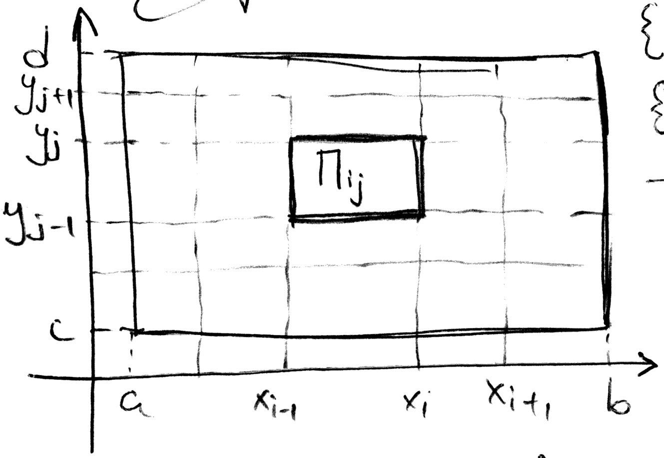
25 | Свеждане на двукратен интеграл към повторен. Смяна на променливите в двукратни интеграли

Т) Нека функция $f(x,y)$ е интегруема върху клетката $\Pi=[a,b]\times[c,d]$ и $\forall x\in[a,b]\ \exists \int_c^d f(x,y)dy$. Тогава функцията $F(x)=\int_c^d f(x,y)dy$ е интегруема върху $[a,b]$ и

$$\iint_\Pi f(x,y)dxdy=\underbrace{\int_a^b\left(\int_c^d f(x,y)dy\right)dx}_{\text{ПОВТОРЕН ИНТЕГРАЛ}}$$

Доказателство:



$\{x_i\}_{i=1}^n : a=x_0<x_1<\ldots<x_n=b$

$\{y_j\}_{j=1}^m : c=y_0<y_1<\ldots<y_m=d$

$T=\{\Pi_{ij}\} : \Pi_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j];$

$\Delta x_i=x_i-x_{i-1}$

$\Delta y_j=y_j-y_{j-1}$

$m_{ij}=\inf\limits_{\Pi_{ij}} f(x,y);\ M_{ij}=\sup\limits_{\Pi_{ij}} f(x,y)$

$x\in[x_{i-1},x_i];\ y\in[y_{j-1},y_j] \longrightarrow m_{ij}\le f(x,y)\le M_{ij}$

$$\Rightarrow \underbrace{\int_{y_{j-1}}^{y_j} m_{ij}dy}_{=\,m_{ij}\cdot\Delta y_j} \le \int_{y_{j-1}}^{y_j} f(x,y)dy \le \underbrace{\int_{y_{j-1}}^{y_j} M_{ij}dy}_{=\,M_{ij}\Delta y_j}$$

$$\Big\Downarrow \sum_{j=1}^m$$

$$\sum_{j=1}^m m_{ij}\Delta y_j \le \sum_{j=1}^m \int_{y_{j-1}}^{y_j} f(x,y)dy \le \sum_{j=1}^m M_{ij}\Delta y_j \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{m} m_{ij}\Delta y_j \le \underbrace{\int_c^d f(x,y)dy}_{F(x)} \le \sum_{j=1}^{m} M_{ij}\Delta y_j$$

$$M_i = \sup_{[x_{i-1},x_i]} F(x)\ ;\ m_i = \inf_{[x_{i-1},x_i]} F(x_i)$$

$$\Rightarrow \sum_{j=1}^{m} m_{ij}\Delta y_j \le m_i \le M_i \le \sum_{j=1}^{m} M_{ij}\Delta y_j \quad \Big| \cdot \Delta x_i$$

$$\forall x \in [x_{i-1}, x_i]$$

$$\forall i = \overline{1,n}$$

$$\Rightarrow \sum_{j=1}^{m} m_{ij}\Delta x_i \Delta y_j \le m_i\Delta x_i \le M_i\Delta x_i \le \sum_{j=1}^{m} M_{ij}\Delta x_i\Delta y_j$$

$$\Big\Downarrow \sum_{i=1}^{n}$$

$$\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\Delta x_i\Delta y_j}_{s_T(f)} \le \underbrace{\sum_{i=1}^{n} m_i\Delta x_i}_{s_\tau(F)} \le \underbrace{\sum_{j=1}^{n} M_i\Delta x_i}_{S_\tau(F)} \le \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}\Delta x_i\Delta y_j}_{S_T(f)}$$

$$\Rightarrow s_T(f) \le s_\tau(F) \le S_\tau(F) \le S_T(f)$$

$f(x,y)$ - интегруема върху $\Pi \Rightarrow \forall \varepsilon > 0\ \exists T: S_T(f) - s_T(f) < \varepsilon$

$$\Rightarrow 0 \le S_\tau(F) - s_\tau(F) \le S_T(f) - s_T(f) < \varepsilon$$

$\Rightarrow$ От силния критерий $\Rightarrow$ $F(x)$ е интегруема. върху $[a,b]$

$$\sum_{j=1}^{m} m_{ij}\Delta y_j \le F(x) \le \sum_{j=1}^{m} M_{ij}\Delta y_j$$

$$\Rightarrow \int_{x_{i-1}}^{x_i}\sum_{j=1}^{m} m_{ij}\Delta y_j\,dx \le \int_{x_{i-1}}^{x_i} F(x)dx \le \int_{x_{i-1}}^{x_i}\sum_{j=1}^{m} M_{ij}\Delta y_j\,dx$$

$$\Rightarrow \sum_{j=1}^{m} m_{ij}\Delta x_i\Delta y_j \le \int_{x_{i-1}}^{x_i} F(x)dx \le \sum_{j=1}^{m} M_{ij}\Delta x_i\Delta y_j$$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\,\Delta x_i\,\Delta y_j \le \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} F(x)\,dx \le \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}\,\Delta x_i\,\Delta y_j$$

$$\Rightarrow s_T(f) \le \int_a^b F(x)\,dx \le S_T(f) \quad (\mathrm{I})$$

(II). $s_T(f) \le \iint_{\Pi} f(x,y)\,dx\,dy \le S_T(f)$ — $f(x,y)$ е интегруема в Π.

$$-\big(S_T(f) - s_T(f)\big) \le \iint_{\Pi} f(x,y)\,dx\,dy - \int_a^b F(x)\,dx \le S_T(f) - s_T(f)$$

$$\Rightarrow \underbrace{\left| \iint_{\Pi} f(x,y)\,dx\,dy - \int_a^b F(x)\,dx \right|}_{\text{константа}} \le \underbrace{S_T(f) - s_T(f)}_{\text{променлива}} < \varepsilon \;;\; \forall \varepsilon > 0$$

$$\Rightarrow \iint_{\Pi} f(x,y)\,dx\,dy - \int_a^b F(x)\,dx = 0 \Rightarrow \iint_{\Pi} f(x,y)\,dx\,dy = \int_a^b F(x)\,dx$$

Следствие 1: Нека $f(x,y)$ е интегруема върху правоъгълника $\Pi = [a,b]\times[c,d]$ и

1) $\forall x \in [a,b] \;\; \exists \int_c^d f(x,y)\,dy$

2) $\forall y \in [c,d] \;\; \exists \int_a^b f(x,y)\,dx$

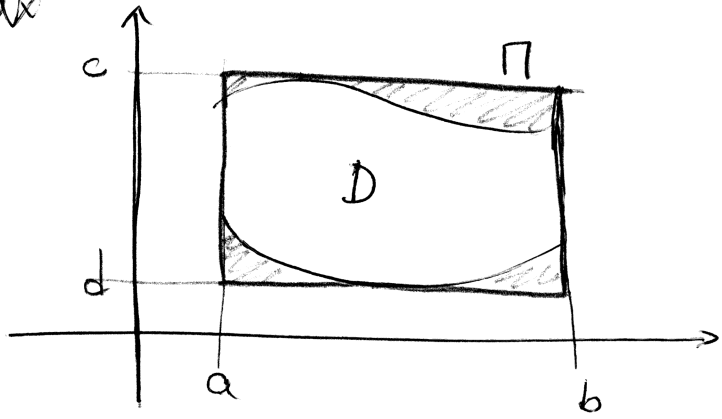
$$\Rightarrow \iint_{\Pi} f(x,y)\,dx\,dy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

Следствие 2: Ако $f(x,y)$ е непрекъсната върху правоъгълника $\Pi = [a,b]\times[c,d]$, то

$$\iint_{\Pi} f(x,y)\,dx\,dy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

$\boxed{T}$ Нека функцията $f(x,y)$ е интегруема върху криволинейния трапец $D: \begin{cases} x\in[a,b] \\ \varphi(x)\le y\le \psi(x) \end{cases}$ където $\varphi(x)$ и $\psi(x)$ са непрекъснати върху $[a,b]$. Тогава, ако за $\forall x\in[a,b]\ \exists \int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy$, то

$$\iint\limits_D f(x,y)\,dy\,dx = \int_a^b\left(\int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy\right)dx$$



Доказателство:

Нека $c=\inf\limits_{[a,b]}\varphi(x)$; $d=\sup\limits_{[a,b]}\psi(x)$

$\Pi=[a,b]\times[c,d]$

$$F(x,y)=\begin{cases} f(x,y) & (x,y)\in D \\ 0 & (x,y)\notin D\end{cases}$$

$$\forall x\in[a,b]\ \exists \int_c^d F(x,y)dy = \int_c^{\varphi} F(x,y)dy + \int\limits_{\varphi(x)}^{\psi(x)} F(x,y)dy + \int\limits_{\psi(x)}^{d} F(x,y)dy =$$

$$= \int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy$$

$$(*)\ \iint\limits_{\Pi} F(x,y)\,dx\,dy = \int_a^b\left(\int_c^d F(x,y)dy\right)dx$$

$$(1)\ \iint\limits_{D\cup(\Pi\setminus D)} F(x,y)\,dx\,dy = \iint\limits_D F(x,y)\,dx\,dy + \underbrace{\iint\limits_{\Pi\setminus D} F(x,y)\,dx\,dy}_{0} =$$

$$= \iint\limits_D f(x,y)\,dx\,dy$$

$$(2)\ \int_a^b\left(\int_c^d F(x,y)dy\right)dx = \int_a^b\left(\int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy\right)dx \Rightarrow$$

$\Rightarrow$ От $(*),(1),(2) \Rightarrow$ $$\iint\limits_D f(x,y)\,dx\,dy = \int_a^b\left(\int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy\right)dx$$