{2x_n^2}} = \frac{\cancel{x_n^3}}{2\sqrt{2}\,\cancel{x_n^3}} \Rightarrow$$



$x_n = y_n \Rightarrow 0$

$$\Rightarrow \frac{1}{2\sqrt{2}}$$

$\Rightarrow$ ф-ята не е диференцуема

ДУ: Ако $\exists$ частните производни и те са непрекъснати $\Rightarrow$ ф-ята е диференцуема

Теорема за диференциране на съставни функции: Ако една ф-я е диференцируема, частните произв. са непрекъснати и ф-ите са непрекъснати.

$F(x,y)$ диф. непр. $\quad \varphi, \psi(t)$ диф. в $t_0 \quad \varphi(t_0) = x_0,\ \psi(t_0) = y_0$

$$Q(t) = F(\varphi, \psi)$$

$$Q'(t) = F'_x(x_0, y_0)\cdot\varphi'(t_0) + F'_y(x_0, y_0)\cdot\psi'(t_0)$$

$\boxed{VI}$ 2.52

(3) $\varphi(t) = t$
$\psi(t) = t$

$$Q(t) = \begin{cases} \frac{t^3}{2t^2} = \frac{t}{2} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

$$Q(t) = \frac{t}{2}$$

$$Q'(t) = \frac{1}{2}$$

$$Q'(0) = 0 \cdot 1 + 0 \cdot 1 = 0$$

$$Q(t) = \frac{1}{2} t \qquad Q'(1) = \frac{1}{2}$$

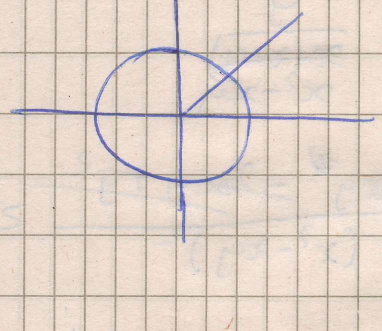
$$Q'(1) = \frac{1}{2} \cdot 1 + 0 \cdot 1 = \frac{1}{2}$$

Д.Г. (1) $f(x,y) = \begin{cases} (x^2+y^2) \sin \frac{1}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

а) непрекъсната в началото
б) не е диференцуема в началото

$$x_n^2 + y_n^2 = \frac{1}{2n\pi}$$

$x_n = y_n$



Дефиниция: Хомогенна ф-я: $t>0$

$F(tx, ty, tz) = t^p f(x, y, z)$

$f(tx, ty) = t^p f(x, y)$ $\quad (x_0, y_0, z_0)$

$x F'_x + y F'_y + z F'_z = p F(x, y, z)$

9) $\boxed{\varphi(t) = \dfrac{F(tx_0, ty_0, tz_0)}{t^p}}$ - хомогенна

$$\varphi'(t) = \frac{(F'_x(tx_0, ty_0, tz_0) \cdot x_0 + F'_y(tx_0, ty_0, tz_0) \cdot y_0 + F'_z() \cdot z_0)\, t^p}{t^{2p}} -$$

$$- \frac{F(\ ) \cdot p t^{p-1}}{t^{2p}} = \frac{p \cdot F(tx_0, ty_0, tz_0) \cdot t^{p-1} - F(tx_0, ty_0, tz_0)\, p\, t^{p-1}}{t^{2p}} = 0$$

$\varphi'(t) = 0 \qquad \varphi(t) = C = \dfrac{F(x_0, y_0, z_0)}{1} \cdot t^p$

$\Rightarrow$ ~~хом~~ $\varphi(t)$ е хомогенна ф-я

5) $z = \ln\left(\frac{1}{x} - \frac{1}{y}\right) \qquad z''_{xx} + z''_{yy} = \frac{1}{x^2}$

$$z'_x = \frac{1}{\frac{1}{x} - \frac{1}{y}}\left(-\frac{1}{x^2}\right) = -\frac{1}{\frac{y-x}{xy}} \cdot \frac{1}{x^2} = -\frac{y}{(y-x)x} = \frac{y}{x^2 - xy}$$

$$z''_{xx} = \frac{-(2x-y)y}{(x^2-xy)^2} = \frac{y^2 - 2xy}{(x^2-xy)^2} =$$

$$= \cancel{\frac{x^2 - 2xy + y^2}{(x^2 - xy)^2}}$$

$$z''_{yy} = \frac{(x^2 - xy) - x(2y - x)}{(x^2-xy)^2} = \frac{x^2 - 2xy}{(x^2 - xy)^2}$$

VI 2.54

$$z''_{xx} + z''_{yy} = \frac{y^2 - 2xy}{(x^2 - xy)^2} + \frac{x^2 - 2xy}{(y^2 - yx)^2} = \frac{(y^4 - 2y^3 + y^2x^2)(y^2 - 2xy)}{(x^4 - 2x^3y + x^2y^2)(y^4 - 2y^3x + x^2y^2)} +$$

$$z'_y = \frac{x}{y^2 - xy}$$

$$+ \frac{(x^4 - 2x^3y + y^2x^2)(x^2 - 2yx)}{(x^4 - 2x^3y + x^2y^2)(y^4 - 2y^3x + x^2y^2)}$$

VII 2.55