— 16.05.2013 г. ~ ДИС ~ упр.

$\frac{1}{n^\alpha} \xrightarrow[\alpha \to 0]{} 0$; $q^n \xrightarrow[-1<q<1]{} 0$; $\sqrt[n]{a} \to 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ (хармоничен при $\alpha = 1$; обобщен харм. при $\alpha \neq 1$?)

$\sum_{n=1}^{\infty} q^n = \frac{1}{1-q}$ $\quad -1<q<1$

Зад 1

$1 + 2q + 3q^2 + \dots + nq^{n-1} + \dots = \sum_{n=1}^{\infty} nq^{n-1}$, $-1<q<1$

Д-д, че ще е сходящ, и да се намери сумата.

$S_n = 1 + 2q + \dots + n.q^{n-1} = 1 + q + \dots + q^{n-1}$ $\to$ I н.

$q + \dots + q^{n-1}$

$q^2 \quad + q^n$

II н.) Тм на Щолц : $a_n \to \infty/0$ и $b_n \to \infty/0$; $\frac{a_n}{b_n}$?
монотон.

Ако b_n – монот. Образ. $\frac{a_n - a_{n-1}}{b_n - b_{n-1}} \to R$, то и $\frac{a_n}{b_n}$ има същата граница.

Пол. $\frac{1}{q} = t$

$S_n = \frac{t^{n-1} + 2t^{n-1} + 3t^{n-3} + \dots + n}{t^{n-1}}$ $\quad$ (числ. $= a_n$, знам. $= b_n$)

За $q>0 \Rightarrow t>1 \Rightarrow t^n \to \infty$ и числ. $\to \infty$

И вече можем да образ. $\frac{a_n - a_{n-1}}{b_n - b_{n-1}}$:

$$\frac{t^{n-1}+2t^{n-2}+3t^{n-3}\dots+n-(t^{n-2}+2t^{n-3}+3t^{n-4}\dots+n-1)}{t^{n-1}-t^{n-2}}=$$

$$=\frac{t^{n-1}+t^{n-2}+t^{n-3}\dots\dots+t+1}{t^{n-1}-t^{n-2}}=\frac{1-t^{n}}{(1-t)t^{n-2}(t-1)}=$$

$$=\frac{1}{(1-t)^2}\left(t^2-\frac{1}{t^{n-2}}\right)=\frac{1}{(1-q)^2}-q^{n-2}\cdot\frac{1}{\left(1-\frac{1}{q}\right)^2}\to\frac{1}{(1-q)^2}$$

$\Rightarrow$ редът е сх. с гран. $\frac{1}{(1-q)^2}$

III н.) $Sn.(1-q)=1+2q+3q^2\dots+nq^{n-1}-q-2q^2\dots-(n-1)q^{n-1}-n.q^n=$

$=1+q+\dots+q^{n-1}-n.q^n$

$Sn=\frac{1-q^{n}}{(1-q)^2}-1-nq^n \to 0$

$*$ $n^k.q^n$; $k>0$ и $-1<q<1$ $\to 0$

Зад.) $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Дали е сх. ДД, че сумата е $\frac{\pi^2}{6}$

$S_n=1+\frac{1}{2}+\frac{1}{3}+\dots$

Зад.) ДД, че редът $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ е сх. и да се намери сумата му

$S_n=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)}$

образ. ф-ията:

$\frac{1}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$

$1=A(x+1)+B.x$ $\quad\quad 1=Ax+A+Bx$

$0=A+B$

$1=A$, $B=-1$

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\Rightarrow S_n = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{n}-\frac{1}{n+1} = 1-\frac{1}{n+1}$$

$\Rightarrow$ ред. е сход. и гран. е 1

$S_n \le 1$, защото ред. е монот. и сх.

$$\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{(n+1)^2} \le S_n \le 1$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2} < 2$$

Тв.: $0 \le a_n \le b_n$. Разгл. $\sum a_n$ и $\sum b_n$, парц. им суми са $\uparrow$

$\Rightarrow S_n(a_n)\uparrow$ и $S_n(b_n)\uparrow$. Ако $\sum b_n$ е сх. $\Rightarrow \sum a_n$ е сх.
рх $\Leftarrow$ рх

(*) Зн., че $\sum_{n=1}^{\infty} \frac{1}{n^2}$ е сх, то $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ при $\alpha \ge 2$ редът е рх.

Зад. $1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\frac{1}{4^\alpha}+\dots+\frac{1}{(2^n-1)^\alpha}+\cancel{\frac{1}{(2^{n+1})^\alpha}} \le$

парц. сума на реда

$\cancel{S_n = (S_2, S_4, S_8, \dots)}$ $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, $\alpha > 1$ — редът

$S_n \le S_{2^n}$

$S_1 \le S_2 \le S_3 \le \dots$

$\underbrace{S_2 \le S_4 \le S_8 \le S_{16}}_{S_{2^n}}$ — ако е изн., то и е изн. ($S_n \le S_{2^n}$)

п. сума $\le 1+\frac{1}{2^\alpha}+\frac{1}{2^\alpha}+\underbrace{\frac{1}{4^\alpha}+\frac{1}{4^\alpha}+\frac{1}{4^\alpha}+\frac{1}{4^\alpha}}_{4}+\underbrace{\frac{1}{8^\alpha}+\frac{1}{8^\alpha}+\dots}_{8}$

$+\frac{1}{(2^{n-1})^\alpha}+\frac{1}{(2^{n-1})^\alpha}+\frac{1}{(2^{n-1})^\alpha}+\dots+\frac{1}{(2^{n-1})^\alpha}$

$S_{2^n} \leq 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{4^{\alpha-1}} + \dots + \frac{1}{(2^{n-1})^{\alpha-1}} = 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{(2^{\alpha-1})^2} + \frac{1}{(2^{\alpha-1})^3} +$

$\dots + \frac{1}{(2^{\alpha-1})^{n-1}} = \dfrac{1-\left(\frac{1}{2^{\alpha-1}}\right)^n}{1-\frac{1}{2^{\alpha-1}}}$ (< S на геом. прогр.)

$\alpha > 1 \; : \; \dfrac{1-\left(\frac{1}{2^{\alpha-1}}\right)^n}{\underbrace{1-\frac{1}{2^{\alpha-1}}}_{>0}} \leq \dfrac{1}{1-\frac{1}{2^{\alpha-1}}}$

$\Rightarrow S_{2^n}$ е огран. ($\leq$ е от число, незав. от n) $\Rightarrow$

S_n е огран. $(S_n < S_{2^n}) \Rightarrow$ редът е сходящ

<u>зад.</u> $\sum \frac{1}{n}$? е разходящ **УПР!**

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n} \leq$?

(*) $\sum \frac{1}{n^\alpha}$, $0 < \alpha < 1$

$\frac{1}{n^\alpha} > \frac{1}{n} \Rightarrow \sum \frac{1}{n^\alpha}$ $\to \alpha > 1$, сх; $\to \alpha \leq 1$, рх.

<u>Методи за решение:</u>

$\sum a_n$ и $\sum b_n$; $0 < a_n, b_n$

$\frac{a_n}{b_n} = c_n \qquad a_n = c_n \cdot b_n$

<u>Тм:</u> Ако $c_n \to A \neq 0$. Тогава $\sum a_n$ и $\sum b_n$ са едновр. сх. / рх.

сх $\leftarrow$ сх

рх $\leftarrow$ рх

зад. 1) $\sum \frac{1}{\sqrt{n(3n+1)}}$

$$\underbrace{\frac{1}{\sqrt{n(3n+1)}}}_{a_n} = \underbrace{\frac{1}{n}}_{b_n} \cdot \underbrace{\frac{1}{\sqrt{3+\frac{1}{n}}}}_{c_n}$$

$\boxed{c_n} \to \frac{1}{\sqrt{3}} \neq 0 \Rightarrow$ Ако b_n е сх $\Rightarrow a_n$ сх

b_n е рх $\Rightarrow a_n$ е рх

2) $\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{n^3}$

$$\underbrace{\frac{\sin\frac{1}{n}}{n^3}}_{a_n} = \overbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}^{\to 1} \cdot \frac{1}{n^4} \text{ - сх} \quad \Rightarrow a_n \text{ е сх.}$$

$\boxed{\text{II}}$ $\sum a_n \cdot b_n$; b_n има гран. $\neq 0 \Rightarrow \sum a_n \cdot b_n \sim \sum a_n$

пр.! 1) $\sum \frac{2\sqrt{n}+1}{(n+1)^2\sqrt{n+1}} \to 2$ $\sim \sum \overbrace{\frac{1}{(n+1)^2}\cdot\frac{n^2}{n^2}}^{\to 1} \sim$

$\sim \sum \frac{1}{n^2}$ е сх.

2) $\sum \left(e^{\frac{1}{n}}-1\right)\sin\frac{1}{\sqrt{n+1}}$

$$\overbrace{\frac{e^{\frac{1}{n}}-1}{\frac{1}{n}}}^{1} \overbrace{\frac{\sin\frac{1}{\sqrt{n+1}}}{1:\sqrt{n+1}}}^{\to 1} \cdot \frac{1}{n} \cdot \overbrace{\frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\sqrt{n}}}^{\to 1} \sim \frac{1}{n^{\frac{3}{2}}} \text{ е сх.}$$

3) $\sum \left(1-\cos \frac{2\pi}{\sqrt[5]{n^2}}\right)$

$\circledast$ $\frac{1-\cos ax}{x^2} \to \frac{a^2}{2}$

$\frac{1-\cos \frac{2\pi}{\sqrt[5]{n^2}}}{\frac{1}{n^{\frac{4}{5}}}} \to 2\pi^2 \cdot \frac{1}{n^{\frac{4}{5}}} \sim \sum \frac{1}{n^{\frac{4}{5}}}$ $-p\times$