23.05.2013г. ДИС - упражнения

1. Критерий на Даламбер

$\sum a_n \qquad a_n > 0$

$$\frac{a_{n+1}}{a_n} \leq q < 1$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell \quad \begin{cases} < 1 & сх \\ > 1 & рх \\ \frac{a_{n+1}}{a_n} \geq 1 & \text{за } \forall n \text{ рх} \end{cases}$$

2. Критерий на Коши

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \ell \qquad \text{ако границата} \quad \begin{matrix} < 1 & сх \\ > 1 & рх \\ = 1 & \end{matrix}$$

при $= 1$ гледаме неравенството

$\sqrt[n]{a_n} \geq 1 \Rightarrow$ реда е рх

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} \sqrt[n]{a} = 1$$

1 след (1) при неприятен случай 3

3

$$\lim_{n\to\infty} \underbrace{n\left(\frac{a_n}{a_{n+1}} - 1\right)}_{d_n} = m \qquad \begin{matrix} > 1 & сх \\ < 1 & рх \\ = & \\ d_n \leq 1 & рх \end{matrix}$$

ако $\lambda_n \to m > 0$

$a_n \to 0$

④ $\sum_{n=1}^{\infty} (-1)^n a_n$, където $a_n > a_{n+1}$

$\forall\, a_n$ са > 0

и $a_n \to 0$

ако тези условия са изп. $\Rightarrow$ че реда е сходящ.

⑤

$\sum |a_n| \Rightarrow \sum a_n$

сх. — сх

рх. —

условно сх — абсолютно сх.

$\rightsquigarrow$

$\sum_{n=1}^{\infty} n^k q^n$

$\sqrt[n]{|n^k q^n|} = \Big(\underbrace{\sqrt[n]{n}}_{\to 1}\Big)^k |q| \to |q| \Rightarrow |q| < 1 \Rightarrow$ редът е аб. сх.

ако $|q| \geq 1$ рх.

$n^k q^n \not\to 0$

задача

$\sum_{n=1}^{\infty} \frac{a^n}{n!}$ $\quad \frac{|a_{n+1}|}{a_n} =$, $a \neq 0$

$= \frac{a^{n+1}}{(n+1)!} : \frac{a^n}{n!} = \frac{a^{n+1} \cdot n!}{a^n (n+1)!} = \frac{|a|}{|n+1|} \longrightarrow 0 \Rightarrow$ от (1) сх.

задача $\sum_{n=1}^{\infty} \underbrace{\frac{2.5.8\ldots[2+3(n-1)]}{1.5.9\ldots[1+4(n-1)]}}_{a_n}$

$$\frac{a_{n+1}}{a_n} = \frac{2.5.8\ldots[2+3(n-1)][2+3n]}{1.5.9\ldots[1+4(n-1)][1+4n]} \cdot \frac{1.5.9\ldots[1+4(n-1)]}{2.5.8\ldots[2+3(n-1)]}$$

$$= \frac{2+3n}{1+4n} \longrightarrow \frac{3}{4} < 1 \Rightarrow \text{сх.}$$

задача

$$\sum_{n=1}^{\infty} \binom{2n}{n}$$

$$\frac{a_{n+1}}{a_n} = \binom{2(n+1)}{n+1} \cdot \cancel{2n} \cdot \frac{1}{\binom{2n}{n}} =$$

$$= \frac{[(2n+2)2n(2n-2)\ldots(2n-n+1)][n(n-1)\ldots 3.2.1]}{[(n+1)n(n-1)\ldots 3.2.1][2n.2(n-1).2(n-2)\ldots(2n-n+1)]} =$$

$$= \frac{(2n+2)(2n+1)}{(n+1)\cancel{2n}(n+1)} = \frac{\cancel{4n}}{n+1} = \cancel{2n+} \longrightarrow 4 > 1 \text{ рх.}$$

задача

$$\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!\,(n+1)} =$$

$$\frac{a_{n+1}}{a_n} =$$

първо по (1) после (3)

отс. усл. сх.

задача

$$\sum_{n=1}^{\infty} \left(\frac{n}{e}\right)^n \frac{1}{n!}$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{e}\right)^{n+1} \cdot \frac{1}{(n+1)!} \cdot \frac{n!\, e^n}{n^n} = \frac{(n+1)^{n+1} \cdot n! \cdot e^n}{e^{n+1}\,(n+1)!\, n^n} =$$

$$= \cancel{(n+1)(n+1)^n\, e^n} \qquad = \frac{(n+1)(n+1)^n}{e\,(n+1)\, n^n} = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{e} \xrightarrow[n\to\infty]{} 1$$

от (3) $\qquad n = \frac{1}{x}$

$$n\left(e\left(\frac{n}{n+1}\right)^n - 1\right) = \frac{1}{x}\left(e\left(\frac{1}{1+x}\right)^{\frac{1}{x}} - 1\right) = \frac{1}{x}\left(e\,(1+x)^{-\frac{1}{x}} - 1\right) =$$

$$= \frac{1}{(1+x)^{\frac{1}{x}}} \cdot \frac{e - (1+x)^{\frac{1}{x}}}{x} \qquad \text{Лопитал}$$

$$\text{В} \quad -e^{\frac{1}{x}\ln|1+x|}\underbrace{\left[-\frac{1}{x^2}\ln|1+x|+\frac{1}{x(x+1)}\right]}_{g(x)}$$

$\to e$

$$g(x) = -\frac{1}{x^2}\ln(1+x) + \frac{1}{x(x+1)} = \frac{-\ln(1+x)(x+1)+x}{x^2(x+1)} \qquad (x+1)\to 1$$

пак Лопитал не минаване на Пеано

Обаче ще трябват поне 2 така ли?

$$\frac{x-(x+1)\ln(x+1)}{x^2(x+1)} \qquad \ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

и тук някъде вече явно се изгуби нишката

$$= \frac{x-(x+1)\left(x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)\right)}{x^2(x+1)} = \frac{x^2-\frac{x^2}{2}+o(x^2)}{x^2} =$$

$$= \frac{1}{2}\,\frac{x^2+o(x^2)}{x^2} = \longrightarrow \frac{1}{2}$$

получихме $\frac{1}{2e}$ $\Rightarrow$ по Раабе-Дюамел

$$\frac{e-\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n \quad \frac{1}{n}}$$

$$\frac{e-e^{\frac{1}{x}\ln(1+x)}}{x} = \frac{e-e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)\right)}}{x} = \frac{-e-e^{1-\frac{x}{2}+\frac{x^2}{3}+o(x^2)}}{x} =$$

$$= \frac{e-e.e^{-\frac{x}{2}+\frac{x^2}{3}+o(x^2)}}{x} = e\,\frac{1-e^{-\frac{x}{2}+\frac{x^2}{2}+o(x^2)}}{\frac{-\frac{x^2}{2}+\frac{x^3}{3}+o(x^2)}{-1}} = \frac{-\frac{x^2}{2}+\frac{x^3}{3}+o(x^2)}{x}$$

$$\frac{e^x-1}{x}\to 1 \qquad \downarrow\; -\frac{1}{2}$$