Тт| Ако $\int_a^{+\infty} f(x)\,dx$ и $\int_a^{+\infty} g(x)\,dx$ са абс. сх. $\Rightarrow$

1) $\int_a^{+\infty} [f(x)+g(x)]\,dx$ е абс. сх.

2) $\int_a^{+\infty} \lambda f(x)\,dx$ е абс. сх. $(\forall \lambda \in \mathbb{R})$

Д-во:

1) $\int_a^{+\infty} f(x)\,dx + \int_a^{+\infty} g(x)\,dx$ са абс. сх. $\Rightarrow$ са сх.

$\int_a^{+\infty} |f(x)|\,dx$ и $\int_a^{+\infty} |g(x)|\,dx \Rightarrow \int_a^{+\infty} (|f(x)|+|g(x)|)\,dx$

т.к. $0 \le |f(x)+g(x)| \le |f(x)| + |g(x)|$ $\forall x \in [a,+\infty)$ и

т.к. $\int_a^{+\infty} (|f(x)|+|g(x)|)\,dx$ е сх $\xrightarrow[\text{сравн.}]{\text{п-п. за}}$ $\int_a^{+\infty} |f(x)+g(x)|\,dx$ е сх

$\Rightarrow \int_a^{+\infty} (f(x)+g(x))\,dx$ е сх. абс.

2) $\int_a^{+\infty} f(x)\,dx$ е абс. сх. $\Rightarrow \int_a^{+\infty} |f(x)|\,dx$ е сх $\Rightarrow$

$\int_a^{+\infty} |\lambda||f(x)|\,dx = \int_a^{+\infty} |\lambda f(x)|\,dx$ е сх $\Rightarrow \int_a^{+\infty} \lambda f(x)\,dx$ е абс. сх.

13. Безкрайни числови редове – сходимост, свойства.

1) 1, 2, 3, 4 ... $\quad 1+2+3+4+5$ $\quad S_1 = 1$

2) 1, −1, 1, −1, ... $\quad 1+(-1)+1+(-1)+\dots = 0$ $\quad S_2 = 0$ – разх. $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1}$ е разх

3) 1, 2, 3, 0, 0, 0, ... $\quad 1+2+3+0+0+\dots = 6$ $\quad S_3 = 6$

4) $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots$ $\quad S_n = 1+\frac{1}{2}+\dots+\frac{1}{2^n}+\dots = 2\left(1-\frac{1}{2^n}\right) = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right) = 2 \to$ сума а на б.ч.р. $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$

Деф| Нека $a_1, a_2, \dots, a_n, \dots$ е б.ч.р

формална сума: $a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$ $\quad$ a_n – общ/неин член на б.ч. ред.

• безкраен числов ред
Нека имаме б.ч. ред $\sum_{n=1}^{\infty} a_n$
$\forall n \in \mathbb{N}: S_n = a_1 + a_2 + \dots + a_n$ $\to$ n-та парциална сума на б.ч.ред $\sum_{n=1}^{\infty} a_n$
$S_1, S_2, \dots, S_n, \dots \to S$
$S = \sum_{n=1}^{\infty} a_n$ – сх – сума на б.ч. ред

* $S_n = 1+2+\dots+n = \frac{(n+1)n}{2}$

$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{n(n+1)}{2} = +\infty \Rightarrow \sum_{n=1}^{\infty} n$ е разх.

Пример: 1) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} + \dots$

$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$

$\frac{1}{k(k+1)} = \frac{k+1-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$\lim\limits_{n\to+\infty} S_n = \lim\limits_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$

$\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

2) $\sum\limits_{n=0}^{\infty} q^n, (q\in R)$

$(q\neq\pm1)\ S_n = 1+q+q^2+\dots+q^{n-1} = \frac{1-q^n}{1-q}$

$\Rightarrow \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \frac{1-q^2}{1-q} = \frac{1}{1-q}\left(1-\lim\limits_{n\to\infty} q^n\right) = \begin{cases}\frac{1}{1-q}, & |q|<1\\ \text{разх}, & |q|\ge 1\end{cases}$

$\lim\limits_{n\to\infty} q^n = \begin{cases} 0, & |q|<1 \\ \infty, & |q|>1\end{cases}$

$q=1 \Rightarrow \sum\limits_{n=0}^{\infty} (-1)^{n-1} = 1+(-1)+1+(-1)+\dots \to$ разх.

$q=-1 \Rightarrow \sum\limits_{n=0}^{\infty} 1^n = \sum\limits_{n=1}^{\infty} 1$ – разх

$S_n = \underbrace{1+1+\dots+1}_{n} = n \to +\infty$

$\sum\limits_{n=0}^{\infty} q^n = \begin{cases}\frac{1}{1-q}, & \text{ако } |q|<1\\ \text{разх}, & \text{ако } |q|\ge 1\end{cases}$

НУ-ЯП | Ако д.т. ред $\sum\limits_{n=1}^{\infty} a_n$ е сх $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$ ($\not\Leftarrow$)

Нека $\sum\limits_{n=1}^{\infty} a_n$ е сх. $\Rightarrow \{S_n\}_{n=1}^{\infty}$ е сх., където $S_n=\sum\limits_{k=1}^{n} a_k$ ($\forall n\in N$)

$\Rightarrow \exists S=\lim\limits_{n\to\infty} S_n$ $\qquad a_n = S_n - S_{n-1}$

$S_n = \underbrace{a_1+a_2+\dots+a_{n-1}}_{S_{n-1}} + a_n$

$\Rightarrow \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (S_n-S_{n-1}) = \lim\limits_{n\to\infty} S_n - \lim\limits_{n\to\infty} S_n = S-S=0$

Пример | $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt n}$, $\forall k<n \Rightarrow \sqrt k<\sqrt n \Rightarrow \frac{1}{\sqrt k} > \frac{1}{\sqrt n}$

$S_n = \frac{1}{\sqrt1}+\frac{1}{\sqrt2}+\dots+\frac{1}{\sqrt n} > \frac{1}{\sqrt n}+\frac{1}{\sqrt n}+\dots+\frac{1}{\sqrt n}+\frac{1}{\sqrt n} = n\cdot\frac{1}{\sqrt n}=\sqrt n$

$\forall n\in N$ $\underset{+\infty}{\underset{\downarrow}{S_n}} > \underset{+\infty}{\underset{\downarrow}{\sqrt n}}$ $n\to\infty$ $\Rightarrow \{S_n\}^{\infty}$ е разх $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt n}$ е разх.

$\Rightarrow a_n = \frac{1}{\sqrt n} \xrightarrow[n\to+\infty]{} 0$

- $\sum\limits_{n=1}^{\infty} n$, $a_n=n\not\to 0 \Rightarrow$ разх.
- $\sum\limits_{n=1}^{\infty} (-1)^{n-1}$, $a_n=(-1)^{n-1}\not\to 0 \Rightarrow$ разх.

Свойство 1 | Ако д.т. редове $\sum\limits_{n=1}^{\infty} a_n$ и $\sum\limits_{n=1}^{\infty} b_n$ са сх $\Rightarrow$

1) $\sum\limits_{n=1}^{\infty} \lambda a_n$ ($\lambda\in R$) е сх. и $\sum\limits_{n=1}^{\infty} \lambda a_n = \lambda\sum\limits_{n=1}^{\infty} a_n$

2) $\sum\limits_{n=1}^{\infty} (a_n+b_n)$ е сх. и $\sum\limits_{n=1}^{\infty}(a_n+b_n)=\sum\limits_{n=1}^{\infty} a_n+\sum\limits_{n=1}^{\infty} b_n$

Д-во:

1) Нека $\sum\limits_{n=1}^{\infty} a_n = S$, и $S_n=\sum\limits_{k=1}^{n} a_k$

обозначаване: $S_n' = \sum_{n=1} \lambda a_k$ – n-та парц. една на бт. ред $\sum_{n=1} \lambda a_n$

$S_n' = \sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k = \lambda S_n$

$\lim_{n\to\infty} S_n' = \lim_{n\to\infty} \lambda S_n = \lambda \lim_{n\to\infty} S_n = \lambda S \Rightarrow$

i) $\sum_{n=1}^{\infty} \lambda a_n$ е сх.

ii) $\sum_{n=1}^{\infty} \lambda a_n = \lambda S = \lambda \sum_{n=1}^{\infty} a_n$

2) Нека $\sum_{n=1}^{\infty} a_n = S$, $\sum_{n=1}^{\infty} b_n = S_1$, $\hat{S}_n = \sum_{k=1}^{n} a_k$, $S_n' = \sum_{n=1}^{n} b_k$

Разгл. $\sum_{n=1}^{\infty} (a_n + b_n)$ и нека $S_n = \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = S_n + \hat{S}_n$

$\lim_{n\to\infty} \hat{S}_n = \lim_{n\to\infty} (S_n + S_n') = \lim_{n\to\infty} S_n + \lim_{n\to\infty} S_n' = S + S_1 \Rightarrow$

i) $\sum_{n=1}^{\infty} (a_n + b_n)$ е сх.

ii) $\sum_{n=1}^{\infty} (a_n + b_n) = S + S_1 = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

<u>Свойство 2</u>| Б. т. ред $\sum_{n=1}^{\infty} a_n$ е сходящ $\Leftrightarrow \sum_{n=N+1}^{\infty} a_n$, к. $N \in \mathbb{N}$

<u>Д-во</u>

Нека $S_n = \sum_{k=1}^{n} a_k$

$\hat{S}_m = \sum_{k=N+1}^{m} a_k \quad (k_m = N+1)$

$\hat{S}_m = \sum_{k=N+1}^{m} a_k = \sum_{k=1}^{m} a_k - \sum_{k=1}^{N} a_k = S_m - S_N$

$\hat{S}_m = S_m - S_N \Rightarrow$ сх. не се нарушава

<u>Свойство 3</u>| Ако редът $\sum_{n=1}^{\infty} a_n$ е сх. и $S = \sum_{n=1}^{\infty} a_n \Rightarrow$ е сх. и редът $\sum_{m=1}^{\infty} b_m$, където $b_m = \sum_{k=n_m+1}^{n_{m+1}} a_k$, $n_0 = 0 < n_1 < n_2 < \dots < n_m < \dots$ $(n_m \in \mathbb{N})$

и $\sum_{m=1}^{\infty} b_m = S$

<u>Пример</u>| $\sum_{n=1}^{\infty} (-1)^{n-1} = \underbrace{1 + (-1)}_{b_1} + \underbrace{1 + (-1)}_{b_2} + \dots$

$\sum_{m=1}^{\infty} b_m = \sum_{n=1}^{\infty} ((-1)^{n-1} + (-1)^n) = \sum_{n=1}^{\infty} 0$ – сх. $\Rightarrow$ обр. тв. не е вярно!

<u>Д-во:</u>

Нека $S_n = \sum_{k=1}^{n} a_k$ и $S = \lim_{n\to\infty} S_n$, т.е. $\sum_{n=1}^{\infty} a_n = S$

Нека $\hat{S}_m = \sum_{k=1}^{m} b_k = \sum_{k=1}^{m} \sum_{k=n_{k-1}+1}^{n_k} a_{n_k} = \sum_{l=1}^{n_m} a_l = S_{n_m}$

$\left.\begin{array}{l} S_{n_1}, S_{n_2}, \dots, S_{n_m} \dots \text{ е подред} \\ S_1, S_2, \dots, S_n, \dots \to S \end{array}\right\} S_{n_m} \to S$, т.е.

$\hat{S}_m \xrightarrow[m\to\infty]{} S \Rightarrow \sum_{m=1}^{\infty} b_m$ е сх. и $\sum_{m=1}^{\infty} b_m = S$

<u>Т (Критерий на Коши)</u>| Б. т. ред $\sum_{n=1}^{\infty} a_n$ е сх $\Leftrightarrow \forall \varepsilon > 0, \exists N = N(\varepsilon)$

$\forall n > N, p \in \mathbb{N} \Rightarrow |a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon$

Д-во: кр. на К. (с Стадия)

$\sum_{n=1}^{\infty} a_n$ е сх $\Leftrightarrow$ сх $\{S_n\}_{n=1}^{\infty} \Leftrightarrow \forall \varepsilon > 0, \exists N = N(\varepsilon): \forall n > N, p \in \mathbb{N} \Rightarrow$

$|S_{n+p} - S_n| < \varepsilon$

$|S_{n+p} - S_n| = |\sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k| = |\sum_{k=n+1}^{n+p} a_k| = |a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon$

Пример | $\sum_{n=1}^{\infty} \frac{1}{n}$, $a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$ разх.

Кр. на Коши (отрицанието му) | $\sum_{n=1}^{\infty} a_n$ е разх $\Leftrightarrow \exists \varepsilon_0 > 0, \forall N,$

$\exists n_0 > N, p_0 \in \mathbb{N}: |a_{n_0+1} + a_{n_0+2} + \dots + a_{n_0+p_0}| \geq \varepsilon_0$

- $\sum_{n=1}^{\infty} \frac{1}{n}$: $\forall N \in \mathbb{N}$, $n_0 = N$, $p_0 = N$,

$$\underbrace{\frac{1}{N+1}}_{a_{n_0+1}} + \underbrace{\frac{1}{N+2}}_{a_{n_0+2}} + \underbrace{\frac{1}{N+3}}_{a_{N+3}} + \dots + \underbrace{\frac{1}{N+N}}_{a_{N+N}} > \underbrace{\frac{1}{2N} + \dots + \frac{1}{2N}}_{N} = N \cdot \frac{1}{2N} = \underset{\varepsilon_0}{\frac{1}{2}}$$

$1 \leq k \leq N$

$\frac{1}{N+k} > \frac{1}{2N} \xRightarrow{\text{кр. Коши}} \sum_{n=1}^{\infty} \frac{1}{n}$ е разх.

хармоничен ред!

14. Редове с неотрицателни членове, признак за сравнение. Крит. на Даламбер. Крит. на Коши. Интегрален критерий на Коши

Д | Редът от вида $\sum_{n=1}^{\infty} a_n$, където $a_n \geq 0$ ($\forall n \in \mathbb{N}$) се нарича ред с неотрицателни членове.

Т | Редът с неотр. членове $\sum_{n=1}^{\infty} a_n$ е сх $\Leftrightarrow \{S_n\}_{n=1}^{\infty}$ е огр.

Д-во:

Нека $S_n = \sum_{k=1}^{n} a_k \Rightarrow S_{n+1} - S_n = a_{n+1} \geq 0 \Rightarrow$

$S_{n+1} \geq S_n$, т.е. $\{S_n\}_{n=1}^{\infty}$ е мон. $\uparrow$ ред.

Тогава д.т.р. $\sum_{n=1}^{\infty} a_n$ е сх. $\Leftrightarrow \{S_n\}_{n=1}^{\infty}$ е сх. $\Leftrightarrow \{S_n\}_{n=1}^{\infty}$ е огр.

Т | (Признак за сравн.) Нека д.т.р. $\sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ удовл. усл.

$0 \leq a_n \leq b_n$ ($\forall n \in \mathbb{N}$). Тогава:

1) ако $\sum_{n=1}^{\infty} b_n$ е сх $\Rightarrow \sum_{n=1}^{\infty} a_n$ е сх