26. VIII. 2013г.   Аналитична Геометрия

② $\vec{a}, \vec{b}$

$|\vec{a}| = |\vec{b}| = 1$

$\measuredangle(\vec{a}, \vec{b}) = \frac{\pi}{3}$

$\overrightarrow{OA} = \vec{a}$

$\overrightarrow{OB} = \vec{a} \times \vec{b}$

$\overrightarrow{OC} = \vec{b} \times (\vec{a} \times \vec{b})$

? $\overrightarrow{OA}, \overrightarrow{OB}$ и $\overrightarrow{OC}$ – ЛНЗ

$V = ?$

$a(a \times b) = (a \times a)b$

$\vec{a}\vec{b}\vec{a} - \frac{1}{2}\vec{a}\vec{b}\vec{b}$

$(a \times b)^2 = a^2b^2 - (\vec{a}\vec{b})^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$(\vec{a} - \frac{1}{2}\vec{b})^2 = \vec{a}^2 + \frac{1}{4}\vec{b}^2 - \vec{a}\vec{b} =$

$= 1 + \frac{1}{4} - \frac{1}{2} = \frac{3}{4}$

$$\overrightarrow{OC} = \vec{b} \times (\vec{a} \times \vec{b}) = \overbrace{(\vec{b}.\vec{b})}^{1}\vec{a} - \underbrace{\overbrace{(\vec{b}\vec{a})}^{|\vec{b}||a|\cos\measuredangle(\vec{b},\vec{a})}}_{\cos\frac{\pi}{3}}\vec{b} = \vec{a} - \frac{1}{2}\vec{b}$$

$$\begin{vmatrix} \overrightarrow{OA}.\overrightarrow{OA} & \overrightarrow{OA}.\overrightarrow{OB} & \overrightarrow{OA}.\overrightarrow{OC} \\ \overrightarrow{OA}.\overrightarrow{OB} & \overrightarrow{OB}.\overrightarrow{OB} & \overrightarrow{OB}.\overrightarrow{OC} \\ \overrightarrow{OA}.\overrightarrow{OC} & \overrightarrow{OC}.\overrightarrow{OB} & \overrightarrow{OC}.\overrightarrow{OC} \end{vmatrix} = \begin{vmatrix} \vec{a}^2 & \overbrace{\vec{a}(\vec{a} \times \vec{b})}^{0} & \vec{a}(\vec{a} - \frac{1}{2}\vec{b}) \\ 0 & (\vec{a} \times \vec{b})^2 & \underbrace{(\vec{a} \times \vec{b})(\vec{a} - \frac{1}{2}\vec{b})}_{0} \\ 1 - \frac{1}{2}\cdot\frac{1}{2} & 0 & (\vec{a} - \frac{1}{2}\vec{b})^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \frac{3}{4} \\ 0 & \frac{3}{4} & 0 \\ \frac{3}{4} & 0 & \frac{3}{4} \end{vmatrix} = \cancel{\frac{9}{16}}\ \frac{9}{16} - \frac{27}{64} = \frac{9}{64} = (\overrightarrow{OA}\,\overrightarrow{OB}\,\overrightarrow{OC})^2$$

$$(\overrightarrow{OA} \times \overrightarrow{OB}).\overrightarrow{OC} = (\vec{a} \times (\vec{a} \times \vec{b}))(\vec{a} - \frac{1}{2}\vec{b}) = [(\vec{a}\vec{b})\vec{a} - (\vec{a})^2\vec{b}](\vec{a} - \frac{1}{2}\vec{b}) =$$

$$= (\frac{1}{2}\vec{a} - \vec{b})(\vec{a} - \frac{1}{2}\vec{b}) = \frac{1}{2}\underbrace{(\vec{a})^2}_{1} - \frac{1}{4}\underbrace{\vec{a}\vec{b}}_{\frac{1}{2}} - \cancel{\underbrace{\vec{a}\vec{b}}_{\frac{1}{2}}} + \cancel{\frac{1}{2}\underbrace{\vec{b}^2}_{1}} = \frac{1}{2} - \frac{1}{8} =$$

$= \frac{3}{8}$

$(\overrightarrow{OA}\,\overrightarrow{OB}\,\overrightarrow{OC})^2 = \Delta$

①

3

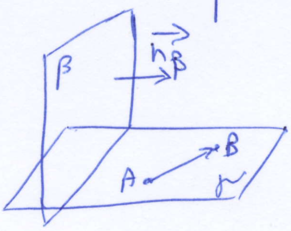
$A(0,0,-1)$

$B(-2,-8,-3)$

$\beta: 3x+4y-z+1=0$

$b: \begin{cases} x=3+3s \\ y=-8+1s \\ z=1-1s \end{cases}, s\in\mathbb{R}$

а) $?\ \gamma: \begin{cases} \ni A \\ \ni B \\ \perp \beta \end{cases}$



I н. $\gamma: \begin{cases} \ni A \\ \parallel \overrightarrow{AB}(-2,-8,-2) \\ \parallel \vec{n}_\beta(3,4,-1) \end{cases}$

$\gamma: \begin{cases} x=0+(-2)k+(3)m \\ y=0+(-8)k+(4)m \\ z=-1+(-2)k+(-1)m \end{cases}$

II н. $(\overrightarrow{AB}\times\vec{n}_\beta)\parallel\vec{n}_\gamma$

$(-2,-8,-2)\times(3,4,-1) = (16, -8, 16) \to \vec{n}_k(2,-1,2)$

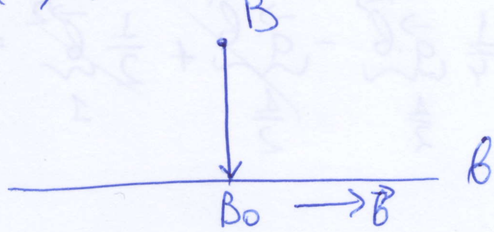
$\begin{vmatrix}-8 & -2\\ 4 & -1\end{vmatrix}$ $\begin{vmatrix}-2 & -2\\ -1 & 3\end{vmatrix}$ $\cancel{\begin{vmatrix}-2 & -8\\ 3 & 4\end{vmatrix}}$

$\Rightarrow \gamma: 2x-1y+2z+D=0$

$A(0,0,-1)\ni\gamma \Rightarrow D=2$

$\Rightarrow \gamma: 2x-y+2z+2=0$

б) $\delta(B,b)=\sqrt{30}$



$B_0(x_0,y_0,z_0)$

$\overrightarrow{BB_0}\perp\vec{b} \Rightarrow \begin{cases}\overrightarrow{BB_0}\cdot\vec{b}=0\\ \overrightarrow{BB_0}=\vec{b}\end{cases}$

$B(-2,-8,-3)$

$B_0\ni b \Rightarrow B_0(\underbrace{3+3s}_{x_0}, -8+s, 1-s)$

$\overrightarrow{BB_0}(5+3s, s, 4-s)$

$\vec{b}(3,+1,-1)$

$\overrightarrow{BB_0}\cdot\vec{b}=3(5+3s)+1(s)-1(4-s)=15+9s+s-4+s=$

$=11+11s=0 \Rightarrow \boxed{s=-1}$

$\overrightarrow{BB_0}(2,-1,5) \Rightarrow |\overrightarrow{BB_0}|^2=(\overrightarrow{BB_0})^2=30$ $\quad |\overrightarrow{BB_0}|=\sqrt{30}$

(5) $k: \underset{}{x^2} + 6xy + y^2 + 18\underset{t}{x} + 6\underset{t}{y} + \underset{t^2}{5} = 0$

1) тип на кривата (броя корените на квадратичната част)

$x^2 + 6xy + y^2 = 0$

$D = 9 - 1 > 0 \Rightarrow$ 2 корена $\Rightarrow$ 2 безкр. точки $\Rightarrow$ хипербола X

разкриване на скобите

1 ~~парабу~~ ~~елипса~~

0 ~~парабола~~

1 безкрат корен $\Rightarrow$ парабула Π

0 корена $\Rightarrow$ елипса $\mathcal{E}$

2)
$$\begin{array}{c|ccc} & x & y & \begin{matrix}1\\ \text{или}\\ t\end{matrix} \\ x & 1 & 3 & 9 \\ y & 3 & 1 & 3 \\ 1 \text{ или } t & 9 & 3 & 5 \end{array}$$
– централни криви (X и $\mathcal{E}$)
(в беловата не се пише само резултата)

$$\begin{pmatrix} 1 & 3 & 9 \\ 3 & 1 & 3 \\ 9 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \neq 0 \end{pmatrix}$$

$$-\left| \begin{array}{l} x + 3y + 9t = 0 \quad | .3 \\ 3x + y + 3t = 0 \end{array} \right. \qquad \begin{cases} -8y - 24t = 0 \\ 8y = -24t \\ y = -3t \end{cases}$$

$3\cancel{x} - 3t + 3t = 0 \Rightarrow x = 0$

$C(0, -3t, t) \Rightarrow \boxed{C(0, -3, 1)}$

$\boxed{C(0, -3)}$

$9 \cdot 0 + 3(-3) + 5 \cdot 1 = -9 + 5 = \boxed{-4}$

3) трансформация $T_1 : OKC : K = O_{xyt} \xrightarrow{T_1} C_{x'y'}$

$$T_1 : \left| \begin{array}{l} C(0, -3) \\ C_{x'} \uparrow\uparrow O_x \\ C_{y'} \uparrow\uparrow O_y \end{array} \right. \qquad T_1 : \begin{cases} x = x' + 0 \\ y = y' - 3 \end{cases}$$

$(x')^2 + 6x'(y'-3) + (y'-3)^2 + 18x' + 6(y'-3) + 5 = 0$

$x'^2 + 6x'y' + y'^2 \boxed{-4} = 0$

4) главни направления

$$\left|\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} - \lambda E\right| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0 \qquad \lambda_1 = -2 \quad \lambda_2 = 4$$

$$\lambda = -2 \Rightarrow \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1, -1)$ дели се на дължината

$$a_1 + b_1 = 0$$
$$a_1 = -b_1 \Rightarrow \boxed{\vec{a}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}$$

$$\lambda = 4 \Rightarrow \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-a_2 + b_2 = 0 \Rightarrow \vec{b}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$a_2 = b_2 = 4$$

5) $T_2 : K' = Cx'y' \xrightarrow{T_2} K'' = Cx''y''$

$$Cx'' \uparrow\uparrow \vec{a}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad (\lambda = -2)$$
$$Cy'' \uparrow\uparrow \vec{b}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad (\lambda = 4)$$

$$T_2 : \begin{cases} x' = \frac{1}{\sqrt{2}}x'' + \frac{1}{\sqrt{2}}y'' \\ y' = \frac{1}{\sqrt{2}}x'' + \frac{1}{\sqrt{2}}y'' \end{cases}$$

$$\left(\frac{1}{\sqrt{2}}x'' + \frac{1}{\sqrt{2}}y''\right)^2 + 6\left(\frac{1}{\sqrt{2}}x'' + \frac{1}{\sqrt{2}}y''\right)\left(-\frac{1}{\sqrt{2}}x'' + \frac{1}{2}y''\right) +$$

$$+\left(-\frac{1}{\sqrt{2}}x'' + \frac{1}{\sqrt{2}}y''\right)^2 - 4 = 0$$

$$\frac{1}{2}\left(x''^2 + y''^2 + \cancel{2x''y''}\right) + \frac{6}{2}(x''+y'')(y''-x'') + \frac{1}{2}\left(x''^2 + y''^2 \cancel{- 2x''y''}\right) - 4 = 0$$

$$\frac{1}{2}(2x''^2 + 2y''^2) + 3(y''^2 - x''^2) - 4 = 0$$

$$x''^2 + y''^2 + 3y''^2 - 3x''^2 - 4 = 0$$

още в началото се махат първите степени.

$$-2x''^2 + 4y''^2 - 4 = 0$$

$$-2x''^2 + 4y''^2 = 4 \quad | : 4$$

$$-\frac{1}{2}x''^2 + y''^2 = 1$$

$$-\frac{x''^2}{2} + \frac{y''^2}{1} = 1 \qquad -\frac{x''^2}{(\sqrt{2})^2} + \frac{y''^2}{1^2} = 1$$

4

①

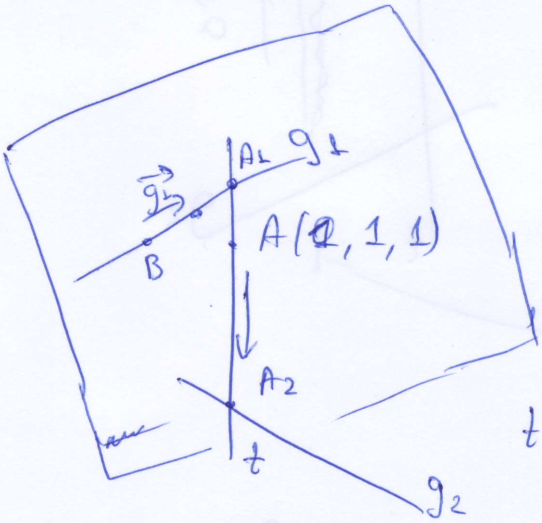
$$g_1 : \begin{cases} x = 3+\lambda \\ y = -1+2\lambda \\ z = 4\lambda \end{cases}$$

Не търсете общи точки и не ги намирайте

$$g_2 : \begin{cases} x = -2+3\mu \\ y = -1 \\ z = 4-5\mu \end{cases}$$

$A = (1, 1, 1)$

1) $\alpha : \begin{cases} \supset g_1 \\ \ni A(1,1,1) \end{cases}$



$$t : \begin{cases} \ni A_1(1,1,1) \\ \ni A_2(\end{cases}$$

$B(3,-1,0)$ > точка ~~вектор~~
$\vec{g_1}(1,2,4)$ и колинеарен вектор или 2 точки

$$\alpha : \begin{cases} \ni A(1,1,1) \\ \parallel \vec{g_1}(1,2,4) \\ \parallel \vec{AB}(2,-2,-1) \end{cases}$$

общо у-ние или ~~координатно~~ параметрично у-ние (едно от 2-те по желание)

$$(\vec{g_1} \times \vec{AB}) \parallel \vec{n_\alpha}$$

$$(1,2,4) \times (2,-2,-1) = \left(\begin{vmatrix} 2 & 4 \\ -2 & -1 \end{vmatrix}, \begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \right) = (6, 9, -6)$$

$\boxed{\vec{n_\alpha}(2,3,-2)}$

$\Rightarrow \alpha : 2x+3y-2z+D=0$

$A(1,1,1) \in \alpha \Rightarrow D=-3$

$\alpha : 2x+3y-2z-3=0$

2) $g_2 \cap \alpha = A_2$

$$\begin{cases} 2x+3y-2z-3=0 \\ \cancel{-2+3\mu=0} \quad x=-2+3\mu \\ y=-1 \\ z=4-5\mu \end{cases}$$

$$2(-2+3\mu)-3-2(4-5\mu)=0$$

$$-4-3-8+\mu(6+10)-3=0$$

$$-18+16\mu=0$$

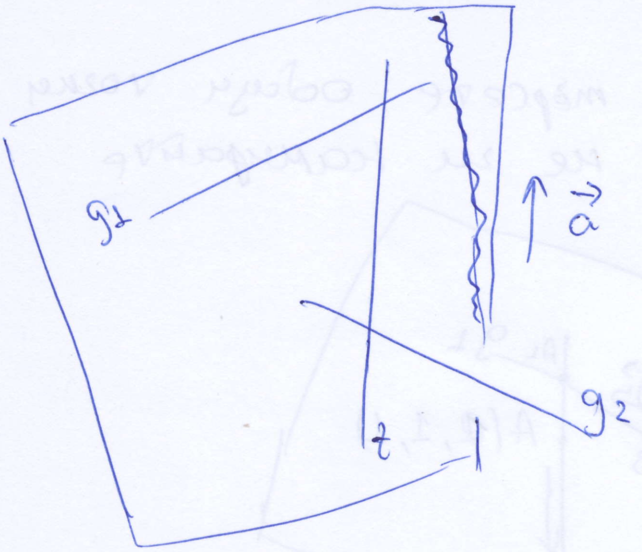
$$16\mu=18$$

не съкращаваме (променят се коорд. на точката)

$$\mu=\frac{18}{16}$$

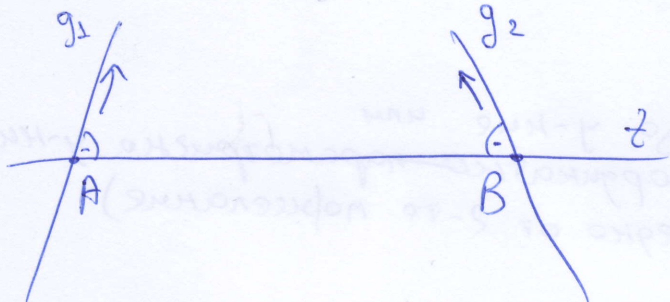
$$\Rightarrow A_2 : \begin{cases} x=-2+\frac{3.18}{16} \\ y=-1 \\ z=4-\frac{5.18}{16} \end{cases}$$

$$\Rightarrow A_2 : \begin{cases} x=\frac{3.18}{16}-2 \\ y=-1 \\ z=4- \end{cases}$$



$$\mathcal{L}: \begin{cases} \supset g_1 \\ \parallel \vec{a} \end{cases}$$

11 Oc



$$g_1: \begin{cases} x = 3+\lambda \\ y = -1+2\lambda \\ z = 4\lambda \end{cases}$$

$$g_2: \begin{cases} x = -2+3\mu \\ y = -1 \\ z = 4-5\mu \end{cases}$$

$$\left| \begin{array}{l} A = t \cap g_1 \Rightarrow A(3+\lambda, -1+2\lambda, 4\lambda) \\ B = t \cap g_2 \Rightarrow B(-2+3\mu, -1, 4-5\mu) \end{array} \right.$$

$\vec{AB} \parallel t \Rightarrow \vec{AB}(-5+3\mu-\lambda, -2\lambda, 4-5\mu-4\lambda)$

$$\left| \begin{array}{l} \vec{AB} \cdot \vec{g_1} = 0 \\ \vec{AB} \cdot \vec{g_2} = 0 \end{array} \right.$$

$\vec{g_1}(1, 2, 4)$
$g_2(3, 0, -5)$

$$\left| \begin{array}{l} (-5+3\mu-\lambda) \cdot 1 - 2\lambda \cdot 2 + (4-5\mu-4\lambda) \cdot 4 = 0 \\ (-5+3\mu-\lambda) \cdot 3 - (2\lambda \cdot 2) \cdot 0 + (-5)(4-5\mu-4\lambda) = 0 \end{array} \right.$$

$$\left| \begin{array}{l} \lambda(-1-4-16) + \mu(3-20) - 5 + 16 = 0 \\ \lambda(-3+20) + \mu(9+25) - 15 - 20 = 0 \end{array} \right.$$

$$\left| \begin{array}{l} -21\lambda - 17\mu + 11 = 0 \\ 17\lambda + 34\mu - 35 = 0 \end{array} \right.$$

$$\underbrace{(-42+17)}_{-25}\lambda \underbrace{-35+22}_{-13} = 0$$

$$\lambda = -\frac{13}{25}$$

$S_{усп.} = |\vec{a} \times \vec{b}|$

$S_{\Delta} = \frac{1}{2} |\vec{a} \times \vec{b}|$

$\vec{a} \times \vec{b} = \vec{c}$

$\vec{c}^{\,2} = |\vec{c}|^2$

~~$|\vec{a}| = |\vec{b}| = 1$~~

$a \times b = 5$

$\lambda \vec{a} + \mu \vec{b} + \nu \vec{c} = \vec{0}$

$(a \times b) \times c = (ac)\vec{b} - \vec{a}(bc)$

$a \times (b \times c) = (ac)\vec{b} - \vec{c}(ab)$

III $|\vec{a}| = |\vec{b}| = 1$

$\angle(\vec{a}, \vec{b}) = \frac{\pi}{3}$

$\overrightarrow{OA} = \vec{a} \times \vec{b}$

$\overrightarrow{OB} = \vec{b} \times (\vec{a} \times \vec{b}) = (\vec{b})^2 \vec{a} - (\vec{a}\vec{b})\vec{b} = \vec{a} - \frac{1}{2}\vec{b}$

$\overrightarrow{OC} = 2\vec{a}$

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a) $(\overrightarrow{OA} \times \overrightarrow{OB}) \cdot \overrightarrow{OC} = \left[(\vec{a} \times \vec{b}) \times (\vec{a} - \frac{1}{2}\vec{b})\right] \cdot 2\vec{a} =$

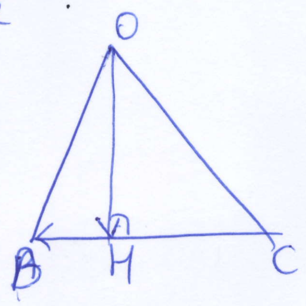
$= \left[(\vec{a} \times \vec{b}) \times \vec{a} - \frac{1}{2}(\vec{a} \times \vec{b}) \times \vec{b}\right] \cdot 2\vec{a} =$

$= 2\left((\vec{a})^2 b - (\vec{a}\vec{b})\vec{a} - \frac{1}{2}\left((\vec{a}\vec{b})\vec{b} - (\vec{b})^2 \vec{a}\right)\right) \cdot \vec{a} =$

$= 2\left[\vec{b} - \frac{1}{2}\vec{a} - \frac{1}{2}\left(\frac{1}{2}\vec{b} - \vec{a}\right)\right]\vec{a} = 2\left[\vec{b} - \cancel{\frac{1}{2}\vec{a}} - \frac{1}{4}\vec{b} + \cancel{\frac{1}{2}\vec{a}}\right]\vec{a} =$

$= 2\left[\frac{3}{4}\underbrace{\vec{b} \cdot \vec{a}}_{\frac{1}{2}}\right] = \frac{\cancel{2} \cdot 3}{4 \cdot \cancel{2}} = \frac{3}{4} \neq 0$

б) ? H →



$\overrightarrow{OH} = ?$

$\overrightarrow{OH} = \overrightarrow{OC} + \overrightarrow{CH} = \overrightarrow{OC} + \lambda \cdot \overrightarrow{CB}$

$\overrightarrow{CB} \perp \overrightarrow{OH}$

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \vec{a} - \frac{1}{2}\vec{b} - 2\vec{a} =$

$= -\vec{a} - \frac{1}{2}\vec{b}$

$\overrightarrow{OH} = \overrightarrow{OC} + \lambda \cdot \overrightarrow{CB} = 2\vec{a} - \lambda(\vec{a} + \frac{1}{2}\vec{b})$

$\overrightarrow{CB} \cdot \overrightarrow{OH} = (-\vec{a} - \frac{1}{2}\vec{b}) \cdot (2\vec{a} - \lambda(\vec{a} + \frac{1}{2}\vec{b})) = 0$

$= 2\vec{a} + \lambda(\vec{a}^{\,2} + \frac{1}{2}\vec{a}\vec{b}) - \vec{a}\vec{b} +$

$+ \lambda \frac{1}{2}(\vec{a}\vec{b} + \frac{1}{2}\vec{b}^{\,2}) = 0$

$-2 + \lambda\left(1 + \frac{1}{4}\right) - \frac{1}{2} + \lambda \cdot \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = 0$

$\frac{5}{4}\lambda + \frac{1}{2}\lambda = \frac{5}{2}$ $\frac{7}{4}\lambda = \frac{5}{2}$ $\lambda = \frac{5}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{7} = \frac{10}{7}$

$$\vec{OH} = 2\vec{a} - \frac{10}{7}\left(\vec{a} + \frac{1}{2}\vec{b}\right)$$

в) т. М - медицентър на $\triangle ABC$

$|\vec{OM}| = ?$ $\quad \vec{OM} = \frac{1}{3}\left(\vec{OA} + \vec{OB} + \vec{OC}\right)$

$$\vec{OM} = \frac{1}{3}\left(\vec{a}\times\vec{b} + \vec{a} - \frac{1}{2}\vec{b} + 2\vec{a}\right)$$

$$\vec{OM} = \frac{1}{3}\left(3\vec{a} - \frac{1}{2}\vec{b} + \vec{a}\times\vec{b}\right)$$

$$|\vec{OM}|^2 = (\vec{OM})^2 = \frac{1}{9}\Big[9\vec{a}^2 + \frac{1}{4}\vec{b}^2 + (\vec{a}\times\vec{b})^2 - 2\cdot 3\vec{a}\cdot\frac{1}{2}\vec{b} - 2\cdot\frac{1}{2}\underbrace{\vec{b}\cdot(\vec{a}\times\vec{b})}_{0} + 2\cdot\underbrace{3\vec{a}(\vec{a}\times\vec{b})}_{0}\Big] =$$

$$= \frac{1}{9}\Big[9 + \frac{1}{4} + \vec{a}^2\vec{b}^2 - (\vec{a}\vec{b})^2 - 3\vec{a}\vec{b}\Big] = \frac{1}{9}\Big[9 + \frac{1}{4} + 1 - \frac{1}{4} - \frac{3}{2}\Big] =$$

$$= \frac{1}{9}\cdot\frac{17}{2} = \frac{17}{18}$$

$$|\vec{OM}| = \sqrt{\frac{17}{18}} = \frac{\sqrt{17}}{3\sqrt{2}} = \frac{\sqrt{34}}{6}$$