Група А

1 зад

а) $\Delta = \begin{vmatrix} 3 & -1 & 2 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 \\ -1 & 4 & 0 \\ 3 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 \\ -13 & 0 & 0 \\ 3 & 1 & 0 \end{vmatrix} =$

$= \begin{vmatrix} 2 & -1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -13 \end{vmatrix} = 2\,1\,{-13} = -26$

б) $\Delta = \begin{vmatrix} 5 & 5 & 5 & 3 \\ 5 & 5 & 3 & 5 \\ 5 & 3 & 5 & 5 \\ 3 & 5 & 5 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 5 & 5 \\ 5 & 3 & 5 & 5 \\ 5 & 5 & 3 & 5 \\ 5 & 5 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 5 & 5 \\ 2 & -2 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{vmatrix} =$

$(-2)^3 \begin{vmatrix} 3 & 5 & 5 & 5 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = (-2)^3 \begin{vmatrix} 3+3\cdot5 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} =$

$= (-2)^3 \cdot 18 = -144$

в) $\begin{vmatrix} x & x & x & x \\ x & 0 & x & x \\ x & x & 0 & x \\ x & x & x & 0 \end{vmatrix} = \begin{vmatrix} x & x & x & x \\ 0 & -x & 0 & 0 \\ 0 & -x & -x & 0 \\ 0 & 0 & 0 & -x \end{vmatrix} = x(-x)^3 = -x^4$

2. а) $A=\begin{pmatrix}1&2\\3&5\end{pmatrix}$, $B=\begin{pmatrix}1&3\\3&8\end{pmatrix}$

$$\left(\begin{array}{cc|cc}1&2&1&3\\3&5&3&8\end{array}\right)\sim\left(\begin{array}{cc|cc}1&2&1&3\\0&-1&0&-1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&2&1&3\\0&1&0&1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&0&1&1\\0&1&0&1\end{array}\right)$$

$$X=\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

$$X^2=\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&2\\0&1\end{pmatrix}$$

$$X^3=\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}1&3\\0&1\end{pmatrix}\Rightarrow$$

$$X^5=\begin{pmatrix}1&5\\0&1\end{pmatrix}$$

б) $A=\begin{pmatrix}2&1\\-1&3\end{pmatrix}$, $B=\begin{pmatrix}2&3\\-1&2\end{pmatrix}$

$$\left(\begin{array}{cc|cc}2&1&2&3\\-1&3&-1&2\end{array}\right)\sim\left(\begin{array}{cc|cc}2&1&2&3\\-7&0&-7&-7\end{array}\right)\sim\left(\begin{array}{cc|cc}7&0&7&7\\2&1&2&3\end{array}\right)\sim$$

$$\sim 7\left(\begin{array}{cc|cc}1&0&1&1\\2&1&2&3\end{array}\right)\sim 7\left(\begin{array}{cc|cc}1&0&1&1\\0&1&0&1\end{array}\right)$$

$$X=7\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

$$X^5=7^5\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}^5=7^5\begin{pmatrix}1&5\\0&1\end{pmatrix}$$

$3_{зад}$ а) $| 2x_1 + 3x_2 - x_3 = 0$

$(2, 3, -1)$

$p \quad q$

$(p, q, 2p+3q)$

базис $- e_1 (1, 0, 2)$ ....

$e_2 (0, 1, 3)$

б) $| 3x_1 + 2x_2 + x_3 = 0$

$(3 \quad 2 \quad 1)$

$p \quad q$

$(p, q, -3p-2q)$

базис $e_1 (1, 0, -3)$

$e_2 (0, 1, -2)$

$4_{зад}$ Ако $\theta \notin U \Rightarrow U$ не е подпространство

$\Rightarrow$ а) $3.0 + 2.0 \neq 1 \Rightarrow \theta \notin U \Rightarrow U$ не е подпространство на $\mathbb{R}^2$

г) $0 - 3.0 \neq 2 \Rightarrow$ — // —

б) $U = \{(x, y) \mid 3x - 2y = 0\}$

$\theta \in U$

Нека $\exists u = (x_1, y_1) : 3x_1 - 2y_1 = 0$

$v = (x_2, y_2) : 3x_2 - 2y_2 = 0$

$3(x_1 + x_2) - 2(y_1 + y_2) \stackrel{?}{=} 0$

$3x_1 + 3x_2 - 2y_1 - 2y \stackrel{?}{=} 0$

$0 + 0 = 0$

$\lambda(x_1, y_1) = (\lambda x_1, \lambda y_1) = 3\lambda x_1 - 2\lambda y_1 = \lambda(3x_1 - 2y_1) = \lambda.0 = 0 \Rightarrow$

$u + v \in U$ и $\lambda u \in U \Rightarrow$

U - подпространство на $\mathbb{R}^2$

б) $U = \{(x, y) \mid 4y + 2 = 2(x+1)\}$

$-2x + 4y = 0$

$\Theta \in U$

Нека $\exists$ $u = (x_1, y_1) : -2x_1 + 4y_1 = 0$

$v = (x_2, y_2) : -2x_2 + 4y_2 = 0$

$-2(x_1 + x_2) + 4(y_1 + y_2) \overset{?}{=} 0$

$-2x_1 + 4y_1 - 2x_2 + 4y_2 \overset{?}{=} 0$

$0 + 0 = 0$

$\lambda u = \lambda(x_1, y_1) = (\lambda x_1, \lambda y_1) = -2\lambda x_1 + 4\lambda y_1 = \lambda(-2x_1 + 4y_1) = \lambda . 0 = 0$

$\Rightarrow u + v \in U$ и $\lambda u \in U \Rightarrow$

U - подпространство на $\mathbb{R}^2$

5 зад а) $\begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases}$ $\mathbb{R}^7$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow r(u) = 2$$

$\dim U = \dim \mathbb{R}^7 - r(u) = 7 - 2 = 5$

б) $\begin{cases} 4x_5 - 10x_7 = 0 \\ 6x_5 - 15x_7 = 0 \end{cases}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 0 & -10 \\ 0 & 0 & 0 & 0 & 6 & 0 & -15 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 2 & 0 & -5 \end{pmatrix} \Rightarrow r(u) = 1$$

$\dim U = \dim \mathbb{R}^7 - r(u) = 7 - 1 = 6$

6 зад а) $\varphi(e_1) = 2e_1 - 3e_2$, $\varphi(e_2) = -2e_1$,
$\varphi(a) = ?$, където $a = (2, -1)$

$\begin{pmatrix} 2 & -2 \\ -3 & 0 \end{pmatrix}$ − координати по стълбове = матрицата на лин. оператор φ.

$$\begin{pmatrix} 2 & -2 \\ -3 & 0 \end{pmatrix} \cdot \underset{a}{\begin{pmatrix} 2 \\ -1 \end{pmatrix}} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} - \varphi(a)$$

б) $\varphi(\underbrace{2e_1 + e_2}_{a_1}) = \underbrace{e_1 + 3e_2}_{b_1}$ $\quad \varphi(\underbrace{e_1 - e_2}_{a_2}) = \underbrace{2e_1 + 3e_2}_{b_2}$

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 3 \\ 1 & -1 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & 1 & 1 & 3 \\ 3 & 0 & 3 & 6 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & -1 & 2 & 3 \\ 3 & 0 & 3 & 6 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -1 & 2 & 3 \\ 1 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -1 & 2 & 3 \\ 0 & -1 & 1 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

$A^t = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ ← матрица на лин. оператор φ

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \varphi(a)$$

? $a = (1\ 2\ 3\ 4)$

Група Б

1 зад. $C \in M_n(F)$, C - скаларна? $\Rightarrow$.

$C = A$

$A = \sum a_{ij} E_{ij}$ $\quad E_{ij}$ са базис на всички матрици $\Rightarrow$ проверяваме дали A комутира с тях

$A \cdot E_{ij} =$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{array}{c} \quad j \\ \begin{pmatrix} 0 & \vdots & 0 \\ & 1 & \\ 0 & \vdots & 0 \end{pmatrix} \end{array} = \begin{array}{c} \quad j \\ \begin{pmatrix} 0 & a_{1i} & 0 \\ & a_{2i} & \vdots \\ \vdots & \vdots & \\ 0 & a_{ni} & 0 \end{pmatrix} \end{array}$$

$E_{ij} \cdot A =$

$$\begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = i \begin{pmatrix} 0 & \dots & 0 \\ a_{j1} & \dots & a_{jn} \\ 0 & \dots & 0 \end{pmatrix}$$

$A \cdot E_{ij} = E_{ij} \cdot A$

$\Rightarrow a_{ii} = a_{jj}$

$a_{ki} = 0$ при $k \neq i$ $\Rightarrow$

$a_{jk} = 0$ при $k \neq j$

за $\forall\, i,j$ $\quad a_{kl} = 0,\ k \neq l \Rightarrow a_{11} = a_{22} = \dots = a_{nn} = \lambda$

$$A = \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & \dots & \lambda \end{pmatrix} = \lambda E \Rightarrow A \text{ е скаларна} \Rightarrow C \text{ - скаларна}$$

2 зад.

2) $a_1 = (1, 2, 3, 4)$

$a_2 = (7, 14, 20, 27)$

$a_3 = (5, 10, 16, 19)$

$v = (2, \lambda, 5, 5)$

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 7 \\ 14 \\ 20 \\ 27 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 10 \\ 16 \\ 19 \end{pmatrix} = \begin{pmatrix} 2 \\ \lambda \\ 5 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 7 & 5 & 2 \\ 2 & 14 & 10 & \lambda \\ 3 & 20 & 16 & 5 \\ 4 & 27 & 19 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 7 & 5 & 2 \\ 0 & 0 & 0 & \lambda-4 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & -1 & -3 \end{array}\right) \Rightarrow$$

$\Rightarrow \lambda - 4 = 0, \ \lambda = 4$

$$\left(\begin{array}{ccc|c} 1 & 7 & 5 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 12 & 0 & 7 \\ 0 & -1 & 1 & -1 \\ 0 & 2 & 0 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 12 & 0 & 7 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{array}\right) \sim$$

$$\sim \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -17 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right) \\ \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \end{array} \end{array} \Rightarrow \quad \begin{array}{l} \lambda_1 = -17 \\ \lambda_3 = 1 \\ \lambda_2 = 2 \end{array}$$

$v = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = -17 a_1 + 2 a_2 + a_3$

3 зад.

$$\begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & -1 & -1 \\ 1 & -1 & 1-\lambda & -1 \\ 1 & -1 & -1 & 1-\lambda \end{pmatrix} \xrightarrow{\text{събираме 1 ред към останалите}} \begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 2-\lambda & 2-\lambda & 0 & 0 \\ 2-\lambda & 0 & 2-\lambda & 0 \\ 2-\lambda & 0 & 0 & 2-\lambda \end{pmatrix}$$

При $\lambda = 2 \Rightarrow 2-\lambda = 2-2 = 0 \Rightarrow r = 1$

При $\lambda = -2 \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 4 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ 4 & 0 & 0 & 4 \end{pmatrix} \Rightarrow r = 3$

При $\lambda \neq \pm 2 \quad r = 4$

4 зад. $x^3 - 5x^2 + 8x + \lambda$

④ $x_1 + x_2 = x_1 x_2$

$\lambda = ?$

От формулите на Виет $\Rightarrow$

① $x_1 + x_2 + x_3 = 5$

② $x_1x_2 + x_1x_3 + x_2x_3 = 8$

③ $x_1x_2x_3 = \lambda$

От ② и ④ $\Rightarrow$

$x_1 + x_2 + x_3(x_1 + x_2) = 8$

$(x_1 + x_2)(1 + x_3) = 8$

От ① $\Rightarrow x_1 + x_2 = 5 - x_3$

$(5 - x_3)(1 + x_3) = 8$

$x_3^2 - 4x_3 + 3 = 0$

$x_3' = 1 \qquad x_3'' = 3$

I При $x = 1$

$1^3 - 5 + 8 + \lambda = 0$

$\lambda = -4$

II При $x = 3$

$3^3 - 5 \cdot 3^2 + 8 \cdot 3 + \lambda = 0$

$\lambda = -6$

5 зад. $f = x^n - 1$

$g = x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$

Корените на g - $0, 1, 1$

Остатък $r = ax^2 + bx + c$

0 - еднократен корен, 1 - двукратен корен $\Rightarrow$

$$\begin{cases} f(0) = r(0) \\ f(1) = r(1) \\ f'(1) = r'(1) \end{cases}$$

$$\begin{cases} -1 = c \\ 0 = a + b + c \\ n = 2a + b \end{cases}$$

$$\begin{cases} a + b = -1 \\ 2a + b = n \end{cases}$$

$a = n - 1$

$b = 2 - n$

$$r = (n-1)x^2 - (n-2)x - 1$$

6 зад. $\Sigma = \frac{x_1^2}{1+x_1} + \frac{x_2^2}{1+x_2} + \frac{x_3^2}{1+x_3}$

$$\frac{x_i^2 + 1 - 1}{1 + x_i} = \frac{x_i^2 - 1}{x_i + 1} + \frac{1}{1 + x_i} = x_i - 1 - \frac{1}{-1 - x_i}$$

$$\Sigma = x_1 - 1 - \frac{1}{-1-x_1} + x_2 - 1 - \frac{1}{-1-x_2} + x_3 - 1 - \frac{1}{-1-x_3} =$$

$= x_1 + x_2 + x_3 - 3 - \left[\frac{1}{-1-x_1} + \frac{1}{-1-x_2} + \frac{1}{-1-x_3} \right]$

$f = x^3 + px + q$

От формулите на Виет

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 x_2 + x_1 x_3 + x_2 x_3 = p \\ x_1 x_2 x_3 = -q \end{cases}$$

$$S = 0 - 3 - \frac{(-1-x_2)(-1-x_3) + (-1-x_1)(-1-x_3) + (-1-x_2)(-1-x_3)}{[(-1)-x_1][(-1)-x_2][(-1)-x_3)} =$$

$$= -3 - \frac{1 + x_1 + x_2 + x_1 x_2 + 1 + x_1 + x_3 + x_1 x_3 + 1 + x_2 + x_3 + x_2 x_3}{f(-1)} =$$

$$= -3 - \frac{3 + 2(x_1 + x_2 + x_3) + x_1 x_2 + x_1 x_3 + x_2 x_3}{-1 - p + q} =$$

$$= -3 - \frac{3 + p}{q - p - 1} =$$

$$= \frac{-3q + 3p + 3 - 3 - p}{q - p - 1} = \frac{2p - 3q}{q - p - 1}.$$