16 | Формула на Тейлор за функция на две променливи

$\boxed{T}$ Нека $f(x,y)$ е дефинирана върху $B_\delta(x_0,y_0)$ и има непрекъснати частни производни до $(n+1)$-ви ред.

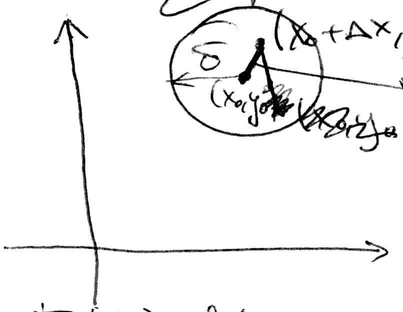
$$\Rightarrow \forall (x,y)=(x_0+\Delta x, y_0+\Delta y)\in B_\delta(x_0,y_0),\ \exists\, 0<\theta<1:$$

$$f(x,y)=f(x_0+\Delta x, y_0+\Delta y)=f(x_0,y_0)+\frac{1}{1!}\left(\frac{\partial f(x_0,y_0)}{\partial x}\Delta x+\frac{\partial f(x_0,y_0)}{\partial y}\Delta y\right)+$$

$$+\frac{1}{2!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{2\}}f(x_0,y_0)+\dots$$

$$\dots+\frac{1}{n!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{n\}}f(x_0,y_0)+\frac{1}{(n+1)!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{n+1\}}f(x_0+\theta\Delta x, y_0+\theta\Delta y)$$

Доказателство:



$\forall t\in[0,1]:(x_0+t\Delta x, y_0+t\Delta y)\in[(x_0,y_0),(x,y)]$

$F(t)=f(x_0+t\Delta x, y_0+t\Delta y)$ – дефинирана върху $[0,1]$

$x(t)=x_0+t\Delta x;\ y(t)=y_0+t\Delta y$

$\Rightarrow F(t)=f(x(t),y(t)) \to$ има производни до $(n+1)$-ви ред $\Rightarrow$

$\Rightarrow$ Можем да приложим формула на Тейлор за F.

$\Rightarrow t\in[0,1],\ \exists\, 0<\theta<1:$

$$F(t)=F(0)+\frac{1}{1!}F'(0).t+\frac{F''(0)}{2!}t^2+\dots+\frac{1}{n!}F^{(n)}(0)t^n+\frac{F^{(n+1)}(\theta t)}{(n+1)!}.t^{n+1}$$

$$F(0)=f(x_0+0\Delta x, y_0+0\Delta y)=f(x_0,y_0)$$

$$F'(t)=\left[f(x_0+t\Delta x, y_0+t\Delta y)\right]'=\frac{\partial f(x_0+t\Delta x, y_0+t\Delta y)}{\partial x}.\frac{dx(t)}{dt}+$$

$$+\frac{\partial f(x_0+t\Delta x, y_0+t\Delta y)}{\partial y}.\frac{dy(t)}{dt}=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y$$

$$F'(0)=\frac{\partial f(x_0,y_0)}{\partial x}\Delta x+\frac{\partial f(x_0,y_0)}{\partial y}\Delta y=\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{1\}}f(x_0,y_0)$$

$$F''(t)=(F'(t))'=\left(\frac{\partial f(x_0+t\Delta x,y_0+t\Delta y)}{\partial x}\Delta x+\frac{\partial f(x_0+t\Delta x,y_0+t\Delta y)}{\partial y}\Delta y\right)'=$$

$$=\frac{\partial^2 f(x_0+t\Delta x,y_0+t\Delta y)}{\partial x^2}\Delta x^2+\frac{2\partial^2 f(x_0+t\Delta x,y_0+t\Delta y)}{\partial x\partial y}\Delta x\Delta y+\frac{\partial^2 f(x_0+\Delta x,y_0+\Delta y)}{\partial y^2}\Delta y^2$$

$$F''(0)=\frac{\partial^2 f(x_0,y_0)}{\partial x^2}\Delta x^2+2\frac{\partial^2 f(x_0,y_0)}{\partial x\partial y}\Delta x\Delta y+\frac{\partial^2 f(x_0,y_0)}{\partial y^2}\Delta y^2=\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{2\}}f(x_0,y_0)$$

по индукция $\longrightarrow \forall k\le n$

$$F^{(k)}(0)=\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{k\}}f(x_0,y_0)$$

$$F^{(n+1)}(\theta t)=\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{n+1\}}f(x_0+\theta t\Delta x,y_0+\theta t\Delta y)$$

$$F(t)=f(x_0,y_0)+\frac{1}{1!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{1\}}f(x_0,y_0).t+\dots+\frac{1}{n!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{n\}}f(x_0,y_0).t^n+$$

$$+\frac{1}{(n+1)!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{n+1\}}f(x_0+\theta t\Delta x,y_0+\theta t\Delta y)$$

$$F(1)=f(x_0+\Delta x,y_0+\Delta y)$$

$$\|$$

$$\underbrace{f(x_0,y_0)+\sum_{k=1}^{n}\frac{1}{k!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{k\}}f(x_0,y_0)}_{\text{полином на Тейлор за 2 независими величини}}+\frac{1}{(n+1)!}\left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)^{\{n+1\}}f(x_0+\theta\Delta x,y_0+\theta\Delta y)$$