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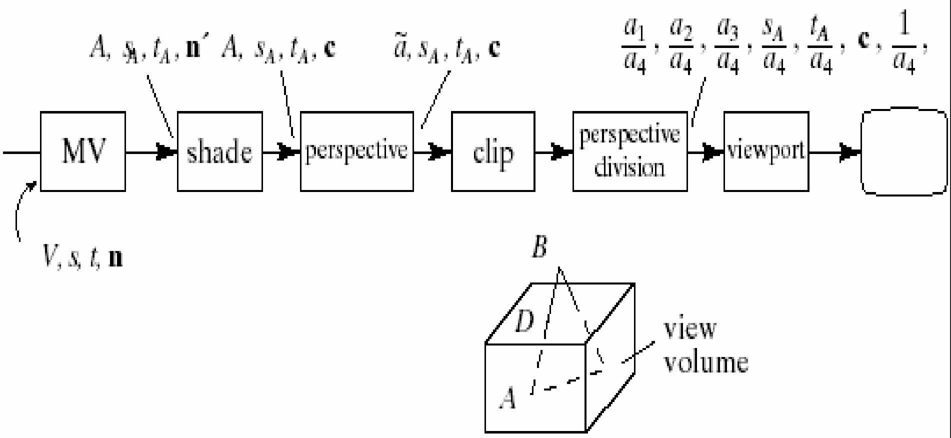
Computer Graphics

Modeling Shapes with Meshes

Slides adapted from F. Hill, S. Kelley Computer Graphics

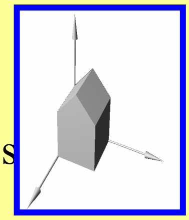
Polygonal Meshes

Recall the Graphics Pipeline

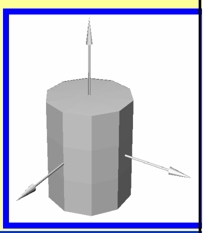


Polygonal Meshes

- *What is a a polygonal mesh* ?
- collection of polygons (faces)
- approximate description of the 3D surfaces

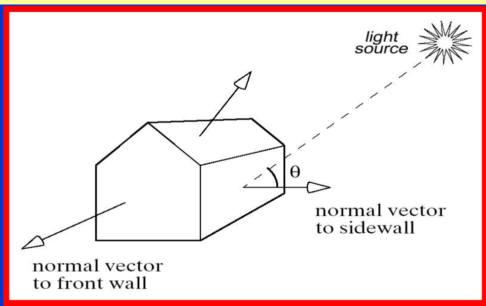


- *Why use meshes?*
- easy to represent (set of vertices) and transform.
- simple properties (a single normal vector,
- a well-defined inside / outside, etc.
- easy to draw
  - using a polygon-fill (scanline) routine
  - mapping textures



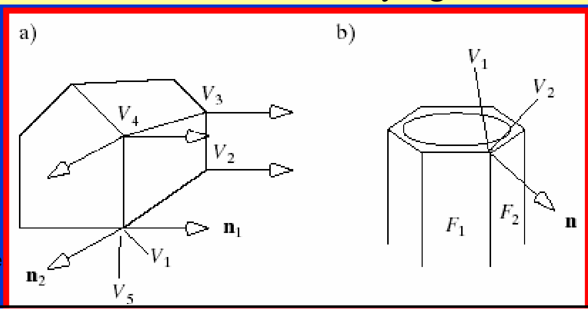
What's included in the description of mesh?

- First thinking: a list of polygons, & directions
- If it represents a solid, faces have inside & outside
- Direction: outward pointing **normal vector**
  - determines its brightness
  - *How do we find normals?*



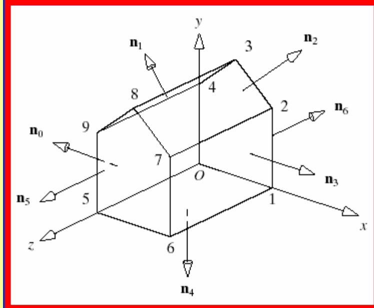
Vertex Normals: Flat and curved faces

- Flat faces: Vertex has several normals *why?*
- Vertices V_1, V_2, V_3, V_4 of side wall use same normal $\mathbf{n}_1$
- Vertex V_5, part of modeling front wall, use normal $\mathbf{n}_2$.
- V_1 & V_5 co-located, but use different normals
- Mesh of curved face – *how many normals for a vertex*
- V_1 of face F_1 & V_2 on face F_2 use the same normal $\boldsymbol{n}$
  - the true normal of the underlying smooth surface.



Defining a Polygonal Mesh

- 3 lists
- Vertices, Outside Normals, Faces
- Example basic barn:
  - 10 vertices
  - 7 Normals
  - 7 polygonal faces
- *How we connect these lists?*
- Each Face indexes into the vertex and normal lists.



Vertex and Normal Lists for the Barn

Normals List			
normal	n_x	n_y	n_z
0	-1	0	0
1	-0.707	0.707	0
2	0.707	0.707	0
3	1	0	0
4	0	-1	0
5	0	0	1
6	0	0	-1

Vertex List			
vertex	x	y	z
0	0	0	0
1	1	0	0
2	1	1	0
3	0.5	1.5	0
4	0	1	0
5	0	0	1
6	1	0	1
7	1	1	1
8	0.5	1.5	10
9	0	1	1

Face List for the Barn

Face	Vertices	Normal
0 (left)	0, 5, 9, 4	0,0,0,0
1 (roof left)	3, 4, 9, 8	1,1,1,1
2 (roof right)	2, 3, 8, 7	2, 2, 2,2
3 (right)	1, 2, 7, 6	3, 3, 3, 3
4 (bottom)	0, 1, 6, 5	4, 4, 4, 4
5 (front)	5, 6, 7, 8, 9	5, 5, 5, 5, 5
6 (back)	0, 4, 3, 2, 1	6, 6, 6, 6, 6

Vertex list of a face: begins with any vertex
Traverse the polygon counterclockwise as seen from outside
How these normals are calculated?

Calculating Normals: Newell Algorithm

- *Cross product method:* $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ vertices of the face
  - $\underline{\mathbf{m}} = (\mathbf{V}_1 - \mathbf{V}_2) \times (\mathbf{V}_3 - \mathbf{V}_2)$; then normalize $\underline{\mathbf{m}}$
- But a given a list of vertices might not be on a plane!
- *Newell:* Traverse the face vertices in CC order
  - Given vertex n, define next vertex: ni = (i+1) mod N.
  - Calculate normal by the Newell Algorithm

$$n_x = \sum_{i=0}^{N-1} (y_i - y_{ni})(z_i + z_{ni})$$

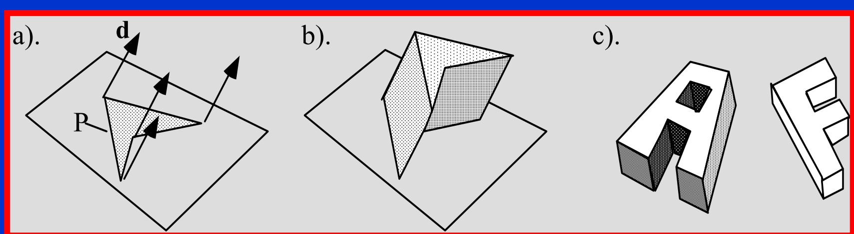
$$n_y = \sum_{i=0}^{N-1} (z_i - z_{ni})(x_i + x_{ni})$$

$$n_z = \sum_{i=0}^{N-1} (x_i - x_{ni})(y_i + y_{ni})$$

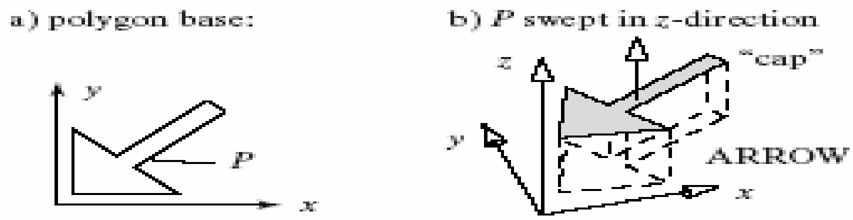
Swept (Extruded) Objects

Prisms

- *How do we create a prism?*
- by extruding a regular polygon along a straight line.
- *What is a "right prism?*
- translation line is perpendicular to the polygon



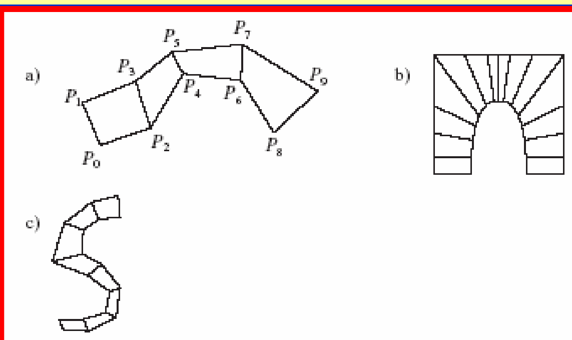
Swept (extruded) Shapes

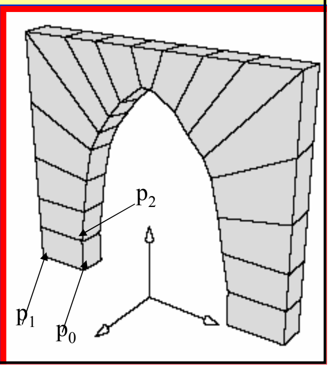


- Base vertices (x_i, y_i, 0); top vertices (x_i, y_i, H).
- (x_i, y_i, H) connected to vertex (x_i, y_i, 0)
- polygon has n sides → n vertical faces
- *How many vertices per vertical face?*
- normals calculated by Newell method

Swept (extruded) Quadstrips

- quad-strip = {p_0, p_1, p_2, …., p_{M-1}}
  - vertices taken in pairs: *odd* ones forming one edge *even* ones forming the other edge.
- *How do we model the gate?*
- front face: quadstrip; extrude the front face along **z**

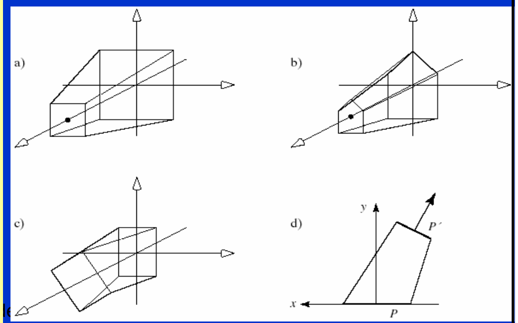




Twisted Extrusion

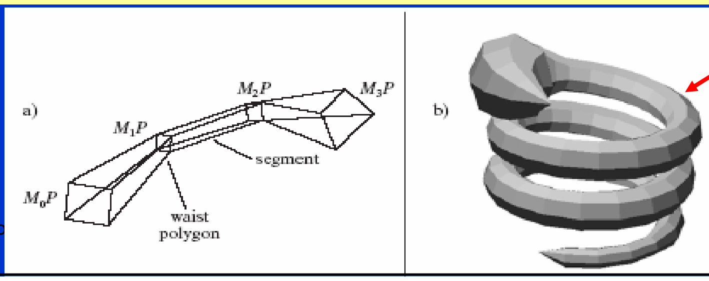
- *How do we get the cut pyramid?*
- *top* scaled (and translated) version of *base* polygon
- *base* $\{p_0, p_1, \ldots, p_{N-1}\}$ → *cap* $\{ Mp_0, Mp_1, \ldots, Mp_{N-1}\}$
- M is some 4 by 4 affine transformation matrix.

- a), b): top is scaled base
- c): top rotated about *z*-axis before extrusion
- d): top *P'* is base P rotated and translated.



Tube or Snake: Sequenced Extrusions

- Snake on the left: sequence of extruded shapes
- Base polygon **P**: $(\boldsymbol{p_0}, \boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3})$
- Initial matrix M_0 positions the start end of the tube
- segments have end polygons $M_i\boldsymbol{P}$ and $M_{i+1}\boldsymbol{P}$,
- *Vertices:* $(M_0\boldsymbol{p_0}, M_0\boldsymbol{p_1}, M_0\boldsymbol{p_2}, M_0\boldsymbol{p_3})$, $(M_1\boldsymbol{p_0}, M_1\boldsymbol{p_1}, M_1\boldsymbol{p_2}, M_1\boldsymbol{p_3})$, $(M_2\boldsymbol{p_0}, M_2\boldsymbol{p_1}, M_2\boldsymbol{p_2}, M_2\boldsymbol{p_3})$, $(M_3\boldsymbol{p_0}, M_3\boldsymbol{p_1}, M_3\boldsymbol{p_2}, M_3\boldsymbol{p_3})$

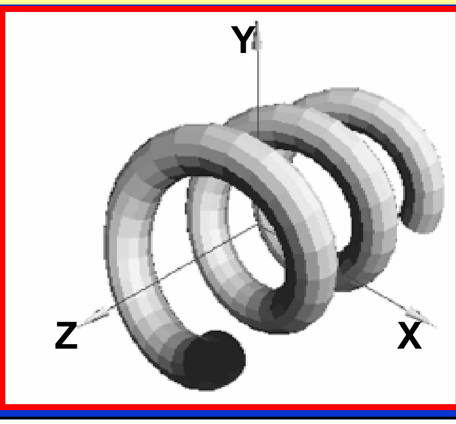


How do we get this?

Wrapping the snake around a space curve

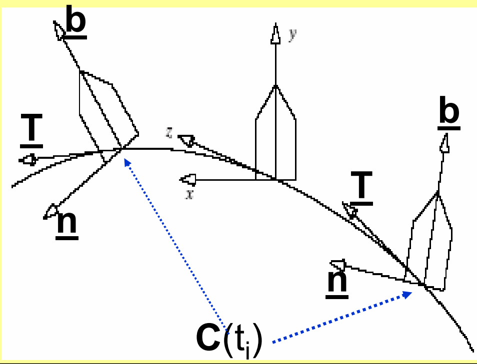
- Space curve **C**(t); e.g. helix
- *what is the shape in the X, Y plane?*
- **C**(t) = (cos(t), sin(t), bt)
- Example Snake: A decagon wrapped around the helix

- Mesh: Get **C**(t_i) for a series of values of t_i
- Rotate & move copies of base polygon to **C**(t_i)
- Using *Frenet Frame*



What is the Frenet Frame

- A local coord frame on a curve at **C**(t_i)
  - ((**n**(t_i), **b**(t_i), **T**(t_i))
  - **T**(t_i) tangent to the curve
  - **n**(t_i), **b**(t_i), **T**(t_i) orthonormal
- **T**(t_i) ~ d**C**(t_i)/dt
  - tangent to curve *why?*
- Normalize **T**(t_i)
- Construct **n**(t_i), **b**(t_i)
  - (later)

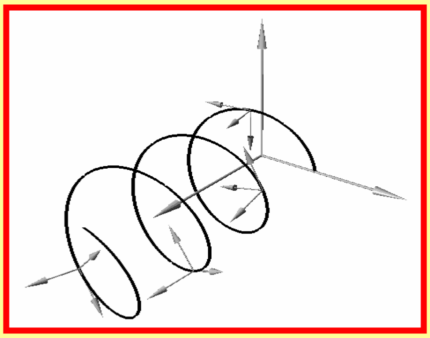


Wrapping the snake around a space curve (2)

- *How do we rotating and Move the polygon to $C(t_i)$*
- Base polygon at X, Y plane, around **O**
- Construct $M(t_i)$ that:
- 1. transforms the vectors **i**, **j**, **k**, to **n**(t_i), **b**(t_i), **T**(t_i)
  - this also rotates the base polygon to its new orientation
- 2. Translates a point at origin, **O,** to **C**(t_i)
  - this also moves the base polygon to its new position
- $M(t_i)$: 4 column matrix: (**n**(t_i)| **b**(t_i)| **T**(t_i)| $C(t_i)$) exercise
- Apply $M(t_i)$ for each i on base polygon

Example Calculation the Frenet Frame vectors

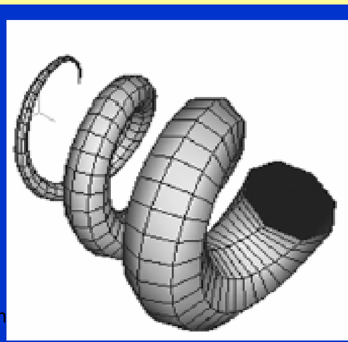
- **C**(t) = (cos (t), sin(t), bt) (a helix) *how much is T(t)?*
- d**C**/dt = (-sin(t), cos(t), b)
  - **T**(t) = $(1+b^2)^{-1}$(-sin(t), cos(t), b) the tangent vector
- We need another vector **a**, then calculate **b** = **T**x**a**
  - Take **a** = d^2**C**(t)/dt^2
    - **a** = (-cos(t), -sin(t), 0).
  - Then: **b** = **T**x**a**/|**T**x**a**|
- *How to find **n**?*
- **n** = **b**x**T**



Example: A Sea Shell

- Polygon around helix with t-dependent scaling
- First *scale* the base polygon (matrix below)
- Then transform by M(t) = ($\mathbf{\underline{n}}(t)$| $\mathbf{\underline{b}}(t)$| $\mathbf{\underline{T}}(t)$| $\boldsymbol{C}(t)$)
  - For each value of $t = t_i$

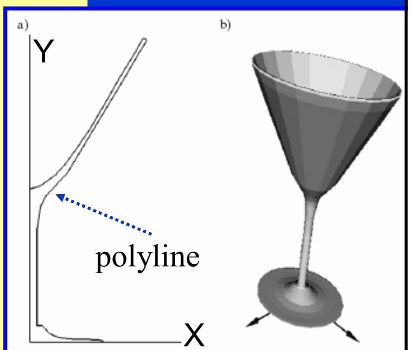
$$M' = M\begin{pmatrix} g(t) & 0 & 0 & 0 \\ 0 & g(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Discrete Surfaces of Revolution

- Polyline: $\mathbf{P}_j = (x_j, y_j, 0, 1)^T$ j=1..n
- Rotate all **Pj** around Y axis by θ_i
- $\theta_i = 2\pi i/K$, i = 0,1,.. K-1
- $\mathbf{Q}_{j,i} = M(\theta_i)P_j$

$$\tilde{M} = \begin{pmatrix} \cos(\vartheta_i) & 0 & \sin(\vartheta_i) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\vartheta_i) & 0 & \cos(\vartheta_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



- $\mathbf{Q}_{j,i} = (x_j\cos(\theta_i), y_j, -x_j\sin(\theta_i), 1)^T$

Mesh Approximation to Smooth Surfaces

Review: Forms of Plane equation

- **B** a point in the plane; **n** vector normal to the plane
- **P** = (x, y, z, 1)^T general point in the plane
- What's *Point Normal Form* of plane equation:
- **n**•(**P** – **B**) = 0

- $n_x x + n_y y + n_z z - D = 0$ (where D = **n**•B)
- Define: $F(x, y, z) \equiv n_x x + n_y y + n_z z - D$
- *Implicit Form of plane equation* : F(x, y, z) = 0
- Alternatively: define F(**P**) ≡ **n**•(**P** – **B**)
  - *Implicit Form of plane equation*: F(**P**) = 0
- F(x,y,z) and F(**P**) Linear functions (for a plane)

Review: Parametric Formula of planar patch

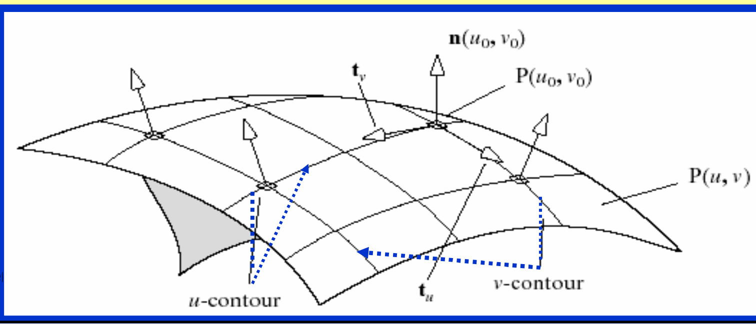
- *What is the Parametric Formula of a planar patch?*
- $\boldsymbol{P}(u, v) = \boldsymbol{C} + \underline{\mathbf{a}}u + \underline{\mathbf{b}}v$
  - $\boldsymbol{C}$ point, $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ vectors, u and v are in [0, 1].
- $x(u, v) = C_x + \mathbf{a}_x u + \mathbf{b}_x v$
- $y(u, v) = C_y + \mathbf{a}_y u + \mathbf{b}_y v$
- $z(u, v) = C_z + \mathbf{a}_z u + \mathbf{b}_z v$
- *Fixed* $u = u_0$ $\boldsymbol{P}(u_0, v) = \boldsymbol{C} + \underline{\mathbf{a}}u_0 + \underline{\mathbf{b}}v$ line along $\underline{\boldsymbol{b}}$

General Smooth Surfaces

- *What is the Implicit Form of the equation of a surface ?*
  - For some $F()$: $F(x, y, z) = 0$ or $F(\mathbf{P}) = 0$
- If inside/outside to surface is meaningful, then for point **Q**:
- $F(\mathbf{Q}) < 0$ Q is inside the object
- $F(\mathbf{Q}) = 0$ Q is in (or on) the surface
- $F(\mathbf{Q}) > 0$ Q is outside
- *Parametric Formula* $\boldsymbol{P}(u, v) = (X(u, v), Y(u, v), Z(u, v))$
  - with u and v restricted to suitable intervals.
- u-contour lines: constant $u = u_0$ varying v
- v-contour lines: constant $v = v_0$, varying u

The Normal vectors of a surface

- $\underline{\mathbf{p}}(u, v) \equiv \mathbf{P}(u, v)–(0, 0, 0, 1)^T = (x(u,v),y(u,v),z(u,v))$
- $\underline{\mathbf{t}}_{\mathbf{u}} = \lim[\underline{\mathbf{p}}(u+du, v) – \underline{\mathbf{p}}(u, v)]/du$ → ??
- $\underline{\mathbf{t}}_{\mathbf{v}} = \lim[\underline{\mathbf{p}}(u, v+dv) – \underline{\mathbf{p}}(u, v)]/dv$ → ??
- $\underline{\mathbf{n}}(u_0, v_0) \equiv (\partial\underline{\mathbf{p}}/\partial u)\ \mathbf{X}\ (\partial\underline{\mathbf{p}}/\partial v)\ |_{u_0, v_0}$
- also $\underline{\mathbf{n}}'(x,y,z) \equiv ((\partial F/\partial x, (\partial F/\partial y), (\partial F/\partial z))|x_0,y_0,z_0$

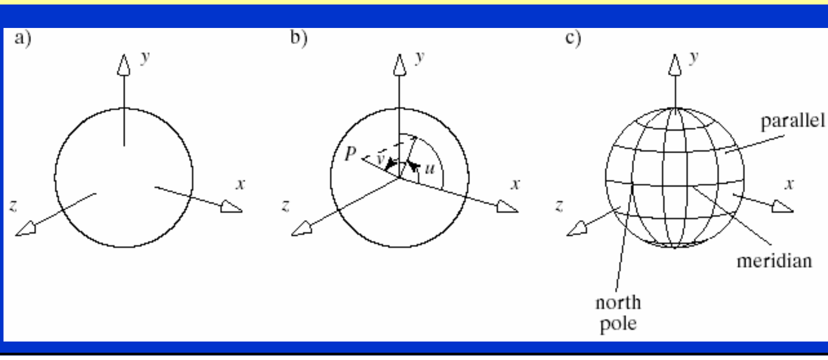


Applying affine transformation to a surface

- Surface points are given by the parametric formulae
- $\mathbf{P}(u,v) = (x(u,v), y(u,v), z(u,v))$
- or by the implicit form equation $F(\mathbf{P}) = 0$
- Under a transformation M
  - $\mathbf{P}(u, v) \rightarrow M\mathbf{P}(u, v)$
  - $F(\mathbf{P}) \rightarrow F(M^{-1}\mathbf{P})$ exercise
  - $\underline{\mathbf{n}}(u,v) \rightarrow M^{-1}(\underline{\mathbf{n}}(u,v))$. exercise

Normal Vector Example: Generic Sphere (1)

- Center (0, 0, 0), radius 1; Poles at z = -1, +1
- $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$
- or: $F(\mathbf{P}) \equiv |\mathbf{P} - \mathbf{O}|^2 - 1 = 0$
- $\mathbf{P}(u, v) = (\cos(v)*\cos(u), \cos(v)*\sin(u), \sin(v))$
- $0 \le u \le 2\pi$, $-\pi/2 \le v \le \pi/2$; *what are u and v?*



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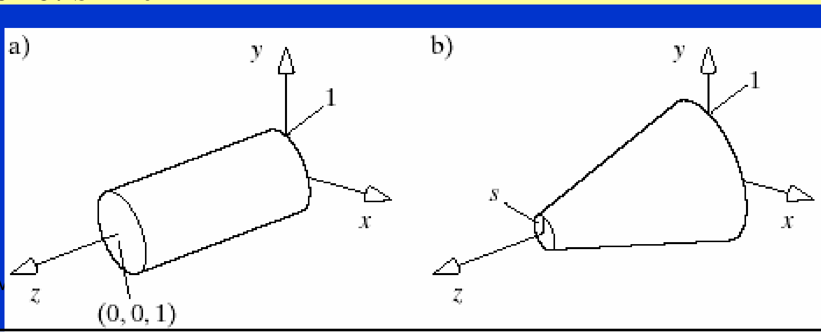
What is the normal to a centric sphere?

- explicit form
- $\underline{\mathbf{n}}'(x,y,z) \equiv ((\partial F/\partial x), (\partial F/\partial y), (\partial F/\partial z))|x_0,y_0,z_0$
  - $\underline{\mathbf{n}}' = 2(x, y, z) = 2\underline{\mathbf{r}}$
  - radially outward. Not normalized
- *parametric form:*
- $\underline{\mathbf{n}}(u_0, v_0) \equiv (\partial \underline{\mathbf{p}}/\partial u) \mathbf{X} (\partial \underline{\mathbf{p}}/\partial v) |_{u_0, v_0}$
  - $\underline{\mathbf{n}}(u_0, v_0) = -\cos(v)\underline{\mathbf{p}}(u, v)$,
  - $-\cos(v)$ will disappear when we normalize **n**
  - $\mathbf{p}(u, v)$ also radially outward.

Generic Tapered Cylinder

- Axis: *z*-axis; z from 0 to 1
  - Circular base & top: radiuses 1 & s
  - $x^2 + y^2 = r(x, y) = (1 + (s-1)z)^2$ Exercise
- generic cylinder: s =1
- *How do we get a cone?*
- Cone: $s = 0$



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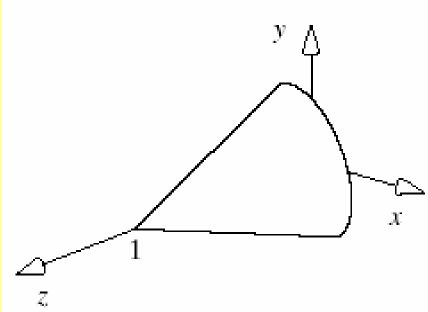
Generic Tapered Cylinder (2)

- Implicit form of the equation of wall
  - $F(x, y, z) \equiv x^2 + y^2 - (1 + (s-1)z)^2 = 0 \quad 0 < z < 1$
- Parametric form:
- $\mathbf{P}(u, v) = ((1 + (s-1)v)\cos(u), (1+(s-1)v)\sin(u), v)$ Exercise
- normal vector to wall $\mathbf{n}(x, y, z) = (x, y, -(s - 1)(1+ (s - 1)z))$
  - parametric form $\mathbf{n}(u, v) = (cos(u), sin(v), 1 - s)$ Exercise
  - generic cylinder normal: $(cos(u), sin(u), 0)$
- Cap: $z = 1, \ x^2 + y^2 < s^2$
  - $\boldsymbol{P}(u, v) = (\rho\, cos(\alpha), \beta\, sin(\alpha), 1)$ for ρ in $[0, s]$

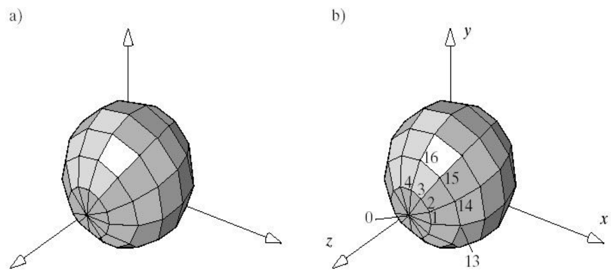
Generic Cone: Tapered Cylinder with s=0

- Wall: $F(x, y, z) = x^2 + y^2 - (1 - z)^2 = 0$
- for $0 < z < 1$;
- parametric form:
  - $\boldsymbol{P}(u, v) = ((1-v)\, cos(u), (1- v)\, sin(u), v)$
- u in $[0, 2\pi]$ and v in $[0, 1]$.
- normal vector to the wall
  - $(x, y, 1-z)$ (s = 0)



Mesh for the Generic Sphere

- imagine longitude and latitude lines:
- Longitude Lines: fixed $u_i = 2\pi i/N_1$, $i = 0, 1, ..., N_1 -1$
- Latitude Lines: fixed $v_j = -\pi/2 + \pi\, j/N_2$, $j = 0, 1, ...,N_2$
  - $j = 0$ south pole; $j = N_2$ north pole
  - $(N_2-1)/2$ below the equator, $(N_2-1)/2$ above the equator
- Example:
- $N_1 = 12$ longitudes
- $N_2 = 9$, 8 latitudes
- 96 + 2 points



Mesh for the Generic Sphere (2)

- vertex list - enumeration of all $P(u_i, v_j)$ from south pole up
- $\mathbf{P}_0$ south pole $\mathbf{P}(u_0, v_0 = -\pi/2)$
- Then $\{\mathbf{P}(u, v) \mid u_i\ i = 1, 12\ ;\ v_1 = -\pi/2 + \pi/9\}$ latitude 1
- Then $\{\mathbf{P}(u, v) \mid u_i\ i = 1, 12\ ;\ v_2 = -\pi/2 + 2\pi/9\}$ latitude 2
- … Then $\{P(u, v) \mid u_i\ i = 1, 12\ ;\ v_8 = -\pi/2 + 8\pi/9\}$...
- Then P_{97} north pole: $P(u_0, v_9 = +\pi/2)$
- Normal list:
  - $\underline{\mathbf{n}}_k$, the normal for the sphere at P_k
  - For a sphere $\underline{\mathbf{n}}_k = \mathbf{P}_k - \mathbf{O}$

Mesh for the Generic Sphere (3)

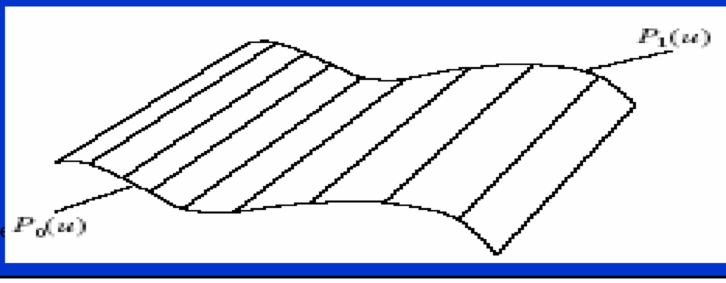
- Face list:
- first 12 faces: bottom triangles (near south pole)
- Next 12 quadrilaterals above the triangles
- Next 12 quadrilaterls, etc.
- The first 3 entries (triangles) in the face list:

number of vertices:	3	3	3	...
vertex indices:	0 1 2	0 2 3	0 3 4	...
normal indices:	0 1 2	0 2 3	0 3 4	...

vertex 0 is the south pole

Ruled Surfaces

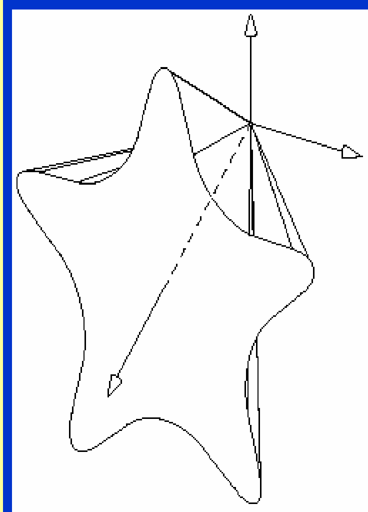
- Ruled Surface: through every point passes at least one straight line lying entirely on the surface.
  - Made by moving the ends of a straight line along curves, $\mathbf{P}_0(u)$, $\mathbf{P}_1(u)$
- Every point on the surface is an affine combination of 2 points, $\mathbf{P}_0(u)$, $\mathbf{P}_1(u)$ one on each of the curves



Ruled Surfaces: Cone

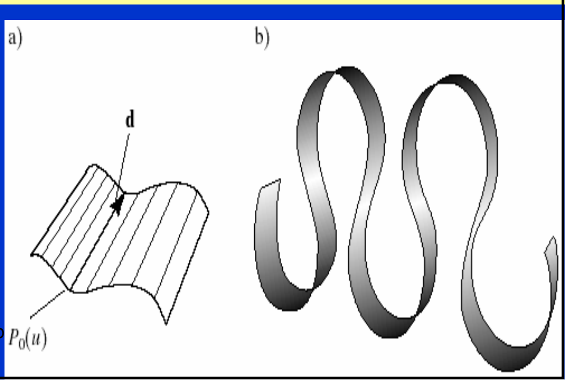
- A cone is a ruled surface
- one of the curves, $\boldsymbol{P}_0(u)$, is a *single* point
- $\boldsymbol{P}_0(u) = \boldsymbol{P}_0$, the apex of the cone
- second curve: $\boldsymbol{P}_1(\mathrm{u})$
- ***Surface:*** $\boldsymbol{P}(u, v)$

$$= v\boldsymbol{P}_1(\mathrm{u}) + (1-v)\boldsymbol{P}_0$$



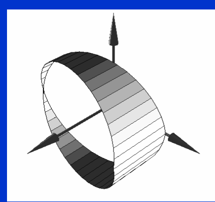
Ruled Surfaces: Cylinder

- $P_1(u)$ is a translated version of $P_0(u)$
  - $P_1(u) = P_0(u) + \mathbf{d}$, for some vector **d**.
- general point in the cylinder: $P(u, v) = P_0(u) + \underline{\mathbf{d}}v$
- $\mathbf{P}_0$(u) any curve, must be in a plane
- **d** need not be perpendicular to the plane

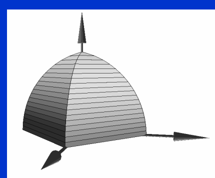


More Ruled Surfaces

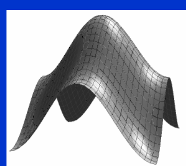
- Möbius strip (has only one edge).



- roof made up of four ruled surfaces.



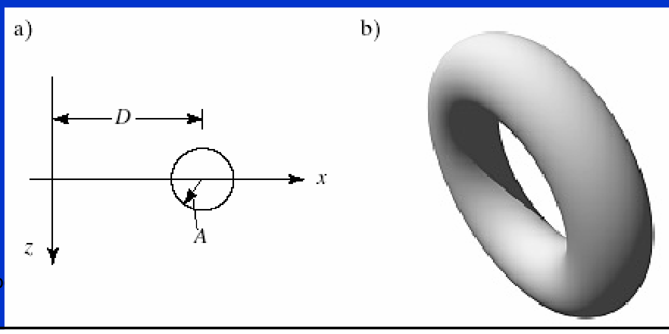
- Coons patch



Surface of revolution example: Torus

- *Profile*: a circle in x,z plane, center at (D, 0, 0) on the x axis
- **C**(v) = (D + A cos(v), 0, A sin(v)) A = radius of circle
- Rotate about the z-axis
- Any point on the torus: $P(u, v)$ =

  $((D + A\ cos(v))\ cos(u), (D + A\ cos(v))\ sin(u), A\ sin(v))$



Surfaces of Revolution

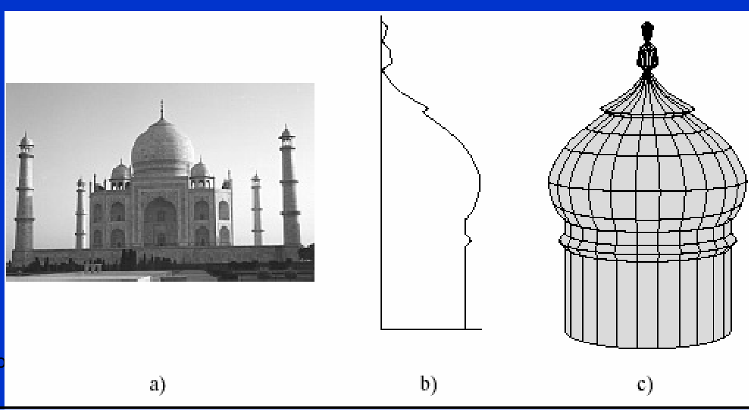
- *profile curve* **C**(v) - rotate around an axis.
- example **C**(v) in the x, z plane **C**(v)= (x(v), 0, z(v))
  - Rotate around the z axis.
- Rotate point (x(v), 0, z(v)) by angle u
  - → ((x(v)cos(u), x(v)sin(u), z(v)).
- **P**(u, v) = (x(v)cos(u), x(v)sin(u), z(v))
- <u>**n**</u>(u, v) ≡ (∂<u>**p**</u>/∂u) **x** (∂<u>**p**</u>/∂v) |
  - (∂<u>**p**</u>/∂u = (-x(v)sin(u), x(v)cos(u), 0)
  - (∂<u>**p**</u>/∂v) = (x'(v)cos(u), x'(v)sin(u), z'(v)
  - **n** (u, v) = x(v) [z'(v)cos(u), z'(v)sin(u), -x'(v)].

Constructing a mesh for surface of revolution

- choose a set of u and v values, $\{u_i\}$ and $\{v_j\}$,
- and compute a vertex at each pair $P(u_i, v_j)$
- compute a normal direction $\mathbf{n}(u_i, v_j)$.
- Build polygonal faces by joining four adjacent vertices with straight lines.

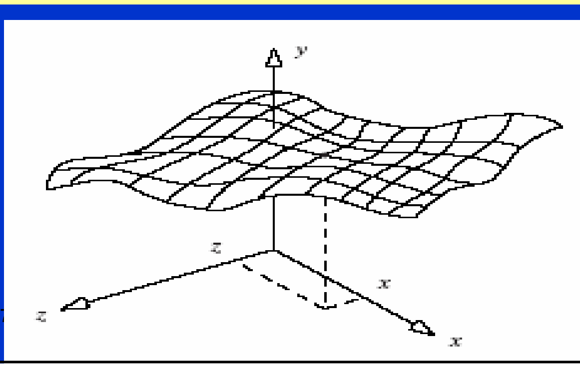
Example

- A model of the dome of the Taj Mahal in Agra, India
- Profile is given as a set of points
- We'll later replace the points by Bezier curve



Surfaces which are Functions of Two Variables

- Single valued height function y = f(x, z)
- Define parameters: $u = x$, $v = z$, $y(u,v) = f(u,v)$
- Parametric formula: $\mathbf{P}(u,v) = (u, f(u, v), v)$
- $\underline{\mathbf{n}}(u, v) = (1, \partial f/\partial u, 0)\ \mathbf{X}\ (0, \partial f/\partial v, 1) = (\partial f/\partial u, -1, \partial f/\partial v)$



Example

- $y = \text{sync}(x,z) \equiv \sin(\text{sqrt}(x^2 + z^2))/\text{sqrt}(x^2 + z^2)$
- $\mathbf{P}(u, v) = (u, \sin(\text{sqrt}(u^2 + v^2))/\text{sqrt}(u^2 + v^2), v)$