14 Частни производни от по-висок ред. Теорема за смесените производни. Диференциали от по-висок ред.

Нека $f(x,y)$ е дефинирана върху $U \subset \mathbb{R}^2$ - отворено множество,
$\exists\ \dfrac{\partial f}{\partial x}$ и $\dfrac{\partial f}{\partial y}$ за $\forall (x,y) \in U$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

D $\dfrac{\partial^2 f}{\partial x \partial y}$ и $\dfrac{\partial^2 f}{\partial y \partial x}$ се наричат **СМЕСЕНИ ЧАСТНИ ПРОИЗВОДНИ.**

T Нека $f(x,y)$ е дефинирана върху $U(x_0, y_0)$ и върху $U(x_0, y_0)$ $\exists\ f''_{xy}(x,y)$ и $f''_{yx}(x,y)$ и тези смесени производни са непрекъснати в точката (x_0, y_0). Тогава: $f''_{xy}(x_0,y_0) = f''_{yx}(x_0,y_0)$.

Доказателство.

$* \ \Delta_x f(x,y) = f(x+\Delta x, y) - f(x,y)$

$* \ \Delta_y f(x,y) = f(x, y+\Delta y) - f(x,y)$

$\Delta_{xy} f(x,y) = \Delta_y \Delta_x f(x,y) = [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)] - [f(x+\Delta x, y) - f(x,y)]$ *

$\Delta_{yx} f(x,y) = \Delta_x \Delta_y f(x,y) = [f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)] - [f(x, y+\Delta y) - f(x,y)]$

$\Delta_{xy} f(x,y) = \Delta_{yx} f(x,y)$

Нека $\varphi(x) = \Delta_y f(x, y_0) = f(x, y_0+\Delta y) - f(x, y_0)$ и разглеждаме $\varphi(x)$ върху интервала с краища x_0 и $x_0 + \Delta x$.

$\Delta\varphi(x) = \varphi(x_0+\Delta x) - \varphi(x_0) \Rightarrow$ По т-мата на Лагранж

$\exists\ 0 < \theta_1 < 1$:

$\Delta\varphi = \varphi'(x_0+\theta_1\Delta x)\Delta x = (f'_x(x_0+\theta_1\Delta x, y_0+\Delta y) - f'_x(x_0+\theta_1\Delta x, y_0))\Delta x =$

$\underset{(*)}{=} f''_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)\Delta x\Delta y.$

(*) от Лагранж $\exists\, 0<\theta_2<1$

$\Rightarrow \Delta_{xy} f(x_0,y_0) = f''_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)\Delta x\Delta y \quad (1)$

аналогично: $\Delta_{yx} f(x_0,y_0) = f''_{yx}(x_0+\theta_4\Delta x, y_0+\theta_3\Delta y)\Delta x\Delta y \quad (2)$

$\hookrightarrow \psi(y) = \Delta_x f(x_0,y) = f(x_0+\Delta x, y) - f(x_0,y)$

$\hookrightarrow$ в интервала с краища y_0 и $y_0+\Delta y$

$\exists\, \theta_3, \theta_4 \in (0,1): \Delta_{xy} f(x_0,y_0) = f''_{xy}(x_0+\theta_4\Delta x, y_0+\theta_3\Delta y)\Delta x\Delta y.$

$\Rightarrow f''_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)\Delta x\Delta y = f''_{yx}(x_0+\theta_4\Delta x, y_0+\theta_3\Delta y)\Delta x\Delta y$

$\Rightarrow f''_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y) = f''_{yx}(x_0+\theta_4\Delta x, y_0+\theta_3\Delta y)$

f''_{xy} и f''_{yx} са непрекъснати в $(x_0,y_0) \Rightarrow$

$\Rightarrow \lim\limits_{(\Delta x,\Delta y)\to(0,0)} f''_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y) = f''_{xy}(x_0,y_0)$

$\lim\limits_{(\Delta x,\Delta y)\to(0,0)} f''_{yx}(x_0+\theta_4\Delta x, y_0+\theta_3\Delta y) = f''_{yx}(x_0,y_0) \Rightarrow f''_{xy}(x_0,y_0) = f''_{yx}(x_0,y_0)$

Диференциал от по-висок ред

Нека $f(x,y)$ - диференцируема

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y \longrightarrow \begin{array}{l}\Delta x = dx\\ \Delta y = dy\end{array}$$

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

$$\delta(df(x,y)) = \left(df(x,y)\right)'_x \delta x + \left(df(x,y)\right)'_y \delta y =$$

$$= \left(\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy\right)'_x \delta x + \left(\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy\right)'_y \delta y =$$

$$= \frac{\partial^2 f}{\partial x}dx\,\delta x + \frac{\partial^2 f}{\partial y\partial x}dy\,\delta x + \frac{\partial^2 f}{\partial y\partial x}dx\,\delta y + \frac{\partial^2 f}{\partial y^2}dy\,\delta y$$

$\boxed{D}$ $x, y \in \mathbb{R}^n$, т.е. $x(x_1 \dots x_n)$, $y(y_1 \dots y_n)$

$$A(x,y) = A(x_1 \dots x_n, y_1 \dots y_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j$$ – билинейна форма

Ако $x_i = y_j \Rightarrow A(x,x) =$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$ – квадратична форма

$$d(df(x,y)) = \frac{\partial^2 f}{\partial x^2} dx^2 + \left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x} \right) dx dy + \frac{\partial^2 f}{\partial y^2} dy^2$$

$\boxed{D}$ Втори диференциал

при $f''_{xy} = f''_{yx} \Rightarrow d^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2 =$

$$= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^{\{2\}} f(x,y)$$

$\boxed{D}$ n-ти диференциал

$$d^n f = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^{\{n\}} f(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^n f}{\partial x^{n-k} \partial y^k} dx^{n-k} dy^k$$